LENGTH OF ARC AS INDEPENDENT ARGUMENT FOR HIGHLY ECCENTRIC ORBITS

E.V. BRUMBERG
Institute of Applied Astronomy, 197042 St. Petersburg

(Received 11 April, 1990; accepted 2 December, 1991)

Abstract. For analytic step regulation in numerical integration of highly eccentric orbits it is proposed to use the orbital arc length of a moving particle as independent argument.

Key words: Numerical integration, fictitious time, step regulation.

1. Introduction

In integrating highly eccentric orbits the physical time t is often replaced by the fictitious time s

$$dt = Q\,ds\ , \tag{1}$$

with Q being some function of coordinates and velocity components of moving particle (Janin, 1974; Nacozy, 1981). In particular, it is aimed thereby to ensure the more uniform distribution of orbital points for equidistant values of argument (analytic step regulation) and to make possible the use of fixed-step numerical integration methods. The function most generally employed is

$$Q = c_k\, r^k \tag{2}$$

with $k = 1$ (eccentric anomaly), $k = 3/2$ (elliptic anomaly) and $k = 2$ (true anomaly). c_k is usually chosen so that the variation of 2π for s corresponds to one period P of the Keplerian motion in physical time

$$c_k = \frac{1}{2\pi} \int_0^P r^{-k}\, dt\ . \tag{3}$$

A more complicated function Q has been proposed in (Ferrandiz et al., 1987; see also Ferrandiz, 1986)

$$Q = r^{3/2}(a_0 + a_1\, r)^{-1/2}\ , \tag{4}$$

with a_0 and a_1 being constants or functions of Keplerian elements. In the general case, with non-zero a_0 and a_1, the function (4) leads to the generalized elliptic anomaly s. By a suitable choice of a_0 and a_1 one may ensure a sufficiently uniform distribution of points of specific orbit for equal intervals of s.

However, the same purpose may be achieved directly by adopting for s the orbital arc length of a particle. The aim of this short note is to call attention to the possibility of such a choice. Needless to say, the length of arc is sometimes used as an independent argument in theoretical investigations but, it seems to us, its use for numerical integration has not been discussed in the literature.

2. Keplerian Motion

In terms of eccentric anomaly g the rectangular orbital coordinates of the moving particle have the values

$$X = a(\cos g - e) \,, \qquad Y = a(1 - e^2)^{1/2} \sin g \,, \tag{5}$$

with a and e being the semi-major axis and eccentricity, respectively.

The radius-vector is determined by the expression

$$r = a(1 - e \cos g) \,. \tag{6}$$

The relation with time is given by the Kepler equation

$$g - e \sin g = n(t - T) \,. \tag{7}$$

T is the time of pericentre passage, n is the mean motion. The length of arc s satisfies the differential equation

$$\frac{ds}{dg} = \left[\left(\frac{dX}{dg} \right)^2 + \left(\frac{dY}{dg} \right)^2 \right]^{1/2} = a(1 - e^2 \cos^2 g)^{1/2} \,. \tag{8}$$

Reckoned from the pericentre, the length of arc is

$$s = a \int\limits_0^g (1 - e^2 \cos^2 u)^{1/2} \, du = a \int\limits_{\pi/2}^{g+\pi/2} (1 - e^2 \sin^2 u)^{1/2} \, du$$

$$= a \left[E(g + \pi/2, e) - E(e) \right] \,. \tag{9}$$

We use here the standard notation for incomplete and complete elliptic integrals of the second kind

$$E(u,k) = \int_0^u (1 - k^2 \sin^2 u)^{1/2}\, du\,,$$

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 u)^{1/2}\, du\,.$$

Therefore, using elliptic Jacobi functions of modulus e one has

$$\begin{aligned} g &= am\Big(\frac{s}{a} + E\Big) - \frac{\pi}{2}\,, \\ \sin g &= -cn\Big(\frac{s}{a} + E\Big)\,, \qquad \cos g = sn\Big(\frac{s}{a} + E\Big)\,. \end{aligned} \tag{10}$$

Needless to say, the quantity s may be represented in different forms, for example

$$s = E\Big(\arcsin \frac{\sin g}{(1 - e^2 \cos^2 g)^{1/2}}\,, e\Big) - \frac{e^2 \sin g \cos g}{(1 - e^2 \cos^2 g)^{1/2}}\,. \tag{11}$$

From (7) and (8) it follows that:

$$\frac{dt}{ds} = \frac{dt}{dg}\,\frac{dg}{ds} = \frac{1}{na}\Big[\frac{1 - e \cos g}{1 + e \cos g}\Big]^{1/2}\,. \tag{12}$$

Using Kepler's third law $n^2 a^3 = GM$ and denoting by $-h_k$ the Keplerian energy of particle

$$h_k = \frac{GM}{2a}\,, \tag{13}$$

one may reduce (12) with the aid of (6) to the form (1) with the function

$$Q = r^{1/2}(2GM - 2h_k\, r)^{-1/2}\,. \tag{14}$$

Expressed in this form the transformation (1) is applicable to any type of Keplerian motion. In the case of elliptic motion one may demand that one period in physical time t would correspond to the variation of a new argument s^* by 2π. This means the description of relation (1) in the form

$$dt = Q^*\, ds^* \tag{1a}$$

with

$$Q^* = cQ\,, \qquad ds^* = c^{-1}\, ds\,, \qquad c = \frac{2a}{\pi}\, E(e)\,. \tag{15}$$

3. Perturbed Motion

Let us consider the equations of perturbed motion in the standard form (Stiefel and Scheifele, 1971; Janin, 1974)

$$\ddot{\mathbf{x}} + \frac{GM}{r^3}\,\mathbf{x} = \mathbf{F} - \frac{\partial V}{\partial \mathbf{x}}\,, \tag{16}$$

with $V = V(\mathbf{x}, t)$ being a perturbing potential and $\mathbf{F}$ being a non-conservative perturbing force. If perturbations are sufficiently smooth and small, then the step of numerical integration will again be determined by the eccentricity of the Keplerian orbit. Therefore, the transformation (1) with the value (14) may be useful. h_k is to be meant there as

$$h_k = \frac{GM}{r} - \frac{\mathbf{v}^2}{2}\,, \tag{17}$$

where $\mathbf{v}$ is the velocity of the particle. h_k satisfies the equation

$$\dot{h}_k = \Big(\frac{\partial V}{\partial \mathbf{x}} - \mathbf{F}\Big)\,\mathbf{v}\,. \tag{18}$$

As noted in (Stiefel and Scheifele, 1971) it is suitable to use the complete energy $-h$

$$h = h_k - V\,, \tag{19}$$

satisfying the equation

$$\dot{h} = -\,\frac{\partial V}{\partial t} - \mathbf{F}\mathbf{v}\,. \tag{20}$$

Transforming Equation (16) to the equations of the first order and introducing the independent variable s one obtains

$$\frac{d\mathbf{x}}{ds} = Q\mathbf{v}\,, \tag{21}$$

$$\frac{d\mathbf{v}}{ds} = Q\Big(-\frac{GM}{r^3}\,\mathbf{x} + \mathbf{F} - \frac{\partial V}{\partial \mathbf{x}}\Big)\,, \tag{22}$$

$$\frac{dt}{ds} = Q\,, \tag{23}$$

$$\frac{dh}{ds} = -Q\Big(\frac{\partial V}{\partial t} + \mathbf{F}\mathbf{v}\Big)\,. \tag{24}$$

These equations should be complemented by algebraic relations (14), (17) and (19). The definition of s implies that

$$Q = (\mathbf{v}^2)^{-1/2} . \tag{25}$$

In virtue of (17) this determination coincides with (14). Therefore, Equations (21)–(24) may be deduced without using the solution of Kepler problem. Moreover, in using (25) Equation (24) becomes unnecessary for solving Equations (21)–(23). The energy h may be determined, if needed, by the finite formulae (17) and (19).

To improve stabilization of numerical integration the variable t is often replaced by the time element τ which changes linearly with s in the unperturbed motion. In our case this may be done using the standard technique by Nacozy (1981) on the basis of the Kepler equation (7) and explicit expressions as (11). But the interrelation between τ and t will involve elliptic functions making computation of right-hand members of the equations of perturbed motion more complicated. The efficiency of such transformation may be tested only by actual calculations.

4. Numerical Tests

The efficiency of the transformation (1) and (25) has been tested by several examples. We have used the Bulirsch-Stoyer integrator as implemented in (Press *et al.*, 1989). Integration was performed on a 12 MHz 286 IBM AT computer. To avoid the scaling factors we have used as independent arguments the mean anomaly $M = nt$ and the length of arc s^* expressed in radians. In each version we have tried to set the optimal parameters of integrator in order to perform the 'clean' comparison of M- and s^*-integrations. It is to be noted that the M-version is much more sensitive to the choice of initial step of integration than the s^*-version.

In the results reproduced here and dealing with unperturbed motion we give the integrator accuracy (EPS), the number of steps (NS), the number of the right-hand side computation (NRHS), the computing time (CPT) and the final accuracies in position vector $|\Delta\mathbf{x}|$ and velocity $|\Delta\mathbf{v}|$. All these data are given for one revolution of a satellite. The satellite under consideration was HEOS I with the orbital elements (Janin, 1974):

$$\begin{aligned} a &= 118363.47\,\text{km} \\ e &= 0.942572319 \\ i &= 28^\circ.16096 \\ \Omega &= 185^\circ.07554 \\ \omega &= 270^\circ.07151 \\ M_0 &= 0^\circ . \end{aligned}$$

The period of revolution of this satellite is 4.69 days.

The results obtained are as follows:

Argument	EPS	NS	NRHS	CPT [s]	$|\Delta \mathbf{x}|$ [km]	$|\Delta \mathbf{v}|$ [km/s]
M	1.E−12	34	2234	8.9	0.3 E−05	0.2 E−08
s^*	1.E−12	27	1771	7.1	0.1 E−0.7	0.1 E−10
M	1.E−14	47	3119	12.5	0.4 E−07	0.3 E−10

The last line shows that to obtain in the M-version the same final accuracy as in the s^*-version one needs much more calculating.

Finally, it is interesting to note that even for quasi-circular Navstar orbit ($e = 0.01$) the s^*-version turns out to be a little more advantageous than the M-version as seen from the following data:

Argument	EPS	NS	NRHS	CPT [s]	$|\Delta \mathbf{x}|$ [km]	$|\Delta \mathbf{v}|$ [km/s]
M	1.E−12	13	877	3.5	0.4 E−07	0.5 E−11
s^*	1.E−12	11	763	3.1	0.3 E−0.8	0.4 E−12

5. Conclusion

Analytic step regulation for highly eccentric orbits may be achieved rather simply by choosing as independent argument the arc length s determined by differential Equation (1) with value (14). In unperturbed motion the dependence of s on t is expressed by means of elliptic functions. Equations (21)–(24) of perturbed motion may prove very useful for numerically integrating highly eccentric orbits.

Acknowledgements

The author is indebted to Prof. S. Ferraz-Mello and referees for valuable remarks.

References

Ferrandiz, J.M.: 1986, 'Linearization in Special Cases of Perturbed Keplerian Motions', *Celes. Mech.* **39**, 23.

Ferrandiz, J.M., Ferrer, S. and Sein-Echaluce, M.L.: 1987, 'Generalized Elliptic Anomalies', *Celes. Mech.* **40**, 315.

Janin, G.: 1974, 'Accurate Computation of Highly Eccentric Satellite Orbits', *Celes. Mech.* **10**, 451.

Nacozy, P.E.: 1981, 'Time Elements in Keplerian Orbital Elements', *Celes. Mech.* **23**, 173.

Stiefel, E.L. and Scheifele, G.: 1971, *Linear and Regular Celestical Mechanics*, Springer, Berlin-Heidelberg-New York.

Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T.: 1989, *Numerical Recipes. The Art of Scientific Computing* (FORTRAN Version). Cambridge University Press, Cambridge-New York-Port Chester-Melbourne-Sydney.