THE LISSAJOUS TRANSFORMATION

I. BASICS

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Abstract. A new canonical transformation is proposed to handle elliptic oscillators, that is, Hamiltonian systems made of two harmonic oscillators in a 1-1 resonance. Lissajous elements pertain to the ellipse drawn with a light pen whose coordinates oscillate at the same frequency, hence their name. They consist of two pairs of angle-action variables of which the actions and one angle refer to basic integrals admitted by an elliptic oscillator, namely, its energy, its angular momentum and its Runge-Lenz vector. The Lissajous transformation is defined in two ways: explicitly in terms of Cartesian variables, and implicitly by resolution of a partial differential equation separable in polar variables. Relations between the Lissajous variables, the common harmonic variables, and other sets of variables are discussed in detail.

Key words: Dynamics – Hamiltonian systems – perturbations – reduction – isotropic oscillators

Certaines oeuvres méritent qu'on saisisse
tous les prétextes pour témoigner,
même sans nuances,
de la gratitude qu'on leur doit.
ALBERT CAMUS, *Préface á l'édition allemande des Poésies de René Char.*

1. Introduction

We call *elliptic oscillators* the dynamical systems represented by Hamiltonians of the type

$$\mathcal{H}_0 = \tfrac{1}{2}(X^2 + Y^2) + \tfrac{1}{2}\omega^2(x^2 + y^2) \tag{1}$$

in a phase space made of the Cartesian coordinates (x, y) and their conjugate momenta (X, Y); the parameter ω is assumed to be strictly positive. Perturbed elliptic oscillators with Hamiltonians of the type

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{V}(x, y, X, Y; \epsilon) \tag{2}$$

are among the systems most studied in non-linear dynamics, especially in molecular spectroscopy, galactic dynamics and celestial mechanics.

For the sake of convenience, we look at x and y as the Cartesian coordinates of a particle in a plane; we take X and Y to be the components of its

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of its velocity, and we take the independent variable t to stand as a time scale. The parameter ω may then be interpreted as a frequency measured on that time scale. With those conventions, all identities linking time, coordinates, velocities and frequency must show homogeneity with respect to the dimensional scaling

$$(x, y, X, Y, t, \omega) \rightarrow (\lambda x, \lambda y, \lambda X/\tau, \lambda Y/\tau, \tau t, \omega/\tau). \tag{3}$$

This criterion of homogeneity proves very useful in checking hand and computer calculations, and this is one of the reasons why, rather than announcing that the physical units have been chosen to make $\omega = 1$, we accept to drag that parameter along the whole exposition. Besides, as the fourth installment in these series of notes will demonstrate, there are circumstances where we need to admit that the Lissajous transformation is not simply a transformation but a one-parameter family of transformations.

The Lie derivative associated with $\mathcal{H}_0$ is the partial differential operator

$$L_0 : F \rightarrow (F, \mathcal{H}_0)$$

mapping F onto its Poisson bracket to the right with $\mathcal{H}_0$. The *kernel* of L_0 is the set $\ker(L_0)$ of functions F such that $L_0(F) = 0$; the *image* of L_0, the set $\mathrm{im}(L_0)$ of functions F of the form $F = L_0(G)$. Normalization of a Hamiltonian of type

$$\mathcal{H}(\boldsymbol{p}, \boldsymbol{P}, \epsilon) = \sum_{n \geq 0} \epsilon^n \mathcal{H}_n(\boldsymbol{p}, \boldsymbol{P}),$$

we recall, is a one-parameter family of canonical transformations

$$\nu : (\boldsymbol{p}', \boldsymbol{P}', \epsilon) \rightarrow (\boldsymbol{p}, \boldsymbol{P})$$

that changes $\mathcal{H}$ into a function

$$\nu^{\#}\mathcal{H}(\boldsymbol{p}', \boldsymbol{P}', \epsilon) = \mathcal{H}\Big(\boldsymbol{p}(\boldsymbol{p}', \boldsymbol{P}', \epsilon), \boldsymbol{P}(\boldsymbol{p}', \boldsymbol{P}', \epsilon), \epsilon\Big)$$

in the kernel of L_0. From an operational standpoint, three critical issues must be faced before undertaking a normalization:

1. one must identify the algebra $\mathcal{A}$ of functions involved at any order of the normalization;
2. one must verify that $\mathcal{A}$ is the direct sum $\ker(L_0|\mathcal{A}) \oplus \mathrm{im}(L_0|\mathcal{A})$;
3. one must prefigure the topological structure subjacent to the orbital space after reduction.

In the Cartesian variables (x, y, X, Y), the Lie derivative associated with the elliptic oscillator (1) is represented by the partial differential operator

$$L_0 = X\,\frac{\partial}{\partial x} + Y\,\frac{\partial}{\partial y} - \omega^2\left(x\,\frac{\partial}{\partial X} + y\,\frac{\partial}{\partial Y}\right). \tag{4}$$

Thus, when the perturbation is a power series in ϵ with coefficients in the algebra $\mathcal{P}$ of real polynomials in (x, y, X, Y), the normalization takes place entirely within that algebra. To characterize the vector space $\ker(L_0|\mathcal{P})$,[1] we begin with examining under what conditions the linear form $\alpha X + \beta Y + \omega(\gamma x + \delta y)$ is an eigenvector of $L_0|\mathcal{P}$. Clearly, the identity

$$L_0(\alpha X + \beta Y + \omega\gamma x + \omega\delta y) = \lambda(\alpha X + \beta Y + \omega\gamma x + \omega\delta y)$$

is satisfied either when $\lambda = i\omega$, $\gamma = i\alpha$ and $\delta = i\beta$ or when $\lambda = -i\omega$, $\gamma = -i\alpha$ and $\delta = -i\beta$ (The symbol i designates here the imaginary unit $\sqrt{-1}$). In sum, the combinations $X + i\omega x$ and $Y + i\omega y$ are eigenvectors for the eigenvalue $i\omega$ while $X - i\omega x$ and $Y - i\omega y$ are eigenvectors for the eigenvalue $-i\omega$. To capitalize on this basic property of L_0, it seems appropriate to introduce the canonical transformation[2]

$$\begin{aligned} z &= \frac{1}{\sqrt{2}}(x - iX/\omega), \qquad & w &= \frac{1}{\sqrt{2}}(y - iY/\omega), \\ Z &= \frac{1}{\sqrt{2}}(X - i\omega x), \qquad & W &= \frac{1}{\sqrt{2}}(Y - i\omega y). \end{aligned} \tag{5}$$

All critical issues when normalizing a Hamiltonian that is a power series over $\mathcal{P}$ are best addressed in the complex variables (z, w, Z, W). For, in those variables,

$$\mathcal{H}_0 = i\omega(zZ + wW),$$

and the Lie derivative becomes the differential operator

$$L_0 = i\omega\Big(z\frac{\partial}{\partial z} - Z\frac{\partial}{\partial Z} + w\frac{\partial}{\partial w} - W\frac{\partial}{\partial W}\Big).$$

There follows immediately that

$$L_0(z^\alpha w^\beta Z^\gamma W^\delta) = i\omega(\alpha + \beta - \gamma - \delta) z^\alpha w^\beta Z^\gamma W^\delta. \tag{6}$$

This identity characterizes the kernel of L_0 restricted to the algebra $\mathcal{C}$ of polynomials in (z, w, Z, W) as the vector subspace generated by the monomials $z^\alpha w^\beta Z^\gamma W^\delta$ for which

$$\alpha + \beta - \gamma - \delta = 0. \tag{7}$$

In particular, the monomials zZ, wW, zW and wZ belong to $\ker(L_0|\mathcal{C})$. Furthermore, for $n = \alpha + \beta + \gamma + \delta$, condition (7) is satisfied only if n is

[1] The symbol $L_0|\mathcal{P}$ designates the restriction of the operator L_0 to the set $\mathcal{P}$.

[2] The presence of $\sqrt{2}$ in (5) may cause minor, but irritating, complications for some algebraic processors. In which case, the transition from real to complex variables should be expressed as the symplectic linear transformation:

$$z = (x - iX/\omega), \qquad Z = (X - i\omega x), \qquad w = \tfrac{1}{2}(y - iY/\omega), \qquad W = \tfrac{1}{2}(Y - i\omega y).$$

even. In which case, setting $m = n/2$, we must have that $\alpha+\beta = \gamma+\delta = m$, hence

$$z^\alpha w^\beta Z^\gamma W^\delta = \begin{cases} (zZ)^\alpha (wW)^\beta & \text{when } \alpha = \gamma, \\ (zZ)^\gamma (wW)^\beta (zW)^k & \text{when } \alpha = \gamma + k, \\ (zZ)^\alpha (wW)^\delta (wZ)^k & \text{when } \gamma = \alpha + k. \end{cases}$$

Thus is proved that $\ker(L_0|\mathcal{C})$ is the subalgebra of $\mathcal{C}$ generated by the monomials zZ, wW, zW and wZ [For another proof see, e.g., (Cushman and Rod 1982)]. There follows also that the normalization, when $\mathcal{V}$ is a power series in ϵ with coefficients in $\mathcal{C}$, is nothing else but a canonical transformation

$$(z', w', Z', W', \epsilon) \to (z, w, Z, W)$$

to change (1) into a power series whose coefficients belong to the algebra of complex polynomials in $z'Z'$, $w'W'$, $z'W'$ and $w'Z'$.

Again, according to (6), $\mathrm{im}(L_0|\mathcal{C})$ is generated by the terms $z^\alpha w^\beta Z^\gamma W^\delta$ whose exponents do not satisfy condition (7). Not only does (6) prove that $\mathcal{C}$ is the direct sum $\ker(L_0|\mathcal{C}) \oplus \mathrm{im}(L_0|\mathcal{C})$, it also supplies a very simple rule for building a polynomial G in $\mathcal{C}$ such that $L_0(G) = F$ when F belongs to $\mathrm{im}(L_0|\mathcal{C})$. Normalization in the complex variables (5) turns into an exercise in polynomial algebra.

Physicists, nonetheless, think spontaneously of a normalization as a *symmetrization*, that is, as a way of approximating a Hamiltonian with a power series invariant with respect to the group (sic!) of transformations

$$\begin{aligned} x &= \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_0{}^n(x') = x' \cos\omega\epsilon + (X'/\omega)\sin\omega\epsilon, \\ y &= \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_0{}^n(y') = y' \cos\omega\epsilon + (Y'/\omega)\sin\omega\epsilon, \\ X &= \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_0{}^n(X') = X' \cos\omega\epsilon - \omega x'\sin\omega\epsilon, \\ Y &= \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_0{}^n(Y') = Y' \cos\omega\epsilon - \omega y'\sin\omega\epsilon. \end{aligned} \tag{8}$$

In that perspective, to say that a function F belongs to $\ker(L_0)$ is equivalent to saying that it admits (8) as a group of symmetries.

Normalization has a second, no less important geometric facet: it is a *reduction* (Meyer 1973; Marsden and Weinstein 1974). In the process of normalization from a perturbed to a "symmetrized" integrable Hamiltonian, the number of degrees of freedom has fallen by one unit. The paradigm of a reduction in traditional mechanics is the *elimination* of a cyclic coordinate, say p. In which case, the textbooks continue, its conjugate moment P is an integral, and the equations of motion split into

1. a dynamical system derived from the original Hamiltonian with one degree of freedom less than the original one because p is to be ignored, and P to be held as a parameter,
2. a quadrature determining the cyclic coordinate p as a function of time.

Upon analyzing this paradigm more closely from a geometric standpoint, one will recognize that ignoring p and holding P as a parameter amounts to partitioning the phase space into leaves consisting of all states for which P has a given value, collecting into classes all points within that leaf which are images of one another by the canonical transformations generated by the integral P, and then "reducing" the phase space on each leaf P = constant by handling each class as an individual phase state.

The Lissajous transformation was invented to set the physicist's intuition at ease. Orbits of the elliptic oscillator which, by the way, are exactly the orbits of the group (8), are ellipses possibly collapsing to straight line segments. Two of the Lissajous variables, L and G, serve to measure the size and shape of the ellipses, a third one, g, fixes their position in the plane. The fourth one, ℓ, acts as a clock by which to time the journey of the particle on its trajectory. The symmetries in (8) correspond to a shift of origin on the ℓ-scale. They lead naturally to realize that the equivalence classes introduced by the reduction consist of ellipses of a given size at a given inclination in the position plane (to be paired, of course, with the corresponding hodograph in the velocity plane). The reduced equations tell how, on the average, the perturbation affects the length of the axes, and sets them into rotation or libration.

Now that computer algebra is becoming widespread, we speculate that the reader is concerned, like we are, about manipulating Lissajous variables by symbolic processors. For this reason, we present an explicit definition (Section 2). We pay special attention to a number of details which should not be overlooked lest we drag extensive calculations into a quagmire of lethal complications. This version has been tested extensively with our algebraic processor *MAO* [Deprit and Miller (1986; 1989)] and also with *Mathematica* (Wolfram 1988). Yet, this polished version says nothing to justify the transformation. We remedy this oversight in Section 3 where, as we did originally, we trace the Lissajous transformation back to the Hamilton-Jacobi equation derived from the Hamiltonian of an elliptic oscillator.

A number of authors before us, e.g., (Braun 1973; Kummer 1976; Churchill *et al.* 1978) have proposed elements which, in their opinion, were better suited than the common harmonic ones to map the orbital space after reduction. As a help in relating the Lissajous variables to other sets, we present in full the formulas needed to express our variables in terms of the usual harmonic variables (Section 4).

Lissajous variables serve as cylindrical coordinates on the orbital spheres after reduction. In that regard, they grant privileged status to a particular great circle on such spheres, and to the diameter perpendicular to it. In Section 5, we build two variants of the Lissajous transformation with a view of determining different systems of cylindrical coordinates on the reduced orbital spheres.

2. Explicit Definition

The Lissajous transformation

$$\lambda : (\ell, g, L, G; \omega) \to (x, y, X, Y) : D \to \mathbf{R}^4$$

is defined in the domain

$$D = [0, 2\pi[\times[0, 2\pi[\times\{L > 0\} \times \{|G| \leq L\} \tag{9}$$

by the equations

$$\boxed{\begin{aligned}
x &= \sqrt{\frac{L+G}{2\omega}}\cos(\ell+g) - \sqrt{\frac{L-G}{2\omega}}\cos(\ell-g),\\
y &= \sqrt{\frac{L+G}{2\omega}}\sin(\ell+g) + \sqrt{\frac{L-G}{2\omega}}\sin(\ell-g),\\
X &= \omega\frac{\partial x}{\partial \ell} = -\sqrt{\frac{\omega(L+G)}{2}}\sin(\ell+g) + \sqrt{\frac{\omega(L-G)}{2}}\sin(\ell-g),\\
Y &= \omega\frac{\partial y}{\partial \ell} = \sqrt{\frac{\omega(L+G)}{2}}\cos(\ell+g) + \sqrt{\frac{\omega(L-G)}{2}}\cos(\ell-g).
\end{aligned}} \tag{10}$$

Should we be of a mind to satisfy the scaling requirement (3), we would assign to L and G the dimensions length^2 time^{-1} of an angular momentum; the angles ℓ and g would, of course, be dimensionless.

On account of the conditions in (9), we introduce a function $a(L, G, \omega)$ and a function $b(L, G, \omega)$ (optionally, both with the dimension of a length) such that

$$L = \tfrac{1}{2}\omega(a^2 + b^2) \qquad \text{and} \qquad G = \omega ab \qquad \text{with} \quad a > 0. \tag{11}$$

Then, if we set

$$s = \tfrac{1}{2}(a + b) \qquad \text{and} \qquad d = \tfrac{1}{2}(a - b).$$

we find that

$$\omega s^2 = \tfrac{1}{2}(L + G) \qquad \text{and} \qquad \omega d^2 = \tfrac{1}{2}(L - G),$$

and we remove the irrational expressions from (10). As a result, it proved very convenient, especially when the calculations are done by computer, to introduce the Lissajous transformation in the form

$$\begin{aligned} x &= s\cos(g+\ell) - d\cos(g-\ell), \\ y &= s\sin(g+\ell) - d\sin(g-\ell), \\ X &= -\omega\Big(s\sin(g+\ell) + d\sin(g-\ell)\Big), \\ Y &= \omega\Big(s\cos(g+\ell) + d\cos(g-\ell)\Big). \end{aligned} \tag{12}$$

The Lissajous transformation is everywhere regular in the domain defined by the conditions (9). Indeed, singularities occur solely at the points where the partial derivatives

$$\frac{\partial s}{\partial L} = \frac{\partial s}{\partial G} = \frac{1}{4\omega s}, \qquad \frac{\partial d}{\partial L} = -\frac{\partial d}{\partial G} = \frac{1}{4\omega d}, \tag{13}$$

are infinite, which happens only for $s = 0$ or $d = 0$, hence outside the domain specified in (9). Inside (D), however, the determinant of the Jacobian matrix

$$\frac{\partial(x, y, X, Y)}{\partial(\ell, g, L, G)}$$

is found to be equal to 1. Given a quadruple (x, y, X, Y) in the regularity domain, one obtains its Lissajous coordinates as follows:

1. one calculates the actions L and G from the formulas

$$\omega L = \tfrac{1}{2}\left(X^2 + Y^2 + \omega^2(x^2 + y^2)\right), \qquad G = xY - yX;$$

It is worth noting that, if (x, y) are position coordinates and (X, Y), velocity components of a particle, then, according to the preceding formula, G is the angular momentum of that particle per unit of mass.

2. then one computes the auxiliary quantities

$$2\omega s = \sqrt{2\omega(L+G)}, \qquad 2\omega d = \sqrt{2\omega(L-G)};$$

3. finally, using the relations

$$\begin{aligned} 2\omega s\sin(\ell+g) &= \omega y - X, & 2\omega d\sin(\ell-g) &= X + \omega y, \\ 2\omega s\cos(\ell+g) &= \omega x + Y, & 2\omega d\cos(\ell-g) &= Y - \omega x, \end{aligned}$$

one determines the angles ℓ and g without ambiguity.

Formulas in step 1 of the inversion process gives a hint as to the fundamental role the following quadratic forms

$$
\begin{aligned}
I_0 &= \frac{1}{4\omega}\Big(X^2+Y^2+\omega^2(x^2+y^2)\Big) &&= \frac{i}{2}(wW+zZ),\\
I_1 &= \frac{1}{4\omega}\Big(Y^2-X^2+\omega^2(y^2-x^2)\Big) &&= \frac{i}{2}(wW-zZ),\\
I_2 &= -\frac{1}{2\omega}(XY+\omega^2 xy) &&= -\frac{i}{2}(zW+wZ),\\
I_3 &= \frac{1}{2}(xY-yX) &&= \frac{1}{2}(zW-wZ),\\
J_1 &= \frac{1}{4\omega}\Big(X^2+Y^2-\omega^2(x^2+y^2)\Big) &&= \frac{1}{4\omega}\left(Z^2+W^2-\omega^2(z^2+w^2)\right),\\
J_2 &= \frac{1}{2}(xX+yY) &&= \frac{i}{4\omega}\left(Z^2+W^2+\omega^2(z^2+w^2)\right).
\end{aligned}
\tag{14}
$$

play in the study of the Lissajous transformation and of its relations with other sets proposed to analyze perturbed elliptic oscillators.

The frequency ω has been introduced here and there throughout (14) so that, if one adheres to the dimensional conventions summarized in (3), all these quadratic forms have the dimension legnth^2 time^{-1} of an angular momentum per unit of mass. Note also that, in contrast to the first four quadratic forms whose role has long been recognized in the theory of the elliptic oscillator [see, e.g., (McIntosh 1959)], the last two forms do not seem to have ever been mentioned.

The six functions in (14) are not independent. As one can readily check,

$$I_1^2+I_2^2=J_1^2+J_2^2=I_0^2-I_3^2.$$

On the other hand, the identities

$$\det\frac{\partial(I_0,I_1,J_1,J_2)}{\partial(x,y,X,Y)}=I_2I_3,\qquad \det\frac{\partial(I_1,I_2,J_1,J_2)}{\partial(x,y,X,Y)}=I_0I_1,$$

show that the Jacobian matrix

$$\frac{\partial(I_0,I_1,I_2,I_3,J_1,J_2)}{\partial(x,y,X,Y)}$$

is of rank four. Therefore one can express these forms in terms of four independent variables. Indeed, replacing the Cartesian variables by their expressions in terms of the Lissajous variables yields the formulas

$$
\begin{aligned}
I_0 &= \tfrac{1}{2}L, & I_1 &= \omega s d\cos 2g, & J_1 &= \omega s d\cos 2\ell,\\
I_3 &= \tfrac{1}{2}G, & I_2 &= \omega s d\sin 2g. & J_2 &= \omega s d\sin 2\ell.
\end{aligned}
\tag{15}
$$

From these identities there results that the Lissajous variables are determined unambiguously from the Cartesian variables as follows:

$$L = 2I_0, \qquad \cos 2g = \frac{I_1}{\sqrt{I_0^2 - I_3^2}}, \qquad \cos 2\ell = \frac{J_1}{\sqrt{I_0^2 - I_3^2}},$$
$$G = 2I_3, \qquad \sin 2g = \frac{I_2}{\sqrt{I_0^2 - I_3^2}}, \qquad \sin 2\ell = \frac{J_2}{\sqrt{I_0^2 - I_3^2}}.$$

Inversely, we find that

$$I_1 = \tfrac{1}{2}\sqrt{L^2 - G^2}\cos 2g, \qquad J_1 = \tfrac{1}{2}\sqrt{L^2 - G^2}\cos 2\ell,$$
$$I_2 = \tfrac{1}{2}\sqrt{L^2 - G^2}\sin 2g, \qquad J_2 = \tfrac{1}{2}\sqrt{L^2 - G^2}\sin 2\ell.$$

The functions I_1, I_2 and I_3 determine a transformation

$$(x, y, X, Y) \longmapsto (I_1, I_2, I_3)$$

mapping the three-dimensional sphere

$$S^3(L) \,:\, X^2 + Y^2 + \omega^2(x^2 + y^2) = 2\omega L$$

in the phase space $(\omega x, \omega y, X, Y)$ onto the two-dimensional sphere

$$S^2(L) \,:\, I_1^2 + I_2^2 + I_3^2 = \tfrac{1}{4}L^2$$

(Hopf 1931). In that perspective, it is interesting to note that G and $2g$ appear as cylindrical coordinates on $S^2(L)$: G is the elevation above the plane (I_1, I_2) whereas $2g$ is the longitude reckoned in that plane from the axis I_1. We shall show in Section 5 how to modify the Lissajous transformations in order to obtain different cylindrical coordinate systems on the sphere $S^2(L)$. The issue is not without importance considering that, for a normalized perturbed elliptic oscillator, the reduced phase space is partitioned into a collection of spheres $S^2(L)$.

At present, though, we shall show how the Lissajous transformation stands out as the one most intimately related with the geometry of an orbit for the elliptic oscillator. From putting (12) in the form

$$x = u\cos g - v\sin g, \qquad X = U\cos g - V\sin g,$$
$$y = u\sin g + v\cos g, \qquad Y = U\sin g + V\cos g,$$

after introducing the intermediate quantities

$$u = b\cos\ell, \qquad U = \omega\,\partial u/\partial\ell = -\omega b\sin\ell,$$
$$v = a\sin\ell, \qquad V = \omega\,\partial v/\partial\ell = \omega a\cos\ell,$$

we make clear that, in the space (x, y, X, Y), when $b \neq 0$, the curves ($L =$ constant, $G =$ constant, $g =$ constant) consist of pairs of ellipses:

- one ellipse $\gamma(L,G,g)$ in the coordinate plane (x,y), centered at the origin with semi-axes of length $|b|$ and a along the u- and v-axes respectively, the u-axis of symmetry being inclined at an angle g over the x-axis;
- and one ellipse $\Gamma(L,G,g)$ in the moment plane (X,Y), likewise centered at the origin with semi-axes of length $\omega|b|$ and ωa along the U- and V-axes respectively, the U-axis of symmetry being inclined at an angle g over the X-axis.

These curves, as Lissajous (1849) demonstrated in his laboratory, are precisely the figures drawn by a light point whose Cartesian coordinates are in harmonic vibration at the same frequency, hence the name we have given the transformation.

The geometric model explains why the Lissajous transformation is singular when either $s=0$ or $d=0$, that is equivalently, when $|b|=a$. For, in that case, both ellipses γ and Γ being circles, the inclination g is undetermined.

When $b=0$, then $s=d=a/2$ and

$$\begin{aligned}
x &= -\ a\sin g\sin\ell = a\cos(g+\pi/2)\sin\ell,\\
y &= \ a\cos g\sin\ell = a\sin(g+\pi/2)\sin\ell,\\
X &= -\omega a\sin g\cos\ell = \omega a\cos(g+\pi/2)\cos\ell,\\
Y &= \ \omega a\cos g\cos\ell = \omega a\sin(g+\pi/2)\cos\ell.
\end{aligned}$$

Thus the ellipses $\gamma(L,0,g)$ collapse into straight line segments of length $2a$ centered at the origin and inclined over the x-axis by an angle $g+\pi/2$ while the ellipses $\Gamma(L,0,g)$ reduce to line segments of length $2\omega a$, likewise centered at the origin and inclined by the same angle over the X-axis.

Let us now assume that the space on which the Lissajous transformation acts is a symplectic manifold, and that the Cartesian variables define a chart in which all mutual Poisson brackets (x,y), etc., are zero save (x,X) and (y,Y) supposed to be equal to 1. By straightforward differentiations, one checks readily that substituting the definitions (10) in those Poisson brackets does not alter their values. In other words, the Lissajous transformation is canonical.

It might be worth noting, incidentally, how using (12) rather than (10) simplifies the evaluation of Poisson brackets. Indeed, on account of (13) for any functions p and q in the Cartesian variables,

$$\begin{aligned}
(p,q) = \frac{1}{4\omega}\Bigg[&\frac{\partial p}{\partial \ell}\left(\frac{1}{s}\frac{\partial q}{\partial s}+\frac{1}{d}\frac{\partial q}{\partial d}\right)-\frac{\partial q}{\partial \ell}\left(\frac{1}{s}\frac{\partial p}{\partial s}+\frac{1}{d}\frac{\partial p}{\partial d}\right)\\
&+\frac{\partial p}{\partial g}\left(\frac{1}{s}\frac{\partial q}{\partial s}-\frac{1}{d}\frac{\partial q}{\partial d}\right)-\frac{\partial q}{\partial g}\left(\frac{1}{s}\frac{\partial p}{\partial s}-\frac{1}{d}\frac{\partial p}{\partial d}\right)\Bigg]
\end{aligned}$$

So are circumvented complications arising from square roots.

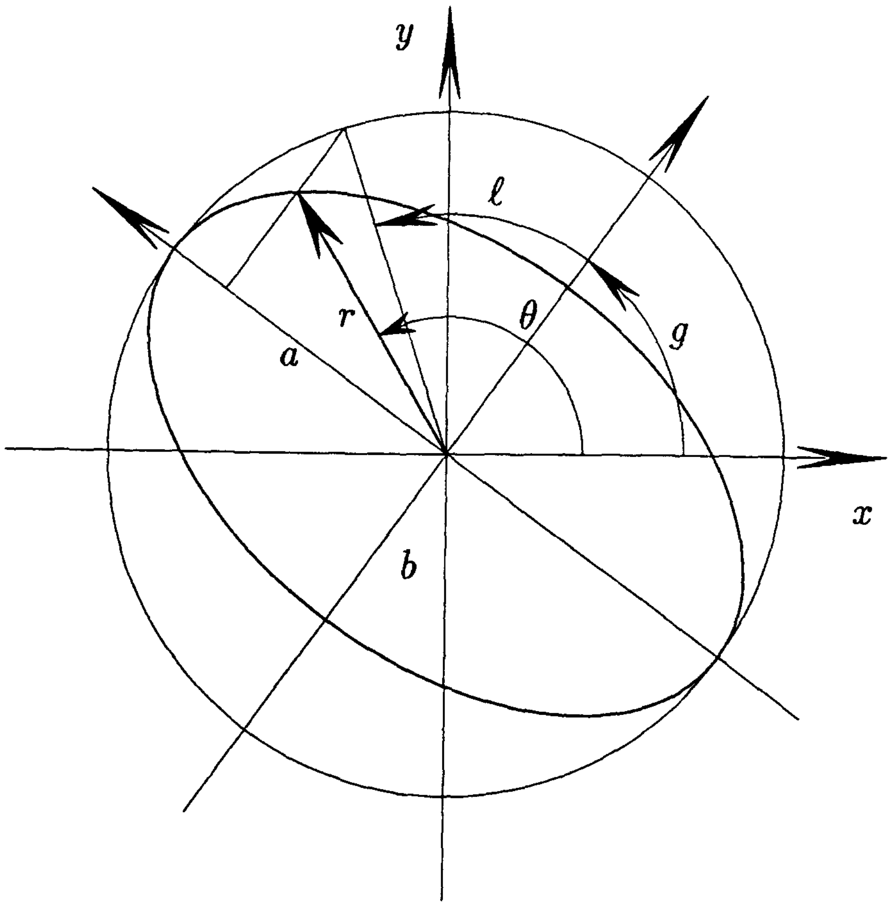

Fig. 1. Geometric meaning of the Lissajous variables.

That the Lissajous transformation is not a Mathieu — or gauge-free transformation — results from the differential identity

$$X\,dx + Y\,dy - L\,d\ell - G\,dg = dJ_2.$$

3. Implicit Definition

It is time now we explain how we came to the definition (10). Originally it was obtained in the manner familiar to specialists of classical mechanics from a *generating* function that is a complete solution of a partial differential equation. Since the original derivation has been in private circulation for some time now, it will make for clarity if we retrace our steps.

Passing from Cartesian to polar variables by the canonical transformation

$$\begin{aligned} x &= r\cos\vartheta, & X &= R\cos\vartheta - \frac{\Theta}{r}\sin\vartheta, \\ y &= r\sin\vartheta, & Y &= R\sin\vartheta + \frac{\Theta}{r}\cos\vartheta, \end{aligned} \tag{16}$$

we change the Hamiltonian (1) into the function

$$\mathcal{H}_0 = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2} + \omega^2 r^2\right);$$

then, as we adopt the method of Poincaré (1893, vol. 2, pp. 315–342) rather than that of Hamilton and Jacobi, we look for a function $\mathcal{S} \equiv \mathcal{S}(r, \vartheta, L, G)$ and a function $\mathcal{K} \equiv \mathcal{K}(L, G)$ which satisfy identically the relation

$$\frac{1}{2}\left[\left(\frac{\partial \mathcal{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathcal{S}}{\partial \vartheta}\right)^2 + \omega^2 r^2\right] = \mathcal{K}(L, G).$$

We adopt $\mathcal{K} = \omega L$; by virtue of this selection, L is restricted to be > 0. Furthermore, we separate the coordinates r and ϑ by setting

$$\mathcal{S} = G\vartheta + \mathcal{W}(L, G, r).$$

Under these assumptions, $\mathcal{W}$ is to be determined so as to satisfy the relation

$$\frac{1}{2}\left[\left(\frac{\partial \mathcal{W}}{\partial r}\right)^2 + \frac{G^2}{r^2} + \omega^2 r^2\right] = \omega L.$$

For $\mathcal{W}$ we select the quadrature

$$\mathcal{W} = \int_P^r \sqrt{Q}\, d\bar{r}$$

whose integrand is such that

$$Q \equiv Q(r, L, G) = 2\omega L - \omega^2 r^2 - G^2/r^2,$$

the lower limit $P \equiv P(L, G)$ being a root of the equation $Q = 0$, i.e., such that $Q(P, L, G) = 0$.

Clearly, the quartic equation $Q = 0$ has real roots if and only if $L^2 \geq G^2$. In which case, with the functions a and b of L and G defined in (11), we factorize Q into the product

$$Q = \frac{\omega^2}{r^2}(a^2 - r^2)(r^2 - b^2).$$

For the lower limit P we choose the root b.

The transformation generated by $\mathcal{S}$ is deduced from the implicit equations

$$\begin{aligned} R &= \frac{\partial \mathcal{S}}{\partial r} = \sqrt{Q}, & \ell &= \frac{\partial \mathcal{S}}{\partial L} = \omega \int_b^r \frac{d\bar{r}}{\sqrt{Q}}, \\ \Theta &= \frac{\partial \mathcal{S}}{\partial \vartheta} = G, & g &= \frac{\partial \mathcal{S}}{\partial G} = \vartheta - G \int_b^r \frac{d\bar{r}}{r^2\sqrt{Q}}. \end{aligned} \tag{17}$$

The quadrature giving ℓ is performed by introducing an angle $E(L, G, r)$ such that

$$r^2 = a^2 \sin^2 E + b^2 \cos^2 E, \tag{18}$$

and the quadrature giving g, by introducing an angle $f(L, G, r)$ such that

$$\frac{1}{r^2} = \frac{\sin^2 f}{a^2} + \frac{\cos^2 f}{b^2}. \tag{19}$$

Then, after a few elementary manipulations, the implicit equations yield that

$$\ell = E \qquad \text{and} \qquad g = \vartheta - f.$$

There remains now to check that the transformation is identical to the one defined in Section 2. The proof is easy as far as the coordinates (x, y) are concerned, but slightly more intricate for the velocities (X, Y). From (19), we deduce that

$$r^2 \sin^2 f = a^2 \frac{r^2 - b^2}{a^2 - b^2} \qquad \text{and} \qquad r^2 \cos^2 f = b^2 \frac{a^2 - r^2}{a^2 - b^2};$$

in these expressions, we substitute (18) to obtain that

$$r^2 \sin^2 f = a^2 \sin^2 E \qquad \text{and} \qquad r^2 \cos^2 f = b^2 \cos^2 E;$$

finally, since $E = \ell$, we find that

$$r \sin f = a \sin \ell \qquad \text{and} \qquad r \cos f = b \cos \ell.$$

Introducing these expressions in (16) will reproduce the first two equations in the explicit definitions (10). As for the velocities, we begin by observing that

$$Q = \omega^2 \frac{(a^2 - b^2)^2}{a^2 b^2} r^2 \sin^2 f \cos^2 f$$

hence the value for the radial moment

$$R = \sqrt{Q} = \omega \frac{a^2 - b^2}{ab} r \sin f \cos f.$$

A few algebraic manipulations will then yield the identities

$$R\sin f + \frac{\Theta}{r}\cos f = \omega\frac{a}{b}r\cos f = \omega a\cos\ell = \omega\frac{\partial}{\partial\ell}r\sin f,$$

$$R\cos f - \frac{\Theta}{r}\sin f = -\omega\frac{b}{a}r\sin f = -\omega b\sin\ell = \omega\frac{\partial}{\partial\ell}r\cos f.$$

The same way as we just did for the coordinates, we introduce these expressions in (16), and we recover the last two equations in the system (10).

4. Harmonic Variables

We undertake now to relate the Lissajous variables to the elements hitherto most often used by physicists in dealing with elliptic oscillators.

At the outset of Section 3, we took advantage of the fact that the harmonic oscillators composing an elliptic oscillator are in 1-1 resonance. In that case, the rotations

$$x = x'\cos\epsilon - y'\sin\epsilon, \qquad X = X'\cos\epsilon - Y'\sin\epsilon,$$
$$y = x'\sin\epsilon + y'\cos\epsilon, \qquad Y = X'\sin\epsilon + Y'\cos\epsilon,$$

leave the Hamiltonian (1) invariant, hence the angular momentum G is an integral. The Lissajous transformation suits well the dynamics of the elliptic oscillator by raising this integral to the condition of an action variable next to the energy $\omega\, L$. By contrast, Poincaré's harmonic variables shift the emphasis to a different group of symmetries, one that is shared not only by elliptic oscillators but by any couple of harmonic oscillators whether they are in resonance or not. This is the group made of pairs of rotations

$$x = x'\cos\epsilon' - (X'/\omega)\sin\epsilon', \qquad y = y'\cos\epsilon'' - (Y'/\omega)\sin\epsilon'',$$
$$X = \omega x'\sin\epsilon' + X'\cos\epsilon', \qquad Y = \omega y'\sin\epsilon''. + Y'\cos\epsilon'',$$

separately in the Lagrangian planes (x, X) and (y, Y) with amplitudes ϵ' and ϵ'' respectively. They stem from the fact that the Hamilton-Jacobi equation for the elliptic oscillator is separable in Cartesian variables.

From Poincaré's standpoint, this means that the partial differential identity

$$\frac{1}{2}\left[\left(\frac{\partial\mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial\mathcal{S}}{\partial y}\right)^2 + \omega^2(x^2+y^2)\right] = \mathcal{K}$$

is satisfied by choosing $\mathcal{K} = \omega(\Phi + \Psi)$ while decomposing $\mathcal{S}$ into a sum $\mathcal{X} + \mathcal{Y}$ whose components should verify the partial differential identities

$$\left(\frac{\partial\mathcal{X}}{\partial x}\right)^2 + \omega^2 x^2 = 2\omega\Phi, \qquad \left(\frac{\partial\mathcal{Y}}{\partial y}\right)^2 + \omega^2 y^2 = 2\omega\Psi.$$

For $\mathcal{X}$ we choose the quadrature

$$\mathcal{X} = \int_0^x \sqrt{2\omega\Phi - \omega^2\bar{x}^2}\, d\bar{x}.$$

An elementary integration yields immediately that

$$\mathcal{X} = 2\Phi \arcsin\left(x\sqrt{\frac{\omega}{2\Phi}}\right) + \frac{1}{2}x\sqrt{2\omega\Phi - \omega^2 x^2}.$$

But, by introducing the function $\zeta(\Phi, x)$ such that $x = \sqrt{2\Phi/\omega}\sin\zeta$, we get the simpler expression:

$$\mathcal{X} = \Phi(\zeta + \cos\zeta\sin\zeta).$$

The solution $\mathcal{X}$ is to be used as the generator of a canonical transformation $(\phi, \Phi) \to (x, X)$ defined by the implicit equations

$$X = \frac{\partial\mathcal{X}}{\partial x}, \qquad \phi = \frac{\partial\mathcal{X}}{\partial\Phi},$$

or, since

$$\frac{\partial\zeta}{\partial x} = \frac{1}{\cos\zeta}\sqrt{\frac{\omega}{2\Phi}}, \qquad \text{and} \qquad \frac{\partial\zeta}{\partial\Phi} = -\frac{\sin\zeta}{2\Phi\cos\zeta},$$

by the implicit equations

$$\begin{aligned} X &= \frac{\partial\mathcal{X}}{\partial x} = \frac{\partial\mathcal{X}}{\partial\zeta}\frac{\partial\zeta}{\partial x} = \sqrt{2\Phi\omega}\sin\zeta, \\ \phi &= \frac{\partial\mathcal{X}}{\partial\Phi} = \zeta + \cos\zeta\sin\zeta + \frac{\partial\mathcal{X}}{\partial\zeta}\frac{\partial\zeta}{\partial\Phi} = \zeta. \end{aligned}$$

In a similar manner, we attach the harmonic elements ψ and Ψ to the Cartesian coordinates y and Y. In sum, without having had to introduce a time dependent characteristic, as would have happened had we resorted to Hamilton-Jacobi formalism [see, e.g., (Pars 1965, pp. 279–281)], we obtain the classical canonical transformation

$$\mu : (\phi, \psi, \Phi, \Psi) \longmapsto (x, y, X, Y)$$

defined by the equations

$$\begin{aligned} x &= \sqrt{2\Phi/\omega}\sin\phi, & y &= \sqrt{2\Psi/\omega}\sin\psi, \\ X &= \sqrt{2\omega\Phi}\cos\phi, & Y &= \sqrt{2\omega\Psi}\cos\psi. \end{aligned} \tag{20}$$

Upon substituting these values for Cartesian variables in the quadratic forms (14), we obtain the formulas expressing the Lissajous variables in terms of the harmonic variables, namely

$$
\begin{aligned}
I_0 &= \tfrac{1}{2}L = \tfrac{1}{2}(\Phi + \Psi),\\
I_3 &= \tfrac{1}{2}G = \sqrt{\Phi\Psi}\sin(\phi - \psi),\\
I_1 &= \omega s d\cos 2g = \tfrac{1}{2}(\Psi - \Phi),\\
I_2 &= \omega s d\sin 2g = -\sqrt{\Phi\Psi}\cos(\phi - \psi),\\
J_1 &= \omega s d\cos 2\ell = \tfrac{1}{2}(\Phi\cos 2\phi + \Psi\cos 2\psi),\\
J_2 &= \omega s d\sin 2\ell = \tfrac{1}{2}(\Phi\sin 2\phi + \Psi\sin 2\psi).
\end{aligned}
\tag{21}
$$

From the preceding formulas, we deduce that

$$
\begin{aligned}
s\sqrt{\omega} &= \sqrt{\tfrac{1}{2}(\Phi + \Psi) + \sqrt{\Phi\Psi}\sin(\phi - \psi)},\\
d\sqrt{\omega} &= \sqrt{\tfrac{1}{2}(\Phi + \Psi) - \sqrt{\Phi\Psi}\sin(\phi - \psi)},\\
\omega s d &= \sqrt{\tfrac{1}{4}(\Phi^2 + \Psi^2) + \tfrac{1}{2}\Phi\Psi\cos 2(\phi - \psi)}.
\end{aligned}
$$

These formulas complete the conversion from harmonic to Lissajous variables.

For as long as perturbed elliptic oscillators have been studied has it been felt that the usual harmonic variables are not the best suited for the problem. Time and again new coordinates are proposed. Finding the correspondences among these sets is somewhat intricate. The preceding analysis points to a systematic way of cutting through the complications: consider the set made of the Hopf variables I_0, I_1, I_2, I_3, the quadratic forms J_1 and J_2 and the functions

$$
\begin{aligned}
K_1 &= \frac{1}{4\omega}(Y^2 - X^2 - \omega^2 y^2 + \omega^2 x^2), & K_3 &= \tfrac{1}{2}(xX - yY),\\
K_2 &= \frac{1}{2\omega}(XY - \omega^2 xy), & K_4 &= \tfrac{1}{2}(xY + yX),
\end{aligned}
$$

and express these 10 forms in one set of variables and then in the other. There will usually result enough relations to determine unambiguously one set of variables in terms of the other. We shall apply this methodology in the next section while introducing two variants of the Lissajous transformation.

The complexity of the conversion formulas (21) masks the simple relation that exits between the Lissajous transformation λ and the Poincaré transformation μ. To establish that link, we introduce two elementary canonical transformations:

1. $\sigma : (\ell', g', L', G') \longmapsto (\phi, \psi, \Phi, \Psi)$ defined by the equations

$$\phi = \ell' + g', \qquad \Phi = \tfrac{1}{2}(L' + G'),$$
$$\psi = \ell' - g', \qquad \Psi = \tfrac{1}{2}(L' - G')$$

2. $\chi : (x, y, X, Y) \longmapsto (p, q, P, Q)$ defined by the equations

$$p = \frac{1}{\sqrt{2}}\left(y - \frac{X}{\omega}\right), \qquad P = \frac{1}{\sqrt{2}}(Y + \omega x),$$
$$q = \frac{1}{\sqrt{2}}\left(y + \frac{X}{\omega}\right), \qquad Q = \frac{1}{\sqrt{2}}(Y - \omega x).$$

It is readily seen then that

$$(\chi \circ \lambda)^{\#}(p) = (\mu \circ \sigma)^{\#}(p) = \sqrt{\frac{L' + G'}{\omega}} \sin(\ell' + g'),$$

$$(\chi \circ \lambda)^{\#}(q) = (\mu \circ \sigma)^{\#}(q) = \sqrt{\frac{L' - G'}{\omega}} \sin(\ell' - g'),$$

$$(\chi \circ \lambda)^{\#}(P) = (\mu \circ \sigma)^{\#}(P) = \sqrt{\omega(L' + G')} \cos(\ell' + g'),$$

$$(\chi \circ \lambda)^{\#}(P) = (\mu \circ \sigma)^{\#}(Q) = \sqrt{\omega(L' - G')} \cos(\ell' - g').$$

Thus is proved that $\chi \circ \lambda = \mu \circ \sigma$, or that the diagram below is commutative.

$$\begin{tikzcd}
(\ell', g', L', G') \arrow[r, "\lambda"] \arrow[d, "\sigma"] & (x, y, X, Y) \arrow[d, "\chi"] \\
(\phi, \psi, \Phi, \Psi) \arrow[r, "\mu"] & (p, q, P, Q)
\end{tikzcd}$$

(N. B. Except for a swap of coordinates (p, q) with moments (P, Q), $\chi \circ \lambda$ is the transformation that Ferrer *et al.* (1990) attribute to Vorob'ev and Zaslavskiĭ.

5. Variants

The Lissajous transformation, we have noted in Section 2, defines a map of cylindrical coordinates on the sphere $S^2(L)$ involving the plane

(I_1, I_2) and the axis I_1 as its reference elements. With the transformation

$$\lambda' = \chi \circ \lambda = \mu \circ \sigma \, : \, (\ell', g', L', G') \longmapsto (x, y, X, Y),$$

we hold a different map of cylindrical coordinates on $S^2(L)$, one in which the plane (I_2, I_3) and the axis I_2 constitute the reference objects. This we can see most easily by substituting the equations of the transformation

$$\boxed{\begin{aligned} x &= \sqrt{\frac{L' + G'}{\omega}} \sin(\ell' + g'), \\ y &= \sqrt{\frac{L' - G'}{\omega}} \sin(\ell' - g'), \\ X &= \sqrt{\omega(L' + G')} \cos(\ell' + g'), \\ Y &= \sqrt{\omega(L' - G')} \cos(\ell' - g') \end{aligned}}$$

into the Hopf's quadratic forms. A simple calculation will thus show that

$$I_0 = \tfrac{1}{2}L', \quad I_2 = -\tfrac{1}{2}\sqrt{L'^2 - G'^2} \cos 2g', \quad K_2 = \tfrac{1}{2}\sqrt{L'^2 - G'^2} \cos 2\ell',$$
$$I_1 = -\tfrac{1}{2}G', \quad I_3 = \tfrac{1}{2}\sqrt{L'^2 - G'^2} \sin 2g', \quad K_4 = \tfrac{1}{2}\sqrt{L'^2 - G'^2} \sin 2\ell'.$$

So it appears that G' measures the elevation above the plane (I_2, I_3) and that $2g'$ serves as the longitude in that plane as reckoned from the axis I_2. This change of reference elements is the effect of the transformation χ which it is readily checked transforms the function I_3 into the function $-I_1$ while it sends I_2 onto $-I_3$ and I_3 onto I_2.

Expectedly, these properties of the coordinates (G, g) or (G', g') stem from commutativity properties of the transformations λ and λ' themselves.

Any smooth function $F(x, y, X, Y)$ may serve as the generator of an infinitesimal contact transformation $\lambda_F \; : \; (x, y, X, Y) \longmapsto (x', y', X', Y')$ to be defined by the equations

$$x' = x + \epsilon(x, F) = x + \epsilon L_F(x), \quad X' = X + \epsilon(X, F) = X + \epsilon L_F(X),$$
$$y' = y + \epsilon(y, F) = y + \epsilon L_F(y), \quad Y' = Y + \epsilon(Y, F) = Y + \epsilon L_F(Y),$$

the operator L_F designating the Lie derivative associated with F, i.e., the derivative in the direction of the vector field

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, -\frac{\partial F}{\partial X}, -\frac{\partial F}{\partial Y}.$$

By exponential continuation, λ_F gives rise to a canonical transformation

$$\Lambda_F : (x, y, X, Y) \longmapsto (x', y', X', Y')$$

defined by the equations

$$\begin{aligned}
x' &= (\exp \epsilon L_F)(x) = \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_F^n(x),\\
y' &= (\exp \epsilon L_F)(y) = \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_F^n(y),\\
X' &= (\exp \epsilon L_F)(X) = \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_F^n(X),\\
Y' &= (\exp \epsilon L_F)(Y) = \sum_{n\geq 0} \frac{\epsilon^n}{n!} L_F^n(Y).
\end{aligned}$$

(N. B. Λ_F is known in some quarters of celestial mechanics as the Lie-Hori transformation generated by F.)

The canonical transformation $\Lambda_0 = \Lambda_{I_0}$ is already mentioned in (8). The transformation Λ_1

$$\begin{aligned}
x' &= x\cos\epsilon - \frac{X}{\omega}\sin\epsilon, & y' &= y\cos\epsilon + \frac{Y}{\omega}\sin\epsilon,\\
X' &= X\cos\epsilon + \omega x\sin\epsilon, & Y' &= Y\cos\epsilon - \omega y\sin\epsilon
\end{aligned}$$

generated by I_1 is similar to it in that it imparts a rotation in both Lagrangian planes (y, Y) with amplitude ϵ, except that the rotations in (x, X) and (y, Y) are of opposite sense. The transformation $\Lambda_2 = \Lambda_{I_2}$,

$$\begin{aligned}
x' &= x\cos\epsilon + \frac{Y}{\omega}\sin\epsilon, & y' &= y\cos\epsilon + \frac{X}{\omega}\sin\epsilon,\\
Y' &= Y\cos\epsilon - \omega x\sin\epsilon, & X' &= X\cos\epsilon - \omega y\sin\epsilon,
\end{aligned}$$

defines a rotation in the plane (x, Y) paired with a rotation with same amplitude and same sense in the plane (y, X). Finally, as one should expect when the generator is the angular momentum, $\Lambda_3 = \Lambda_{I_3}$ is the usual pair of rotations,

$$x' = x\cos\epsilon + y\sin\epsilon, \qquad X' = X\cos\epsilon + Y\sin\epsilon,$$
$$y' = y\cos\epsilon - x\sin\epsilon, \qquad Y' = Y\cos\epsilon - X\sin\epsilon,$$

in the plane of coordinates and in the plane of moments.

These considerations can facilitate our understanding and consequent extension of the concept of a Lissajous transformation. They lead naturally to enquire about the effect some of these mappings have on the Lissajous variables. The main conclusions are encapsulated in the accompanying diagrams.

$$\begin{tikzcd}
(x,y,X,Y) \arrow[r, "\Lambda_0(\epsilon)"] & (x',y',X',Y') \\
(\ell,g,L,G) \arrow[u, "\lambda"'] \arrow[r] & (\ell+\epsilon,g,L,G) \arrow[u, "\lambda"']
\end{tikzcd}$$

$$\begin{tikzcd}
(x,y,X,Y) \arrow[r, "\Lambda_3(\epsilon)"] & (x',y',X',Y') \\
(\ell,g,L,G) \arrow[u, "\lambda"'] \arrow[r] & (\ell,g+\epsilon,L,G) \arrow[u, "\lambda"']
\end{tikzcd}$$

It is readily checked, indeed, that

- changing the mean anomaly ℓ into $\ell+\epsilon$ amounts to applying the transformation Λ_0 to the Cartesian variables;
- changing the argument of pericenter g into $g+\epsilon$ amounts to applying the transformation Λ_3 to the Cartesian variables.

Analogous behavior for the transformation λ' translates into similar diagrams.

$$\begin{tikzcd}
(x,y,X,Y) \arrow[r, "\Lambda_0(\epsilon)"] & (x',y',X',Y') \\
(\ell',g',L',G') \arrow[u, "\lambda'"'] \arrow[r] & (\ell'+\epsilon,g',L',G') \arrow[u, "\lambda'"']
\end{tikzcd}$$

$$\begin{array}{ccc} (x,y,X,Y) & \xrightarrow{\Lambda_1(\epsilon)} & (x',y',X',Y') \\ \uparrow \lambda' & & \uparrow \lambda' \\ (\ell',g',L',G') & \longrightarrow & (\ell',g'+\epsilon,L',G') \end{array}$$

Under what circumstances would one wish to perform the normalization of a perturbed elliptic oscillator in the variables (ℓ', g', L', G') rather than in the Lissajous variables? It depends on what the phase flow turns out to be on the reduced orbital spheres. In the regions where it looks like a circulation about diameter $I_1 = I_2 = 0$, the Lissajous variables are likely to be the best suited because the angle g through its secular variation will measure the rotation or circulation index about the equilibria at $I_3 = \pm L/2$. By the same token, should the reduced phase flow look on the whole as a circulation around an equilibrium at the extremities of the axis I_1, one might find it more convenient to operate in the coordinates (ℓ', g', L', G') rather than in Lissajous variables irrespective of their intuitive or geometric meanings.

Just as the Lissajous transformation can be modified to insure commutativity with Λ_1, it can be turned around to make of the plane (I_3, I_1) and of the axis I_2 the reference elements of a cylindrical map on the reduced orbital spheres $S^2(L)$. The key factor in the conversion is the canonical transformation $\zeta : (x, y, X, Y) \longmapsto (p, q, P, Q)$ for which

$$p = y, \qquad q = -X/\omega, \qquad P = Y, \qquad Q = \omega x.$$

Its action on the Hopf quadratic forms has the desired effect because it converts I_3 into $-I_2$ (besides changing I_2 into $-I_3$ and I_1 into $-I_1$). We compose the Lissajous transformation λ with ζ to make the transformation

$$\lambda'' = \zeta \circ \lambda \, : \, (\lambda'', g'', L'', G'') \longmapsto (x, y, X, Y)$$

defined by the equations

$$\boxed{\begin{aligned}
x &= \sqrt{\frac{L''+G''}{2\omega}}\sin(\ell''+g'') + \sqrt{\frac{L''-G''}{2\omega}}\sin(\ell''-g''),\\
y &= \sqrt{\frac{L''+G''}{2\omega}}\sin(\ell''+g'') - \sqrt{\frac{L''-G''}{2\omega}}\sin(\ell''-g''),\\
X &= \sqrt{\frac{\omega(L''+G'')}{2}}\cos(\ell''+g'') + \sqrt{\frac{\omega(L''-G'')}{2}}\cos(\ell''-g''),\\
Y &= \sqrt{\frac{\omega(L''+G'')}{2}}\cos(\ell''+g'') - \sqrt{\frac{\omega(L''-G'')}{2}}\cos(\ell''-g'').
\end{aligned}}$$

That the transformation λ'' establishes the angle g'' as the longitude in the plane (I_3, I_1) reckoned from the axis I_3 can be readily seen from the formulas below in which the Hopf's quadratic forms are expressed in terms of the doubly primed variables.

$$\begin{aligned}
I_0 &= \tfrac{1}{2}L'',\\
I_2 &= \tfrac{1}{2}G'',\\
I_1 &= -\tfrac{1}{2}\sqrt{L''^2 - G''^2}\cos 2g'',\\
I_3 &= -\tfrac{1}{2}\sqrt{L''^2 - G''^2}\sin 2g'',\\
K_1 &= -\tfrac{1}{2}\sqrt{L''^2 - G''^2}\cos 2\ell'',\\
K_3 &= \tfrac{1}{2}\sqrt{L''^2 - G''^2}\sin 2\ell''.
\end{aligned}$$

The commutativity diagrams below will make clear how the variant λ'' differs in nature from the Lissajous transformation and the mapping λ'.

$$\begin{array}{ccc}
(x, y, X, Y) & \xrightarrow{\Lambda_0(\epsilon)} & (x', y', X', Y')\\
\Big\uparrow{\scriptstyle \lambda''} & & \Big\uparrow{\scriptstyle \lambda''}\\
(\ell'', g'', L'', G'') & \longrightarrow & (\ell''+\epsilon, g'', L'', G'')
\end{array}$$

$$\begin{tikzcd}
(x,y,X,Y) \arrow[r, "\Lambda_2(\epsilon)"] & (x',y',X',Y') \\
(\ell'',g'',L'',G'') \arrow[u, "\lambda''"] \arrow[r] & (\ell'',g''+\epsilon,L'',G'') \arrow[u, "\lambda''"]
\end{tikzcd}$$

One will often start the analysis of a perturbed elliptic oscillator in the Lissajous variables. Not unfrequently will it turn out that, in some regions of the parameters, the flow for the averaged system is a rotation about one of the axes I_1, I_2 or I_3. One woud then be well advised to take advantage of that situation by adopting the appropriate variant of the Lissajous variables. Skipping that transposition usually ends up complicating considerably the algebraical developments at higher orders. Take for instance (Ghikas 1989)

$$\mathcal{V} = \beta(x^4 + y^4) + 2\alpha x^2y^2 + \delta(x^6 + y^6) + 3\gamma x^2y^2(x^2 + y^2).$$

It has been found that, the terms of degree 6 beong omitted, this perturbed elliptic oscillator after normalization is degenrate, i.e., admits non isolated equilibria in the following three cases:

1. when $\alpha = 0$, in which case the general flow corresponds to a differential rotation about the I_1-axis, all points on the great circle $I_1 = 0$ being stationary. The Lissajous variant λ' would be more approirate in this case;
2. when $\alpha = 3\beta$ because ten the averaged flow is that of rotation about the I_2-axis with all points on the great circle $I_2 = 0$ being equilibria. One should use the variant λ'' rather than the Lissajous transformation λ.
3. when $\alpha = \beta$ since then the flow is a differential rotation about I_3 which leaves the creat circle $I_3 = 0$ invariant. There is no reason here for changing variables.

Having chosen the most convenient variables, one can then pursue the analysis to the sixth degree in x and y to ascertain how the additional perturbations remove the degenracies. With the mimimum amount of algebraic developments, one will be able to establish, for instance, that the averaged system is in roation about the I_1-axis only when $\alpha = \gamma = 0$, that it is in rotation about the I_2-axis only when $\alpha = 3\beta$ and $\gamma = 5\delta$, and, finally, that it is in rotation about the I_3-axis only when $\alpha = \beta$ and $\gamma = \delta$ (Deprit and Ferrer, 1990).

6. Conclusions

There is a close analogy between the Delaunay and the Lissajous transformations. Both are intimately tied to a class of geometric figures and the way one parametrizes their equations. Much like the Delaunay transformation offers the most intuitive way of handling perturbed Keplerian systems, so does the Lissajous transformation in regard to perturbed elliptic oscillators. It is reasonable to relate intrinsic characteristics of the unperturbed orbits and the effects of perturbations. This gives the physicist a high degree of control over the parameters of the system and direct information, for example, over the causes of bifurcations.

Parts II and III in this series of communications will show how to use the Lissajous transformation for normalizing perturbed elliptic oscillators, and what kind of global information they readily bring out after averaging.

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