THE ALMOST CONSTANT-SPEED TWO-BODY PROBLEM WITH RESISTANCE

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Abstract. The almost constant-speed motion of a mass acted upon by a Newtonian attraction and a resisting force is treated. The equation of orbit is derived for a specific type of resistance which covers the familiar case of Danby's drag$(=\alpha r^{-2})$ whilst the vector invariants are obtained by direct operation on the vector form of the equation of motion.

Key words: Artificial satellite, drag effect.

1. Introduction

The classical two-body problem, in which the moving, less massive body is further acted upon by a resisting force, has been an important topic for investigators, especially during the last decades, because it is applicable to the study of motion of artificial satellites in the upper regions of the Earth's atmosphere. The resulting motion has been extensively discussed by means of either closed or analytic solutions of the equations of motion (Mittleman and Jezewski, 1982, 1983; Leach, 1987; Danby, 1962; Brouwer and Hori, 1961 etc.). In most studies the resistance is proportional to the velocity and inversely proportional to the square of the radial distance.

Undoubtedly, in the totality of orbits there exist some which are described with a speed that varies so slowly that its rate of change with time is approximately equal to zero. These orbits may be found if, together with the equations of motion, one considers the additional condition $\mathrm{d}^2s/\mathrm{d}t^2 \cong 0$. In order to avoid this apparently difficult task, we reformulate the problem and seek the solution of the conditioned equations of motion. The study also deals with the vector invariants of this particular motion and is accomplished with the exposition of the general solution in a simple compact form which is derived by means of one of these constant vectors.

2. Statement of the Problem and a Scalar Invariant Related to Moment of Momentum

The force field consists of the mutual attraction of the two bodies $C(\mathbf{r})\mathbf{r}$ and a tangential resistance $R(\mathbf{r}, \mathbf{v})\mathbf{v}$, where C and R are scalar functions and $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ are, respectively, the radius vector, the velocity and the acceleration of the moving mass with respect to the attracting centre. Hence, the equation of motion in vector form is

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given by

$$\mathbf{a} + R(\mathbf{r}, \mathbf{v})\mathbf{v} + C(\mathbf{r})\mathbf{r} = \mathbf{0} \tag{1}$$

As we demonstrate below, the orbit of (1) is planar and we use plane polar coordinates (r, θ).

If, according to the assumption about the rate of change of speed, we expand v in Taylor series of the polar angle θ, we have

$$v = v_0 + \left.\frac{\mathrm{d}v}{\mathrm{d}\theta}\right|_{\theta_0}(\theta - \theta_0) + \left.\frac{\mathrm{d}^2 v}{2\,\mathrm{d}\theta^2}\right|_{\theta_0}(\theta - \theta_0)^2 + \cdots \tag{2}$$

where θ_0 is the initial value of θ and $v_0 = v(\theta_0)$. Neglecting the second and higher order terms and denoting by ε the small quantity $(\mathrm{d}v/\mathrm{d}\theta)_{\theta_0}$ we may approximate the speed by

$$v = v_0 + \varepsilon(\theta - \theta_0). \tag{3}$$

As the motion is expected to be a spiral, we see that for small values of ε ($<10^{-5}$) the moving mass must revolve many times about the origin before the quantity $|v - v_0|$ starts to increase perceptibly.

From Equation (3) the tangential and normal components of acceleration are, respectively

$$\mathbf{a}_T = \dot{v}\mathbf{T} = \varepsilon\dot{\theta}\mathbf{T} \quad \text{and} \quad \mathbf{a}_N = \frac{v^2}{\rho}\mathbf{N} = \frac{1}{\rho}\left[v_0 + \varepsilon(\theta - \theta_0)\right]^2\mathbf{N}$$

where $\mathbf{T}$, $\mathbf{N}$ are the unit vectors along the tangent and the normal to it and ρ is the radius of curvature. Introducing these expressions into Equation (1) and taking into account the relation

$$\mathbf{e}_r = \frac{\mathbf{r}}{r} = -\sin\varphi\mathbf{T} - \cos\varphi\mathbf{N} \tag{4}$$

where φ is the angle between the radial distance r and the distance P of the origin O from the tangent (Figure 1), we obtain the equations of motion in the form

$$\varepsilon\dot{\theta} + Rv - Cr\sin\varphi = 0, \tag{5a}$$

$$\frac{v^2}{\rho} - Cr\cos\varphi = 0. \tag{5b}$$

The evaluation of φ is found in the Appendix.

Since we examine a particular case of the two-body problem with resistance, it is useful to derive the basic aspects of the motion. We shall show that at any time the moving mass lies in a plane. Taking the cross product of Equation (1) with the vector $\mathbf{r}$

$$\mathbf{r}\times\mathbf{a} + R\mathbf{r}\times\mathbf{v} + C\mathbf{r}\times\mathbf{r} = \mathbf{0}. \tag{6}$$

With $\mathbf{H} = \mathbf{r}\times\mathbf{v}$, the above equation takes the form

$$\dot{\mathbf{H}} + R\mathbf{H} = \mathbf{0} \tag{7}$$

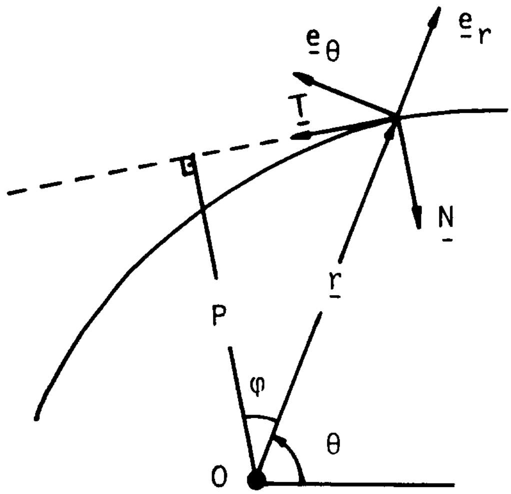

Fig. 1.

from which we readily obtain

$$\mathbf{H} \times \dot{\mathbf{H}} = \mathbf{0}$$

(the dot denotes differentiation with respect to time). This expression admits a first vector invariant which is the constant direction $\mathbf{e}_H = \mathbf{H}/H$ of the moment of momentum H. Therefore the motion is planar. This fact enables us to simplify Equation (7) by writing

$$\dot{H} + RH = 0. \tag{7a}$$

We see that in the presence of a resisting force, the magnitude of the moment of momentum is not conserved, although its direction is. As the magnitude H depends on the function R, we suppose, in order to carry out the integration of Equation (7a), that the resistance is a continuous function of H, θ, and $\dot{\theta}$ which may be written as a product of separable functions

$$R = R_1(H)R_2(\theta)\dot{\theta}. \tag{8}$$

Then, on substituting in (7a) we obtain

$$\int \frac{\mathrm{d}H}{HR_1(H)} = -\int R_2(\theta)\,\mathrm{d}\theta. \tag{9}$$

Apart from the well known Danby drag model (1962) which, for $R_1 = H^{-1}$ and $R_2 = \alpha$ (constant), gives

$$H = r^2\dot{\theta} = h - \alpha\theta \tag{10}$$

(h being the constant of integration), Gorringe and Leach (1988) propose a new, more general, type of resistance, which has the advantage that it covers the case of the drag function $R = \alpha r^{-2}$. More precisely they consider the resistance

$$R = H^{-1}(\alpha + b\cos\theta)\dot{\theta}, \quad \alpha, b \text{ const.} \tag{11}$$

which yields

$$H = r^2\dot{\theta} = h - \alpha\theta - b\sin\theta. \tag{12}$$

Notice that neglecting both constants α and b we have the simple form $H = h$ which, in association with the unit vector $\mathbf{e}_H$, furnishes the known integral $\mathbf{H} = \text{constant}$, which fits into any central field.

3. The Equation of the Orbit

The derivation of the equation of the orbit is reflected in our choice of the functions R and C. Adopting the form of the Newtonian central attraction

$$C = \frac{\mu}{r^3} \tag{13}$$

where μ is the gravitational constant, we have from Equation (5a) that

$$\varepsilon\dot{\theta} + Rv - \frac{\mu}{r^2}\sin\varphi = 0. \tag{14}$$

We now prove that

$$\sin\varphi = -\frac{\mathrm{d}r}{\mathrm{d}s} \tag{15}$$

where s is arc-length.
If we differentiate the expression $r^2 = \mathbf{r}\cdot\mathbf{r}$ we have

$$\frac{\mathbf{r}}{r}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = 1$$

or, since $\mathbf{T} = \mathrm{d}\mathbf{r}/\mathrm{d}s$

$$\mathbf{e}_r\cdot\mathbf{T}\frac{\mathrm{d}s}{\mathrm{d}r} = 1. \tag{16}$$

But as

$$\mathbf{e}_r\cdot\mathbf{T} = \cos(\tfrac{1}{2}\pi + \varphi) = -\sin\varphi$$

the desired relation (15) is obtained immediately from (16). Thus Equation (14) becomes

$$\varepsilon\dot{\theta} + Rv + \frac{\mu}{r^2}\frac{\mathrm{d}r}{\mathrm{d}s} = 0 \tag{17}$$

or, after changing the independent variable from s to θ via the operator

$$\frac{\mathrm{d}}{\mathrm{d}s}(\ ) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\ )\frac{\dot{\theta}}{v}$$

and using Equation (8)

$$\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{1}{\mu}(\varepsilon v + R_1 R_2 v^2). \tag{18}$$

On using the substitution (3) and neglecting terms of order ε^2, we finally obtain

$$\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{v_0}{\mu}[\varepsilon + R_1 R_2 v_0 + 2\varepsilon R_1 R_2(\theta - \theta_0)]. \tag{19}$$

This is the differential equation of the problem in hand whose solution is obtained analytically provided that both products $R_1 R_2$ and $R_1 R_2(\theta - \theta_0)$ are integrable. For example, if we consider the resistance (11) with $b = 0$, we have $R_1(H) = H^{-1} = (h - \alpha\theta)^{-1}$, $R_2(\theta) = \alpha$ (the Danby drag) and then, after carrying out the integration

$$\frac{1}{r} - \frac{1}{r_0} = -\frac{v_0}{\mu}\left\{\varepsilon(\theta - \theta_0) + \left(v_0 + \frac{2\varepsilon}{\alpha}H_0\right)\ln\frac{h - \alpha\theta}{h - \alpha\theta_0}\right\} \tag{20}$$

where $r_0 = r(\theta_0)$ and $H_0 = H(\theta_0)$.

The second equation of system (5), as may easily be proved, is an identity.

4. The Vector Invariants of Motion

We attempt now to derive the Laplace-like vector invariant by taking the cross product of Equation (1) with $\lambda\mathbf{H}$. The factor λ is a scalar function of time which can establish the time-dependence of the invariant. Now

$$\lambda\mathbf{a} \times \mathbf{H} + \lambda R\mathbf{v} \times \mathbf{H} + \lambda C\mathbf{r} \times \mathbf{H} = \mathbf{0} \tag{21}$$

and if we take into account the expression $\dot{\mathbf{H}} = -R\mathbf{H}$ and the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda\mathbf{v} \times \mathbf{H}) = \dot{\lambda}\mathbf{v} \times \mathbf{H} + \lambda\mathbf{a} \times \mathbf{H} + \lambda\mathbf{v} \times \dot{\mathbf{H}}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda\mathbf{v} \times \mathbf{H}) + (2\lambda R - \dot{\lambda})\mathbf{v} \times \mathbf{H} + \lambda C\mathbf{r} \times \mathbf{H} = \mathbf{0}. \tag{22}$$

In order to eliminate the second term, we require

$$\dot{\lambda} = 2R\lambda.$$

This equation, in view of (7), becomes

$$\dot{\lambda}/\lambda = -2\dot{H}/H$$

with solution

$$\lambda = 1/H^2. \tag{23}$$

We notice here the difference between this expression for λ and the corresponding result $v = H^{-1}$ given by Jezewski and Mittleman (1983), where v there represents our function λ. This difference is due to the fact that they have used the single term vR in expression (22), arriving at a differential equation for the function v differing by a factor 2 from the relevant equation for the function λ.

If we introduce this expression for λ into Equation (22) and put $C = \mu r^{-3}$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{v}\times\mathbf{H}}{H^2}\right) + \frac{\mu}{H^2}\frac{\mathbf{r}\times\mathbf{H}}{r^3} = \mathbf{0}. \tag{24}$$

Recognizing that

$$\frac{\mathbf{r}\times\mathbf{H}}{r^3} = -\dot{\theta}\mathbf{e}_\theta$$

where $\mathbf{e}_\theta$ is the unit vector pointing in the normal direction $\dot{\mathbf{e}}_r$, we may integrate Equation (24) to achieve

$$\frac{\mathbf{v}\times\mathbf{H}}{H^2} - \mu\int\frac{\mathbf{e}_\theta}{H^2}\,\mathrm{d}\theta = \mathbf{J} \tag{25}$$

where $\mathbf{J}$ is a constant vector which is the analogue of the Laplace–Runge–Lenz vector. In order to evaluate the second term of the integral, we put

$$H^{-2}(\theta) = \frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} + g \tag{26}$$

where the function $g = g(\theta)$ has the form

$$g(\theta) = A\cos\theta + B\sin\theta$$

with

$$A = -\int_{\theta_0}^{\theta}\frac{\sin\eta\,\mathrm{d}\eta}{H^2(\eta)}, \qquad B = \int_{\theta_0}^{\theta}\frac{\cos\eta\,\mathrm{d}\eta}{H^2(\eta)}.$$

Integrating by parts and using the well known relations $\mathrm{d}\mathbf{e}_r = \mathbf{e}_\theta\,\mathrm{d}\theta$, $\mathrm{d}\mathbf{e}_\theta = -\mathbf{e}_r\,\mathrm{d}\theta$ we have

$$\begin{aligned}\int\frac{\mathbf{e}_\theta}{H^2}\,\mathrm{d}\theta &= \int(g''+g)\mathbf{e}_\theta\,\mathrm{d}\theta = \int\mathbf{e}_\theta\,\mathrm{d}g' + \int g\mathbf{e}_\theta\,\mathrm{d}\theta = g'\mathbf{e}_\theta - \int g'\,\mathrm{d}\mathbf{e}_\theta + \int g\mathbf{e}_\theta\,\mathrm{d}\theta\\ &= g'\mathbf{e}_\theta + \int\mathbf{e}_r\,\mathrm{d}g + \int g\mathbf{e}_\theta\,\mathrm{d}\theta = g'\mathbf{e}_\theta + g\mathbf{e}_r - \int g\,\mathrm{d}\mathbf{e}_r + \int g\mathbf{e}_\theta\,\mathrm{d}\theta = g'\mathbf{e}_\theta + g\mathbf{e}_r.\end{aligned}$$

This result, together with the relations

$$\mathbf{v} = [v_0 + \varepsilon(\theta - \theta_0)]\mathbf{T}$$
$$\mathbf{N} = -\cos\varphi\,\mathbf{e}_r - \sin\varphi\,\mathbf{e}_\theta \tag{27}$$
$$H = rv\sin(\tfrac{1}{2} + \varphi) = Pv$$

after some rearrangement gives

$$\mathbf{J} = \left(\frac{1}{r} - \mu g\right)\mathbf{e}_r + \left(\frac{\tan\varphi}{r} - \mu g'\right)\mathbf{e}_\theta. \tag{28}$$

Obviously this vector is normal to the first vector invariant $\mathbf{e}_H$. As a consequence the cross product of these vectors is also an invariant

$$\mathbf{K} = \mathbf{J}\times\mathbf{e}_H = \left(\frac{\tan\varphi}{r} - \mu g'\right)\mathbf{e}_r - \left(\frac{1}{r} - \mu g\right)\mathbf{e}_\theta. \tag{29}$$

The vector integral (28) enables us to take the relationship between r and θ that expresses the equation of orbit, instead of Equation (20). Indeed, if we consider that θ is measured from the axis of the vector $\mathbf{J}$, then

$$\mathbf{J}\cdot\mathbf{e}_r = J\cos\theta = \frac{1}{r} - \mu g$$

and

$$r = \frac{1}{J\cos\theta + \mu g}. \tag{30}$$

As is expected, since Equation (30) is a particular case of the two-body problem with resistance, this closed form is identical to that derived by Leach (1987).

5. Illustrative Examples

Figures 2–7 illustrate examples of motion (20) for several values of v_0 and the same constant $\mu = 10$. Parameter h was chosen equal to 10 for five orbits and equal to 25 for the last orbit. Factor α in Danby's drag expression was represented by four different values, and $\varepsilon = 10^{-6}$.

Comparing Figures 2, 3, 4, 6 we see that the drag effect is more pronounced as factor α goes from 0.01 to 1. Also we remark that the spiral becomes more dense as v_0 decreases (Figures 4, 5). The same happens when h increases (Figures 6, 7).

Appendix: Computational Sequence for Angle φ

From (15) and (27) we have

$$\frac{p^2}{r^2} + \left(\frac{dr}{d\theta}\right)^2\frac{\dot{\theta}^2}{v^2} = \cos^2\varphi + \sin^2\varphi = 1$$

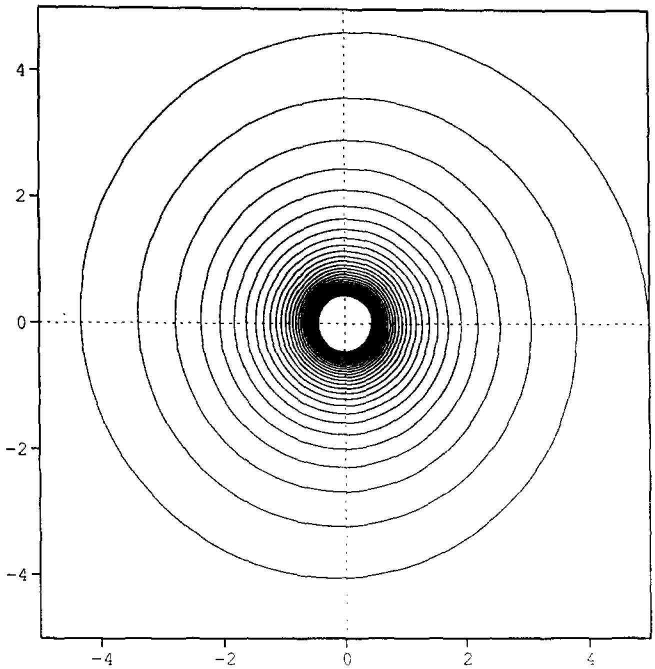

Fig. 2. Orbit for $h = 10$, $\alpha = 0.01$, $v_0 = 10$.

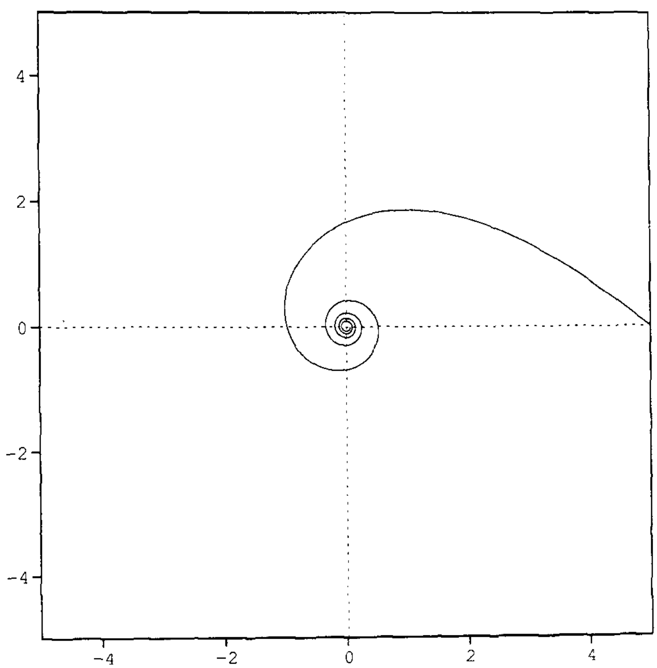

Fig. 3. Orbit for $h = 10$, $\alpha = 0.25$, $v_0 = 10$.

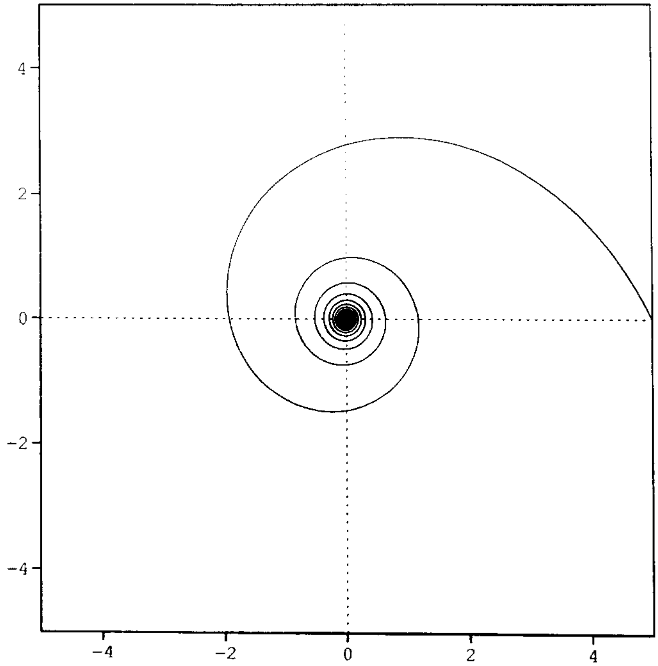

Fig. 4. Orbit for $h = 10$, $\alpha = 0.10$, $v_0 = 10$.

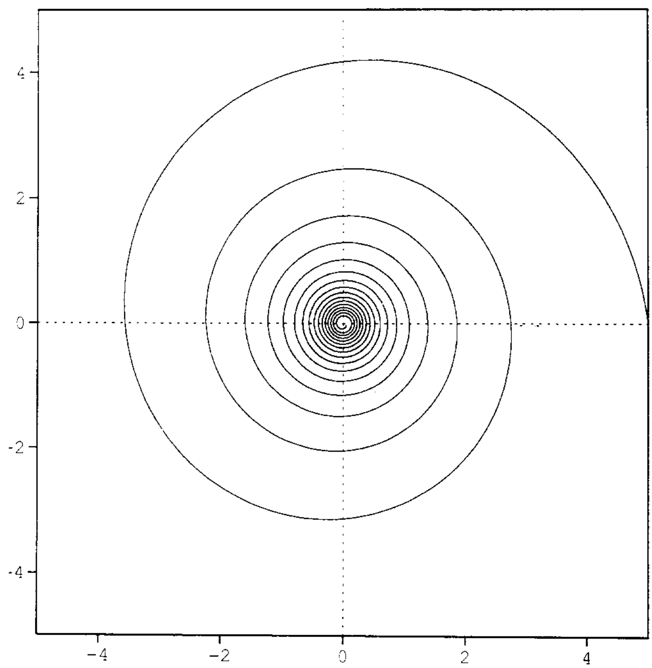

Fig. 5. Orbit for $h = 10$, $\alpha = 0.10$, $v_0 = 5$.

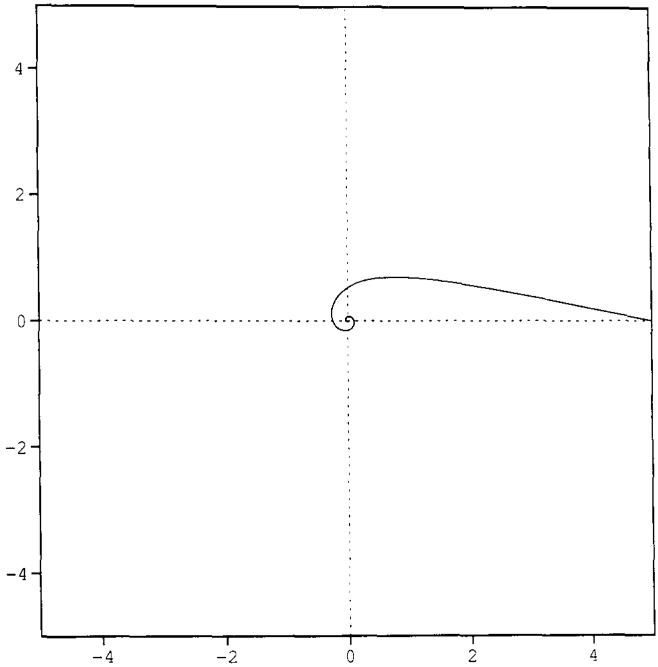

Fig. 6. Orbit for $h=10$, $\alpha=1.00$, $v_0=10$.

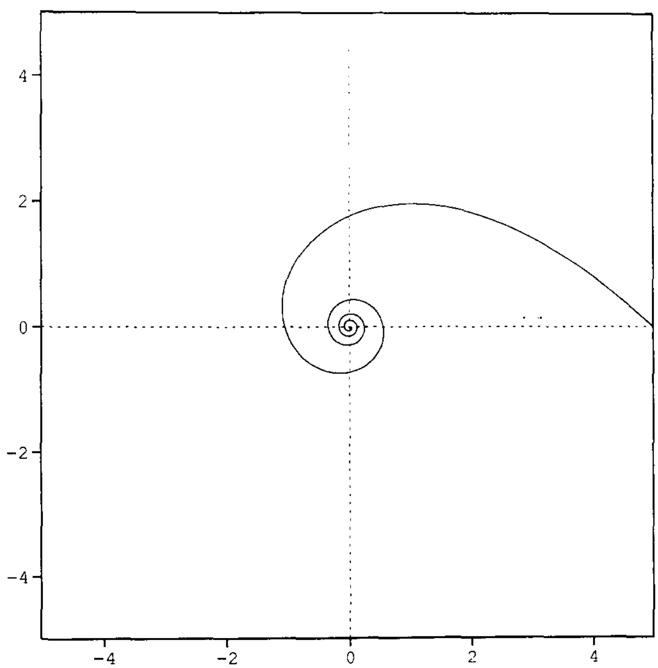

Fig. 7. Orbit for $h=25$, $\alpha=1.00$, $v_0=7.5$.

or

$$\frac{P^2v^2}{r^2} + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \dot{\theta}^2 = v^2. \tag{A.1}$$

Since $H = r^2\dot{\theta}$ the above is equivalent to

$$\frac{H^2}{r^2} + \frac{H^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = v^2$$

or

$$1 + \frac{1}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{v^2r^2}{H^2}.$$

If we now make the substitution $vr = H/\cos\varphi$

$$1 + \frac{1}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{1}{\cos^2\varphi}$$

and finally, after solving in φ and using Equation (19),

$$\cos^2\varphi = \left\{1 + \frac{v_0^2r^2}{\mu^2}[\varepsilon + R_1R_2v_0 + 2\varepsilon R_1R_2(\theta - \theta_0)]^2\right\}^{-1} \tag{A.2}$$

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