# ON THE USE AND ABUSE OF NEWTON'S SECOND LAW FOR VARIABLE MASS PROBLEMS

ANGEL R. PLASTINO* and JUAN C. MUZZIO**
*Facultad de Ciencias Astronómicas y Geofisicas de la Universidad Nacional de La Plata*

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**Abstract.** We clarify some misunderstandings currently found in the literature that arise from improper application of Newton's second law to variable mass problems. In the particular case of isotropic mass loss, for example, several authors introduce a force that actually does not exist.

**Key words:** variable mass problems, restricted problem.

Although masses are regarded as constant for the great majority of problems investigated by dynamical astronomy, there are some cases of interest where masses change with time. Several authors have studied the two-body problem with variable masses in connection with the evolution of binary systems (see, e.g., Hadjidemetriou, 1963; and also Hadjidemetriou, 1967, for an excellent review). Also, some interesting work has been devoted to the restricted three-body problem with variable mass primaries (see Horedt, 1984, for references on this subject). Besides, important efforts have been done in connection to the effects of galactic mass loss in galactic dynamics (see, for example, Richstone and Potter, 1982).

Despite the fact that variable mass dynamics has been an active research field for many years, we still find in the literature wrong applications of Newton's second law in this context. For example, Shrivastava and Ishwar (1983), Singh and Ishwar (1984), and Das *et al.* (1989), who analyzed the restricted three-body problem when the mass of the infinitesimal body varies, and Saslaw (1985), who discussed the virial theorem for a collection of bodies of variable mass, incorrectly applied Newton's second law (or the equivalent Lagrange's equations) to deal with the variable masses and obtained erroneous results.

The formulation of Newton's second law as the time derivative of the momentum:

$$\bar{F} = \frac{\mathrm{d}(m\bar{v})}{\mathrm{d}t} \tag{1}$$

where $\bar{F}$ is the force acting on a particle of mass $m$ that moves with velocity $\bar{v}$, allows the possibility of a variable mass and it was popular in textbooks decades ago because it was then in the vogue to consider that, according to special relativity, mass depended on velocity (e.g., Sears, 1958). Since then, it has been recognized

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** Member of the Carrera del Investigador Cientifico del Consejo Nacional de Investigaciones Cientificas y Técnicas de la República Argentina.

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that the concept of mass dependence on velocity can be misleading, and it is no longer favored (see Adler, 1987, for an excellent discussion on this subject).

If we consider the simple case of a variable mass, and we write Newton's second law as:

$$\bar{F} = m\,\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} + \bar{v}\,\frac{\mathrm{d}m}{\mathrm{d}t} \tag{2}$$

we can easily see that it violates the relativity principle under Galilean transformations. When $\bar{F}$ is zero, in particular, Equation (2) implies that the particle will remain at rest in a system where it is originally at rest, but it will be accelerated by the "force" $-\bar{v}\,\mathrm{d}m/\mathrm{d}t$ in a system where the particle moves with velocity $\bar{v}$!

To solve this apparent paradox, let us concentrate on the case of interest for problems of classical dynamical astronomy, that is, where the mass of a body does not actually 'change' but, rather, either part of the original mass is ejected or, conversely, the body gains mass through the capture of debris from its environment.

As it is well known (see Sommerfeld, 1952), the general equation of motion for a body whose mass $m$ varies according to any of the above mechanisms is

$$m(t)\,\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \bar{F} + \bar{u}\,\frac{\mathrm{d}m}{\mathrm{d}t}\,, \tag{3}$$

where $\bar{v}$ is the velocity of its center of mass. $\bar{F}$ is the sum of all the external forces, and $\bar{u}$ is the relative velocity of escaping (or incident) mass with respect to the center of mass of the body. Equation (3) is actually invariant under Galilean transformations.

It is plain from Equation (3) that, in the case of isotropical mass loss (and it must be emphasized that we mean isotropical in a system that moves with the body), the total contribution from $\bar{u}\,\mathrm{d}m/\mathrm{d}t$ terms will be zero. So, in this case, the correct equation will be

$$m(t)\,\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \bar{F}\,. \tag{4}$$

This fact had been recognized long ago by Mescerskii (1897; according to Hadjidemetriou, 1963). However, there are still some authors: Shrivastava and Ishwar (1983), Singh and Ishwar (1984), Das *et al.* (1989) and Saslaw (1985), that will use Equation (1) in the case of isotropical mass loss.

Shrivastava and Ishwar (1983) (to be shortened herefrom SI) consider a modified version of the circular restricted three-body problem. They assume that the main bodies have constant masses $m_1$ and $m_2$, respectively, while the small body $m$ loses mass isotropically. SI use Lagrange's formulation in deriving the equations of motion of the small body. So, before discussing the SI paper, let us consider Lagrange's version of Equation (3) following the approach of Lichtenegger (1984).

Let $\bar{r}$ be a vector whose components $X$, $Y$, $Z$ are the Cartesian coordinates of the particle in an inertial reference frame; $q_1$, $q_2$, $q_3$ stand for any set of generalized coordinates for the particle. We assume that the force $\bar{F}$ in Equation (3) is derivable from a potential $\phi(\bar{r}, t)$

$$\bar{F} = -\bar{\nabla}\,\phi\,. \tag{5}$$

Then, it is easy to see that Lagrange's equations (equivalent to (3)) are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i\,, \quad i = 1,2,3\,, \tag{6}$$

where the Lagrangian $L$ has the usual form:

$$L = \tfrac{1}{2}\,mv^2 - \phi \tag{7}$$

and the generalized forces $Q_i$ are

$$Q_i = \dot{m}(\dot{\bar{r}} + \bar{u}) \cdot \frac{\partial \bar{r}}{\partial q_i}\,, \quad i = 1,2,3\,. \tag{8}$$

The forces $Q_i$ take into account the momentum that each layer of ejected mass takes with it.

Now, following SI notation, let $(x, y, z)$ stand for a reference frame rotating with angular velocity $w$, where the principal masses $m_1$ and $m_2$ have constant coordinates $(-a, 0, 0)$ and $(b, 0, 0)$, respectively. The Lagrangian corresponding to the small body $m$ will be

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mw(x\dot{y} - y\dot{x}) + \\ Gm\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) + \tfrac{1}{2}mw^2(x^2 + y^2)\,, \tag{9}$$

where

$$\rho_1^2 = (x + a)^2 + y^2 + z^2\,, \quad \rho_2^2 = (x - b)^2 + y^2 + z^2 \tag{10}$$

and $G$ is the gravitation constant.

It follows from (6), (8), and (9) that the concomitant (Lagrange) equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q_x\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = Q_y\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = Q_z\,, \tag{11}$$

where

$$Q_x = \dot{m}(\dot{x} - wy + u_x)\,,$$

$$Q_y = \dot{m}(\dot{y} + wx + u_y)\,,$$

$$Q_z = \dot{m}(\dot{z} + u_z)\,, \tag{12}$$

where $u_x$, $u_y$, $u_z$ denote the components of $\bar{u}$ along the instantaneous coordinate axes of the rotating frame. In the case of isotropical mass loss the total contribution from the terms $\dot{m}u_x$, $\dot{m}u_y$, $\dot{m}u_z$ in the generalized forces vanish.

The problem with SI is that, while using the correct Lagrangian (9), they do not include in their Lagrange's equations the generalized forces $Q_x$, $Q_y$, $Q_z$. They just equate them to zero (Equations (5) of SI). It is easy to show, replacing (9) in (11), that the resulting equations of motion in the isotropical case, for the small body, are identical to the usual equations of motion of the restricted three-body problem (with constant mass). Instead, the SI equations of motion have a spurious term equivalent to the term $\bar{v}\ \mathrm{d}m/\mathrm{d}t$ in (2) (see Equations (6) of SI paper).

Singh and Ishwar (1984) consider the effect of small perturbations in the Coriolis and the centrifugal forces on the location of the equilibrium points in the circular restricted three-body problem with variable mass. They use the same (wrong) Lagrange's equations of SI (Equations (1)–(4) of Singh and Ishwar, 1984).

Das *et al.* (1989) study the elliptical restricted three-body problem under the assumption that the principal masses are constant, while the small body loses mass isotropically. They also do not include the forces $Q_x$, $Q_y$, $Q_z$ in their Lagrange's equations (Equations (1)–(3) of Das *et al.*, 1989).

Alternatively, Saslaw (1985) tries to derive the virial theorem for an $N$-body system where each body loses mass isotropically. In doing so, he begins by writing down the equations of motion in the form (Equation (9.20) of Saslaw, 1985):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m^{(\alpha)}\, v_i^{(\alpha)}\right) = -G\, m^{(\alpha)} \sum_{\substack{\beta\neq\alpha \\ \beta=1}}^{N} \frac{m^{(\beta)}(x_i^{(\alpha)} - x_i^{(\beta)})}{|\bar{x}^{(\alpha)} - \bar{x}^{(\beta)}|^3}\,, \qquad \begin{matrix} i = 1,2,3 \\ \alpha,\beta = 1,\ldots,N \end{matrix} \tag{13}$$

where, following Saslaw's notation, $G$, $m^{(\alpha)}$, and $V_i^{(\alpha)}$ stand respectively for the gravitation constant and, for each particle $\alpha$, the $i$ cartesian coordinate and velocity component. From Equation (13) Saslaw derives the following form of Jacobi's tensorial equation (Equation (9.28) of Saslaw)

$$\tfrac{1}{2}\,\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = \tfrac{1}{2}\,\frac{\mathrm{d}J_{ij}}{\mathrm{d}t} + 2T_{ij} + W_{ij}\,, \tag{14}$$

where the inertia tensor $I_{ij}$ is given by:

$$I_{ij} = \sum_{\alpha=1}^{N} m^{(\alpha)}\, x_i^{(\alpha)}\, x_j^{(\alpha)} \tag{15}$$

the kinetic energy tensor by:

$$T_{ij} = \tfrac{1}{2} \sum_{\alpha=1}^{N} m^{(\alpha)}\, v_i^{(\alpha)}\, v_j^{(\alpha)} \tag{16}$$

the potential energy tensor by:

$$W_{ij} = -\frac{G}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} [m^{(\alpha)} m^{(\beta)} (x_i^{(\alpha)} - x_i^{(\beta)})\,(x_j^{(\alpha)} - x_j^{(\beta)})]\, r_{\alpha\beta}^{-3}\,, \tag{17}$$

where

$$r_{\alpha\beta}^2 = (x_1^{(\alpha)} - x_1^{(\beta)})^2 + (x_2^{(\alpha)} - x_2^{(\beta)})^2 + (x_3^{(\alpha)} - x_3^{(\beta)})^2$$

and the mass variation tensor by:

$$J_{ij} = \sum_{\alpha=1}^{N} \dot{m}^{(\alpha)}\, x_i^{(\alpha)}\, x_j^{(\alpha)}\,. \tag{18}$$

Taking the time average of (14), and assuming that $\dot{m}(t)$ does not increase as fast as $t$ for $t \to \infty$, Saslaw claims that if $x_i$ and $v_i$ remain bound, then, the virial theorem adopts the usual form:

$$2\ <T_{ij}> + <W_{ij}> = 0\,. \tag{19}$$

However, if we start from the correct equations of motion for particles that lose mass isotropically:

$$m^{(\alpha)} \frac{\mathrm{d}v_i^{(\alpha)}}{\mathrm{d}t} = -G\, m^{(\alpha)} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{m^{(\beta)}(x_i^{(\alpha)} - x_i^{(\beta)})}{|\bar{x}^{(\alpha)} - \bar{x}^{(\beta)}|^3} \tag{20}$$

it is straightforward to show, by applying twice the time derivate to Equation (15), that Equation (14) should be replaced by:

$$\tfrac{1}{2}\,\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = \tfrac{1}{2}\,\frac{\mathrm{d}J_{ij}}{\mathrm{d}t} + 2T_{ij} + W_{ij} + Q_{ij}\,, \tag{21}$$

where

$$Q_{ij} = \tfrac{1}{2} \sum_{\alpha=1}^{N} \dot{m}^{(\alpha)} [\dot{x}_i^{(\alpha)}\, x_j^{(\alpha)} + \dot{x}_j^{(\alpha)}\, x_i^{(\alpha)}]\,. \tag{22}$$

If we take the time average of (21), making the same assumptions as Saslaw we arrive to

$$2\ <T_{ij}> + <W_{ij}> + <Q_{ij}> = 0 \tag{23}$$

which has the extra term $< Q_{ij} >$ that does not appear in the virial theorem with constant masses.

We may conclude emphasizing that Newton's second law is valid for constant mass only. When the mass varies due to accretion or ablation, Equation (3) (or the ones of Lichtenegger, 1984) should be used. It is worthwhile recalling that when a body loses mass isotropically (in a system that moves with the body) no additional 'force' should appear; in other words, if we consider the restricted three-body problem, the motion of the infinitesimal body will not be altered if it loses mass isotropically.

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