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RATE OF CONVERGENCE OF HERMITE–FEJÉR POLYNOMIALS FOR FUNCTIONS WITH DERIVATIVES OF BOUNDED VARIATION

R. BOJANIC (Columbus, Ohio) and F. CHENG (Lexington, Kentucky)

1. Introduction

Let f be a function defined on $[-1,1]$. The Hermite–Fejér interpolation polynomial $\mathbf{H}_n(f,x)$ of f, based on the zeros

$$x_{kn} = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k=1,2,\dots,n \tag{1.1}$$

of the Chebysev polynomial $T_n(x)=\cos(n\cos^{-1}x)$, is defined by

$$\mathbf{H}_n(f,x)=\sum_{k=1}^{n} f(x_{kn})(1-x_{kn}x)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2. \tag{1.2}$$

It was proved by L. Fejér [1] that $\mathbf{H}_n(f,x)$ converges uniformly to $f(x)$ if $f(x)$ is a continuous function on $[-1,1]$. The rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ when $f(x)$ is a continuous function has been extensively studied before ([2]--[9]). A survey of various quantitative estimates of the rate of convergence can be found in [9], where it was proved that

$$|\mathbf{H}_n(f,x)-f(x)| \leqq \frac{C_1}{n}T_n^2(x)\sum_{k=1}^{n}\left[W_f\left(\frac{(1-x^2)^{1/2}}{k}\right)+W_f\left(\frac{1}{k^2}\right)\right]+$$
$$+C_2W_f\left(\frac{|T_n(x)|}{n}\right).$$

Here C_1 and C_2 are positive constants and W_f is the *modulus of continuity* of f.

The behavior of $\mathbf{H}_n(f,x)$ when $f\in \mathbf{BV}[-1,1]$ (i.e., f is of *bounded variation* on $[-1,1]$) was studied by Bojanic and Cheng [10]. It was proved that if $f\in \mathbf{BV}[-1,1]$ and continuous at $x\in(-1,1)$ then $\mathbf{H}_n(f,x)$ converges to $f(x)$ when n tends to $+\infty$ and the rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ satisfies the following inequality

$$|\mathbf{H}_n(f,x)-f(x)| \leqq \frac{64T_n^2(x)}{n}\sum_{k=1}^{n}\mathbf{V}_{x-\pi/k}^{x+\pi/k}(f)+2\mathbf{V}_{x-\pi|T_n(x)|/2n}^{x+\pi|T_n(x)|/2n}(f) \tag{1.3}$$

where $\mathbf{V}_a^b(f)$ is the *total variation* of f on $[a,b]$. (1.3) can not be improved asymptotically.

However, if x is a point of discontinuity of f where $f(x+) \neq f(x-)$, the sequence $(\mathbf{H}_n(f,x))$ is no longer convergent. This follows from the following observation

$$(1.4)\quad \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} H_n(f,x) = \frac{1}{2}(f(x+)+f(x-)) \pm \frac{1}{2}|f(x+)-f(x-)|\beta(x)$$

where $\beta(x)=1$ if $x=\cos(\alpha\pi)$ and α is irrational, and

$$\beta(x) = \left(\frac{\sin(\pi/2q)}{\pi/2q}\right)^2 \left(1 - \sum_{k=1}^{\infty} \frac{8qk}{(4q^2k^2-1)^2}\right)$$

if $x = \cos(p\pi/q)$.

(1.4) shows that, unlike Fourier series of 2π-periodic functions of bounded variation or Bernstein polynomials of functions of bounded variation which all converge to $(f(x+)+f(x-))/2$ ([11], [12]), the Hermite–Fejér interpolation polynomials of a function of bounded variation converge only if $f(x+)=f(x-)$.

Although smoother functions find more applications in various fields such as computer aided geometric design, computer vision, graphics and image processing, the asymptotic behavior of Hermite–Fejér interpolation polynomials for functions smoother than continuous functions has been studied only for functions with continuous derivatives.

In this paper we shall investigate the asymptotic behavior of Hermite–Fejér polynomials for functions defined the following way

$$(1.5)\qquad f(x) = f(-1) + \int_{-1}^{x} \varphi(t)\,dt, \quad x \in [-1,1]$$

where φ is a function of bounded variation on $[-1,1]$. This class of functions can be described as the class of differentiable functions whose derivatives are of bounded variation and will be denoted by $\mathbf{DBV}[-1,1]$. It is clear that this class of functions is much more general than functions with continuous derivatives. However, as it will be seen in the next section, the asymptotic behavior of Hermite–Fejér interpolation polynomials for functions in this category is also much better than the asymptotic behavior of Hermite–Fejér interpolation polynomials for continuous functions. Results for Bernstein polynomials for functions of this type can be found in [13].

2. Results

Let f be a function in $\mathbf{DBV}[-1,1]$ and $\varphi \in \mathbf{BV}[-1,1]$ so that (1.5) is satisfied. For any $x \in (-1,1)$ such that $x \neq x_{kn}$ for $k=1,2,\dots,n$ we have, from (1.2) and (1.5),

$$\mathbf{H}_n(f,x)-f(x)=\sum_{k=1}^{n}\left(\int_x^{x_{kn}}\varphi(t)dt\right)\mathbf{H}_{kn}(x)= \tag{2.1}$$

$$=-\sum_{x_{kn}<x}\left(\int_{x_{kn}}^{x}\varphi(t)dt\right)\mathbf{H}_{kn}(x)+\sum_{x_{kn}>x}\left(\int_x^{x_{kn}}\varphi(t)dt\right)\mathbf{H}_{kn}(x)$$

where

$$\mathbf{H}_{kn}(x)=(1-xx_{kn})\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2,\quad k=1,2,\dots,n.$$

If we define $\varphi_x(t)$ the following way

$$\varphi_x(t)=\begin{cases}\varphi(t)-\varphi(x-), & t<x\\ 0, & t=x\\ \varphi(t)-\varphi(x+), & t>x\end{cases}$$

then (2.1) can be expressed as

$$\mathbf{H}_n(f,x)-f(x)= \tag{2.2}$$

$$=-\sum_{x_{kn}<x}\left(\int_{x_{kn}}^{x}\varphi_x(t)dt\right)\mathbf{H}_{kn}(x)+\sum_{x_{kn}>x}\left(\int_x^{x_{kn}}\varphi_x(t)dt\right)\mathbf{H}_{kn}(x)$$

$$-\varphi(x-)\sum_{x_{kn}<x}(x-x_{kn})\mathbf{H}_{kn}(x)+\varphi(x+)\sum_{x_{kn}>x}(x_{kn}-x)\mathbf{H}_{kn}(x).$$

Since

$$\varphi(x+)\sum_{x_{kn}>x}(x_{kn}-x)\mathbf{H}_{kn}(x)-\varphi(x-)\sum_{x_{kn}<x}(x-x_{kn})\mathbf{H}_{kn}(x)=$$

$$=\frac{\varphi(x+)-\varphi(x-)}{2}\sum_{k=1}^{n}|x_{kn}-x|\mathbf{H}_{kn}(x)+\frac{\varphi(x+)+\varphi(x-)}{2}\sum_{k=1}^{n}(x_{kn}-x)\mathbf{H}_{kn}(x)$$

and the two summations on the right-hand side of the equation are the Hermite–Fejér interpolation polynomials of $f_x(t)=|t-x|$ and $g_x(t)=t-x$, $-1\leqq t\leqq 1$, respectively, we can further convert (2.2) as follows

$$\mathbf{H}_n(f,x)-f(x)=\frac{\sigma}{2}\mathbf{H}_n(f_x,x)+\frac{\lambda}{2}\mathbf{H}_n(g_x,x)+\mathbf{P}_n(f,x) \tag{2.3}$$

where

$$\sigma = \varphi(x+) - \varphi(x-); \quad \lambda = \varphi(x+) + \varphi(x-) \tag{2.4}$$

and

$$\mathbf{P}_n(f,x) = \tag{2.5}$$
$$= -\sum_{x_{kn}<x} \left(\int_{x_{kn}}^{x} \varphi_x(t)dt \right) \mathbf{H}_{kn}(x) + \sum_{x_{kn}>x} \left(\int_{x}^{x_{kn}} \varphi_x(t)dt \right) \mathbf{H}_{kn}(x).$$

Therefore, evaluation of the rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ is simply a matter of evaluating $\mathbf{H}_n(f_x,x)$, $\mathbf{H}_n(g_x,x)$ and $\mathbf{P}_n(f,x)$.

We shall give estimates for $\mathbf{H}_n(f_x,x)$ and $\mathbf{H}_n(g_x,x)$ first and then use these estimates to get an estimate for the rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$. Since the estimates we get for $\mathbf{H}_n(f_x,x)$ and $\mathbf{H}_n(g_x,x)$ are of some interest, we shall state them as an independent theorem.

THEOREM 1. *If $x \in (-1,1)$ and $x \neq x_{kn}$ for any $k = 1,2,\dots,n$ then*

$$\left| \sum_{k=1}^{n} |x_{kn} - x| \mathbf{H}_{kn}(x) - \frac{2(1-x^2)^{1/2} T_n^2(x)}{\pi} \frac{\log n}{n} \right| \leqq C \frac{|T_n(x)|}{n}, \tag{2.6}$$

$$\left| \sum_{k=1}^{n} (x_{kn} - x) \mathbf{H}_{kn}(x) \right| \leqq C \frac{|T_n(x)|}{n} \tag{2.7}$$

where $C = 7 + \pi$.

The rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ can be estimated as follows.

THEOREM 2. *Let f be a function in $\mathbf{DBV}[-1,1]$ and $\varphi \in \mathbf{BV}[-1,1]$ so that (1.5) is satisfied. Then for any $x \in (-1,1)$ such that $x \neq x_{kn}$ for $k = 1,2,\dots,n$ we have*

$$\left| \mathbf{H}_n(f,x) - f(x) - \frac{\sigma(1-x^2)^{1/2} T_n^2(x)}{\pi} \frac{\log n}{n} \right| \leqq \frac{C(|\sigma| + |\lambda|)}{2} \frac{|T_n(x)|}{n} + \tag{2.8}$$
$$+ \frac{\pi |T_n(x)|}{n} \mathbf{V}_{x-\pi|T_n(x)|/n}^{x+\pi|T_n(x)|/n}(\varphi_x) + \frac{12 T_n^2(x)}{n} \sum_{k=1}^{n} \frac{\mathbf{V}_{x-\pi/k}^{x+\pi/k}(\varphi_x)}{k}$$

where $\mathbf{V}_a^b(\varphi_x)$ is the total variation of φ_x on $[a,b]$, σ and λ are defined in (2.4), and C is defined in Theorem 1.*

*We assume here and in the rest of the paper that f is extended to the entire real line by $f(x) = f(1)$ for $x > 1$ and $f(x) = f(-1)$ for $x < -1$.

If f' is continuous at x, i.e., $\sigma=0$ and $\lambda=2f'(x)$, then (2.8) can be simplified as follows.

$$(2.9)\quad |\mathbf{H}_n(f,x)-f(x)| \leqq \frac{C|f'(x)T_n(x)|}{n}+\frac{\pi|T_n(x)|}{n}\mathbf{V}_{x-\pi|T_n(x)|/n}^{x+\pi|T_n(x)|/n}(f')+ +\frac{12T_n^2(x)}{n}\sum_{k=1}^{n}\frac{\mathbf{V}_{x-\pi/k}^{x+\pi/k}(f')}{k}.$$

The right-hand side of (2.8) converges to zero as $n\to\infty$ since continuity of φ_x at x implies that

$$\mathbf{V}_{x-\beta}^{x+\alpha}(\varphi_x)\to 0\quad (\alpha,\beta\to 0+).$$

Actually, the last term of the right-hand side of (2.8) is $o(\log n/n)$-convergent. Therefore (2.8) can also be expressed as

$$\mathbf{H}_n(f,x)=f(x)+\frac{\sigma(1-x^2)^{1/2}T_n^2(x)}{\pi}\frac{\log n}{n}+o\left(\frac{\log n}{n}\right).$$

Note that all the estimates mentioned in Section 1 for continuous functions or functions of bounded variation are $o(1)$-convergent only.

As far as the precision of (2.9) is concerned, consider the Hermite–Fejér polynomials of the function $f(x)=x^2$ at $x=0$ for even n. Since $T_n(0)=1$ if n is an even integer, we have

$$\mathbf{H}_n(f,0)-f(0)=\sum_{k=1}^{n}\frac{T_n^2(0)}{n^2}=\frac{1}{n}.$$

On the other hand, since $\sigma=\lambda=0$ at $x=0$, it follows from (2.9) that

$$|\mathbf{H}_n(f,0)-f(0)|\leqq\frac{\pi}{n}\mathbf{V}_{-\pi/n}^{+\pi/n}(\varphi_0)+\frac{12}{n}\sum_{k=1}^{n}\frac{\mathbf{V}_{-\pi/k}^{+\pi/k}(\varphi_0)}{k}.$$

since $\varphi(t)=2t$ and $\varphi_0(t)=\varphi(t)$, we have

$$|\mathbf{H}_n(f,0)-f(0)|\leqq\frac{4\pi^2}{n^2}+\frac{12}{n}\sum_{k=1}^{n}\frac{4\pi}{k^2}<\frac{C}{n}$$

for some $C>0$. Hence for the function $f(x)=x^2$ when n is an even integer we have

$$\frac{1}{n}\leqq|\mathbf{H}_n(f,0)-f(0)|\leqq\frac{C}{n}$$

for some positive constant $C>0$. Therefore (2.9) can not be improved asymptotically.

3. Proofs

3.1. PROOF OF THEOREM 1. To prove (2.6), observe that

$$\left|\mathrm{H}_n(f_x,x)-\sum_{k=1}^{n}|x_{kn}-x|(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2\right| \leqq \frac{|x|T_n^2(x)}{n}. \tag{3.1}$$

Therefore, it is sufficient to study the asymptotic behavior of the second term on the left-hand side of (3.1) only.

Let $x=\cos\vartheta$, $0<\vartheta<\pi$, $x_{kn}=\cos\vartheta_{kn}$, $\vartheta_{kn}=(2k-1)\pi/(2n)$, $k=1,2,\dots,$ n, and define

$$\delta_\vartheta(\alpha)=\begin{cases}1, & \text{if } 0<\alpha<\vartheta\\ -1, & \text{if } \vartheta<\alpha<\pi.\end{cases}$$

Then

(3.2)
$$\sum_{k=1}^{n}|x_{kn}-x|(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2=\sum_{k=1}^{n}\delta_\vartheta(\vartheta_{kn})\frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)}.$$

Since $\cos\vartheta_{kn}-\cos\vartheta=(\vartheta-\vartheta_{kn})\sin\overline{\vartheta}_{kn}$ for some $\overline{\vartheta}_{kn}$ between ϑ and ϑ_{kn}, it follows that

$$\left|\sum_{k=1}^{n}\delta_\vartheta(\vartheta_{kn})\frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)}-\sum_{k=1}^{n}\delta_\vartheta(\vartheta_{kn})\frac{\sin\vartheta\cos^2(n\vartheta)}{n^2(\vartheta-\vartheta_{kn})}\right| \leqq$$

$$\leqq \frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\left(\frac{\sin\vartheta}{\cos\vartheta_{kn}-\cos\vartheta}-\frac{1}{\vartheta-\vartheta_{kn}}\right)\right| \leqq$$

$$\leqq \frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\left(\frac{\sin\vartheta}{\sin\overline{\vartheta}_{kn}(\vartheta-\vartheta_{kn})}-\frac{1}{\vartheta-\vartheta_{kn}}\right)\right| \leqq$$

$$\leqq \frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\frac{1}{\vartheta-\vartheta_{kn}}\left(\frac{\sin\vartheta-\sin\overline{\vartheta}_{kn}}{\sin\overline{\vartheta}_{kn}}\right)\right| \leqq$$

$$\leqq \frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\frac{1}{\vartheta-\vartheta_{kn}}\frac{\vartheta-\overline{\vartheta}_{kn}}{\sin\overline{\vartheta}_{kn}}\right| \leqq \frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\frac{1}{\sin\overline{\vartheta}_{kn}}\right|$$

Furthermore, since $|\sin\frac{\vartheta}{2}|\geqq|\vartheta|/\pi$ if $|\vartheta|\leqq\pi$, we have

$$\sin \overline{\vartheta}_{kn} = \left| \frac{\cos \vartheta - \cos \vartheta_{kn}}{\vartheta - \vartheta_{kn}} \right| = \left| \frac{2 \sin \frac{\vartheta + \vartheta_{kn}}{2} \sin \frac{\vartheta - \vartheta_{kn}}{2}}{\vartheta - \vartheta_{kn}} \right| \geqq$$

$$\geqq \left| \frac{2 \sin \frac{\vartheta + \vartheta_{kn}}{2} \frac{\vartheta - \vartheta_{kn}}{\pi}}{\vartheta - \vartheta_{kn}} \right| = \frac{2}{\pi} \left| \sin \left(\frac{\vartheta + \vartheta_{kn}}{2} \right) \right| \geqq \frac{2}{\pi} M(\vartheta)$$

where $M(\vartheta) = \min(\sin \frac{\vartheta}{2}, \sin \frac{\pi - \vartheta}{2})$. Therefore,

(3.3)
$$\left| \sum_{k=1}^{n} \delta_\vartheta(\vartheta_{kn}) \frac{\sin^2 \vartheta \cos^2(n\vartheta)}{n^2(\cos \vartheta_{kn} - \cos \vartheta)} - \sum_{k=1}^{n} \delta_\vartheta(\vartheta_{kn}) \frac{\sin \vartheta \cos^2(n\vartheta)}{n^2(\vartheta - \vartheta_{kn})} \right| \leqq \frac{\pi \cos^2 n\vartheta}{n}$$

by noticing that $M(\vartheta) \geqq \sin \vartheta / 2$.

Let j be the integer such that $\vartheta_{jn} < \vartheta < \vartheta_{j+1,n}$. It is easy to see that

$$j = \left[\frac{n\vartheta}{\pi} + \frac{1}{2} \right].$$

Since $n(\vartheta - \vartheta_{kn}) = \pi(\frac{n\vartheta}{\pi} + \frac{1}{2} - k)$, we have

(3.4)
$$\sum_{k=1}^{n} \delta_\vartheta(\vartheta_{kn}) \frac{\sin \vartheta \cos^2(n, \vartheta)}{n^2(\vartheta - \vartheta_{kn})} =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=1}^{j} \frac{1}{\left(\frac{n\vartheta}{\pi} + \frac{1}{2} - k\right)} - \sum_{k=j+1}^{n} \frac{1}{\left(\frac{n\vartheta}{\pi} + \frac{1}{2} - k\right)} \right) =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=0}^{j-1} \frac{1}{\frac{n\vartheta}{\pi} + \frac{1}{2} - j + k} - \sum_{k=0}^{n-j-1} \frac{1}{\frac{n\vartheta}{\pi} + \frac{1}{2} - j - 1 - k} \right) =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=0}^{j-1} \frac{1}{\frac{n\vartheta}{\pi} + \frac{1}{2} - j + k} + \sum_{k=0}^{n-j-1} \frac{1}{k + 1 - \left(\frac{n\vartheta}{\pi} + \frac{1}{2}\right) + j} \right) =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=0}^{j-1} \frac{1}{\rho(n\vartheta) + k} + \sum_{k=0}^{n-j-1} \frac{1}{1 - \rho(n\vartheta) + k} \right) = \Delta_1 + \Delta_2$$

where

(3.5)
$$\rho(x) = \frac{x}{\pi} + \frac{1}{2} - \left[\frac{x}{\pi} + \frac{1}{2} \right].$$

We shall prove that Δ_1 and Δ_2 are both asymptotically equal to $\sin \vartheta \cos^2 n\vartheta \log n / (n\pi)$.

First, observe that

$$\Delta_1 - \frac{\sin\vartheta\cos^2 n\vartheta \log n}{n\pi} = \delta_{1,1} + \delta_{1,2} \tag{3.6}$$

where

$$\delta_{1,1} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\frac{1}{\rho(n\vartheta)}, \quad \delta_{1,2} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\left(\sum_{k=1}^{j-1}\frac{1}{\rho(n\vartheta)+k} - \log n\right).$$

Since $\cos n\vartheta = (-1)^j \sin(\rho(n\vartheta)\pi)$ where $j = [n\vartheta/\pi + 1/2]$, if follows that

$$|\delta_{1,1}| \leqq \frac{\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.7}$$

On the other hand, it is easy to see that

$$\left|\sum_{k=1}^{j-1}\frac{1}{\rho(n\vartheta)+k} - \log n\right| \leqq 5.$$

Hence,

$$|\delta_{1,2}| \leqq \frac{5\sin\vartheta\cos^2 n\vartheta}{n\pi}. \tag{3.8}$$

Therefore, from (3.6), (3.7) and (3.8) we have

$$\left|\Delta_1 - \frac{\sin\vartheta\cos^2 n\vartheta\log n}{n\pi}\right| \leqq \frac{3\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.9}$$

The evaluation of the asymptotic behavior of Δ_2 can be carried out in a similar way. First, observe that

$$\Delta_2 - \frac{\sin\vartheta\cos^2 n\vartheta\log n}{n\pi} = \delta_{2,1} + \delta_{2,2} \tag{3.10}$$

where

$$\delta_{2,1} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\frac{1}{1-\rho(n\vartheta)},$$

$$\delta_{2,2} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\left(\sum_{k=1}^{n-j-1}\frac{1}{1-\rho(n\vartheta)+k} - \log n\right).$$

Since $\cos n\vartheta = (-1)^j \sin(1-\rho(n\vartheta)\pi)$ where $j = [n\vartheta/\pi + 1/2]$, it follows that

$$|\delta_{2,1}| \leqq \frac{\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.11}$$

On the other hand, it is also easy to see that

$$\left|\sum_{k=1}^{n-j-1} \frac{1}{1-\rho(n\vartheta)+k} - \log n\right| \leqq 5.$$

Hence,

$$|\delta_{2,2}| \leqq \frac{5\sin\vartheta\cos^2 n\vartheta}{n\pi}. \tag{3.12}$$

Therefore, from (3.10), (3.11) and (3.12) we have

$$\left|\Delta_2 - \frac{\sin\vartheta\cos^2 n\vartheta\log n}{n\pi}\right| \leqq \frac{3\sin\vartheta|\cos n\vartheta|}{n}, \tag{3.13}$$

and the estimate (2.6) follows from (3.1), (3.2), (3.3), (3.4), (3.6), (3.9) and (3.13).

To prove (2.7), observe that by using a similar technique we can show that

$$\left|\mathrm{H}_n(g_x,x) - \sum_{k=1}^{n}(x_{kn}-x)(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2\right| \leqq \frac{|x|T_n^2(x)}{n}, \tag{3.14}$$

$$\sum_{k=1}^{n}(x_{kn}-x)(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2 = \sum_{k=1}^{n}\frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)}, \tag{3.15}$$

$$\left|\sum_{k=1}^{n}\frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)} - \sum_{k=1}^{n}\frac{\sin\vartheta\cos^2(n\vartheta)}{n^2(\vartheta-\vartheta_{kn})}\right| \leqq \frac{\pi\cos^2 n\vartheta}{n}, \tag{3.16}$$

where $x=\cos\vartheta$, $0<\vartheta<\pi$, $x_{kn}=\cos\vartheta_{kn}$, $\vartheta_{kn}=(2k-1)\pi/2n$, $k=1,2,\dots,n$, and

$$\sum_{k=1}^{n}\frac{\sin\vartheta\cos^2(n\vartheta)}{n^2(\vartheta-\vartheta_{kn})} = \Delta_1-\Delta_2 \tag{3.17}$$

where Δ_1 and Δ_2 are defined in (3.4). It follows immediately from (3.9) and (3.13) that

$$|\Delta_1-\Delta_2| \leqq \frac{6\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.18}$$

Therefore, (2.7) follows from (3.14), (3.15), (3.16), (3.17) and (3.18).

3.2. Proof of Theorem 2. Since the evaluation of $\mathbf{H}_n(f_x, x)$ and $\mathbf{H}_n(g_x, x)$ is already done in Theorem 1, the only thing we have to do now is the evaluation of $\mathbf{P}_n(f, x)$. The technique used here is similar to the one used in [10].

For any $x \in (-1, 1)$ such that $x \neq x_{kn}$ for $k = 1, 2, \ldots, n$ we have

$$|P_n(f,x)| \leqq \sum_{k=1}^{n} \left| \int_{x_{kn}}^{x} \varphi_x(t)dt \right| \mathbf{H}_{kn}(x) \leqq$$

$$\leqq \sum_{k=1}^{n} \left| \int_{x_{kn}}^{x} \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x)dt \right| \mathbf{H}_{kn}(x) \leqq \sum_{k=1}^{n} |x - x_{kn}| \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x) \mathbf{H}_{kn}(x)$$

where $t_{kn} = |x - x_{kn}|$ and $\mathbf{V}_a^b(\varphi_x)$ is the total variation of φ_x on $[a, b]$. Let $x = \cos\vartheta$, $0 < \vartheta < \pi$, $x_{kn} = \cos\vartheta_{kn}$, $\vartheta_{kn} = (2k-1)\pi/(2n)$, $k = 1, 2, \ldots, n$, and define

$$E_r(n, \vartheta) = \left\{ k : \frac{r\pi}{2n} < |\vartheta - \vartheta_{kn}| \leqq \frac{(r+1)\pi}{2n} \right\}, \ r = 0, 1, \ldots, 2n-1.$$

Then we have

$$|\mathbf{P}_n(f,x)| \leqq \sum_{r=0}^{2n-1} \sum_{k \in E_r(n,\vartheta)} |x - x_{kn}| \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x) \mathbf{H}_{kn}(x).$$

Since $t_{kn} = |x - x_{kn}| \leqq |\vartheta - \vartheta_{kn}| \leqq \pi |T_n(x)|/2n$ if $k \in E_0(n, \vartheta)$ (see [9], p.257) and $E_0(n, \vartheta)$ has at most two elements, it follows that

(3.19)

$$\sum_{k \in E_0(n,\vartheta)} |x - x_{kn}| \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x) \mathbf{H}_{kn}(x) \leqq \frac{\pi |T_n(x)|}{n} \mathbf{V}_{x-\pi|T_n(x)|/2n}^{x+\pi|T_n(x)|/2n}(\varphi_x).$$

On the other hand, since $E_r(n, \vartheta)$ has at most two elements,

$$|x - x_{kn}| \mathbf{H}_{kn}(x) \leqq (1 - \cos\vartheta\cos\vartheta_{kn} + \sin\vartheta\sin\vartheta_{kn}) \frac{T_n^2(x)}{n^2 |\cos\vartheta - \cos\vartheta_{kn}|} \leqq$$

$$\leqq (1 - \cos(\vartheta_{kn} + \vartheta)) \frac{T_n^2(x)}{2n^2 \sin\left(\frac{\vartheta+\vartheta_{kn}}{2}\right) \sin\left|\frac{\vartheta-\vartheta_{kn}}{2}\right|} \leqq$$

$$\leqq 2\sin^2\left(\frac{\vartheta+\vartheta_{kn}}{2}\right) \frac{T_n^2(x)}{2n^2 \sin\left(\frac{\vartheta+\vartheta_{kn}}{2}\right) \sin\left|\frac{\vartheta-\vartheta_{kn}}{2}\right|} \leqq$$

$$\leqq \frac{T_n^2(x)\pi}{n^2|\vartheta - \vartheta_{kn}|} \leqq \frac{2T_n^2(x)}{nr}$$

and $t_{kn} \leqq (r+1)\pi/2n$ if $k \in E_r(n, \vartheta)$, we have

$$(3.20)\qquad \sum_{k\in E_r(n,\vartheta)} |x-x_{kn}|\mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x)\mathbf{H}_{kn}(x) \leqq \frac{4T_n^2(x)}{nr}\mathbf{V}_{x-(r+1)\pi/2n}^{x+(r+1)\pi/2n}(\varphi_x)$$

for $r=1,2,\dots,2n-1$. Therefore, by (3.19) and (3.20),

$$(3.21)\qquad |\mathbf{P}_n(f,x)| \leqq \frac{\pi|T_n(x)|}{n}\mathbf{V}_{x-\pi|T_n(x)|/2n}^{x+\pi|T_n(x)|/2n}(\varphi_x) + \\ +\frac{4T_n^2(x)}{n}\sum_{r=1}^{2n-1}\frac{1}{r}\mathbf{V}_{x-(r+1)\pi/2n}^{x+(r+1)\pi/2n}(\varphi_x).$$

Let $Q(t)=\mathbf{V}_{x-t}^{x+t}(\varphi_x)$. Then

$$(3.22)\qquad \sum_{r=1}^{2n-1}\frac{1}{r}\mathbf{V}_{x-(r+1)\pi/2n}^{x+(r+1)\pi/2n}(\varphi_x) = \\ =\sum_{r=2}^{2n}\frac{1}{r-1}Q\left(\frac{r\pi}{2n}\right) \leqq 2\sum_{r=2}^{2}\frac{1}{r}Q\left(\frac{r\pi}{2n}\right).$$

By virtue of the fact that $Q(t)$ is a non-decreasing function, we have

$$\int_{r\pi/2n}^{(r+1)\pi/2n}\frac{Q(t)}{t}dt \geqq Q\left(\frac{r\pi}{2n}\right)\int_{r\pi/2n}^{(r+1)\pi/2n}\frac{dt}{t} \geqq Q\left(\frac{r\pi}{2n}\right)\log\left(1+\frac{1}{r}\right)$$

or

$$\frac{1}{r}Q\left(\frac{r\pi}{2n}\right) \leqq \frac{3}{2}\int_{r\pi/2n}^{(r+1)\pi/2n}\frac{Q(t)}{t}dt.$$

Hence

$$\sum_{r=2}^{2n}\frac{1}{r}Q\left(\frac{r\pi}{2n}\right) \leqq \frac{3}{2}\int_{\pi/n}^{\pi(2n+1)/2n}\frac{Q(t)}{t}dt.$$

Since $Q(\pi/t)$ is non-decreasing and $Q(\pi/t)=Q(\pi)$ for $0<t\leqq 1$, we have

$$\int_{2n/(2n+1)}^{n}\frac{Q(\pi/t)}{t}dt \leqq \int_{2n/(2n+1)}^{1}\frac{Q(\pi/t)}{t}dt + \int_1^n\frac{Q(\pi/t)}{t}dt \leqq \\ \leqq \frac{Q(\pi)}{2n+1}+\sum_{k=1}^{n}\frac{Q(\pi/k)}{k} \leqq 2\sum_{k=1}^{n}\frac{Q(\pi/k)}{k}$$

and therefore

$$(3.23)\qquad \sum_{r=2}^{2n}\frac{1}{r}Q\left(\frac{r\pi}{2n}\right) \leqq 3\sum_{k=1}^{n}\frac{Q(\pi/k)}{k}.$$

The proof of Theorem 2 follows now from (3.21) and (3.23).

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DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210
USA

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY 40506
USA