A RESTRICTED CHARGED FOUR-BODY PROBLEM

J. CASASAYAS*

Departament de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Universitat de Barcelona, Barcelona, Spain

and

A. NUNES**

Departament de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain

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Abstract. We consider a restricted charged four body problem which reduces to a two degrees of freedom Hamiltonian system, and prove the existence of infinite symmetric periodic orbits with arbitrarily large extremal period. Also, it is shown that an appropriate restriction of a Poincaré map of the system is conjugate to the shift homeomorphism on a certain symbolic alphabet.

1. Introduction and Equations of Motion

Consider two pairs of identical particles of masses $m_1 = m_2 = m, m_3 = m_4 = M$, and charges $e_1 = e_2 = E, e_3 = e_4 = e$, in the following physical setting. The four particles lie in the plane and their relative positions are such that each particle forms an isosceles triangle with the pair of particles which differ from it (see Figure 1). The initial velocities are also symmetric: particles 1 and 2 (resp. 3 and 4) are given initial velocities along the straight line through them and symmetrical with respect to the straight line through particles 3 and 4 (resp. 1 and 2).

Denote by q_i the position vector of the ith particle. From Newton's law, the equations of motion are, taking into account electrical as well as gravitational interactions and choosing appropriate charge units.

$$\begin{aligned}
\ddot{q}_1 &= \frac{Gm^2 - E^2}{m}\frac{q_2 - q_1}{|q_2 - q_1|^3} + \frac{GMm - eE}{m}\frac{q_3 - q_1}{|q_3 - q_1|^3} + \frac{GMm - eE}{m}\frac{q_4 - q_1}{|q_4 - q_1|^3},\\
\ddot{q}_2 &= \frac{Gm^2 - E^2}{m}\frac{q_1 - q_2}{|q_2 - q_1|^3} + \frac{GMm - eE}{m}\frac{q_3 - q_2}{|q_3 - q_2|^3} + \frac{GMm - eE}{m}\frac{q_4 - q_2}{|q_4 - q_2|^3},\\
\ddot{q}_3 &= \frac{GM^2 - e^2}{M}\frac{q_4 - q_3}{|q_4 - q_3|^3} + \frac{GMm - eE}{M}\frac{q_1 - q_3}{|q_1 - q_3|^3} + \frac{GMm - eE}{M}\frac{q_2 - q_3}{|q_2 - q_3|^3},\\
\ddot{q}_4 &= \frac{GM^2 - e^2}{M}\frac{q_3 - q_4}{|q_4 - q_3|^3} + \frac{GMm - eE}{M}\frac{q_1 - q_4}{|q_1 - q_4|^3} + \frac{GMm - eE}{M}\frac{q_2 - q_4}{|q_2 - q_4|^3}.
\end{aligned} \tag{1}$$

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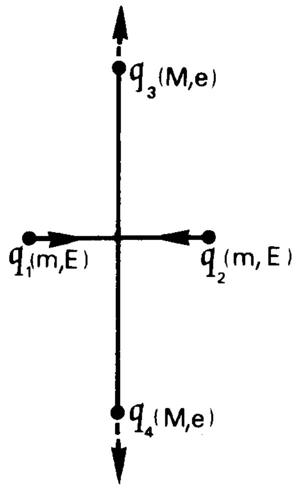

Fig. 1.

It is easy to check that Equations (1) preserve the symmetry of the initial conditions. Therefore, this physical problem has two degrees of freedom, and we may take as independent coordinates, for instance, the oriented half distances between the particles,

$$x = \frac{q_2 - q_1}{2},$$

$$y = \frac{q_3 - q_4}{2}.$$

In these coordinates, Equations (1) become,

$$\begin{aligned} \ddot{x} &= -\frac{Gm^2 - E^2}{4m}\frac{x}{|x|^3} - \frac{2(GMm - eE)}{m}\frac{x}{(x^2 + y^2)^{3/2}}, \\ \ddot{y} &= -\frac{GM^2 - e^2}{4M}\frac{y}{|y|^3} - \frac{2(GMm - eE)}{M}\frac{y}{(x^2 + y^2)^{3/2}}. \end{aligned} \tag{2}$$

We shall now introduce two assumptions on the relative values of the masses and charges of the particles. First, we shall suppose that $GM^2 - e^2 = 0$, that is, that the gravitational and electrostatic interactions between particles 3 and 4 cancel each other. Second, we shall assume that the charges e and E have opposite signs, so that the coefficients of the second terms in Equations (2) are always positive.

With these assumptions and after rescaling time, Equations (2) become:

$$\begin{aligned} \ddot{x} &= \frac{-x}{(x^2 + y^2)^{3/2}} - \frac{Ax}{|x|^3}, \\ \ddot{y} &= -\frac{By}{(x^2 + y^2)^{3/2}}, \end{aligned} \tag{3}$$

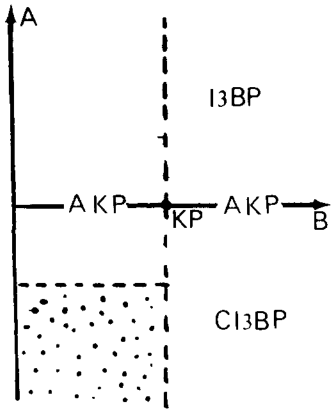

Fig. 2.

where

$$A = \frac{Gm^2 - E^2}{8(GMm - eE)}, \qquad B = \frac{m}{M}.$$

When $B > 1$ Equations (3) coincide with the equations of well-known restricted three body problems: the isosceles three body problem (I3BP) corresponds to $A > 0$ (see Simó-Martínez 1988), and the charged isosceles three body problem (CI3BP) corresponds to $A < 0$ (see Atela). When $A = 0$, $B \neq 1$, Equations (3) are those of the anisotropic Kepler problem (AKP) (see Casasayas–Llibre, 1984 Devaney, 1978). Finally, for $A = 0$, $B = 1$, (3) reduces to the Kepler problem (KP). The parameter space is shown in Figure 2.

Thus, the four body symmetrical setting proposed here gives a new physical interpretation of the I3BP, CI3BP and the AKP, as well as a model system for the strip in parameter space corresponding to $0 < B < 1$.

In this paper, we study system (3) for $B \in (0, 1)$, $A \in (-1, -1/4)$. In terms of the parameters m, M, e and E, the conditions imposed until now are equivalent to:

$$m < M, \quad e = \sqrt{G}M, \quad -E \in (\sqrt{G}(2M + m), \quad \sqrt{G}(8M + m)),$$

or

$$m < M, \quad e = -\sqrt{G}M, \quad E \in (\sqrt{G}(2M + m), \quad \sqrt{G}(8M + m)).$$

Rescaling time again and writing (3) as a first order system, we obtain:

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{p}_x &= \frac{x}{|x|^3} - \frac{ax}{(x^2 + y^2)^{3/2}}, \\ \dot{y} &= bp_y, \\ \dot{p}_y &= -\frac{ay}{(x^2 + y^2)^{3/2}}, \end{aligned} \tag{4}$$

with $a \in (1, 4)$ and $b \in (0, 1)$. Equations (4) have Hamiltonian form, with Hamiltonian function $H(x, y, p_x, p_y)\colon \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$,

$$H(x, y, p_x, p_y) = \frac{p_x^2}{2} + \frac{bp_y^2}{2} + U(x, y),\ U(x, y) = \frac{1}{|x|} - \frac{a}{(x^2 + y^2)^{1/2}}. \tag{5}$$

Note that we consider only positive values of the x coordinate, since the system is symmetric with respect to $x = 0$.

Note also that the only singularity of the system corresponds to the total collision configuration $x = y = 0$. In fact, triple collisions are forbidden by the symmetry of the particle configurations we are studying. As to double collisions, these symmetry restrictions allow only the ones corresponding to $x = 0, y \neq 0$, and $x \neq 0, y = 0$. That is, a collision between particles 1 and 2, or a collision between particles 3 and 4. The first possibility is inaccessible for finite energies, because particles 1 and 2 repel each other. The second possibility does not correspond to a singularity of the system, because the gravitational and electrical field created by particles 3 and 4 cancel each other and so these particles pass through each other naturally.

It is easy to check that for negative energy, $H(x, y, p_x, p_y) = h < 0$, Hill's region is as shown in Figure 3.

We shall study the flow of system (4) and (5) on the negative energy levels, using McGehee's transformation to magnify the singularity and to introduce the total collision manifold.

The main results obtained are stated precisely in Sections 4 (Theorem 11) and 5 (Theorem 22). Theorem 11 concerns the existence of an infinite number of symmetric periodic orbits of the system, and says that its extremal period, that is, the number of extrema of the r coordinate along the orbit, may be arbitrarily large. Theorem 22 says that an appropriate restriction of a Poincaré map is conjugate to the shift homeomorphism on a certain symbolic alphabet. Furthermore, it gives an interpretation of the symbolic sequences in terms of the qualitative behaviour of the orbits.

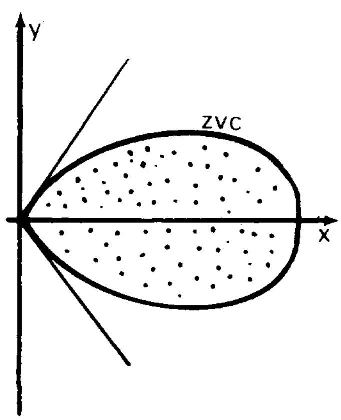

Fig. 3.

2. The Flow on the Total Collision Manifold Λ

Consider McGehee's transformation (see McGehee, 1974):

$$r = (x^2 + b^{-1}y^2)^{1/2},$$

$$\theta = \arctan\left(\frac{b^{-1/2}y}{x}\right),$$

$$v = r^{-1/2}(xp_x + yp_y),$$

$$u = r^{-1/2}(b^{1/2}xp_y - b^{-1/2}yp_x),$$

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = r^{3/2}.$$

The geometrical meaning of the new variables is essentially the following: r and θ are polar type coordinates on the x, y plane, and v, u and τ are, respectively, the rescaled radial velocity, angular velocity and time. Clearly, total quadruple collision corresponds to $r = 0$.

In the new variables, Equations (4) become,

$$\begin{aligned} \bar{r} &= rv, \\ \bar{\theta} &= u, \\ \bar{v} &= \frac{v^2}{2} + u^2 + V(\theta), \\ \bar{u} &= -\frac{uv}{2} - V'(\theta), \end{aligned} \tag{7}$$

where the overbar denotes the derivative with respect to τ,

$$V(\theta) = \frac{-a}{(\cos^2\theta + b\sin^2\theta)^{1/2}} + \frac{1}{\cos\theta},$$

and

$$V'(\theta) = \frac{\mathrm{d}V}{\mathrm{d}\theta}.$$

The energy relation (5) now reads:

$$rh = \frac{u^2 + v^2}{2} + V(\theta). \tag{8}$$

By (8), the total collision manifold Λ on the boundary of every energy level is

$$\Lambda = \{(\theta, v, u) \in S^1 \times \mathbb{R}^2 : u^2 + v^2 = -2V(\theta)\}. \tag{9}$$

From (7), the flow is analytical and invariant on Λ. The description of the flow on Λ and, by analyticity, on its neighbourhood (r small), is the purpose of this section.

Let us start with a lemma:

LEMMA 1. *Consider the function*

$$f_{a,b}(\theta) = -2V(\theta),\ V(\theta) = \frac{-a}{(\cos^2\theta + b\sin^2\theta)^{1/2}} + \frac{1}{\cos\theta},\ a \in (1, 4),\ b \in (0, 1).$$

The following assertions hold:

(*a*) $f_{a,b}(\theta) = f_{a,b}(-\theta), \quad \theta \in [-\pi/2, \pi/2]$.

(*b*) $f_{a,b}(\theta) \geqslant 0$ *iff* $\theta \in [-\arccos c, \arccos c], \quad c = b^{1/2}(a^2 + b - 1)^{-1/2}$.

(*c*) *If* $a(1-b) < 1$, *then* $f'_{a,b}(\theta) = 0$ *iff* $\theta = 0$ *and* $f''_{a,b}(\theta = 0) < 0$.

(*d*) *If* $a(1-b) > 1$, *then* $f'_{a,b}(\theta) = 0$ *iff* $\theta \in \{0, \arccos d, -\arccos d\}$, *where*

$$d = b^{1/2}(b - 1 + (a(1-b)^{2/3}))^{-1/2}.$$

Moreover,

$$f''_{a,b}(\theta = 0) > 0 \text{ and } f''_{a,b}(\theta = \arccos d) = f''_{a,b}(\theta = -\arccos d) < 0.$$

It is easy to check the assertions of Lemma 1 by direct computation. From Lemma 1, the collision manifold Λ is as shown in Figure 4. Note the change in Λ that occurs when $a(1-b)$ passes through the critical value $a(1-b) = 1$.

From equations (7) and (9), the flow on Λ is given by,

$$\begin{aligned} \bar{\theta} &= u, \\ \bar{v} &= \frac{u^2}{2}, \\ \bar{u} &= -\frac{uv}{2} - V'(\theta). \end{aligned} \tag{10}$$

The following proposition determines the equilibrium points of (10) on Λ, as well as their nature.

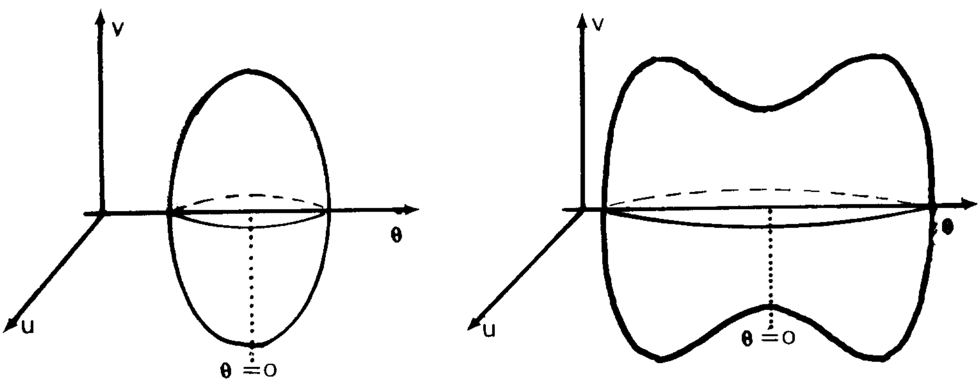

Fig. 4.

PROPOSITION 2. *(a) Suppose $a(1-b)<1$. Then the only equilibrium points on Λ are*

$$p^{\pm}(0)=(\theta=0, u=0, v=\pm\sqrt{-2V(0)}).$$

Moreover, $p^+(0)$ is a hyperbolic sink, and $p^-(0)$ is a hyperbolic source, each being spiral if $a(9-8b)<9$, non-spiral otherwise.

(b) Suppose $a(1-b)>1$. Then the only equilibrium points on Λ are

$$p^{\pm}(\theta_0)=(\theta=\theta_0,\ u=0,\ v=\pm\sqrt{-2V(\theta_0)}),\ \theta_0\in\{0,\ \arccos d,\ -\arccos d\},$$

where

$$d=b^{1/2}(b-1+(a(1-b))^{2/3})^{-1/2}.$$

The points $p^{\pm}(0)$ are hyperbolic saddles. For

$$a^2\left(1-\frac{25b}{24}\right)^3>(1-b)\ (\text{resp. } a^2\left(1-\frac{25b}{24}\right)^3\leqslant(1-b))$$

and

$$\theta_0\in\{\pm\arccos d\},$$

$p^+(\theta_0)$ is a spiral (resp. non-spiral) sink and $p^-(\theta_0)$ a spiral (resp. non-spiral) source.

A flow is called *gradient-like* when there exists a smooth function that increases along all non stationary solutions. From Lemma 1, Proposition 2 and Equations (10), it is easy to obtain the following.

PROPOSITION 3. *The flow on Λ is gradient-like with respect to v.*

Proposition 2(a), together with Proposition 3, fully determines the qualitative behaviour of the global flow on Λ for $a(1-b)<1$, which must be as stated in the next proposition.

PROPOSITION 4. *Suppose $a(1-b)<1$ and let $p^{\pm}(0)$ be the points given by Proposition 2(a). Then, the solutions on Λ tend to $p^-(0)$ (resp. $p^+(0)$) when $\tau\to-\infty$ (resp. $\tau\to+\infty$), encircling Λ infinitely many times if $a(9-8b)<9$.*

Before studying the global flow on Λ for $a(1-b)>1$, let us see that we may derive from Propositions 2 and 3 the version of the classical Euler–Lagrange theorem that applies to this case.

THEOREM 5. *(a) If $a(1-b)<1$, then any collision orbit tends to a collinear configuration.*

(b) If $a(1-b)>1$, then any collision orbit tends either to a collinear configuration or to a certain rhomboidal configuration that depends only on the value of $a(1-b)$.

Proof. From Proposition 3, there are no periodic orbits on Λ, and so any collision orbit must tend to one of the equilibrium points on Λ given by Proposition 2. On the other hand, from (6), $y/x = \sqrt{b}\tan\theta$. Therefore, if $a(1-b) < 1$, the limit of y/x along a collision orbit must be zero. Similarly, if $a(1-b) > 1$, y^2/x^2 always tends either to zero or to $(a(1-b))^{2/3} - 1$. ■

We shall now describe the qualitative behaviour of the flow on Λ when $a(1-b) > 1$. Let $p \in \{p^+(0), p^-(0)\}$ be one of the saddle points on Λ given by Proposition 2(b), and denote by $W^u_+(p)$ (resp. $W^u_-(p)$) the branch of $W^u(p)$ whose intersection with $\Lambda \cap \{u > 0\}$ (resp. $\Lambda \cap \{u < 0\}$) contains points arbitrarily close to p. In a similar way we define $W^s_+(p)$ and $W^s_-(p)$, $p \in \{p^+(0), p^-(0)\}$. Note that, according to the study of the linear approximation of (10) at p, $W^{u;s}_{+;-}$ are well defined.

We shall say that the qualitative behaviour of the flow on Λ is determined if, for every orbit $p(\tau)$, we know $\lim_{\tau\to-\infty} p(\tau)$, $\lim_{\tau\to+\infty} p(\tau)$ and how many complete turns around Λ $p(\tau)$ performs when $\tau \in (-\infty, +\infty)$.

LEMMA 6. *The behaviour of $W^u_+(p^-(0))$ fully determines the qualitative behaviour of the flow on Λ.*

Proof. Since the manifolds $W^{u;s}_{+;-}(p)$, $p \in \{p^+(0), p^-(0)\}$, define an invariant partition of Λ, and since the flow is gradient-like with respect to v (Proposition 3), the qualitative behaviour of every orbit $p(\tau)$ becomes determined once we know that of the invariant manifolds.

Now by Propositions 2 and 3, $W^s_+(p^-(0))$ (resp. $W^s_-(p^-(0))$) must accumulate on the source $p^-(\theta_0)$, $\theta_0 = -\arccos d$ (resp. $\theta_0 = \arccos d$). Moreover, Equations (10) are invariant under the symmetry transformations $T_2(\theta, v, u, \tau) = (-\theta, v, -u, \tau)$ and $T_1(\theta, v, u, \tau) = (\theta, -v, -u, -\tau)$. Then, using the symmetry T_1, $W^u_{+,-}(p^+(0))$ may be obtained from $W^s_{+,-}(p^-(0))$. Also by using T_1, $W^s_-(p^+(0))$ (resp. $W^s_+(p^+(0))$) may be obtained from $W^u_+(p^-(0))$ (resp. $W^u_-(p^-(0))$). Finally, $W^u_-(p^-(0))$ is the image by T_2 of $W^u_+(p^-(0))$. ■

Note that, since particles 3 and 4 are indistinguishable (see Section 1), it is natural to identify the points (θ, v, u) and $(-\theta, v, -u)$. Modulus this identification, and disregarding also the number of times the orbits of (10) pass through $\theta = 0$, the essential information about the flow on Λ reduces to knowing whether $W^u_+(p^-(0))$ and $W^s_+(p^+(0))$ coincide or not.

PROPOSITION 7. *Suppose $a(1-b) > 1$ and let $p^+(0)$ and $p^-(0)$ be the saddles given by Proposition 2. For every value of a there exists a countable set B such that, if $b \in (0,1)\backslash B$ then the unstable manifold of $p^-(0)$ misses the stable manifold of $p^+(0)$.*

Proof. Clearly, the symmetries of Equations (10) imply that if $W^u(p^-(0))$ and $W^s(p^+(0))$ coincide, then $W^u_+(p^-(0)) \cap \{v = 0\}$ must have zero u coordinate, i.e., it must be one of the points $(\theta = \arccos c, v = 0, u = 0)$, $(\theta = -\arccos c, v = 0, u = 0)$, where $c = (b/(a^2 - 1 + b))^{1/2}$ (see Lemma 1).

Now, for a fixed, it is easy to check that arccos c decreases when b increases, while the coordinates of $p^-(0)$ do not depend on b (see Proposition 2(b)). On the other hand, when b increases, the field on Λ defined by (10) is rotated in the negative v direction. In fact, from Equations (10) we have:

$$\operatorname{sign}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}(\theta, v, a, b)\right) = \operatorname{sign} u(\theta, v, a, b), \tag{11}$$

$$\operatorname{sign}\left(\frac{\partial}{\partial b}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}(\theta, v, a, b)\right)\right) = -\operatorname{sign} u(\theta, v, a, b). \tag{12}$$

Then, if we denote by $\theta_0(a, b)$ the θ-coordinate of $W_+^u(p^-(0)) \cap \{v = 0\}$, (11) and (12) imply that $\theta_0(a, b) - \arccos c$ is a strictly increasing function of b for every fixed value of a, and the result follows. ■

3. Singular Orbits

In this section, we shall study the set of orbits which start or end in total collision. We shall call ejection (resp. collosion) orbits those which start (resp. end) in collision.

According to Proposition 5, ejections and collisions must take place either at $\theta = 0$ (collinear configuration) or at $\theta = \pm\arccos d$ (rhomboidal configurations). We shall denote by $E(0)$, $E(+)$ and $E(-)$ (resp. $C(0)$, $C(+)$ and $C(-)$) the ejections (resp. collisions) at $\theta = 0$, $\theta = \arccos d$, and $\theta = -\arccos d$, respectively.

PROPOSITION 8. *Let $H = h < 0$ be a fixed energy level. (a) Suppose $a(1-b) < 1$, $a \in (1, 4)$, $b \in (0, 1)$. Then $E(+) = E(-) = C(+) = C(-) = \emptyset$, and both $E(0)$ and $C(0)$ are one dimensional manifolds.*

(b) Suppose $a(1-b) > 1$, $a \in (1, 4)$, $b \in (0, 1)$. Then $E(0)$ and $C(0)$ are two dimensional manifolds, while $E(+)$, $E(-)$, $C(+)$ and $C(-)$ are one dimensional manifolds.

Proof. Let $p_0 = (r_0 = 0, \theta_0, v_0, u_0 = 0)$ be one of the equilibrium points on Λ given by Proposition 2. Then, $\lambda = v_0$ is an eigenvalue of the linear approximation of (7) at p_0, with associated eigenvector $e_\lambda = (v_0, 0, h, 0)$. It is easy to check that e_λ is tangent to the energy level $H = h$. Therefore, using Proposition 2, p_0 is an hyperbolic equilibrium point of the restriction of (7) to the energy level $H = h$. Consequently, the stable and unstable sets of p_0, $W^u(p_0)$ and $W^s(p_0)$ are immersed submanifolds, and their dimension is determined by the linear approximation. ■

A solution $(r(\tau), \theta(\tau), v(\tau), u(\tau))$ of (7) in $H = h$ is called an *homothetic orbit* at θ_0 if $\theta(\tau) = \theta_0$ for every $\tau \in (-\infty, +\infty)$. We shall now study the ejection-collision orbits which are homothetic.

PROPOSITION 9. *(a) Suppose $a(1-b) < 1$, $a \in (1, 4)$, $b \in (0, 1)$. Then, there exists a homothetic orbit $\gamma(\theta_0)$ at θ_0 if and only if $\theta_0 = 0$. Moreover, $\gamma(0) = E(0) = C(0)$.*

(b) Suppose $a(1-b) > 1$, $a \in (1, 4)$, $b \in (0, 1)$. Then, there exists a homothetic orbit

$\gamma(\theta_0)$ at θ_0 if and only if $\theta_0 \in \{0, \arccos d, -\arccos d\}$, where d is as in Proposition 2. Moreover, $\gamma(0) \subset E(0) \cap C(0)$, $\gamma(\arccos d) = E(+) = C(+)$, $\gamma(-\arccos d) = E(-) = C(-)$.

Proof. The statements are direct consequences of Equations (7) and Propositions 2 and 8. ■

So, to each of the collision configurations of Theorem 5 corresponds a unique homothetic ejection-collision orbit.

4. Periodic Orbits

For a fixed energy value $H = h < 0$, we now look at the flow away from the quadruple collision manifold Λ. We may picture the energy manifold $rh = \frac{1}{2}(u^2 + v^2) + V(\theta)$, which contains the flow for $H = h$, as the interior of the topological sphere Λ in the (θ, v, u) coordinates, see Figure 4. The equations of motion are the last three in (7); the coordinate r may always be read from the energy relation (8).

Given a $2n$-dimensional manifold M, a diffeomorphism R of M such that $R^2 = \mathrm{Id}$ and $\dim(\mathrm{Fix}(R)) = n = \frac{1}{2}\dim(M)$, where Id denotes the identity in M and $\mathrm{Fix}(R)$ the set of the fixed points of R, is called a *reversing involution.* A smooth vector field X on M is said to be R-reversible if $\mathrm{TR}(X) = -X \circ R$, where TR denotes the derivative of R.

Now, recall that the equations of motion (7) possess the symmetries $S_1(r, \theta, v, u, \tau) = (r, \theta, -v, -u, -\tau)$, $S_2(r, \theta, v, u, \tau) = (r, -\theta, -v, u, -\tau)$. In other words, the field given by (7) is S_i-reversible, $i = 1, 2$. Here, we have loosely denoted also by S_i the reversing involution given by the restriction of the symmetry S_i to the phase space, $i = 1, 2$. Moreover, $\mathrm{Fix}(S_1) = \{(r, \theta, v, u)\colon v = u = 0\}$ is the zero velocity curve, and $\mathrm{Fix}(S_2) = \{(r, \theta, v, u)\colon \theta = v = 0\}$ corresponds to orthogonal crossings of the $\theta = 0$ axis.

An orbit $\gamma(\tau)$ of the field given by (7) is called S_i symmetric if and only if $S_i(\gamma(\tau)) = \gamma(\tau)$, $i = 1, 2$. Throughout the rest of this section we shall study the existence of symmetric periodic orbits of (7). Let us start by stating a preliminary result.

PROPOSITION 10. *Let φ_τ be the flow generated by* (7). *Then the following statements hold*: (*a*) *An orbit γ is an S_1-symmetric periodic orbit if and only if it meets the zero velocity curve at two different points $\varphi_0(x) = x$ and $\varphi_T(x)$. Moreover, if T is the smallest positive number with the above property, the period of γ is $2T$.*

(*b*) *An orbit γ is an S_2-symmetric periodic orbit if and only if it crosses the $\theta = 0$ axis orthogonally at two points $\varphi_0(x) = x$ and $\varphi_T(x)$. Moreover, if T is the smallest positive number with the above property, the period of γ is $2T$.*

(*c*) *An orbit γ is an S_1 and S_2-symmetric periodic orbit if and only if it meets the zero velocity curve and crosses orthogonally the $\theta = 0$ axis at the points $\varphi_0(x) = x$ and $\varphi_T(x)$, respectively. Moreover, if T is the smallest positive number with this property, the period of γ is $4T$.*

Proof. We have seen that the field defined by (7) is S_1 and S_2 reversible, and that

$\mathrm{Fix}(S_1)$ coincides with the zero velocity curve, while $\mathrm{Fix}(S_2)$ is the set of points in phase space whose orbits projected in configuration space cross orthogonally the $\theta = 0$ axis. Then, the result follows from the general characterization of the symmetric periodic orbits of vector fields which are reversible with respect to two reversing involutions (see Devaney, 1976). ■

Let $\gamma(\tau) = (r(\tau), \theta(\tau), v(\tau), u(\tau))$, $\tau \in (-\infty, +\infty)$, be a solution of (7). If $r(\tau)$, $\tau \in (-\infty, +\infty)$, has exactly n extremal points (maxima or minima) we shall say that γ has *extremal period* (e.p.)n. Note that a non periodic orbit may have a finite extremal period, as it happens for instance with the homothetic orbit $\gamma(0)$, whose extremal period is $n = 1$.

The main result of this section is the following.

THEOREM 11. *Suppose $a \in (1, 4)$, $b \in (0, 1)$, and $a(9 - 8b) < 9$, and let $n_0 \in \mathbb{N} \cup \{0\}$. Then the following properties hold (see Figure 5): (a) There exist an infinite number of S_1 and S_2-symmetric periodic orbits with extremal period* $8n_0 + 4$.

(*b*) *There exist an infinite number of S_2-symmetric periodic orbits with extremal period* $4n_0 + 2$.

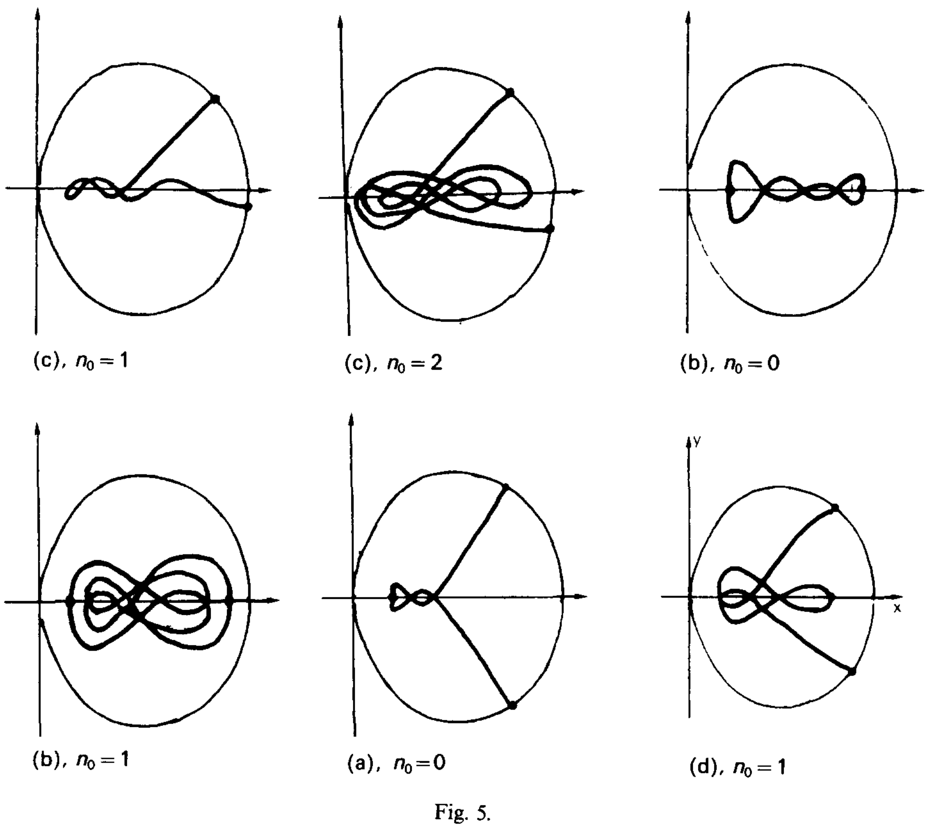

Fig. 5.

(*c*) *There exist an infinite number of* S_1*-symmetric periodic orbits with extremal period* $4n_0$.

(*d*) *There exist an infinite number of* S_1 *and* S_2*-symmetric periodic orbits with extremal period* $8n_0$.

Before proving the theorem, we shall need a few preliminary results.

Let $S = \{(r, \theta, v, u): H(r, \theta, v, u) = h,\ v = 0\}$, $S^+ = \{(r, \theta, v, u) \in S: v' > 0)$ and $S^- = \{(r, \theta, v, u) \in S: v' < 0\}$. From Equations (7), we have $r'' = r'v + rv'$. Therefore, the points of S^+ correspond to minima of r, while the points of S^- correspond to maxima. Also from Equations (7), we see that the set $S^\circ = \{(r, \theta, v, u) \in S: u^2 = -V(\theta)\}$ divides S into three regions, S^1, S^2 and S^3 with $S^1 = S^-$ and $S^2 \cup S^3 = S^+$.

From now on, we shall identify points in S by their coordinates (θ, u). Therefore $(0, 0)$ will denote the point $\gamma(0) \cap S$.

Let $f: A^- \subset S^- \to S^+$ be the function that maps a point $p \in S^-$ onto the point where the time forward orbit through p intersects S^+ for the first time. Similarly, $g: B^- \subset S^- \to S^+$ is the function that maps a point $p \in S^-$ onto the point where the time backward orbit through p intersects S^+ for the first time. Here, A^- and B^- denote the domains of definition of f and g, respectively. The next lemma says that both A^- and B^- coincide with $S^- \backslash \{(0, 0)\}$.

LEMMA 12. *The functions* f *and* g *are both defined in* $S^- \backslash \{(0, 0)\}$.

Proof. We shall prove it only for f (for g the proof is similar). Let $p \in S^- \backslash \{(0, 0)\}$ and let the forward orbit through p be $p(\tau) = (r(\tau), \theta(\tau), v(\tau), u(\tau))$, $\tau \in [0, +\infty)$. Since $v(0) = 0$, $v'(0) < 0$, the set $T^+ = \{\tau \in (0, +\infty): v(\tau) > 0\}$, if it is not empty, satisfies $\inf(T^+) = \tau_0 > 0$. To show that $f(p)$ is well defined, it is enough to prove that $T^+ \neq \emptyset$. If it is not so, then $v(\tau) \leqslant 0$ for every $\tau \in [0, +\infty)$ and therefore $r(+\infty) = \lim_{\tau \to +\infty} r(\tau)$ exists. Now, $r(+\infty)$ cannot be zero. In fact, by Proposition 9, $W^s(p^-(0)) = \gamma(0)$ and, by hypothesis, $p(0) \neq (0, 0)$. Suppose then that $r(+\infty) \neq 0$. By Equations (7), $p(\tau)$ must tend to an equilibrium point out of Λ, which is impossible. ∎

Denote by $E \subset S^+$ the intersection $f(S^- \backslash \{(0, 0)\}) \cap g(S^- \backslash \{(0, 0)\})$. Clearly E is $S^+ \backslash \Lambda$, and we may define on $S^- \backslash \{(0, 0)\}$ a Poincaré map $h = g^{-1} \circ f$.

Consider the following subsets of S:

$$
\begin{aligned}
Z^+ &= \{(\theta, u) \in S: u = 0, \theta > 0\},\\
Z^- &= \{(\theta, u) \in S: u = 0, \theta < 0\},\\
W^+ &= \{(\theta, u) \in S: \theta = 0, u > 0\},\\
W^- &= \{(\theta, u) \in S: \theta = 0, u < 0\}.
\end{aligned}
$$

Note that $Z^+ \cup Z^- \subset S^-$, i.e., the only extremal points of r on the zero velocity curve are maxima.

With this notation we have the following lemma.

LEMMA 13. (*a*) *An orbit through* $p \in (Z^+ \cup Z^-)$ *is* S_1 *and* S_2*-symmetric and periodic* *with e.p.* $8n_0+4$, $n_0 \in \mathbb{N} \cup \{0\}$, *iff* $f(h^{n_0}(p)) \in W^+ \cup W^-$, $h^j(p) \notin W^+ \cup W^-$, $j=1,\ldots,n_0-1$, *and* $f(h^j(p)) \notin W^+ \cup W^-$, $j=0,\ldots,n_0-1$.

(*b*) *An orbit through* $p \in (W^+ \cup W^-) \cap S^-$ *is* S_2*-symmetric and periodic with e.p.* $4n_0+2$, $n_0 \in \mathbb{N} \cup \{0\}$, *iff* $f(h^{n_0}(p)) \in W^+ \cup W^-$, $h^j(p) \notin W^+ \cup W^-$, $j=1,\ldots,n_0-1$, *and* $f(h^j(p)) \notin W^+ \cup W^-$, $j=0,\ldots,n_0-1$.

(*c*) *An orbit through* $p \in (Z^+ \cup Z^-)$ *is* S_1*-symmetric and periodic iff there exists* $n_0 \in \mathbb{N}$ *such that* $h^{n_0}(p) \in (Z^+ \cup Z^-)$. *Furthermore, if* $h^j(p) \notin (Z^+ \cup Z^-)$, $j=1,\ldots,n_0-1$, *its e.p. is* $4n_0$.

(*d*) *An orbit through* $p \in (Z^+ \cup Z^-)$ *is* S_1 *and* S_2*-symmetric and periodic with e.p.* $8n_0$, $n_0 \in \mathbb{N}$, *iff* $h^{n_0}(p) \in W^+ \cup W^-$, $h^j(p) \notin W^+ \cup W^-$, $j=1,\ldots,n_0-1$, *and* $f(h^j(p)) \notin W^+ \cup W^-$, $j=0,\ldots,n_0-1$.

Proof. This is a direct consequence of Proposition 10 and of the definitions of f, h, Z^+, Z^-, W^+ and W^-. ■

Before proving the next auxiliary lemmas we require more notation.

Denote by D_ε a ball in S centered in $(0,0)$ and of radius ε, and by E_ε the set of points of S^+ whose distance to Λ is smaller than ε. Let T denote a curve in $E_\varepsilon \backslash \Lambda$ accumulating to $\Lambda \cap S$.

LEMMA 14. *For* ε *small enough,* $f^{-1}(T) \cap D_\varepsilon$ *is a counterclockwise spiral accumulating in* $(0,0)$. *Similarly,* $g^{-1}(T) \cap D_\varepsilon$ *is a clockwise spiral in* D_ε *accumulating in* $(0,0)$.

Proof. By the tubular flow theorem, and since there are no singular points of (7) other than $p^-(0)$ in $\{v \leqslant 0\}$, the flow will take the curve T, following the solutions on Λ, until it reaches a neighbourhood of $p^-(0)$. In order to study what happens to the points of T when they pass close to $p^-(0)$, we shall use the linear approximation of (7) at $p^-(0)$. By taking ε small enough, this approximation can be made arbitrarily accurate.

From Proposition 2(a), the flow in a neighbourhood of $p^-(0)$ may be written, in appropriate coordinates, in the form,

$$\begin{aligned}
\dot{z} &= -\alpha z,\\
\dot{r} &= \beta r,\\
\dot{\theta} &= \gamma,
\end{aligned}$$

where α, β, $\gamma \in \mathbb{R}^+$, $z=0$ is the unstable manifold of $p^-(0)$ contained in Λ, and $r=0$ corresponds to the stable manifold $\gamma(0)$.

Consider a cylinder C_{δ_1} defined by $r=r^*$, and a segment $r=r^*$, $\theta=\theta^*$, $z \in (0,\delta_1)$. Consider also a disc D' of radius δ_2 transversal to the z-axis at the point z^*. We want to know the image of the segment by the time backwards flow that maps C_{δ_1} onto D'. From,

$$\begin{aligned}
r &= r^* e^{\beta\tau},\\
\theta &= \theta^* + \gamma\tau,\\
z^* &= z e^{-\alpha\tau},
\end{aligned}$$

we have $-\alpha\tau = \log(z^*/z)$, and so,

$$r = r^*(z/z^*)^{\beta/\alpha},$$
$$\theta = \theta^* - (\gamma/\alpha)\log(z^*/z).$$

Therefore, the image of a vertical segment in C_{δ_1} accumulating on Λ is a spiral in D' accumulating on $\gamma(0) \cap D'$. Clearly, the same result holds for any other curve provided that it accumulates on Λ and that the r and θ coordinates of its points are contained in bounded intervals. This proves the result for f^{-1}; for g^{-1} the proof is similar. ■

LEMMA 15. *Let h be the Poincaré map on the disc D_ε. If R_1 and R_2 are two radius in D_ε such that $R_2 = S_2(R_1)$, then $h(R_1)$ and R_2 have an infinite number of intersections.*

Proof. We shall prove that, on the conditions of the lemma, $f(R_1)$ and $g(R_2)$ have an infinite number of intersections. Using the hypothesis and the symmetry S_2 of Equations (7) we have that $f(R_1) = S_2(g(R_2))$.

Now consider one of the two segments of $E_\varepsilon \cap (W^+ \cup W^-)$, T_1. By Lemma 14, $f(R_1)$ intersects T_1 an infinite number of times. On the other hand, T_1 is invariant by S_2. So these points are also points of $g(R_2)$, and the claim is proved. ■

LEMMA 16. *If ε is small enough then $D_1 = h(Z^+ \cap D_\varepsilon)$ and $D_2 = h(Z^- \cap D_\varepsilon)$ are clockwise spirals accumulating in $(0,0)$ and such that $D_1 = S_1 \circ S_2(D_2)$.*

Proof. Using the same arguments as in Lemma 14 the result follows. ■

COROLLARY 17. *Suppose $a \in (1,4)$, $b \in (0,1)$ and $a(9-8b) < 9$. Then the following properties hold: (a) There exist an infinite number of S_1 and S_2-symmetric periodic orbits with extremal period 4.*

(b) There exist an infinite number of S_2-symmetric periodic orbits with extremal period 2.

(c) There exist an infinite number of S_1-symmetric periodic orbits with extremal period 4.

(d) There exist an infinite number of S_1 and S_2-symmetric periodic orbits with extremal period 8.

Proof. This is an immediate consequence of Lemmas 13, 15 and 16. ■

Proof of Theorem 11 Let us start with statement (a). From Corollary 17, we know that it holds for $n_0 = 0$. Using Lemma 13, to prove that (a) holds for $n_0 \geqslant 1$, it is enough to check that the set P_{n_0} defined by

$$P_{n_0} = \left[f(h^{n_0}(Z^+ \cup Z^-)) \cap (W^+ \cup W^-) \right] \Big\backslash \left[\bigcup_{j=0,\dots,n_0-1} f(h^j(Z^+ \cup Z^-)) \right],$$

has an infinite number of points. More precisely, we shall prove that,

$$f^{-1}(P_{n_0}) = \left[h^{n_0}(Z^+ \cup Z^-) \cap f^{-1}(W^+ \cup W^-) \right] \Big\backslash \left[\bigcup_{j=0,\dots,n_0-1} h^j(Z^+ \cup Z^-) \right]$$

has infinite points.

Consider the sets $C_1 = f^{-1}(W^+)$, $C_2 = f^{-1}(W^-)$. By Lemma 14, C_1 and C_2 are counterclockwise spirals accumulating in $(0, 0)$. Moreover, the symmetries imply that $C_2 = S_2 \circ S_1\ (C_1)$, since $W^- = S_2 \circ S_1(W^+)$.

The points of $(C_1 \cup C_2) \cap Z^-$ (resp. $\cap Z^+$) define a partition of Z^- (resp. Z^+) in infinite segments A_k (resp. B_k), $k \in \mathbb{N}$. Let us see that each A_k contains at least one point of $f^{-1}(P_{n_0})$, for every n_0.

Suppose first $n_0 = 1$, and let $A_k = [a_k, a_{k+1}]$, $B_k = [b_{k+1}, b_k]$, $b_i = -a_i, i = k, k+1$. Since $f(a_k)$ and $f(a_{k+1})$ belong to $W^+ \cup W^-$, $h(a_k) = b_k$ and $h(a_{k+1}) = b_{k+1}$ (Lemma 13). Then, by Lemma 16, $h(A_k)$ must curl at least one around $(0, 0)$, and therefore, it must intersect the spirals $f^{-1}(W^+ \cup W^-)$ out of $Z^+ \cup Z^-$ at least once. Hence, $h(A_k) \cap f^{-1}(W^+ \cup W^-)$ contains at least once point and, since k is arbitrary, we conclude that $h(Z^+ \cup Z^-) \cap f^{-1}(W^+ \cup W^-)$ contains infinite points.

Consider now an arbitrary value of $n_0 \geqslant 2$. By an argument similar to the one of the case $n_0 = 1$, there exists a point $p \in \mathrm{Int}(A_k)$ which is the closest point to a_k such that $h^{n_0}(p) \in f^{-1}(W^+ \cup W^-)$. Let $C = \bigcup_{j=0,\ldots,n_0-1} h^j(Z^+ \cup Z^-) \cap f^{-1}(W^+ \cup W^-)$. If we prove that $p \notin C$, then clearly the result follows as in the previous case, because k is arbitrary.

We claim that if $p \in C$ and $h^{n_0}(p) = h^j(q)$ for some $q \in Z^+ \cup Z^-$ and some $j \in \{0, \cdots, n_0 - 1\}$, then we must have $n_0 = K_j + (k-1)/2$, with k odd and greater than or equal to 3. In fact, from Lemma 13, $h^{n_0}(p)$ and $h^j(q)$ are periodic points of h of period $4n_0 + 2$ and $4j + 2$, respectively. Furthermore, we have taken j such that $4j + 2$ is the minimal period of q with respect to h. Then, $4j + 2$ must divide $4n_0 + 2$, which proves the claim.

Let us now prove that $p \notin C$. We shall suppose that $p \in C$ and show that this implies that $h^{n_0}([a_k, p))$ must cross at least once the set $f^{-1}(W^+ \cup W^-)$, which contradicts the hypothesis that p is the closest point to a_k such that $h^{n_0}(p) \in f^{-1}(W^+ \cup W^-)$, and the result follows.

If $p \in C$, we must have $n_0 = j + m(2j+1)$, $m \geqslant 1$, and so, by Lemma 13,

$$h^{n_0}(p) = h^j h^{m(2j+1)}(p) = h^j((-1)^m p),$$
$$h^{n_0}(a_k) = h^j h^{m(2j+1)}(a_k) = h^j((-1)^m a_k).$$

Therefore, $h^{n_0}([a_k, p]) = h^j(D)$, where D is a curve that curls at least once around $(0, 0)$, since it either starts at a_k and ends at p or starts at $-a_k$ and ends at $-p$. Now the image of D by h^j must cross at least once the spirals $f^{-1}(W^+ \cup W^-)$. In fact, $h^j(D)$ must curl at least once around $(0, 0)$ and its intersections with Z^+ or Z^- are ordered approaching $(0, 0)$ (Lemma 16). On the other hand, its end point $h^{n_0}(p)$ lies on $f^{-1}(W^+ \cup W^-)$, and so the result follows.

For (b), (c), and (d), the proof is similar. ∎

REMARK 1. In (Atela), the existence is proved of the orbits given by Theorem 11(b) and (c) for the cases $n_0 = 0$ and $n_0 = 1$, respectively.

REMARK 2. From Proposition 2, we know that for $a \in (1, 4)$, $b \in (0, 1)$, $a(1-b) < 1$ and $a(9 - 8b) \geqslant 9$, the point $p^+(0)$ (resp. $p^-(0)$) is a non spiral hyperbolic sink (resp.

source). Repeating the reasoning of Theorem 11 for this case we find only a finite number of symmetric periodic orbits. In this sense, we may say than an ∞-furcation takes place at $a(9-8b)=9$.

5. Non-Integrability and Chaotic Behaviour

In this Section we shall use McGehee's variables and the transformed equations of motion (7) to prove the non-integrability of the Hamiltonian system (5). More precisely, we show that imposing certain conditions on the parameters a, b, it is possible to construct a Poincaré map F defined on a local surface of section S such that S contains an invariant set I on which F is conjugated to the shift homeomorphism σ over the space of the doubly infinite sequences of symbols of a countably infinite alphabet $\mathscr{A}$. In particular, this implies that system (5) is non-integrable in the sense that there exists no analytic integral of motion independent from the Hamiltonian (see for instance (Moser, 1973). In fact, since the potential $U(x, y)$ in (5) is homogeneous, it is easy to check the non-integrability of (5) as a consequence of Ziglin's theorem (Yoshida, Ziglin, 1983). However, the conjugacy of F and σ is a much stronger result. It implies also the existence of infinitely many hyperbolic periodic orbits, and that the system is chaotic, in the sense that there are orbits whose long time behaviour is quite unpredictable. Furthermore, the symbols of the alphabet $\mathscr{A}$ have a certain dynamical meaning and the conjugacy with the shift σ over $\mathscr{A}$ gives us information on the qualitative features of the chaotic orbits, as in (Moser) and (Devaney, 1978).

Throughout this Section, we shall suppose $a(1-b)>1$. As before, denote by $p^+(\theta_0), p^-(\theta_0)$, $\theta_0 \in \{0, \arccos d, -\arccos d\}$, the equilibrium points on Λ given by Proposition 2(b), and by $W^u(p)$ (resp. $W^s(p)$) the unstable (resp. stable) manifold associated with the equilibrium p. Recall that $W^u(p^+(0))$ and $W^s(p^-(0))$ are two-dimensional, while $W^s(p^+(0))$ and $W^u(p^-(0))$ are one-dimensional and contained in Λ. Denote also by $\gamma(\theta_0)$, $\theta_0 \in \{0, \arccos d, -\arccos d\}$, the homothetic orbits given by Proposition 9(b). We shall also suppose that $W^u(p^-(0))$ misses $W^s(p^+(0))$ which, by Proposition 7, happens for almost every value of the parameters a, b.

LEMMA 18. *The manifolds $W^u(p^+(0))$ and $W^s(p^-(0))$ intersect transversally along $\gamma(0)$.*

Proof. Denote by S^x the surface in the energy level $H=h$ defined by $\{v=x\}$, and let $\gamma(0)=(\bar{r}(\tau), \theta=0, \bar{v}(\tau), u=0)$. It may be easily seen from (7) that:

$$\bar{r}(\tau)=\frac{2v_0^2}{h}\frac{e^{v_0\tau}}{(1+e^{v_0\tau})^2}, \qquad \bar{v}(\tau)=\frac{v_0(1-e^{v_0\tau})}{1+e^{v_0\tau}}$$

where $v_0=\sqrt{-2V(0)}$ is the v coordinate of $p^+(0)$. Since $\gamma(0)\in W^u(p^+(0))$ and $\gamma(0)$ intersects transversally every S^v, $|v|<v_0$, the set $W^u(p^+(0)) \cap S^v \cap \mathscr{U}$, where $\mathscr{U}$ is a neighbourhood of $\gamma(0)$, is non-empty and coincides with the graph of a certain differentiable function $\Phi_v(\theta)$. Denote by $\varphi(v)$ the function $D\Phi_v(0)$, i.e., $\varphi(v)$ is the slope of the straight line defined in S^v by the tangent to $\Phi_v(\theta)$ at $\theta=0$, and let $\Psi(\tau)=\varphi(\bar{v}(\tau))$.

The linear approximation at $p^+(0)$ yields

$$\Psi(-\infty) = \varphi(v_0) = \arctan\left(\frac{v_0}{4} + \sqrt{\frac{v_0^2}{\sqrt{6}} - V''(0)}\right) \in (0, \pi/2).$$

On the other hand, the variational equations along $\gamma(0)$ determine the equation of motion of Ψ,

$$\frac{d\Psi}{d\tau} = (-1 + a(1-b))\cos^2\Psi - \sin^2\Psi - \frac{v(\tau)}{2}\cos\Psi\sin\Psi. \tag{13}$$

Using (13) and the techniques introduced in (Casasayas and Llibre, 1984) the transversality of the homothetic orbit $\gamma(0)$ may be easily proved. Let $\psi^*(\tau)$ denote the unique value of ψ in $(0, \pi/2)$ such that $(d\psi/d\tau)(\psi^*\tau) = 0$. It is easy to check that $\psi^*(\tau)$ is monotonically increasing with τ, $\tau \in (-\infty, 0)$, and that $(d\psi/d\tau)(\psi, \tau) > 0$ (resp. < 0) for $\psi(\tau) < \psi^*(\tau)$ (resp. $\psi(\tau) > \psi^*(\tau)$). Now, we claim that the solution $\psi(\tau)$ of (13) must be monotonically increasing. Clearly, for τ small enough, we have $\psi(\tau) < \psi^*(\tau)$. Suppose then that there exist τ_0 and ε such that $\psi(\tau_0) = \psi^*(\tau_0)$ and $\psi'(\tau) < 0$ for $\tau \in (\tau_0, \tau_0 + \varepsilon)$. Then, $\psi(\tau_0 + \varepsilon) < \psi(\tau_0) = \psi^*(\tau_0) < \psi^*(\tau_0 + \varepsilon)$, which implies $\psi'(\tau_0 + \varepsilon) < 0$, a contradiction, and the claim is proved. Therefore, we must have $\psi(\tau) \in (\psi(-\infty), \psi^*(\tau))$ and hence $\psi(\tau) \in (\psi(-\infty), \pi/2)$ for $\tau \in (-\infty, 0]$. Since the symmetry S_1 maps $W^u(p^+(0))$ onto $W^s(p^-(0))$ we have that the angle $\Delta\psi(\tau)$ between the straight lines defined in $S^{v(\tau)}$ by the tangents to $W^u(p^+(0)) \cap S^{v(\tau)}$ and to $W^s(p^-(0)) \cap S^{v(\tau)}$ at $\theta = 0$ verifies

$$\Delta\psi(\tau) = \psi(\tau) + \psi(-\tau) \tag{14}$$

and so $\Delta\psi(0) \in (0, \pi)$. Finally, from (13) and (14), if $\Delta\psi(\tau^*) = k\pi$, for some $k \in Z$ and some $\tau^* \in (-\infty, +\infty)$, then $\Delta\psi(\tau) = k\pi$ for every τ, and the result follows. ■

Let S be the surface in the energy level $H = h$ defined by $\{v = 0\}$, and let x_0 (resp. x_1, x_2) be given by $\gamma(0) \cap S$ (resp. $\gamma(-\arccos d) \cap S$, $\gamma(\arccos d) \cap S$). Denote by D_i a neighbourhood of x_i in S, $i = 0, 1, 2$. Denote also by B (resp. A) the set $W^s(p^-(0)) \cap S$ (resp. $W^u(p^+(0)) \cap S$). Let f be the map that takes $p \in D_0$ to $p' \in D_1 \cup D_2$, where the forward time orbit through p crosses for the first time S with $v' < 0$. Fixing D_1 and D_2 and taking D_0 small enough, it is easy to check, using the description of the flow on Λ given in Section 2, that f is defined in $D_0 \backslash B$. Denote by A_μ, $\mu \in (-1, 1)$ a foliation of D_0 by curves parallel to A, $A_0 = A$. For simplicity, we shall take local coordinates x, y in D_0 such that $D_0 = \{(x, y) \in [-1, 1] \times [-1, 1]\}$, $A_\mu = \{(x, y) \in D_0 : x = \mu\}$, $B = B_0 = \{(x, y) \in D_0 : y = 0\}$ (notice that, by Lemma 18, this is always possible). With this notation we have the following lemma.

LEMMA 19. *For every value of a, b such that $(a(1 - 25b/24))^3 > a(1 - b) > 1$, the set $f(A_\mu)$ is formed by two spirals, one in D_1 and the other in D_2, accumulating on x_1 and x_2,* *respectively. Moreover,*

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}f(A_\mu(y))\right| \to \infty \textit{ when } y \to 0$$

Proof. As in the proof of Lemma 14, we shall follow the segment $A_\mu(y)$, $y \in [-1, 1]$, by the flow and, by the tubular flow theorem, restrict our study to a neighbourhood of the singular points.

Consider first the point $p^-(0)$ (all the orbits starting in D_0 will follow close to the homothetic orbit $\gamma(0)$ and pass close to $p^-(0)$). In a neighbourhood of $p^-(0)$, the linear approximation may be written using appropriate coordinates

$$\begin{aligned} \dot{z} &= -\lambda_3 z \\ \dot{x} &= -\lambda_1 x \\ \dot{y} &= \lambda_2 y, \end{aligned} \tag{15}$$

where $\lambda_i > 0$, $i = 1, 2, 3$, $x = y = 0$ is the direction of the homothetic orbit $\gamma(0)$ and $z = 0$ the tangent plane to Λ at $p^-(0)$. Consider two surfaces of section, Σ_{z^*} at $z = z^*$ and Σ_{y^*} at $y = y^*$, and let us follow the points of $A_\mu(y) \cap \Sigma_{z^*}$, $y > 0$, until they cross Σ_{y^*}. From (15), we have that the orbits through these points cross Σ_{y^*} along the curve

$$\begin{aligned} x &= \mu(y/y^*)^{\lambda_3/\lambda_2} \\ z &= z^*(y/y^*)^{\lambda_1/\lambda_2} \end{aligned} \tag{16}$$

Since Σ_{y^*} is transversal to $W^u(p^-(0))$, which dies in $p^+(\arccos d)$, the other singular point in whose neighbourhood the orbits through $A_\mu(y)$, $y > 0$, will pass is the point $p^+(\arccos d)$. The linear approximation in a neighbourhood of $p^+(\arccos d)$ is of the form

$$\begin{aligned} \dot{r} &= -\alpha r \\ \dot{\theta} &= \beta \\ \dot{z} &= \lambda z, \end{aligned} \tag{17}$$

$\alpha, \beta, \lambda > 0$. Consider two surfaces of section Σ_{r^*} at $r = r^*$ and Σ_{z^*} at $z = z^*$. The image by the flow of the points of $A_\mu(y)$, $y > 0$, will cross Σ_{r^*} in a curve $(r^*, \theta(y), z(y))$ diffeomorphic to (16) and such that $z \to 0$ and $\theta \to \theta_0$ when $y \to 0$. Using (17) to follow the orbits through this curve until they reach Σ_{z^*}, we obtain

$$\begin{aligned} r &= r^*(z(y)/z^*)^{\alpha/\lambda} \\ \theta &= \theta(y) + \frac{\beta}{\lambda}\log(z^*/z(y)), \end{aligned} \tag{18}$$

which is a spiral in Σ_{z^*} with $r \to 0$ when $y \to 0$. Moreover, (18) implies that $|(\mathrm{d}/\mathrm{d}y)f(A_\mu(y))|$ is of the order of $z^{-1}(\mathrm{d}z/\mathrm{d}y)$, and, since $z \to 0$ when $y \to 0$ as a power of y (see (16)), the result follows. ■

Let g denote the map that takes $p \in D_0$ to $p' \in D_1 \cup D_2$, where the backward time orbit through p crosses for the first time S with $v' < 0$. Using the symmetry of the system

$S_1:(r,\theta,v,u,\tau)\to(r,\theta,-v,-u,-\tau)$, and considering a foliation $B_\mu, \mu\in(-1,1)$ with $B_\mu=S_1(A_\mu)=\{(x,y)\in D_0: y=\mu\}$, $B_0=B$, the following lemma may be directly obtained from Lemma 19.

LEMMA 20. *For every value of a and b such that* $(a(1-25b/24))^3>a(1-b)>1$, *the set* $g(B_\mu(x))$, $x\in[-1,1]$, *is formed by two spirals, one contained in* D_1 *and the other in* D_2, *accumulating on* x_1 *and* x_2, *respectively. Moreover,*

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}g(B_\mu(x))\right|\to\infty \text{ when } x\to 0.$$

In order to prove the main theorem of this section, we shall need two other auxiliary lemmas.

Denote by $a_1(\mu)$ and $a_2(\mu)$ (resp. $b_1(\mu)$ and $b_2(\mu)$) the spirals given by Lemma 19 (resp. Lemma 20). Clearly, $b_i(\mu)=S_1(a_i(\mu))$, $i=1,2$. Therefore, for $i\in\{1,2\}$, $a_i(\mu)$ and $b_i(\mu)$ intersect each other whenever they cross the segment Z in S given by $\{u=0\}$.

LEMMA 21. *Let a and b be as in Lemmas* 19 *and* 20. *Then the intersections of* $a_i(\mu)$ *and* $b_i(\mu)$, $i=1,2$, *on the segment Z are transversal for almost every value of the parameters a, b.*

Proof. Consider one of the spirals given by Lemmas 19 and 20 and let $\phi_n(a,b)$ denote the angle between Z and the tangent to the spiral in its nth intersection with Z.

Since $\phi_n(a,b)$ is, up to diffeomorphisms, a logarithmic spiral, $\lim_{n\to\infty}\phi_n(a,b)$ exists. By analyticity, it is enough to prove that $\lim_{n\to\infty}\phi_n(a,b)$ is not equal to $\pi/2$ for every value of a, b in the conditions of the lemma. If it were so, then for a, b such that $a(1-b)>1$, $(a(1-25b/24))^3-(a(1-b))\in(-\varepsilon,0]$, i.e. when the singular points $p^{\pm}(\pm\arccos d)$ are non-spiral hyperbolic sinks and sources in Λ, the curve $f(A_\mu(y))$ would still have to cross Z infinitely many times in a neighbourhood of x_1 and x_2. But, repeating the reasoning of the proof of Lemma 19, it is easy to check that for these values of the parameters the curve $f(A_\mu(y))$ intersects Z transversally at x_1 and x_2. ■

LEMMA 22. *Let R denote a radius in* $D_i, i\in\{1,2\}$. (*a*) $g(D_0\backslash A)\cap R$ *is formed by an infinite family of intervals* $R_j, j\in N$, *accumulating on* x_i *when* $j\to+\infty$.

(*b*) *Let* T_j *denote the unit vector tangent to R in a point of* R_j. *Then* $|dg^{-1}(T_j)|\to\infty$ *when* $j\to\infty$.

(*c*) $g^{-1}(R_j)$, $j\in N$, *is an infinite family of arcs which tend* C^1 *to A when* $j\to\infty$.

Proof. Statement (a) is an immediate consequence of Lemma 20. To prove (b) and (c) we shall repeat the procedure of the proof of Lemma 19 and follow R_j by the flow as it passes close to the singular points $p^-((-1)^i\arccos d)$ and $p^+(0)$. Consider the linear approximation in a neighbourhood of $p^-((-1)^i\mathrm{arcos}\, d)$,

$$\begin{aligned}\dot r&=\alpha r\\ \dot\theta&=\beta\\ \dot z&=-\lambda z,\end{aligned}\tag{19}$$

α, β, $\lambda>0$, and the surfaces of section Σ_{z^*} at $z=z^*$ and Σ_{r^*} at $r=r^*$.

Let $f_1 : \Sigma_{z^*} \to \Sigma_{r^*}$ be the map defined by the flow of (19). The image by f_1 of a radius R in Σ_{z^*} is the curve

$$\theta = \theta^* + \frac{\beta}{\alpha}\log\frac{r^*}{r}, \qquad z = z^*\left(\frac{r}{r^*}\right)^{\lambda/\alpha}, \tag{20}$$

i.e., it is a curve in Σ_{r^*} spiraling as it tends to $z = 0$.

From (20), we have that $(\xi_1, \eta_1) = \mathrm{d}f_1(T_j)$ verifies

$$\frac{|\eta_1|}{|\xi_1|} \sim r^{\lambda/\alpha} \sim z \tag{21}$$

In a neighbourhood of $p^+(0)$, the linear approximation is

$$\begin{aligned} \dot{x} &= -\lambda_1 x \\ \dot{y} &= \lambda_2 y \\ \dot{z} &= \lambda_3 z, \end{aligned} \tag{22}$$

$\lambda_1, \lambda_2, \lambda_3 > 0$, and we shall consider the map defined by the flow of (22), $f_2 : \Sigma_{x^*} \to \Sigma_{z^*}$, with Σ_{x^*} and Σ_{z^*} surfaces of section transversal to the x-axis and the z-axis, respectively. Using (22) we have

$$(x, y) = f_2(y, z) = (x^*(z/z^*)^{\lambda_1/\lambda_3}, y(z^*/z)^{\lambda_1/\lambda_3}). \tag{23}$$

Let ξ_1' and η_1' be vectors tangent to Σ_{x^*} along the y-axis and the z-axis, respectively. Suppose that $|\eta_1'| \leqslant z|\xi_1'|$ and let $(\xi_2, \eta_2) = \mathrm{d}f_2(\xi_1', \eta_1')$. Then, (23) implies

$$\frac{|\xi_2|}{|\eta_2|} \sim z^{(\lambda_1 + \lambda_2)/\lambda_3} \tag{24}$$

Now, (20), (23) and (24) prove (c), while (b) follows from differentiating (20) and (23). ∎

We can now state the main theorem of this section. Denote by $h = g^{-1} \circ f$ the Poincaré map associated to system (5) defined in $Q \subset D_0$, where Q may possibly be empty. Denote also by A_k the space of the doubly infinite sequences of integers greater than or equal to k, and by σ the shift homeomorphism defined in A_k.

THEOREM 23. *For almost every value of a and b, such that $a(1-b) > 1$ and $(a(1-25b/24))^3 > a(1-b)$, the following statements holds.*

(a) *Q is non-empty and contains an invariant set I.*
(b) *There exists $k \in \mathbb{N}$ such that h on I is conjugated to σ on A_k.*
(c) *The integers of a sequence $s \in A_k$ correspond to the number of consecutive crossings of the axis $\theta = \arccos d$, $\theta = -\arccos d$ performed by the orbit associated to s through the conjugacy given by (b) when $\tau \in (-\infty, +\infty)$.*

Proof. To prove this result it is enough to show that the Horseshoe Theorem (see Moser, 1973) applies to the system we are studying. We shall check that our system verifies the conditions of this theorem in the form stated in (Devaney, 1978), following

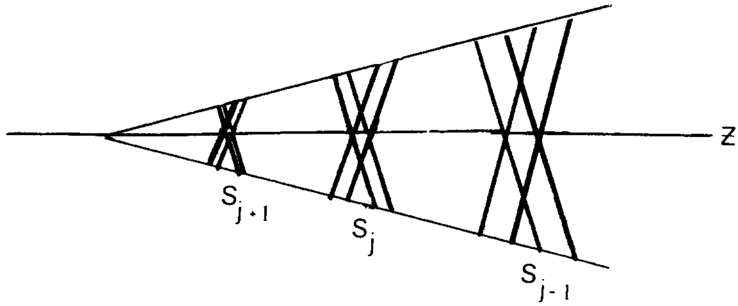

Fig. 6.

the procedure used in this paper to prove a similar result for the Anisotropic Kepler Problem.

Denote by C_i a sector around Z in D_i, $i = 1, 2$, and let $E \subset Q$ be the set of points p such that $f(p) \in g(D \backslash A) \cap (C_1 \cup C_2)$.

If $h(A_\mu \cap E)$ is an infinite family of curves which tend to B in the C^1 topology and $|(d/dy)h(A_\mu(y))| \to \infty$ when $y \to 0$, then the Horseshoe theorem holds and the result follows.

Now, by Lemmas 19, 20 and 21, the sets $f(D_0 \backslash B) \cap C_i, g(D_0 \backslash A) \cap C_i$, $i = 1, 2$, are as shown in Figure 6, and the images of the foliation of D_0 by the curves A_μ and B_μ intersect transversally in C_i. Hence, the image by f of a curve A_μ intersects transversally infinitely many times – once on each set S_j – all the leaves of the image by g of $D_0 \backslash A$ foliated by B_μ (see Figure 6). Then, $g^{-1}(f(A_\mu) \cap S_j)$ is a vertical curve in D_0, and it must tend to A when $j \to \infty$, because, when $x \to 0$, $\psi(B_\mu(x))$ tends to x_i.

Denote now by $\xi(p), \eta(p)$ the unit vectors tangent at p to the foliations B_μ and A_μ, respectively, and by p and p' two points in E such that $h(p) = p'$. Let ω_1 be the unit tangent vector to Z. Since the spiral $g(B_\mu(x))$ intersects Z transversally, taking C_i, $i = 1, 2$, small enough, ω_1 and $\omega_2 = dg(\xi(p'))$ form a basis of $T_{f(p)}C_i$. Now $df(\eta(p)) = a(p)\omega_1 - \omega_2$, where $|a(p)| \to \infty$ when $y(p) \to 0$, $y(p)$ denoting the y coordinate of p, since $|\omega_2| \to \infty$ when $y(p) \to 0$ (Lemma 20).

Then, $dg^{-1}(df(\eta(p))) = a(p)dg^{-1}(\omega_1) - \xi(p')$. *By Lemma* 22, $|dg^{-1}(\omega_1)| \to \infty$ when $y(p) \to 0$. Hence $|dh(\eta(p))| \to \infty$ when $y(p) \to 0$. ■

REMARK 3. We have proved the existence of symmetric periodic orbits only in the case $a(1 - b) < 1$, see Section 4. However, using the techniques introduced in (Casasayas and Llibre, 1984) it is also possible to prove the existence of infinitely many S_1 or S_2 symmetric periodic orbits when $a(1 - b) \geqslant 1$.

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