International Journal of Fracture 63: 59–73, 1993.

Use of J-integral as fracture parameter in simplified analysis of bonded joints

P. FRAISSE and F. SCHMIT
ETCA/CREA, 94114 Arcueil, France

Received 25 March 1992; accepted in final form 14 March 1993

Abstract. The simplified analytical approaches based on beam or plates theories are commonly used to solve the stress field in bonded laminates. However, to be correctly applied, these methods require an appropriate fracture criterion. In this paper, the use of J-integral as a fracture parameter in these simplified analytical approaches is discussed. After examining its path independence, the J-integral is calculated along two particular paths showing first that this integral is equal to the product of the strain energy at the end of the joint (i.e. at the debond tip) by its thickness. This relationship reveals the partitioning of the opening mode I and the shearing mode II. Secondly, the general expression of J as a function of the loading conditions is derived. It is shown that this parameter can be related to the strain energy release rate in the cases of small scale yielding conditions and for usual fracture mechanics specimens.

Nomenclature

J	= Rice integral
W	= Strain energy density
$\mathbf{T}$	= Tension vector
s	= Curvilinear abscissa
$\mathbf{u}$	= Displacement vector
U, V	= x, y-components of the displacement
$\boldsymbol{\sigma}$	= Stress tensor
h_0	= Adhesive thickness
τ, σ_y	= Shear and normal stresses in the adhesive
γ, ε_y	= Shear and normal strains in the adhesive
h_i	= Adherend thickness
N_i, Q_i	= Normal and shear forces in the adherend i
M_i	= Bending moment in the adherend i
u_i, v_i	= x, y displacements of the neutral line
β_i	= Bending rotation
σ_{xx}, τ_{xy}	= Components of stress tensor in the adherends
$\varepsilon_{xx}, \gamma_{xy}$	= Components of strain tensor in the adherends

1. Introduction

As the technology of adhesives continues to develop rapidly, designing methods are needed in order to predict the ultimate properties of bonded joints. To do this, many analytical methods have been developed to calculate the stress and strain distributions in the adhesive layer. The most widely known of these models is Volkersen's analysis of the tensile lap shear joint [1], in which it is assumed that the two adherends are subject to a pure tension and the joint is in a state of simple shear stress. As this model is not very realistic, it has not found its way into many practical applications. Later, Goland and Reissner [2] considered the problem of the single lap joint and also took into consideration the bending effects and the opening stress in the joint. Delale and Erdogan [3] then extended this analysis to the general case of an arbitrary geometry. Numerous extensions to cases where the joint is subjected to elastoplastic [4] or viscoelastic [5] loadings are now found in the literature. However, these models are usually associated with simplistic maximum stress or strain fracture criteria, which restricts their interest.

Simultaneously, several authors suggested the use of Linear Elastic Fracture Mechanics (LEFM) approaches to study the fracture of bonding laminates [6–8]. Previously used for classical laboratory specimens, these methods were recently applied to more complex loading conditions with the modes I and II partitioning analysis [9, 10]. However, LEFM approaches are valid only for slender specimens and the effects of the adhesive thickness and rigidity on the G_I/G_{II} ratio are ignored.

Some authors have proposed the J-integral as a failure criterion for specific geometries (DCB specimens [11], Volkersen joint [12]), and recently for more general configurations [13]. This offers two advantages. Firstly, it can be calculated on a path located far from the crack front, and therefore far from the singular zone. Secondly, its use is not restricted to the case of linear elastic materials. However, the use of this parameter as a fracture criterion requires certain assumptions, in particular its path independence, which are not necessarily verified by the simplified fields used in the analytical methods. But it is proved correct that this integral is path independent, if it passes through an interface parallel to the crack plane.

In the present article, we propose to examine the conditions under which the use of J-integral is valid, and we attempt to establish its general expression in the framework of these analytical approaches. To do this, after a brief review, we calculate the J-integral for closed paths in the adherends and in the adhesive, using Goland and Reissner's stress analysis generalized to the case where the mechanical behavior of the adhesive is nonlinear. We then determine the general expressions of J for two particular paths, in order to relate this parameter to the local stress field and to the loading conditions. Finally, we apply the results obtained to classical fracture mechanics specimens.

2. Review: the Rice integral

The path integral J has been first introduced by Rice [14] in order to describe the fracture of nonlinear materials. This integral is defined by

$$J = \int_{\Gamma} W\,\mathrm{d}y - \mathbf{T}\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\,\mathrm{d}s, \quad \mathbf{T} = \boldsymbol{\sigma}\cdot\mathbf{n}, \tag{1}$$

in which Γ is a path surrounding the crack, s is the curvilinear abscissa, $\mathbf{T}$ is the traction vector defined according to the outward normal along Γ, $\mathbf{u}$ is the displacement vector and W is the deformation energy density, expressed by

$$W = \int_0^{\varepsilon} \sigma_{ij}\,\mathrm{d}\varepsilon_{ij}, \tag{2}$$

$\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{n}$ the vector normal to the path. Rice has shown that this parameter is path independent if the stress field derives from a potential (W) and if the equilibrium equations are verified in Ω (see Fig. 1). In that case, the J-integral could be related to the stress field near a crack tip, allowing its direct evaluation. Furthermore, a proof was presented showing that, in many cases, this parameter was a promising fracture criterion, being consistent with the Griffith energy balance approach. We note that the first hypothesis (i.e. the stress field derives from a potential) underlies the existence of an explicit relation between the components of the stress and strain tensors. As Rice demonstrated, this makes J applicable for the nonlinear elastic bodies and for structures subjected to particular histories of plastic loading. In the case

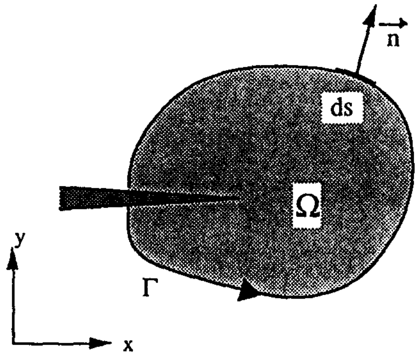

Fig. 1. A path surrounding a crack tip – Notations.

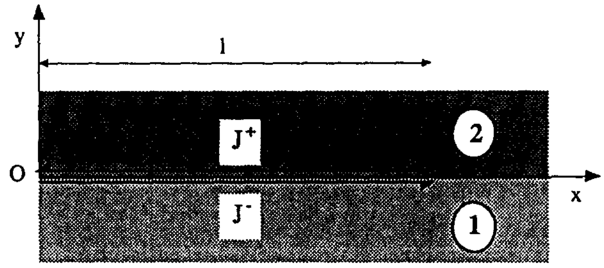

Fig. 2. A zero thickness path passing through an interface parallel to the crack plane.

of bonded structures, these hypotheses must be reviewed in detail: firstly, any path crossing the interfaces must be considered; secondly, the models commonly used for the stress analysis in these structures do not satisfy *a priori* the equilibrium equations; at last, the specific geometry of the bonded joints imposes some restrictive conditions on the stress field.

3. Multilayer geometry

We have seen that J is path independent for an elastic domain. The bonded structures consist of several materials, and it is necessary to verify that the above assumptions remain valid when the domain contains one or more interfaces. Of course, in the case of bonded structures, if the crack propagates parallel to the interface plane, it is sufficient to consider a path of zero thickness passing through an interface parallel to the crack plane (Fig. 2). The energy density term does not appear in this case, due to the fact that the integration is taken along the x-axis, and the J-integral is written

$$J = J^+ + J^- = -\int_0^1 \mathbf{T}_1 \frac{\partial \mathbf{u}_1}{\partial x} \mathrm{d}x - \int_1^0 \mathbf{T}_2 \frac{\partial \mathbf{u}_2}{\partial x}(-\mathrm{d}x). \tag{3}$$

We then see that, if the forces and displacements are continuous through the interface, this integral is zero. So we can say that, for a crack parallel to the interfaces of a multilayer structure, the J-integral is path independent.

4. Calculation of *J*-integral in the framework of Goland and Reissner approximations

4.1. Review: Goland and Reissner analysis

Let us briefly recall the basis of Goland and Reissner's analysis. Let us consider a bonded structure of unit width, subjected to in-plane loading. In this analysis, the adherends, indexed 1 and 2, are assumed to be elastic and to verify the Saint-Venant assumptions. The adhesive layer, of thickness h_0, is subjected to normal and shear stresses (σ_y, τ), thicknesswise [3]. The equilibrium conditions applied to an elementary cross section of each adherend yield

$$\frac{dN_1}{dx} = -\tau; \quad \frac{dQ_1}{dx} = -\sigma_y; \quad \frac{dM_1}{dx} = Q_1 - \frac{h_1}{2}\tau,$$
$$\frac{dN_2}{dx} = \tau; \quad \frac{dQ_2}{dx} = \sigma_y; \quad \frac{dM_2}{dx} = Q_2 - \frac{h_2}{2}\tau, \tag{4}$$

in which $N_i(x)$ and $Q_i(x)$ are the normal and the shear forces. The $M_i(x)$ are the bending moments (Fig. 3). The displacements and the rotations are related to the forces and the moments by the elastic constitutive equations

$$\frac{du_i}{dx} = C_i N_i; \quad \frac{dv_i}{dx} + \beta_i = \frac{Q_i}{B_i}; \quad \frac{d\beta_i}{dx} = -D_i M_i, \quad i = 1, 2, \tag{5}$$

with

$$C_i = \frac{1 - \nu_i^2}{h_i E_i}; \quad B_i = \tfrac{5}{6} h_i G_i; \quad D_i = \frac{12(1 - \nu_i^2)}{h_i^3 E_i} \quad \text{in planar strains,}$$

$$C_i = \frac{1}{h_i E_i}; \quad B_i = \tfrac{5}{6} h_i G_i; \quad D_i = \frac{12}{h_i^3 E_i} \quad \text{in planar stresses,}$$

where (u_i, v_i) is the displacement of the neutral line and β_i is the rotation of the ith adherend. E_i, ν_i and G_i are the elastic constants of the ith adherend. From the adhesive layer, due to the fact that it is very thin and according to the assumption of small rotations, the deformation

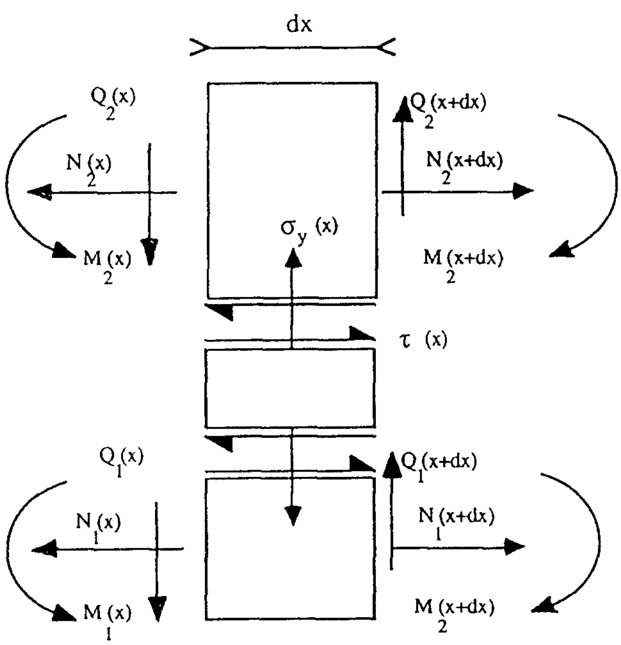

Fig. 3. Forces, moment and adhesive stresses in the adherend element and in the adhesive layer.

field is written

$$h_0\varepsilon_y = (v_2 - v_1),$$

$$h_0\gamma = u_2 - (\tfrac{1}{2}h_2)\beta_2 - u_1 - (\tfrac{1}{2}h_1)\beta_1. \tag{6}$$

From the latter, it follows

$$\frac{\mathrm{d}V}{\mathrm{d}x} \ll \gamma, \tag{7}$$

where V is the displacement along the y-direction in the adhesive joint. Moreover, in the equilibrium problem, we find that the in-plane tensile stress (σ_x) and shear stress are negligible through the entire vertical section of the joint. In other words, the junction is modeled by a tension-shear foundation and not by a tridimensional material. Therefore, the adhesive behavior can be defined by two one-dimensional constitutive equations:

$$\begin{cases}\tau = \tau(\varepsilon_y, \gamma), \\ \sigma_y = \sigma_y(\varepsilon_y, \gamma),\end{cases} \tag{8}$$

providing, in the elastic case

$$\begin{cases}\tau = \mu\gamma, \\ \sigma_y = (\lambda + 2\mu)\varepsilon_y,\end{cases} \tag{9}$$

λ and μ being the Lamé constants. For the elongation mode, we consider than $\varepsilon_{xx} = \varepsilon_{zz} = 0$, because, in general, the adhesive rigidity is very small compared to the substrate. By differentiating these expressions several times, and substituting (4) and (5) in them, we get a system of two differential equations in σ_y and τ, which can be solved in simple cases [3].

In accordance with the above, we observe that the approximations of Goland and Reissner's analysis may pose some problems for the computation of J. Namely, the equilibrium hypothesis is technically delicate because the field in the adhesive as it is proposed was not derived from the equilibrium equations and has no reasons to verify them. Therefore, it is useful to calculate J in these approximations.

4.2. Case of a closed path within an adherend

Let us first consider a closed, rectangular path of length l and height h, surrounding a beam (or a plate) element on a two-dimensional foundation (Fig. 4); to simplify notations, we have omitted the subscripts relative to the substrate. In this element, the stresses and displacements are given by

$$U(x, y) = u(x) - y\beta(x); \quad V(x, y) = v(x),$$

$$\sigma_{xx} = \frac{N}{h} + y\frac{DM}{hC}, \quad \int_{-h/2}^{h/2} \tau_{xy}\,\mathrm{d}y = Q, \tag{10}$$

where $u(x)$ and $v(x)$ are the x–y displacements of the neutral line and β, the rotation.

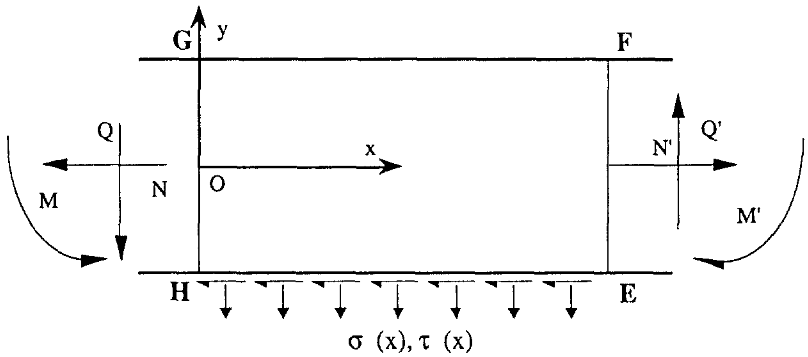

Fig. 4. Substrate element on a two-dimensional foundation.

According to the Saint-Venant hypotheses, the normal stress in the y-direction is assumed to be negligible compared with the other stress tensor terms. For this path, the J-integral can be written

$$J = J_{HE} + J_{EF} + J_{FG} + J_{GH}, \tag{11}$$

with

$$J_{HE} = \int_0^1 \left[\tau \left[\frac{\partial u}{\partial x} + \frac{h \partial \beta}{2 \partial x} \right] + \sigma_y \frac{\partial u}{\partial x} \right] \mathrm{d}x,$$

$$J_{EF} = \int_{-h/2}^{h/2} \left[W - \sigma_{xx} \frac{\partial}{\partial x} [u - \beta y] - \tau_{xy} \frac{\partial v}{\partial x} \right] \mathrm{d}y \quad (x = 1),$$

$$J_{FG} = 0$$

$$J_{GH} = - \int_{-h/2}^{h/2} \left[W - \sigma_{xx} \frac{\partial}{\partial x} [u - \beta y] - \tau_{xy} \frac{\partial v}{\partial x} \right] \mathrm{d}y \quad (x = 0). \tag{12}$$

Let us calculate J_{GH}; considering (4) and (5),

$$\int_{-h/2}^{h/2} W \, \mathrm{d}y = \int_{-h/2}^{h/2} \left[\int_0^\varepsilon \sigma_{xx} \, \mathrm{d}\varepsilon_{xx} + \tau_{xy} \, \mathrm{d}\gamma_{xy} \right] \mathrm{d}y$$

$$= \int_{-h/2}^{h/2} \left[hC \int_0^\sigma \sigma_{xx} \, \mathrm{d}\sigma_{xx} + \frac{h}{B} \int_0^\tau \tau_{xy} \, \mathrm{d}\tau_{xy} \right] \mathrm{d}y$$

$$= \frac{hC}{2} \left[h \frac{N^2}{h^2} + \frac{D^2 M^2}{h^2 C^2} \frac{h^3}{12} \right] + \frac{Q^2}{2B} = \frac{1}{2} \left[CN^2 + \frac{Q^2}{B} + DM^2 \right], \tag{13}$$

(since $C/M = h^3/12$).

Moreover

$$\int_{-h/2}^{h/2}\left[\sigma_x\frac{\partial}{\partial x}[u-\beta y]+\tau\frac{\partial v}{\partial x}\right]\mathrm{d}y=\int_{-h/2}^{h/2}\left[\frac{1}{hC}[CN+yDM]^2+\frac{Q}{h}\left[\frac{Q}{B}-\beta\right]\right]\mathrm{d}y$$

$$=CN^2+\frac{Q^2}{B}+DM^2-Q\beta. \tag{14}$$

Finally, J_{GH} is written

$$J_{GH}=\frac{1}{2}\left[CN^2+\frac{Q^2}{B}+DM^2\right]-Q\beta. \tag{15}$$

Similarly

$$J_{EF}=-\frac{1}{2}\left[CN'^2+\frac{Q'^2}{B}+DM'^2\right]-Q'\beta'. \tag{16}$$

Moreover, using relations (4) and (6)

$$J_{HE}=\int_0^1\left[CN\frac{\mathrm{d}N}{\mathrm{d}x}-DM\left[Q-\frac{\mathrm{d}M}{\mathrm{d}x}\right]+\frac{\mathrm{d}Q}{\mathrm{d}x}\left[\frac{Q}{B}-\beta\right]\right]\mathrm{d}x \tag{17}$$

$$=\frac{1}{2}\left[CN'^2+\frac{Q'^2}{B}+DM'^2-CN^2-\frac{Q^2}{B}-DM^2\right]+J' \tag{18}$$

with

$$J'=\int_0^1\left[-DMQ-\beta\frac{\mathrm{d}Q}{\mathrm{d}x}\right]\mathrm{d}x.$$

The integration gives

$$\int_0^1\beta\frac{\mathrm{d}Q}{\mathrm{d}x}dx=[\beta Q]_0^1-\int_0^1 Q\frac{\mathrm{d}\beta}{\mathrm{d}x}dx=[\beta Q]_0^1-\int_0^1 DMQ\,\mathrm{d}x \tag{19}$$

whence

$$J'=\beta Q-\beta'Q'. \tag{20}$$

Finally, from (15), (16), (18) and (20), we do find that the J-integral is zero.

4.3. Case of a closed path within the joint

Let us now consider a closed, rectangular path located in the joint, of length l, passing along the interfaces between the adhesive and the adherends (Fig. 5). For this path, the J-integral is

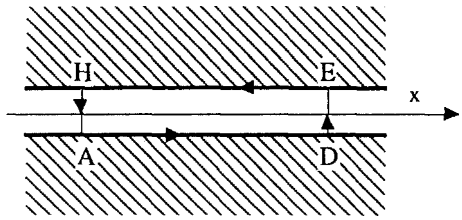

Fig. 5. A closed path within the joint.

written

$$J = J_{AD} + J_{DE} + J_{EH} + J_{HA} \tag{21}$$

with

$$J_{AD} = -\int_0^1 \mathbf{T}\frac{\partial \mathbf{u}}{\partial x}\,\mathrm{d}x = \int_0^1 \left[\tau(x)\frac{\partial U(x,-h_0/2)}{\partial x} + \sigma_y(x)\frac{\partial V(x,-h_0/2)}{\partial x}\right]\mathrm{d}x,$$

$$J_{EH} = -\int_1^0 T\frac{\partial \mathbf{u}}{\partial x}(-\mathrm{d}x) = -\int_0^1 \left[\tau(x)\frac{\partial U(x,h_0/2)}{\partial x} + \sigma_y(x)\frac{\partial V(x,h_0/2)}{\partial x}\right]\mathrm{d}x,$$

$$J_{DE} = h_0\, W(1) - \tau(l)\int_{-h_0/2}^{h_0/2} \frac{\partial V(l)}{\partial x}\,\mathrm{d}y = h_0\, W(1) - h_0\tau(1)V'(1),$$

$$J_{HA} = \quad h_0\, W(0) + r(0)\int_{h_0/2}^{h_0/2} \frac{\partial V(0)}{\partial x}\,\mathrm{d}y = -\, k_{,,}\, W(0) + \; h_0\tau(0)V'(0), \tag{22}$$

in which V'(1) et V'(0) are the mean values of $\partial V/\partial x$ through the thickness of the joint, at x = l and x = 0, respectively. Let us now group the integrals J_{AD} and J_{EH} together

$$J_{AD} + J_{EH} = -\int_0^1 \left[\tau(x)\frac{\partial}{\partial x}[U(x,h_0/2) \quad U(x,-h_0/2)]\right.$$

$$\left. + \;\sigma_y(x)\frac{\partial}{\partial x}[V(x,h_0/2) - V(x,-h_0/2)]\right]\mathrm{d}x \tag{23}$$

or

$$J_{AD} + J_{EH} = -\, h_0\int_0^1 \left[\tau(x)\frac{\partial \gamma}{\partial x} + \;\sigma_y(x)\frac{\partial \varepsilon_y}{\partial x}\right]\mathrm{d}x. \tag{24}$$

Equation (24) should be written as

$$J_{AD} + J_{EH} = h_0\int_0^{\varepsilon(0)} \tau\,\mathrm{d}\gamma + \sigma_y\,\mathrm{d}\varepsilon_y - h_0\int_0^{\varepsilon(1)} \tau\,\mathrm{d}\gamma + \sigma_y\,\mathrm{d}\varepsilon_y = h_0 W(0) \quad h_0 W(1). \tag{25}$$

This variable change is possible if the stress field in the joint derives from a potential (W). We note that in the Goland and Reissner analysis, this hypothesis is far less restrictive, since the usual plastic constitutive equations of the adhesive reduce to two simple laws ($\tau(\varepsilon_y, \gamma)$ and $\sigma_y(\varepsilon_y, \gamma)$), which are explicit in most cases, as when Tresca or modified Tresca-Coulomb criteria are used. We therefore deduce that

$$J = h_0 \tau(0) V'(0) - h_0 \tau(1) V'(1) \tag{26}$$

The simplified stress analysis of Goland and Reissner gives a nonzero J-integral for the closed path in the adhesive joint. However, according to (7) (low-rotation assumption), we can expect $\partial V/\partial x$ to be very small, so that the J-value for a closed path will be small. This point will be discussed further on.

4.4. *Computation of J for a debonded specimen*

Let us presently consider a bonded structure with a debond or a crack in the adhesive layer. We propose to compute the integral for two particular paths (Fig. 6). For the first, located within the joint, we have

$$J = J_{O'A} + J_{AD} + J_{DE} + J_{EH} + J_{HO}. \tag{27}$$

Since the section AH is unloaded ($J_{O'A} = J_{HO} = 0$), and according to relations (22) and (25)

$$J = h_0 W(0) - h_0 \tau V'(1). \tag{28}$$

Moreover, the strain energy density W is of the same order of magnitude as $\sigma_y \varepsilon_y + \tau\gamma$, so that (7) yields

$$h_0 \tau V'(1) \ll h_0 \tau\gamma \sim h_0 W(1) \leqslant W(0), \tag{29}$$

since the energy density in the joint is maximum in the vicinity of the debond tip. Hence (28) provides:

$$J = h_0 W(0). \tag{30}$$

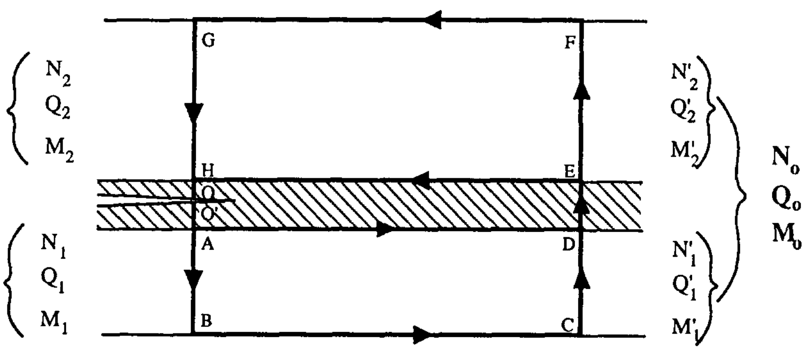

Fig. 6. Schematic representation of a debonded specimen.

To conclude, this computation shows that in the Goland and Reisner analysis, the J-integral is nearly path independent, and is equal to the product of the joint thickness by the energy density at the bond termination. Relation (30) also reveals the mode I and mode II components

$$J_{\mathrm{I}} = h_0 \int_0^{\varepsilon_y(0)} \sigma_y \, d\varepsilon_y; \quad J_{\mathrm{II}} = h_0 \int_0^{\gamma(0)} \tau \, d\gamma. \tag{31}$$

J can also be computed as a function of the loading conditions. For that, let us now consider the path $ABCDEFGH$ shown in Fig. 6. The portions BC and FG are located on the external faces of the adherends and are free of any force; we can write

$$J = J_{AB} + J_{CD} + J_{DE} + J_{EF} + J_{GH}. \tag{32}$$

The expressions for these components have already been determined, and are given by (15) and (16). Finally, we have

$$J = \frac{1}{2}\left[C_1(N_1^2 - N_1'^2) + \frac{Q_1^2 - Q_1'^2}{B_1} + D_1(M_1^2 - M_1'^2) + C_2(N_2^2 - N_2'^2) \right.$$
$$\left. + \frac{Q_2^2 - Q_2'^2}{B_2} + D_2(M_2^2 - M_2'^2) \right] - Q_1\beta_1 + Q_1'\beta_1' - Q_2\beta_2 + Q_2'\beta_2' + J_{DE}, \tag{33}$$

with $J_{DE} = h_0 W(1)$. We note that in general, the computation of J requires the knowledge of the loading conditions and the constitutive equations of the adhesive ($h_0 W(1)$ term). However, for a slender bonded structure, since the stresses in the adhesive layer decrease exponentially from the ends of the bond [3], the cross section $CDEF$ can be chosen far from the crack, so that the joint section DE is not loaded. Therefore

$$J_{DE} \sim 0. \tag{34}$$

We find in that case the relationship obtained by Fernlund and Spelt [13], with additional terms (Q_i^2/B_i and $Q_i'^2; B_i$) due to the shear energy stored in the adherends, which is taken into account here. Thus, for a slender specimen, the only contributions come from the cross sections in the adherends. In that case, expression (33) can be improved upon. Due to the fact that the stress state is zero in the joint on section DE, the displacements of the upper and lower surfaces of the adhesive layer and their derivatives are equal. Hence, for the section DE, we can write [15]

$$v_2 = v_1; \quad \beta_1' - \frac{Q_1'}{B_1'} = \beta_2' - \frac{Q_2'}{B_2'}; \quad D_1 M_1' = D_2 M_2'$$

$$u_2 - (\tfrac{1}{2}h_2)\beta_2 = u_1 + (\tfrac{1}{2}h_1)\beta_1; \quad C_1 N_1' - (\tfrac{1}{2}h_1) D_1 M_1' = C_2 N_2' + (\tfrac{1}{2}h_2) D_2 M_2', \tag{35}$$

$$N_0 = N_1' + N_2'; \quad Q_0 = Q_1' + Q_2',$$

$$M_0 = M_1' + N_1' \frac{C_1}{C_1 + C_2}\left[\frac{h_1 + h_2}{2}\right] + M_2' - N_2' \frac{C_2}{C_1 + C_2}\left[\frac{h_1 + h_2}{2}\right], \tag{36}$$

in which N_0, Q_0, M_0 are the loads and the moment applied on the whole bonded structure. The J-integral can be written in the form

$$J = \frac{1}{2}\left[C_1 N_1^2 + C_2 N_2^2 - C_0 N_0'^2 + \frac{Q_1^2}{B_1} + \frac{Q_2^2}{B_2} - \frac{Q_0^2}{B_0} \right.$$

$$\left. + D_1 M_1^2 + D_2 M_2^2 - D_0 M_0^2 \right] - Q_1 \beta_1 - Q_2 \beta_2 + Q_0 \beta_0, \tag{37}$$

in which C_0, D_0 and B_0 are the tensile, bending and shear stiffnesses of the sandwich composed of two adherends perfectly bonded together

$$\frac{1}{C_0} = \frac{1}{C_1} + \frac{1}{C_2}; \quad \frac{1}{D_0} = \frac{(h_1 + h_2)^2}{4(C_1 + C_2)} + \frac{1}{D_1} + \frac{1}{D_2}; \quad B_0 = B_1 + B_2. \tag{38}$$

We therefore get a simple expression of the J-integral as a function of the geometry, elastic constants of the adherends and loading conditions. A few examples are given in Appendix I to show that this expression can be used to work back to the strain energy release rate for most classical fracture mechanics specimens. Of course, the use of J-integral becomes far more useful in the case of nonlinear adhesives or when the loaded part spreads on the whole overlap (i.e. when (37) is not valid). It is shown in Appendix II that relationship (33) allows finding in a simple way the result obtained by Hu et al. [12, 16] for a tensile lap shear specimen bonded with an elastic-perfectly plastic adhesive.

5. Conclusion

In this article, we have computed J for a bonded structure verifying the following assumptions

- the adherends are likened to elastic beams subjected to low levels of rotation;
- the stresses in the joint are assumed to be uniform thicknesswise;
- the two one-dimensional constitutive equations of the adhesive ($\sigma_y(\gamma, \varepsilon_y)$ and $\tau(\gamma, \varepsilon_y)$) are explicit and do not depend on the loading history.

Under such conditions, the J-integral is shown to be nearly path independent. By computing it along two particular paths, the following results were established. Firstly, this integral is equal to the product of the strain energy at the end of the joint (i.e. at the debond tip) by its thickness. This result clearly reveals the contributions of the opening mode I due to the tensile stress normal to the debond plane and the shearing mode II due to in-plane shear stresses. By calculating J around a path located far from the crack tip, we have shown here that, in the cases of small scale yielding conditions and for usual slender fracture mechanics specimens, the expression for J can be used to work back to the strain energy release rate expressions calculated from the beam theory.

At last, we should note that the hypothesis on the adhesive constitutive laws is not very restrictive, since the behavior of the adhesive is defined by two independent one-dimensional constitutive equations. This makes J-integral applicable to nonlinear elastic materials and to monotonic plastic loadings. Nevertheless the use of J-integral as a fracture parameter remains questionable in the case of nonmonotonic plastic loadings or when the crack propagates after plasticization, since unloading then occurs in the adhesive.

In conclusion, we think the J-integral is a promising fracture parameter for designing bonded structures, because, with the simplified analytical approaches, the most important difficulty is to have a realistic fracture criterion. The use of the J-integral solves this problem because the critical values of the J-integral can be determined by simple fracture mechanics tests. The other interest is that the present analysis makes possible the use of more compact specimens for which the linear fracture mechanics is not valid.

Appendix I: J-Integral computation for fracture mechanics specimens

Here we use relation (37) to compute the energy release rate for particular geometries. To simplify the expressions, we assume that the structures are subjected to in-plane stresses.

DCB specimen (Fig. A.1)

The loading conditions here are

$$\begin{aligned} &N_1 = N_2 = 0; \quad M_1 = M_2 = 0; \quad Q_1 = -Q_2 = P; \quad \beta_1 = -\beta_2 = -\tfrac{1}{2}DPa^2, \\ &N_0 = 0; \quad M_0 = 0; \quad Q_0 = 0; \quad \beta_0 = 0; \end{aligned} \tag{A.1}$$

or

$$J = \frac{Q_1^2}{B_1} + \frac{Q_2^2}{B_2} - Q_1 B_1 - Q_2\beta_2 = \frac{P^2}{B} + DP^2a^2 = \frac{P^2}{E}\left[\frac{12a^2}{h^3} + \frac{6E}{5hG}\right]. \tag{A.2}$$

CLS specimen under pure tensile stress (or DCLS specimen) (Fig. A.2)

$$\begin{aligned} &N_1 = P; \quad N_2 = 0; \quad M_1 = M_2 = 0; \quad Q_1 = Q_2 = 0; \quad \beta_1 = \beta_2 = 0, \\ &N_0 = P; \quad M_0 = 0; \quad Q_0 = 0; \quad \beta_0 = 0; \end{aligned} \tag{A.3}$$

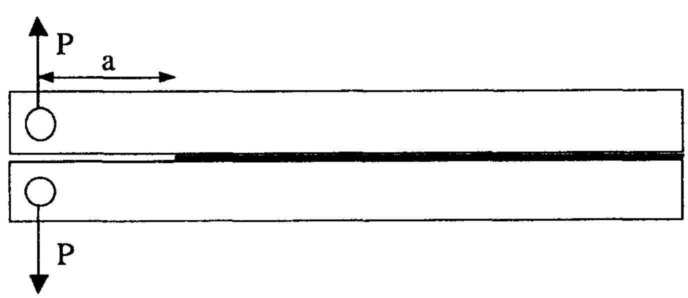

Fig. A.1. Double Cantilever Beam Specimen (DCB).

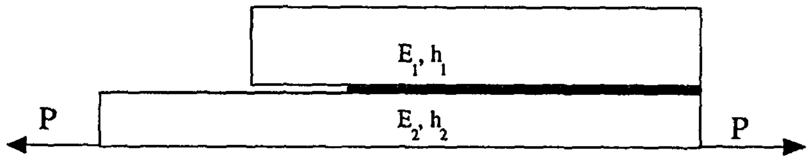

Fig. A.2. Crack lap shear specimen under pure tensile stress (or DCLS specimen).

or

$$J = [C_1 N_1^2 - C_0 N_0'^2] = \tfrac{1}{2}P^2\left[\frac{1}{E_1 h_1} - \frac{1}{E_1 h_1 + E_2 h_2}\right]. \tag{A.4}$$

MMF specimen (Fig. A.3)

$$N_1 = N_2 = M_1 = M_2 = 0; \quad Q_1 = \tfrac{1}{2}P; \quad Q_2 = 0; \quad \beta_0 - \beta_1 = -\tfrac{1}{4}P[D_1 a^2 + D_0(w^2 - a^2)],$$

$$N_0 = 0 : M_0 = \tfrac{1}{2}PW; \quad Q_0 = \tfrac{1}{2}P.$$

Here, we have $h_1 = h_2$ and $E_1 = E_2$, or $B_0 = 2B_1\ a$, $dD_1 = 8D_0$, whence

$$J = \tfrac{1}{8}P^2\left[\frac{1}{B_1} - \frac{1}{B_0} - W^2 D_0^2\right] + \tfrac{1}{2}P(\beta_0 - \beta_1) = \frac{21}{16}\frac{P^2 a^2}{Eh^3} + \frac{3}{40}\frac{P^2}{EGh}. \tag{A.5}$$

Symmetrical CLS specimen under pure tensile stress (Fig. A.4)

$$N_1 = P; \quad N_2 = 0; \quad M_1 = M_2 = 0; \quad Q_1 = Q_2 = 0; \quad \beta_1 = \beta_2 = 0,$$

$$N_0 = P; \quad M_0 = P\delta; \quad Q_0 = 0; \quad \beta_0 = 0; \tag{A.6}$$

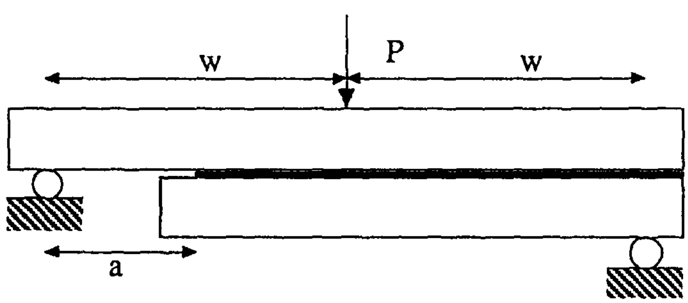

Fig. A.3. Mixed mode flexure specimen (MMF).

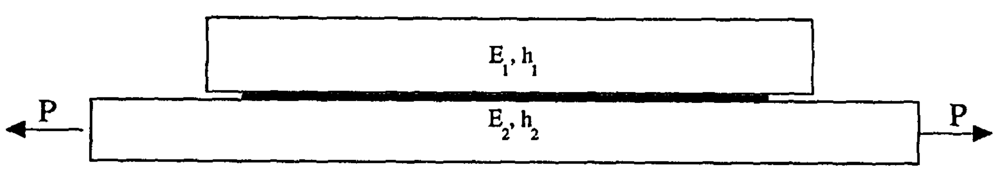

Fig. A.4. Symmetrical crack lap shear specimen under pure tensile stress.

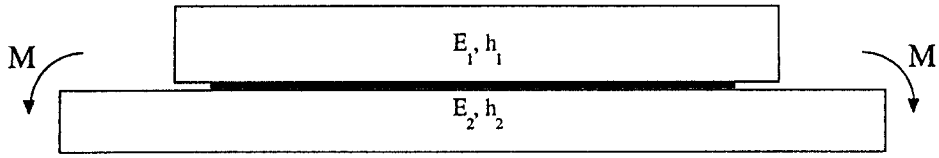

Fig. A.5. Symmetrical crack lap shear specimen under bendings stress.

in which δ is the distance between the neutral lines of beam 1 and the neutral line of the whole structure

$$\delta = \frac{C_1}{C_1 + C_2}\left[\frac{h_1 + h_2}{2}\right] = \frac{E_2 h_2 (h_1 + h_2)}{E_1 h_1 + E_2 h_2}, \tag{A.7}$$

thus

$$J = \tfrac{1}{2}[C_1 N_1^2 - C_0 N_0'^2 - D_0 M_0^2] = \tfrac{1}{2}P^2\left[\frac{1}{E_1 h_1} - \frac{1}{E_1 h_1 + E_2 h_2} - D_0 \delta^2\right] \tag{A.8}$$

Symmetrical CLS specimen under bending stress (Fig. A.5)

$$N_1 = N_2 = 0; \quad M_1 = M; \quad M_2 = 0; \quad Q_1 = Q_2 = 0; \quad \beta_1 = \beta_2 = 0,$$
$$N_0 = 0; \quad M_0 = M; \quad Q_0 = 0; \quad \beta_0 = 0; \tag{A.9}$$

thus

$$J = \tfrac{1}{2}M^2[D_1 - D_0] \tag{A.10}$$

In these computations, the structures are assumed to be of unit width. We then confirm the results found in the literature, by dividing the expressions obtained by the square of the specimen width.

Appendix II

Let us consider a tensile lap shear specimen bonded with an elastic-perfectly plastic adhesive (Fig. A.6). The constitutive equation in shear of the adhesive can be written as:

$$\begin{cases} \gamma < \gamma_p : \tau = G\gamma, \\ \gamma \geqslant \gamma_p : \tau = \tau_p. \end{cases} \tag{A.11}$$

We consider a path surrounding the plastic zone and crossing both adherends. For this path, (33) is written

$$J = \tfrac{1}{2}[C_1(N_1^2 - N_1'^2) - C_2 N_2'^2] + J_{DE}, \tag{A.12}$$

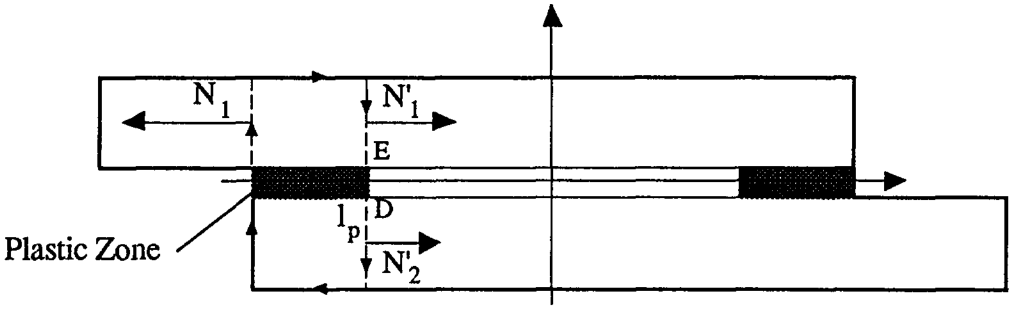

Fig. A.6. Tensile lap shear specimen with partial yielding of the adhesive layer.

with

$$N_1' = N_1 - \tau_p l_p; \quad N_2' = \tau_p l_p; \quad C_1 = C_2 = \frac{1}{Eh}; \quad J_{DE} = h_0 \int_0^{\varepsilon} \tau(l_p)\, \mathrm{d}\gamma,$$

where l_p is the length of the plastic zone. Thus

$$J = \frac{1}{2Eh}[N_1^2 - [N_1 - \tau_p l_p]^2 - \tau_p^2 l_p^2] + J_{DE}$$

$$= \frac{N_1 \tau_p l_p}{Eh} - \frac{\tau_p^2 l_p^2}{Eh} + \frac{h_0 \tau_p^2}{2G}. \tag{A.13}$$

Therefore, we obtain the expression derived from Hu et al. [16]. We note that (A.11) is only valid when the adhesive is partially plasticized.

References

1. O. Volkersen, *Luftfahrtforshung* 15 (1938) 4–47.
2. M. Goland and E. Reissner, *Journal of Applied Mechanics* A11 (1944) 17–27.
3. F. Delale, F. Erdogan and M.N. Aydinoglu, *Journal of Composite Materials* 15 (1981) 249–267.
4. L.J. Hart-Smith, *Developments in Adhesives-2*, A.J. Kinloch (ed) Applied Science Publishers, London (1981) 1–44.
5. F. Delale and F. Erdogan, *Journal of Composite Materials* 5 (1981) 560–581.
6. S. Mostovoy and E.J. Ripling, *Journal of Applied Polymer Science* 15 (1971) 641–654.
7. T.R. Brussat, S.T. Chiu and S. Mostovoy, 'Fracture mechanics for structural adhesive bonds', Air Force National Laboratory, AFML-TR-77-163 (1977).
8. A.J. Kinloch, *Adhesion and Adhesives, Science and Technology*, Chapman and Hall, London (1990) Chap 7.
9. W.S. Johnson, *Journal of Testing and Evaluation* 15(6) (1987) 303–324.
10. J.G. Williams, *International Journal of Fracture* 36 (1988) 101–108.
11. S.E. Yamada, *Engineering Fracture Mechanics* 27 (1987) 315–328.
12. G.K. Hu, 'Etude théorique et expérimentale du comportement, de l'endommagement et de la rupture des assemblages collés', Thèse de l'Ecole Centrale de Paris, 1991.
13. G. Fernlund and J.K. Spelt, *Engineering Fracture Mechanics* 40 (1991) 119–132.
14. J.R. Rice, *Journal of Applied Mechanics, Transactions of the ASME* June (1968) 379–386.
15. P. Fraisse, 'Caractérisation mécanique et ultrasonore de structures collées – Cas des assemblages Verre/Epoxy', Thèse de l'université Paris 6, 1991.
16. G.K. Hu, F. Schmit and D. Francois, *Journal of Adhesion*, to be published.