LARGE SCALE CHAOS AND MARGINAL STABILITY IN THE SOLAR SYSTEM

JACQUES LASKAR*
CNRS, Astronomie et Systèmes Dynamiques, Bureau des Longitudes,
3 rue Mazarine, 75006 Paris, France
E-mail: laskar@bdl.fr

Abstract. Large scale chaos is present everywhere in the solar system. It plays a major role in the sculpting of the asteroid belt and in the diffusion of comets from the outer region of the solar system. All the inner planets probably experienced large scale chaotic behavior for their obliquities during their history. The Earth obliquity is presently stable only because of the presence of the Moon, and the tilt of Mars undergoes large chaotic variations from 0° to about 60°. On billion years time scale, the orbits of the planets themselves present strong chaotic variations which can lead to the escape of Mercury or collision with Venus in less than 3.5 Gyr. The organization of the planets in the solar system thus seems to be strongly related to this chaotic evolution, reaching at all time a state of marginal stability, that is practical stability on a time-scale comparable to its age.

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1. Introduction

Since the seminal work of Poincaré (1892-99) on the non integrability of the three-body problem and the existence of heteroclinic intersections, completed by the results of Kolmogorov (1954) and Arnold (1963a-b), it is known that in general, in celestial bodies dynamics, many non regular motions will appear besides the quasiperiodic orbits confined on invariant tori in the phase space. In practice, positive Lyapunov exponents, reflecting the exponential divergence of nearby orbits will be detected during long time numerical integrations of these non regular orbits lying in the chaotic zones.

As a result of this exponential divergence, a practical limit will arise for the possibility of making precise predictions for the motion of these celestial bodies. If this limit is much larger than the age of the solar system, the motion still could be very well approximated by a regular solution and the chaotic behavior will not be sensible.

On the other case, one needs to give up the program of Laplace which was to determine with the ultimate precision the motion of all the objects of the heaven. This may be of no physical consequences if the chaotic region where the motions wanders is practically confined in a narrow region over the age of the solar system. For a planet, it would just mean for example that the orientation of the orbit or its position on this orbit is not known. Much more important are the cases of extended chaos, when the diffusion of the action like variables is sensible over the considered period. In this case, the orbit will explore a large portion of the phase space, and significant physical changes may occur. It could mean changes in semi major axis, eccentricity, or inclination.

Despite the pioneered work of (Hénon and Heiles, 1964) in galactic dynamics, the exhibition of these large scale chaotic behaviors in the real solar system is very recent, and most of the work reported here was done in the last few years. This leads to a completely new vision of celestial mechanics which in the previous decade was considered in astronomy as an old and dusty field, uniquely concerned by the determination of more and more precise paths for already well known objects. Indeed, until very recently, most people assumed that everything was regular and smooth in the solar system, and the motion of the planets was considered as the paradigm of regularity.

In fact, many objects in the solar system present large scale chaotic behavior, and the analysis of their possible evolution over long times changed profoundly the understanding of the evolution of the solar system, inducing also many new elements for the understanding of its formation.

2. Minor bodies in the solar system

The solar system is crowded with a multitude of small objects : asteroids, comets, small satellites. Many of them are supposed to be the remains of the material of the primitive solar system which did not contributed to the formation of the planets. They are of great interest for the understanding of the formation of the solar system, as their material should not have changed much since the primordial state of the solar system. This leads in the recent years to numerous studies on their dynamics, aiming to the understanding of their dynamical evolution since the formation of the solar system.

From a practical point of view, these small bodies can usually be studied with simplified models. Their very small masses do not perturb the remaining part of the solar system and, as we are more interested by their collective behavior than by the very precise orbit of a single of them. On the other hand, for the same reason, it will be necessary to understand the considered dynamics in a global way, in the large part of the phase space which will correspond to the numerous possible initial conditions of these small bodies.

Apart from the review of (Wisdom, 1987b), the papers of (Ferraz-Mello, 1994, Farinella *et al.*, 1994) can be consulted for an overview of the recent developments in the understanding of the Kirkwood gaps in the asteroid belt, and the delivery of meteorites to the Earth. The dynamical studies on comets have been reviewed in (Fernández, 1994).

2.1. The chaotic motion of Hyperion

The first striking example of chaotic behavior in the solar system was given by the chaotic tumbling of Hyperion, a small satellite of Saturn which strange rotational behavior was detected during the encounter of the Voyager spacecraft with Saturn (Wisdom, Peale, Mignard, 1984). This example, which dynamics can be reduced to the perturbed pendulum one, provides a simple illustration of the arising of chaotic behavior in the vicinity of a resonance. This study will apply more generally to any satellite of irregular shape in the vicinity of spin-orbit resonance (Wisdom, 1987a).

The equations of motion for the orientation of a satellite S orbiting around a planet P on a fixed elliptical orbit of semi major axis a and eccentricity e are given by the Hamiltonian

$$H = \frac{y^2}{2} - \frac{3}{4}\frac{B-A}{C}\left(\frac{a}{r(t)}\right)^3 \cos 2(x - v(t))$$

where $r(t)$ is the distance from the planet to the satellite, x gives the orientation of the satellite with respect to a fixed direction (here the direction of periapse), $y = dx/dt$ is its conjugate variable, v is the true anomaly of

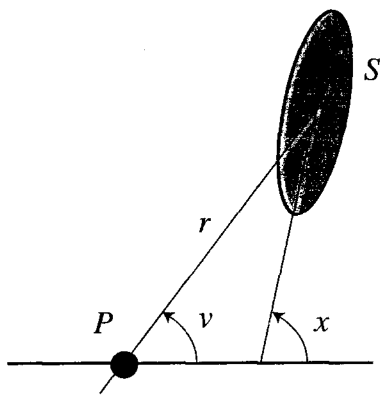


Figure 1. The position of the satellite S around the planet P is defined by the distance r and the angle v from the direction of periapse (true anomaly). The angle x provides the orientation of the satellite.

the satellite, and $A < B < C$ are the principal moments of inertia of the satellite (Fig.1). The associated equations of motion are

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}; \qquad \frac{dx}{dt} = \frac{\partial H}{\partial y} .$$

The unit of time is taken such that the mean motion $n = 1$. When expanding the Hamiltonian with respect to the eccentricity (e) which is supposed to be small, and retaining only the terms of first order in eccentricity, one obtains

$$H = \frac{y^2}{2} - \frac{\alpha}{2}\cos 2(x-t) + \frac{\alpha e}{4}[\cos(2x-t) - 7\cos(2x-3t)] \qquad (1)$$

with $\alpha = 3(B-A)/2C$.

If S has a rotational symmetry, $\alpha = 0$, and the hamiltonian is reduced to $H_0 = y^2/2$. The satellite rotates with constant velocity $dx/dt = y_0$. When the orbit is circular, the problem is also integrable as H_0 is reduced to the first two terms of Eq. (1).

$$H_0 = \frac{y^2}{2} - \frac{\alpha}{2}\cos 2(x-t) \qquad (2)$$

This motion will be similar to the simple pendulum motion, with possibility of libration of the satellite around the direction of the planet (spin-orbit

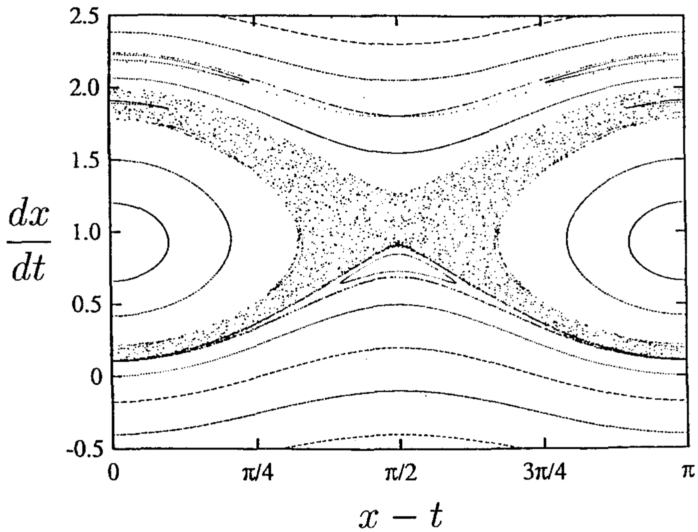


Figure 2. Surface of section in the phase space of Deimos, a small satellite of Mars. $x - t$ defines the orientation of the satellite and dx/dt its rotational velocity. A small chaotic zone appears in the vicinity of the separatrix of the unperturbed problem ($e = 0$) (Wisdom, 1987b).

resonance occurs), or circulation motion for large values of the initial angular rotational velocity of the satellite.

In the general case, $\alpha e \neq 0$, and the hamiltonian H_0 of (2) is perturbed by the remaining terms of (1). At the transition between librational motion and rotational motion of the satellites, appears a small chaotic zone. This can be observed in a section of the phase space portrait of Phobos orientation motion when orbiting around Mars (Fig.2).

When the size of the perturbation αe increases, resonant zones corresponding to the various possible resonant terms $\cos 2(x - t), \cos(2x - t), \cos(2x - 3t)$ will overlap (Chirikov, 1979), giving rise to large scale chaotic motion.

This is the case for Hyperion (Fig.3), where $\alpha e \approx 0.039$. The resulting effect is that the rotational motion of Hyperion is not regular, and it becomes impossible to adjust any periodic or quasiperiodic model to its lightcurve (Klavetter, 1989). The consideration of this chaotic motion was necessary to explain the observations of Hyperion, and this example demonstrated that significant physical phenomenon on the solar system could result from this complicated chaotic dynamics.

It should be noticed that the models used in these computations are of two degrees of freedom. In this case, according to KAM theory, invariant tori of dimension 2 may subsist and will divide the phase space. Thus, although

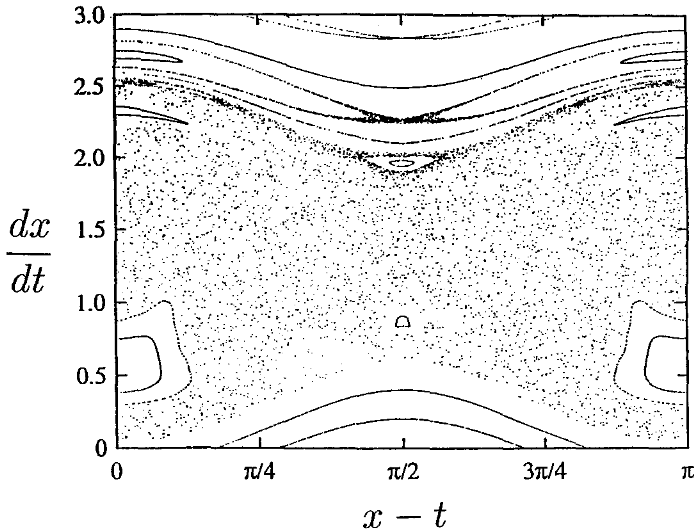


Figure 3. Surface of section in the phase space of Hyperion, an outer satellite of Saturn. When the perturbation parameter αe is large, like in the case of Hyperion, many of the resonances overlap, giving rise to a large chaotic zone of irregular motion (Wisdom, 1987b).

an orbit may be chaotic, it can be bounded for all time by these invariant surfaces (Celletti, 1990a,b). In the complete model, the addition of the extra degrees of freedom leaves place for the possibility of diffusion although this diffusion may be extremely small.

2.2. The Kirkwood gaps

The distribution of the asteroids, the minor planets which orbits lay primarily between Mars and Jupiter, has puzzled astronomers for many decades, since Kirkwood observed in 1867 that they are not randomly distributed. Indeed, when plotting the number of asteroids against their semi major axis, one can observe gaps and accumulations (Fig.4). Kirkwood noticed that these gaps coincide with commensurabilities of mean motion with Jupiter, and the extend of the gaps also coincide with the libration zones of the resonances (Dermott and Murray, 1983). It was thus thought that these gaps result from the effect of these resonances, but although the first numerical integrations of asteroids placed inside the resonance lead to some increase of their eccentricities (Froeschlé and Scholl, 1977), no satisfying explanation was given. Later on, using a simplified model of two degrees of freedom (an averaged planar problem where the asteroid is uniquely submitted to the perturbation of Jupiter orbiting on a fixed ellipse), Wisdom (1983, 1985), inspired by the work of Chirikov (1979), managed to integrate the orbits

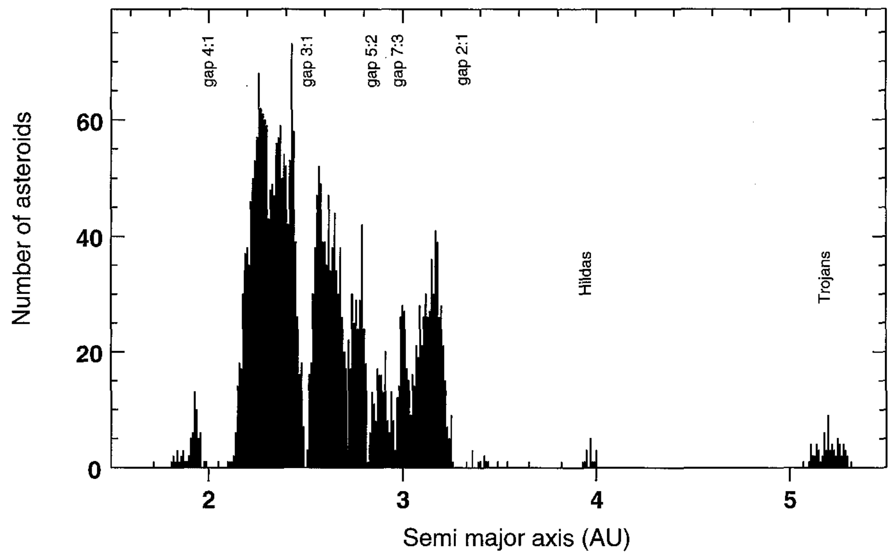


Figure 4. Histogram of the number of asteroids plotted against their semi-major axis showing the location of gaps and accumulations.

approximately over much extended time and showed that in the vicinity of the 3/1 resonance, there exists a chaotic zone which can be easily observed in a Poincaré surface of section of the trajectories, corresponding to the successive intersections with a plane of fixed argument of perihelion. An orbit starting in this chaotic zone can have a moderate eccentricity for a long time, but it could enter in some other branch of this chaotic zone, which would lead to a large increase of the eccentricity, sufficient to cross Mars orbit. A possible close encounter with this planet can then expel the asteroid from its primitive orbit. The location and extent of the chaotic zone related to the 3/1 resonance is in good agreement with the 3/1 asteroid gap. The understanding of this complex dynamics, which differs very much from the ordered motion of the integrable problems, thus allowed to obtain convincing explanation for one famous problem of celestial mechanics.

Since the work of Wisdom, which applies more specifically to the 3/1 gap, many other studies analyzed the possible chaotic behavior in the vicinity of other asteroidal resonances involving more complicated interactions, and models of many degrees of freedom. In particular, in their analysis of the 2/1 and 3/2 resonances, Morbidelli and Moons (1993) needed to take into account the spatial problem and the secular resonances due to the slow

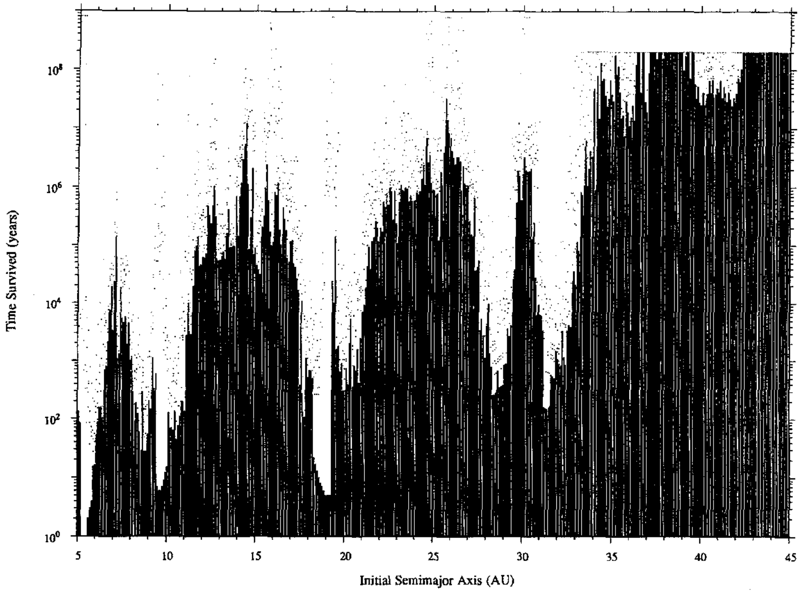


Figure 5. The time survived by each test particle as a function of initial semimajor axis. For each semimajor axis bin, six test particles were started at different longitudes. The vertical bars mark the minimum of the six termination times. The spikes at 5.2, 9.5, 19.2, and 30.1 AU, at the semimajor axes of the planets (Jupiter, Saturn, Uranus and Neptune), correspond to test particle in librating in Trojan or horseshoelike orbits before close encounter. Interior to Neptune the integration extends to 800 Myr; exterior to Neptune to 200 Myr. Beyond about 43 AU all the test particles survive the full integration (Holman and Wisdom, 1993).

precession of Jupiter's orbit under planetary perturbations. Moreover, using the same model, they demonstrated that the overlap of the secular resonances inside the 3/1 libration region provides a more efficient mechanism for the depletion of this gap than the one originally proposed by Wisdom (Moons and Morbidelli, 1994).

2.3. The Chaotic Motion of the Comets and the Dynamics of the Kuiper Belt

The asteroids are not the only small bodies of the solar system which can be subject to chaotic motion. Indeed, many cometary orbits are chaotic.

When Halley's comet came to visit us in 1985, several numerical integrations were carried out to retrace its orbit over all the extent of the observations, that is beyond 163 BC, date of the most ancient observation of this comet (Stephenson *et al.*, 1984). After such a long time, all the different numerical integrations showed different behavior, and their accuracy was questioned. In fact, these divergences were later on explained by the analysis of Chirikov and Vecheslavov (1989), which demonstrated that the motion of Halley's comet could be chaotic, and thus practically unpredictable after 29 revolutions. Indeed, it can be shown that, due to the perturbation of Jupiter, there exists a large chaotic zone for nearly parabolic comets orbits, which extends up to the Oort cloud (Petrosky, 1986, Sagdeev and Zaslavsky, 1987, Natenzon *et al.*, 1990). This chaotic behavior of Halley's comet was later on confirmed by direct numerical integration (Froeschlé and Gonczi, 1988). More generally, most of the long period comets have chaotic orbits, where the chaotic behavior results from repeated close encounters with the planets.

The existence of the Oort's cloud could explain the observation of the long period comets, but the distribution of the inclination of the short period comet lead to the hypothesis of the existence of an other source of comets, the Kuiper belt, located beyond Neptune close to the planetary plane (Kuiper, 1951, Fernandez, 1980)). In order to study this hypothesis, and as the integration of the outer solar system becomes accessible to long time computations, many efforts have been conducted recently to understand the dynamics of small particles in the outer solar system. From these studies, which consist mainly in the numerical integration of thousand of massless particle in the outer solar system, it was found that apart from some special locations, like the trojans Lagrangian points of Jupiter (fig. 4), there was practically no stable orbits which could last for more than 1 Gyr among the outer planets (Duncan *et al.*, 1989, Gladman and Duncan, 1990, Holman and Wisdom, 1993, Levison and Duncan, 1993). On the contrary, there exist some stable orbits at about 40 AU and further, where some planetesimal could last for a long time (Levison and Duncan, 1993). Close to them, unstable regions exist which will provide from time to time, by chaotic diffusion, planetesimal which would enter a more internal part of the solar system, and could be captured temporarily into resonance, as a short period comet (Torbett and Smoluchovski, 1990). During some recent observation campaign, several of these transneptunian objects where observed, at the location of the supposed Kuiper belt, at about 40 AU (see Luu, 1994 for a review of this search). In this case again, the understanding of the non regular orbits, which can thus explore a large part of the solar system, gave some insight of the observed distribution of the short period comets.

These findings are also in good agreement with the scenario of formation of the solar system including a phase where planetesimals are present everywhere (Safronov, 1969). Indeed, as it was forecasted by Kuiper, in the outer solar system the removal of the planetesimals non accreted to form the planets can probably be explained by the gravitational perturbations of the large planets while some remaining bodies are actually present in the stable regions of the outer solar system where these perturbations decrease.

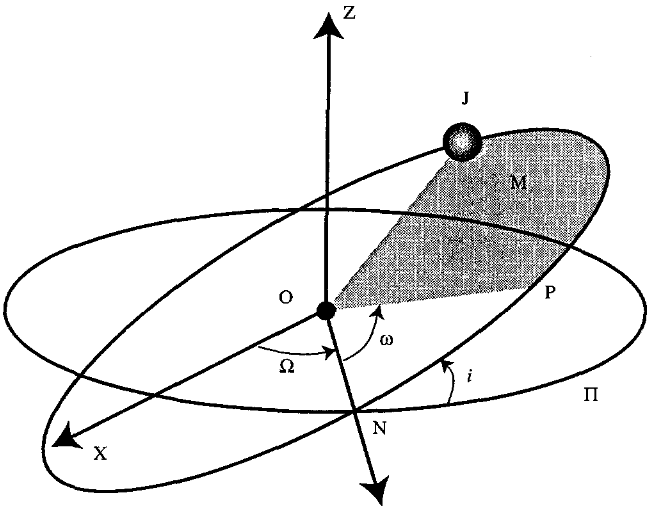


Figure 6. Elliptical elements. At any given time, a planet (J) can be considered to move on an elliptical orbit, with semimajor axis a and eccentricity e, with the sun at one focus (O). The orientation of this ellipse with respect to a fixed plane Π, and a direction of reference OX, is given by three angles: The inclination i, the longitude of the node Ω, and the longitude of perihelion $\varpi = \Omega + \omega$, where ω is the argument of perihelion (P). The position of the planet on this ellipse is given by the mean longitude $\lambda = M + \varpi$, where M (mean anomaly) is an angle which is proportional to the area OPJ (third Kepler's law).

3. The chaotic motion of the planets

The first studies of chaotic motion in the solar system concerned small objects, with simplified dynamical models which could often be reduced to two degrees of freedom. With these simplifications, it was possible to describe their global dynamics, and to study their chaotic zones, which gave rise to new insight in the organization and evolution of the solar system. But chaotic behavior is not confined to the small bodies of the solar system, and concern also the main celestial objects, the planets. Despite the outstanding work of Poincaré, the discovery of the non regular behavior of the actual planets is very recent, as it requested the possibility to study the evolution of the actual solar system over a very long time, which was only achieved in the last few years.

3.1. Historical introduction

The problem of the stability of the solar system has fascinated astronomers and mathematicians since antiquity, when it was observed that among the seemingly fixed stars, there were also "wandering stars"—the planets. Efforts were first focused on finding a regularity in the motion of these wanderers, so their movement among the fixed stars could be predicted. For Hipparcus and Ptolemy, the ideal model was a combination of uniform circular motions, the epicycles, which were adjusted over the centuries to conform to the observed course of the planets. Astronomy had become predictive, even if its models were in continual need of adjustment.

From 1609 to 1618, Kepler fixed the planets' trajectories: having assimilated the lessons of Copernicus, he placed the Sun at the center of the universe and, based on the observations of Tycho Brahe, showed that the planets describe ellipses around the Sun. At the end of a revolution, each planet found itself back where it started and so retraced the same ellipse. Though seductive in its simplicity, this vision of a perfectly stable solar system in which all orbits were periodic would not remain unchallenged for long.

In 1687 Newton announced the law of universal gravitation. By restricting this law to the interactions of planets with the Sun alone, one obtains Kepler's phenomenology. But Newton's law applies to all interactions: Jupiter is attracted by the Sun, as is Saturn, but Jupiter and Saturn also attract each other. There is no reason to assume that the planets' orbits are fixed invariant ellipses, and Kepler's beautiful regularity is destroyed.

In Newton's view, the perturbations among the planets were strong enough to destroy the stability of the solar system, and divine intervention was required from time to time to restore planets' orbits to their place. Moreover, Newton's law did not yet enjoy its present status, and astronomers wondered if it was truly enough to account for the observed movements of bodies in the solar system.

The problem of solar system stability was a real one, since after Kepler, Halley was able to show, by analyzing the Chaldean observations transmitted by Ptolemy, that Saturn was moving away from the Sun while Jupiter was moving closer. By crudely extrapolating these observations, one finds that six million years ago Jupiter and Saturn were at the same distance from the Sun. In the 18th century, Laplace took up one of these observations, which he dated March 1st, 228 BC: *At 4:23 am, mean Paris time, Saturn was observed "two fingers" under Gamma in Virgo.* Starting from contemporary observations, Laplace hoped to calculate backward in time using Newton's equations to arrive to this 2000 year-old observation.

The variations of planetary orbits were such that, in order to predict the planets' positions in the sky, de Lalande was required to introduce artificial "secular" terms in his ephemeris tables. Could these terms be accounted for by Newton's law?

The problem remained open until the end of the 18th century, when Lagrange and Laplace correctly formulated the equations of motion. Lagrange started from the fact that the motion of a planet remains close, over a short duration, to a Keplerian ellipse, and so had the notion to use this ellipse as the basis for a coordinate system (Fig.6). Lagrange then wrote the differential equations that govern the variations in this elliptic motion under the effect of perturbations from other planets, thus inaugurating the methods of classical celestial mechanics. Laplace (1772) and Lagrange (1776), whose work converged on this point, calculated secular variations, in other words long-term variations in the planets' semi-major axes under the effects of perturbations by the other planets. Their calculations showed that, up to first order in the masses of the planets, these variations vanish. Poisson (1809) and Haretu (1885) later showed that this result remains true through second order in the masses of the planets, but not through third order.

This result seemed to contradict Ptolemy's observations from antiquity, but by examining the periodic perturbations between Jupiter and Saturn, Laplace discovered a quasi-resonant term $(2\lambda_{Jupiter} - 5\lambda_{Saturn})$ in their longitudes. This term has an amplitude of 46′50″ in Saturn's longitude, and a period of about 900 years. This explains why observations taken in 228 BC and then in 1590 and 1650 could give the impression of a secular term.

Laplace (1785) then calculated many other periodic terms, and established a theory of motion for Jupiter and Saturn in very good agreement with 18th century observations. Above all, using the same theory, he was able to account for Ptolemy's observations to within one minute of arc, without additional terms in his calculations. He thus showed that Newton's law was in itself sufficient to explain the movement of the planets throughout known history, and this exploit no doubt partly accounted for Laplace's determinism.

Laplace showed that the planets' semi-major axes undergo only small oscillations, and do not have secular terms. At the same time, the eccentricity and inclination of planets' trajectories are also very important for solar system stability. If a planet's eccentricity changes appreciably, its orbit might cut through another planet's orbit, increasing the chances of a close encounter which could eject it from the solar system.

Laplace (1784) revisited his calculations, taking into account only terms of first order in the perturbation series, and showed that the system of equations describing the mean motions of eccentricity and inclination in a planetary system with k planets may be reduced to the system of linear differential equations with constant coefficients

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ \vdots \\ z_k \\ \zeta_1 \\ \vdots \\ \zeta_k \end{bmatrix} = \sqrt{-1}\begin{bmatrix} A_k & 0_k \\ 0_k & B_k \end{bmatrix}\begin{bmatrix} z_1 \\ \vdots \\ z_k \\ \zeta_1 \\ \vdots \\ \zeta_k \end{bmatrix} \tag{3}$$

where $z = e \exp\sqrt{-1}\varpi, \zeta = \sin(i/2) \exp\sqrt{-1}\Omega$, A_k and B_k are (k,k) matrices with real coefficients which depends only on the planetary masses and semi major axis; 0_k is the (k,k) null matrix. Using the invariance of the angular momentum

$$C = \sum_{i=1}^{k} m_i \sqrt{\mu_i a_i (1 - e_i^2)} \cos i_i \tag{4}$$

and retaining only the terms of degree 2 in eccentricity and inclination, and arguing that the eccentricity and inclination evolutions are decoupled in the linear equations, Laplace deduced that the quantities

$$\sum_{i=1}^{k} m_i \sqrt{a_i} e_i^2$$

$$\sum_{i=1}^{k} m_i \sqrt{a_i} \sin^2 i_i/2$$

should be constant, and thus there cannot exist polynomial or exponential terms in the solutions of these linear equations. Therefore, he deduced that all the eigenvalues g_i of A and s_i of B are real and distinct, and the solutions of this linear secular system are quasiperiodic expressions of the form

$$z_i = \sum_{j=1}^{k} \alpha_{ij} \ e^{ig_j t}$$

$$\zeta_i = \sum_{j=1}^{k} \beta_{ij} \ e^{is_j t}$$

where α_{ij} and β_{ij} are complex quantities. The values of the secular frequencies g_i and s_i, computed in the more complete semi-analytical solution of (Laskar, 1990) are given in table I.

The variations in eccentricity thus reduce to a superposition of uniform circular motions (Fig.7) of frequencies g_i and s_i. The inclinations and eccentricities of the orbits are therefore subject to only small variations about their mean values (in fact, this was really established by Le Verrier for the whole solar system). It must be stressed that Laplace's solutions are very different from Kepler's, because the orbits are no longer fixed. They are subject to a double precessionary motion with periods ranging from about 45 000 to a few million years (table I): precession of the perihelion, which is the slow rotation of the orbit in its plane, and precession of the nodes, which is the rotation of the plane of the orbit in space.

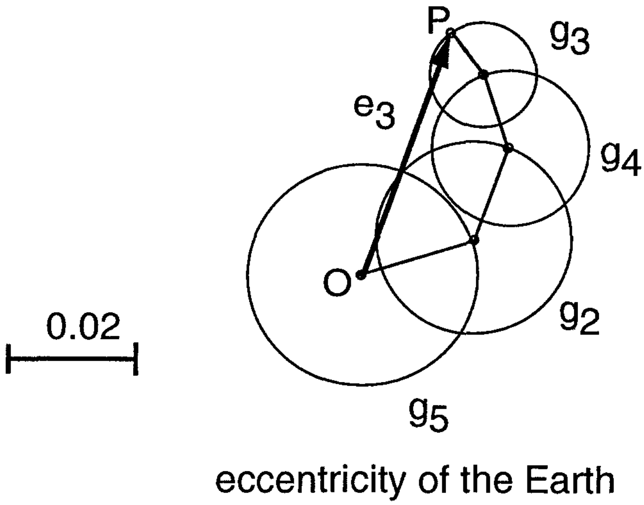
Table I

Fundamental frequencies of the precession motion of the solar system (excluding Pluto). These values are taken as the mean values over 20 million years of from the recent solution La90. For the inner planets, due to chaotic diffusion, the frequencies can change significantly with time (Laskar, 1990).

	ν (" /yr)	period (yr)
g_1	5.596	231 000
g_2	7.456	174 000
g_3	17.365	74 600
g_4	17.916	72 300
g_5	4.249	305 000
g_6	28.221	45 900
g_7	3.089	419 000
g_8	0.667	1 940 000
s_1	- 5.618	230 000
s_2	- 7.080	183 000
s_3	-18.851	68 700
s_4	-17.748	73 000
s_5	0.000	
s_6	-26.330	49 200
s_7	- 3.005	431 000
s_8	- 0.692	1 870 000

The work of Laplace concerns only the linear approximation of the secular motion of the planets. In modern language, one can say that Laplace demonstrated that the origin (planar and circular motions) is an elliptical fixed point in the secular phase space which is obtained after averaging over the mean longitudes. Later, Le Verrier (1856), famed for his discovery in 1846 of the planet Neptune through calculations based on observations of irregularities in the movement of Uranus, took up Laplace's calculations and considered the effects of higher order terms in the series. He showed that these terms produce significant corrections and that Laplace's and Lagrange's calculations "could not be used for an indefinite length of time." He then challenged future mathematicians to find exact solutions, without approximations. The difficulty posed by "small divisors" showed that the convergence of the series depended on initial conditions, and the proof of the stability of the solar system remained an open problem.

But Poincaré (1892–99) formulated a negative response to Le verrier's question. In so doing he rethought the methods of celestial mechanics along the lines of Jacobi's and Hamilton's work. Poincaré showed that it is not possible to integrate the equations of motion of three bodies subject to mutual interaction, and not possible to find an analytic solution representing



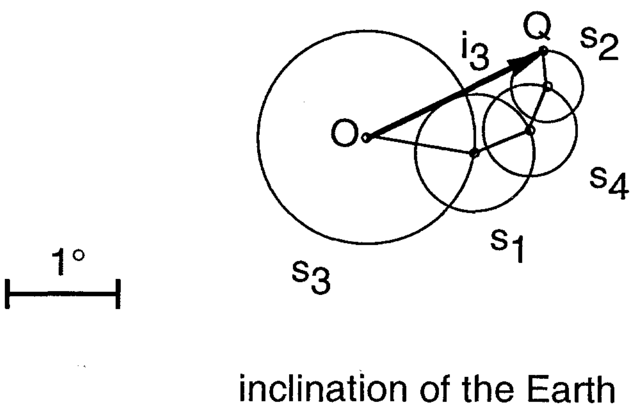


Figure 7. The solutions of Laplace for the motion of the planets are combinations of circular and uniform motions with frequencies the precession frequencies g_i and s_i of the solar system (Table I). The eccentricity e_3 of the Earth is given by OP, while the inclination of the Earth with respect to the invariant plane of the solar system (i_3) is OQ (Laskar, 1992b).

the movement of the planets valid over an infinite time interval, since the perturbation series used by astronomers to calculate the movement of the planets are not convergent on an open set of initial conditions.

Kolmogorov (1954) reexamined this problem and demonstrated that in a perturbed non degenerated Hamiltonian system, among the non regular solutions described by Poincaré, there still exist some quasiperiodic trajectories lying on isolated tori in the phase space. This result was completed by Arnold (1963a) who demonstrated that for a sufficiently small perturbation, the set of invariant tori foliated by quasiperiodic trajectories is of strictly positive measure, tending to unity when the perturbation decreases to zero. Moser (1962) established the same kind of results for less stronger conditions which did not require the analyticity of the Hamiltonian. These theorems are known generically as KAM theorems, and have been employed in various fields. Unfortunately, they do not apply directly to the planetary problem which presents proper degeneracy (the unperturbed Hamiltonian depends only on the semi major axes, and not on the other action variables related to eccentricity and inclination). This led Arnold to extend the proof of existence of invariant tori, taking into account this phenomenon of proper degeneracy. He then applied his theorem explicitly to a planar planetary system of two planets with a ratio of the semi major axis close to zero, demonstrating the existence of quasi periodic trajectories for sufficiently small values of the planetary masses and eccentricities (Arnold 1963b). This result was recently extended to more general planetary systems of two planets (Robutel, 1995).

Arnold's results motivated many discussions; indeed, as the quasiperiodic KAM tori are isolated, an infinitely small change in the initial conditions can change the solution from being stable for all time, to a chaotic orbit. Moreover, as the planetary system is of more than two degrees of freedom, none of the KAM tori separates the phase space, leaving the possibility for chaotic trajectories to travel large distances in the phase space. In fact, several subsequent results demonstrated that very close to a KAM tori, the diffusion of the solutions is very slow (Nekhoroshev, 1977, Giorgilli *et al.*, 1989, Lochak, 1993, Morbidelli and Giorgilli, 1995), and can be negligible over very long time, eventually as long as the age of the universe.

Although the actual masses of the planets are much too large for these results to apply directly to the solar system, it was generally supposed that the scope of these mathematical results extends much further than their actually proven bounds, and until very recently it was generally assumed that the solar system was stable over its lifetime, "by any reasonable acceptation of this term".

In the past few years, the problem of solar system stability has advanced considerably, due largely to the help provided by computers which allow extensive analytic calculations and numerical integrations over model time scales approaching the age of the solar system, but also due to a better understanding of the underlying dynamics, resulting from the expansion of the overall field of dynamical systems theory.

3.2. Numerical integrations

The motion of the planets of the solar system has a very privileged status; indeed, it is one of the best modelized problems in physics, and its study can be practically reduced to the study of the behavior of the solutions of its gravitational equations (Newton's equation completed with relativistic corrections for the most inner planets), neglecting all dissipation, and treating the planets as point masses, except in the case of the Earth, where for more precise results, one likes to take into account the perturbation introduced by the existence of the Moon.

The mathematical complexity of this problem, despite its apparent simplicity is daunting, and has been a challenge for mathematicians and astronomers since its formulation three centuries ago. Since the work of Poincaré, it is also known that the analytical perturbative methods which were used in planetary computations for nearly two centuries cannot provide good approximations of the solutions over infinite time. Moreover, as stated above the stability results obtained by Arnold (1963) do not apply to realistic planetary systems.

Since the introduction of computers, numerical integration of the planetary equations appeared as a straightforward way to overcome this complexity of the solutions, but has always been bounded until now by the available computer technology. The first long time numerical studies of the solar system were limited to the motion of the outer planets, from Jupiter to Pluto (Cohen *et al.*, 1973, Kinoshita and Nakai, 1984). Indeed, the more rapid the orbital movement of a planet, the more difficult it is to numerically integrate its motion. To integrate the orbit of Jupiter, a step-size of 40 days will suffice, while a step-size of 0.5 days is required to integrate the motion of the whole solar system using a conventional multistep integrator.

The project LONGSTOP (Carpino *et al.*, 1987, Nobili *et al.*, 1989) used a CRAY computer to integrate the system of outer planets over 100 million years. At about the same time, calculations of the same system were carried out at MIT on a parallel computer specially designed for the task over even longer periods, corresponding to times of 210 and 875 million years (Applegate *et al.*, 1986, Sussman and Wisdom, 1988). This latter integration showed that the motion of Pluto is chaotic, with a Lyapunov time (the inverse of the Lyapounov exponent) of 20 million years. But since the mass of Pluto is very small, (1/130 000 000 the mass of the Sun), this does not induce macroscopic instabilities in the rest of the solar system, which appeared relatively stable in these studies.

3.3. Chaos in the Solar System

The numerical integrations can give very precise solutions of the trajectories, but are limited by the short stepsize necessary for the integration of the whole solar system and it should be stressed, that until 1991, the only available numerical integration of a realistic model of the full solar system was the numerically integrated ephemeris DE102 of JPL (Newhall *et al.*, 1983) which spanned only 44 centuries.

My approach to to this problem was different, and more in the spirit of the analytical works of Laplace and Le Verrier. Indeed, since these pioneered works, the *Bureau des Longitudes**, has traditionally been the place for development of analytical planetary theories based on classical perturbation series (Brumberg and Chapront, 1973, Bretagnon, 1974, Duriez, 1979). Implicitly, these studies assume that the motion of the celestial bodies is regular and quasiperiodic. The methods used are essentially the same ones which were used by Le Verrier, with the additional help of the computers for symbolic computations. Indeed, such methods can provide very satisfactory approximations of the solutions of the planets over several thousand years, but they will not be able to give answers to the question of the stability of the solar system over time span comparable to its age. This difficulty which is known since Poincaré is one of the reasons which motivated the previously quoted long time direct numerical integrations of the equations.

However, the theoretical results of Arnold (1963) supported the idea that it may have been possible with the help of computer algebra to extend very much the scope of the classical analytical planetary theories, but this revealed to be hopeless when considering the whole solar system, because of severe convergence problems encountered in the Birkhoff normalization of the secular system of the inner planets (Laskar, 1984). This difficulty which revealed to be inherent to this complicated system led me to proceed in two very distinct steps: a first one, purely analytical, consists on the averaging of the equations of motion over the rapid angles, that is the motion of the planets along their orbits. Indeed, from all the achievements of classical celestial mechanics obtained since the XIXth century, it could be forecasted that no severe problems would occur during this first step involving only possible resonances among the orbital motion of the planets.

This averaging process was conducted in a very extensive way, without neglecting any term, up to second order with respect to the masses, and through degree 5 in eccentricity and inclination, conducting to the truncated secular equations of the solar system on the form

* The *Bureau des Longitudes* was founded the 7 messidor year III (june 25, 1795) in order to develop Astronomy and Celestial Mechanics. Its founding members were Laplace, Lagrange, Lalande, Delambre, Méchain, Cassini, Bougainville, Borda, Buache, Caroché.

$$\frac{d\alpha}{dt} = \sqrt{-1}\,(\mathcal{A}\,\alpha + \Phi_3(\alpha,\bar{\alpha}) + \Phi_5(\alpha,\bar{\alpha})) \tag{5}$$

where $\alpha = (z_1, \ldots, z_8, \zeta_1, \ldots, \zeta_8)$, and $\mathcal{A}$ is similar to the linear matrix of Laplace (eq. 3); $\Phi_3(\alpha,\bar{\alpha})$ and $\Phi_5(\alpha,\bar{\alpha})$ gather the terms of degrees 3 and 5.

The system of equations thus obtained comprises some 150000 terms, but it can be considered as a simplified system, as its main frequencies are now the precessing frequencies of the orbits of the planets, and no longer comprises their orbital periods. The full system can thus be numerically integrated with a very large stepsize of about 500 years. Contributions due to the Moon and to the general relativity are added without difficulty (Laskar, 1985, 1986).

This second step, i.e. the numerical integration, is then very efficient because of the symmetric shape of the secular system, and was conducted over 200 millions years in just a few hours on a super computer. The main results of this integration was to reveal that the whole solar system, and more particularly the inner solar system (Mercury, Venus, Earth, and Mars), is chaotic, with a Lyapunov time of 5 million years (Laskar, 1989). An error of 15 meters in the Earth's initial position gives rise to an error of about 150 meters after 10 million years; but this same error grows to 150 million km after 100 million years. It is thus possible to construct ephemerides over a 10 million year period, but it becomes practically impossible to predict the motion of the planets beyond 100 million years.

This chaotic behavior essentially originates in the presence of two secular resonances among the planets: $\theta = 2(g_4 - g_3) - (s_4 - s_3)$, which is related to Mars and the Earth, and $\sigma = (g_1 - g_5) - (s_1 - s_2)$, related to Mercury, Venus, and Jupiter (the g_i are the secular frequencies related to the perihelions of the planets, while the s_i are the secular frequencies of the nodes (table I)). The two corresponding arguments change several times from libration to circulation over 200 million years, which is also a characteristic of chaotic behavior (fig. 8). It should be stressed that these two combinations of frequencies were not chosen in a random way. In fact, the frequency analysis (Laskar, 1990) of the numerical solutions of the secular system showed that these combinations appear with a large amplitude in the very first terms of the inner planets solutions. Indeed, as soon as one goes further than the linear model, they need to be taken into account.

When these results were published, the only possible comparison was the comparison with the 44 centuries ephemeris DE102, which already allowed to be confident on the results by comparing the slopes of the solutions at the origin (Laskar, 1986, 1990). At the time, there was no possibility to obtain similar results with direct numerical integration. In fact, partly due to the very rapid advances in computer technology, and in particular to the development of workstations, only two years later, Quinn *et al.* (1991) were

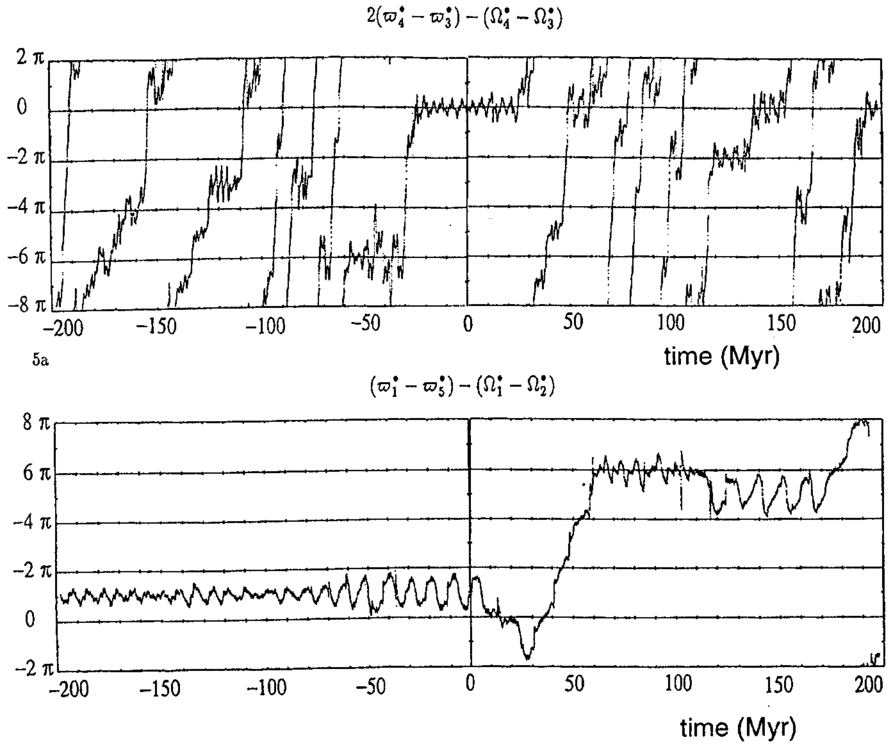


Figure 8. The secular resonances $\theta = 2(g_4 - g_3) - (s_4 - s_3)$ and $\sigma = (g_1 - g_5) - (s_1 - s_2)$. From −200 Myr to +200, the corresponding argument present several transitions from libration to circulation (Laskar, 1992a).

able to publish a numerical integration of the full solar system, including the effects of general relativity and the Moon, which spanned 3 million years in the past (completed later on by an integration from -3Myrs to +3Myrs). Comparison with the secular solution of (Laskar, 1990) shows very good quantitative agreement (fig. 9), and confirms the existence of secular resonances in the inner solar system (Laskar *et al.*, 1992a). Later on, using a symplectic integrator directly adapted towards planetary computations which allowed them to use a larger stepsize of 7.2 days, Sussman and Wisdom (1992) made an integration of the solar system over 100 million years which confirmed the existence of the secular resonances as well as the value of the Liapunov exponent of about 1/5 Myrs for the solar system.

3.4. Planetary evolution over Myr

The planetary eccentricities and inclinations present variations which are clearly visible over a few million of years (fig. 9). Over 1 million years, the perturbation methods of Laplace, and Le verrier (see section 3.1) will give a

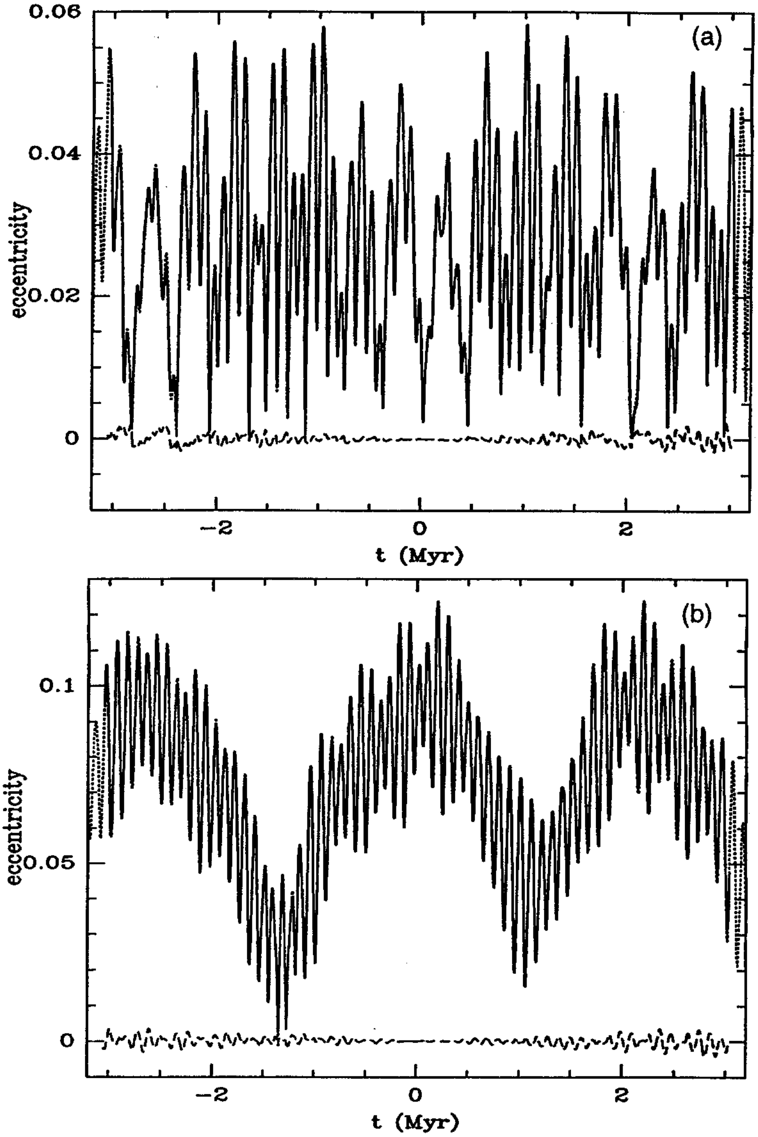


Figure 9. The eccentricity of the Earth (a) and Mars (b) during a 6 Myr timespan centered at the present. The solid line is the numerical solution QTD (Quinn *etal.* 1991), and the dotted line the integration La90 of the secular equations (Laskar, 1990). For clarity, the difference between the two solutions is also plotted (from Laskar, Quinn, Tremaine, 1992).

good account of these variations which are mostly due to the linear coupling present in the secular equations. They involve the precessional periods of the orbits, ranging from 45 000 years to a few million years (Table I). Over sev-

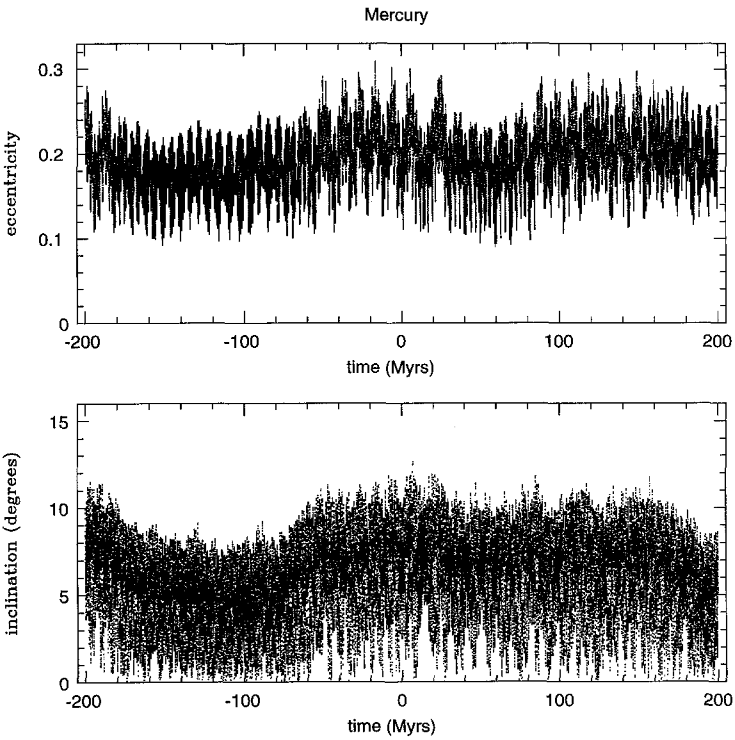


Figure 10. Computed evolution of the eccentricity and inclination of Mercury with respect to time from -200 to +200 million years. On each curves one can see a rapid variation, with periods of about 100 000 years which should correspond to the regular part of the solution as described by Laplace, and a slow variation which reveals the diffusion resulting from the chaotic dynamics (adapted from Laskar, 1992a).

eral hundred million years, the behavior of the solutions for the outer planets (Jupiter, Saturn, Uranus and Neptune) are very similar to the behavior over the first million years, and the motion of these planets appears to be very regular, which was also shown very precisely by means of frequency analysis (Laskar, 1990).

For the Earth, over such time span, the chaotic effect will induce a lost of predictability for the orbit. The additional change of eccentricity resulting from the chaotic diffusion is moderate and may be estimated to about 0.01 for the Earth (Laskar, 1992a,b). The most perturbed planet is Mercury, the effects of its chaotic dynamics being clearly visible over 400 million years (Laskar, 1992a,b) (Fig.10).

It should be stressed that the exponential divergence of the orbits revealed by the computation of the Lyapunov exponent result mostly from the change from libration to circulation of the resonant precession angles, which induce after some time a total indeterminacy of the precessional angles of the orbit, that is its orientation in space. The eccentricity and inclination (which are action like variables) variations due to the chaotic diffusion are much less rapid, and an important question is to estimate their wandering over the life time of the solar system.

3.5. THE CHAOTIC OBLIQUITY OF THE PLANETS

Instabilities of another sort also manifest themselves in the motion of the solar system's planets. These motions are not present in the orbits, but rather in the orientation of the planets' axes of rotation. Because of their equatorial bulge, the planets are subject to torques arising from the gravitational forces of their satellites and of the Sun. This causes a precessional motion, which in the Earth's case has a period of about 26,000 years. Moreover, the obliquity of each planet—the angle between the equator and the orbital plane—is not fixed, but suffers a perturbation due to the secular motion of the planet's orbit. Over one million year period, this variation is only ± 1.3 degrees around the mean value of 23.3 degrees. This may not seem like much, but it is enough to induce variations of nearly 20 percent in the summer insolation received at 65 degrees north latitude (fig. 11). According to Milankovitch theory (see Imbrie 1982), the amount of additional heat received during the summer at high latitudes is an important factor in climate studies, as it melts ice accumulated over the winter and prevents the ice caps from extending their reach. When this insolation is not sufficient, the ice cap extends, inducing a general cooling of the temperature on Earth, and eventually leading to an ice age. Weak variations in the Earth's obliquity are therefore a determining factor in regulating the climate enjoyed by the Earth over the last several million years. The quaternary ice ages constitute significant climatic changes, but were not so severe as to permanently change the conditions for life on the Earth's surface.

The full equations of precession are presented in (Laskar *et al.*, 1993a, b, Laskar and Robutel, 1993). In fact, in order to understand the dynamics of the problem, the very small terms of these equations can be neglected, although they are taken into account in the numerical computations. In the following simplified equations, we shall also neglect the eccentricity of of the Earth and the Moon as well as the inclination of the Moon. This will provide a simple but realistic form for the equations of precession which will allow the curious reader to check directly most of the computations. Using the action variable $X = \cos\varepsilon$, where ε is the obliquity, and the precession angle ψ, the hamiltonian reduces to

$$H(X,\psi,t) = \frac{1}{2}\alpha X^2 + \sqrt{1-X^2}(\mathbf{A}(t)\sin\psi + \mathbf{B}(t)\cos\psi) \qquad (6)$$

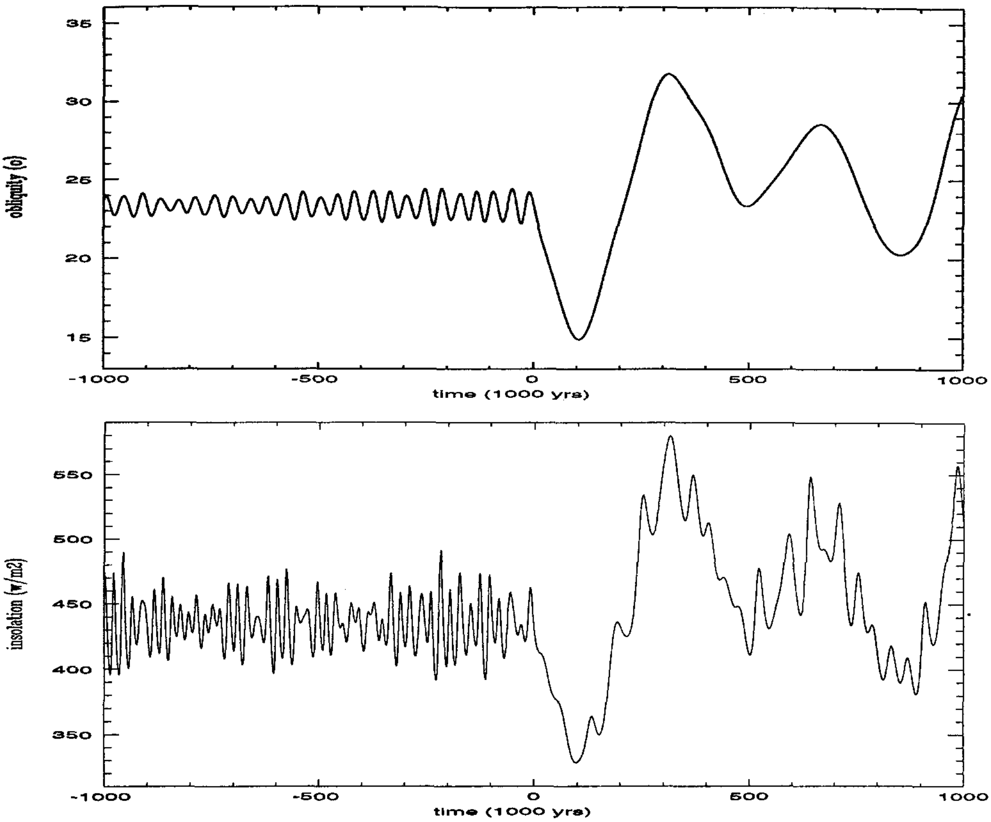


Figure 11. Changes in obliquity (a) and insolation at 65N ($\lambda_d = 120°$) (b) computed in the presence of the Moon from -1Myr to 0, and for 1 Myr after its suppression at $t = 0$ (Laskar *et al.*, 1993a).

where

$$\alpha = \frac{3}{2}\frac{C-A}{C}\frac{1}{\nu}(n_M^2 m_M + n_\odot^2 m_\odot) \tag{7}$$

A and C are the momentum of inertia of the planet (we assume that $B = A$), ν its rotational angular velocity, n_M and $n_\odot$ the mean motions of the Moon and the Sun (around a fixed Earth), m_M and $m_\odot$ their masses. The expression

$$\mathbf{A}(t) + i\mathbf{B}(t) \approx 2\frac{d\zeta}{dt}$$

where $\zeta = \sin i/2 \; e^{i\Omega}$ depends only on the change of inclination of the Earth with respect to a fixed plane and is given by the already computed

Table II
Quasiperiodic approximation of $\mathbf{A}+i\mathbf{B}$ obtained by frequency analysis of the Earth orbital solution over 18 Myr. The 12 Major terms are listed as well as a smaller, well isolated term, due to the perturbation of Jupiter and Saturn. $\mathbf{A}+i\mathbf{B} \approx \sum_{k=1}^{13} \alpha_k e^{i(\nu_k t+\phi_k)}$ (the α_k are expressed in yr^{-1}) (from Laskar *et al.*, 1993b).

k		ν_k ("/yr)	$\alpha_k \times 10^6$	$\phi_k(°)$
1	s_3	-18.8504	1.616070	151.724
2	s_4	-17.7544	0.691588	199.002
3		-18.3016	0.478868	176.641
4	s_6	-26.3302	0.340738	37.294
5	s_1	-5.6128	0.274325	270.479
6		-19.3997	0.286930	305.514
7	s_2	-7.0772	0.237068	9.899
8		-19.1251	0.165838	46.398
9		-6.9564	0.132989	199.316
10		-7.2037	0.112089	176.470
11		-6.8283	0.108391	233.037
12		-5.4892	0.080168	289.422
13	$s_6 - g_6 + g_5$	-50.3021	0.001043	120.161

solution of the solar system from (Laskar, 1990). Although the motion of the solar system is chaotic, for qualitative understanding of the behaviour of the solution, it is convenient to use a quasiperiodic approximation of this quantities over a short time span of a few millions of years (table II):

$$\mathbf{A}(t) + i\mathbf{B}(t) \approx \sum_{k=1}^{N} \alpha_k e^{i(\nu_k t+\phi_k)} .$$

With this approximation, the hamiltonian now reads

$$H = \frac{1}{2}\alpha X^2 + \sqrt{1-X^2}\sum_{k=1}^{N} \alpha_k \sin(\nu_k t + \psi + \phi_k) \tag{8}$$

which is the hamiltonian of an oscillator of frequency αX_0, perturbed by a quasiperiodic external oscillation of small amplitude ($|\alpha_k| << 1$). We will thus obtain a resonance, when $\dot{\psi} \approx \alpha X_0 = \alpha \cos\varepsilon_0 = 50.47"/yr$ will be opposite to one of the frequency ν_k.

When limited to a single term ($N = 1$), this Hamiltonian is integrable (Colombo, 1966). On the contrary, when ($N > 1$), a simple application of Chirikov overlap criterion already allows to forecast the existence of chaotic zone for the obliquity.

In the frequency decomposition of the $\mathbf{A}(t) + i\mathbf{B}(t)$ planetary forcing term, there exists a periodic term of small amplitude related to perturbations exerted by Jupiter and Saturn and of frequency $s_6-g_6+g_5 = -50.30207"/yr$, which could enter into resonance with the precession frequency.

In order to see the effect of this resonance, we slightly changed the value of the dynamical ellipticity $(C - A/C)$ of the Earth, keeping fixed its angular momentum. This is what can happen for example during an ice age, where the redistribution of the ice changes a little the dynamical ellipticity of the Earth. The integration of the obliquity of the Earth in this small resonance showed and increase of the maximum obliquity of the Earth of about 0.5 degrees. This small term could thus be of great importance in the computation of the past insolation of the Earth (Laskar *et al.*, 1993b).

After investigating the effect of this small resonance, we investigated the global dynamics of the obliquity of the Earth, by means of frequency map analysis (Laskar, 1990, 1993a, Laskar *et al.*, 1992, Dumas and Laskar, 1993). Briefly speaking, for a Hamiltonian system with n degrees of freedom close to integrable $H(J_i,\theta_i) = H_0(J_i)+\varepsilon H_1(J_i,\theta_i)$, we shall construct numerically, the frequency map

$$\begin{aligned} F_T : \mathbf{R}^n \times \mathbf{R} &\longrightarrow \mathbf{R}^n \\ (J,\tau) \quad &\longrightarrow \nu(J,\tau) \end{aligned} \tag{9}$$

which associates to the action like variables $(J_i)_{i=1,n}$ and to the starting time τ, the frequency vector $(\nu_i)_{i=1,n}$ obtained numerically with a refined Fourier analysis of the solution of initial conditions (J_i,θ_{i0}) over the finite time interval $[\tau,\tau+T]$ (the initial phases $\theta_i(0) = \theta_{i0}$ are fixed to an arbitrary value).

The regularity of the trajectories can then be monitored by the analysis of the frequency map (9), which allows also to make refined estimates of the chaotic diffusion of the orbit in the phase space. Indeed, the frequencies $(\nu_i)_{i=1,n}$ can be thought as the "best" action variables obtained locally for the given initial condition.

In the case of the present $1 + 15$ degrees of freedom problem, we fix $\tau = 0$, and as the orbital motion of the solar system is not supposed to be perturbed by the orientation of the planets, the frequency map will reduce to a $\mathbf{R} \longrightarrow \mathbf{R}$ map. The regularity of the motion will then be studied directly by looking to the regularity of the frequency curve giving the numerically determined precession frequency for various values of the initial obliquity.

This analysis, which was performed for every 0.1° in obliquity over 18 million years shows immediately that the obliquity of the Earth is presently stable, but reveals also the existence of a very large chaotic region, ranging from 60° to 90° (fig. 12) (Laskar *et al.*, 1993b).

We are far from this chaotic region, and the changes of obliquity of the Earth remains small (23.3° ± 1.3°), but if the Moon were not present, the value of the precession constant α would be roughly divided by 3 (as for the

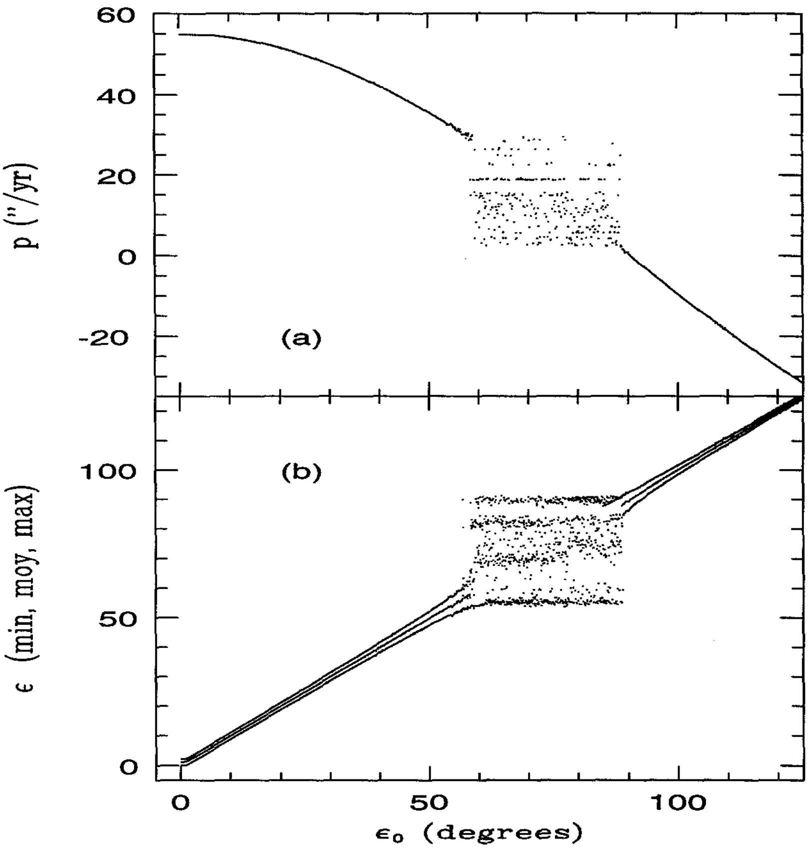


Figure 12. Stability of the rotation axis of the Earth in presence of the Moon. A different numerical integration of the precession equations is made over 18 Myr for each value of the initial obliquity of the Earth ranging from 0 to 125 degrees by 0.1 degree stepsize. For each integration, the minimum, mean, and maximum values of the obliquity are retained (b). A frequency analysis is also performed in order to determine precisely the averaged precession frequency of the Earth over this 18 Myr time span. The regularity of the frequency map (a) reflects the regularity of the motion. In particular, an extended chaotic zone is clearly visible from 60° to 90°. (Laskar *et al.*, 1993a).

ocean tides, the Moon accounts for $\approx 2/3$ in α, and the Sun for $\approx 1/3$). The precession frequency will also be divided by 3, and will be in resonance with the perturbations due to the motion of the orbital plane of the Earth. Even more, many resonances overlap, giving rise to an extended chaotic zone.

We investigated the global stability of the precession of the Earth for many values of its rotation speed ν (it should be noted that the dynamical ellipticity $C - A/C$ is proportional to ν^2), and found that for all primordial rotation period ranging from 12 h to about 48 h, the Earth obliquity would

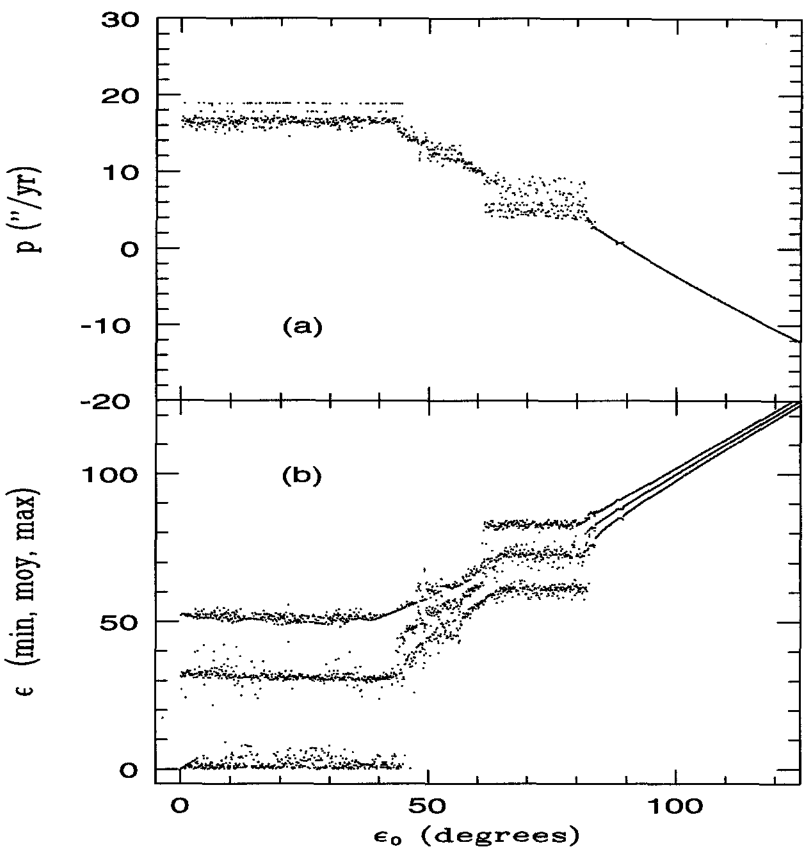


Figure 13. Without the Moon, the chaotic zone revealed by the analysis of the frequency map over 18 Myr (a) extends from 0° to ≈ 85°, for a period of rotation of the Earth of about 20 hours (Laskar *et al.*, 1993a).

suffer very large chaotic variations, from nearly 0° to about 85° (figs. 13, 14), which would probably lead to terrible climate variations on its surface (Laskar *et al.*, 1993a, Laskar 1993b) (typical changes from nearly 0° to about 60° can occur in less than 2 Myr, while transition to higher values of the obliquity should take a much longer time).

In much the same way as described above for the Earth, we studied the stability of the axial orientation of all the principal planets of the solar system. Mercury and Venus are special cases, since—no doubt because of solar tides acting over time—their rotational speeds are now very slow. Venus also possesses a trait that has long intrigued astronomers: it does not rotate in the same direction as the other planets, or in other words, it is upside down.

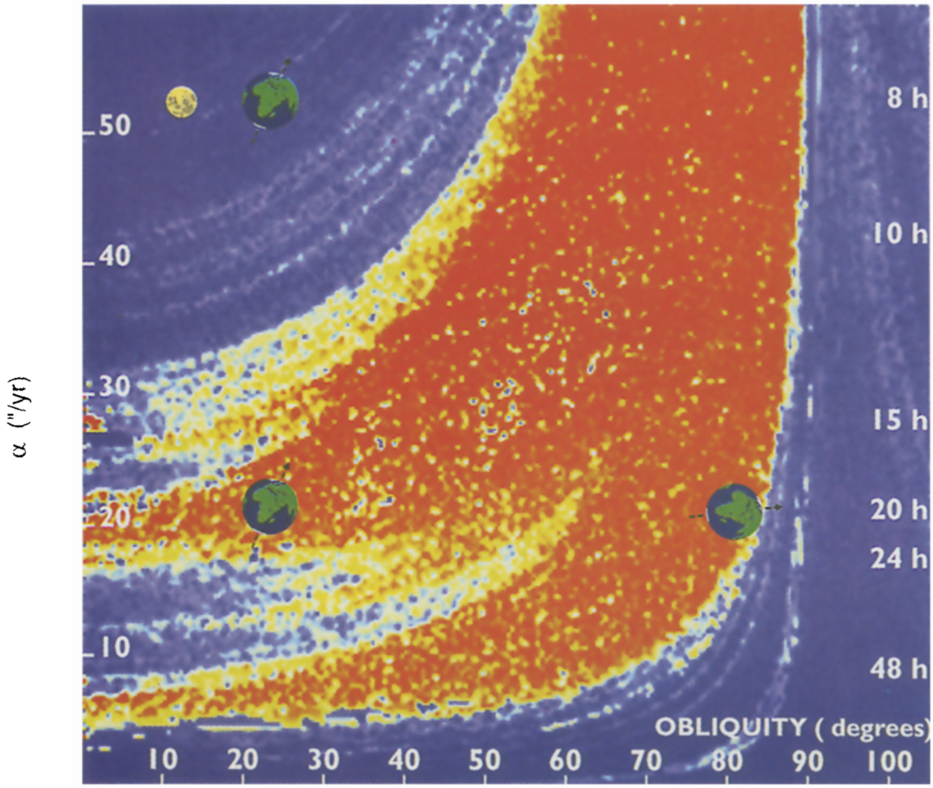


Figure 14. The zone of large scale chaotic behavior for the obliquity of The Earth (without the Moon) for a wide range of spin rate. The precession constant α is given on the left in arcseconds per year, and the estimate of the corresponding rotation period of the planet on the right, in hours. The blue region corresponds to the stable orbits, where the variations of the obliquity are moderate, while the orange and red zone is the chaotic zone. Indeed, the chaotic motion is estimated by the diffusion rate of the precession frequency measured for each initial condition (ε, α) via numerical frequency map analysis over 36 Myr. In the large chaotic zones visible here, the chaotic diffusion will occur on horizontal lines (α is fixed), and the obliquity of the planet can explore horizontally all the red orange zone. The extend of the chaotic zone should even be larger, when considering the diffusion of the orbits over longer time scale. With the Moon, one can consider that the present situation of the Earth can be represented approximately by the point of coordinates $\varepsilon = 23°, \alpha = 55\ ''/\text{yr}$, which is in the middle of a large zone of regular motion. Without the Moon, for spin period ranging from about 12h to 48h, the obliquity of the Earth would suffer very large chaotic variations ranging from nearly 0° to about 85°. This figure summarizes the results of about 250 000 numerical integrations of the Earth obliquity variations under the whole solar system perturbations for various initial conditions over 36 Myr. (Laskar and Robutel, 1993, Laskar, 1993b).

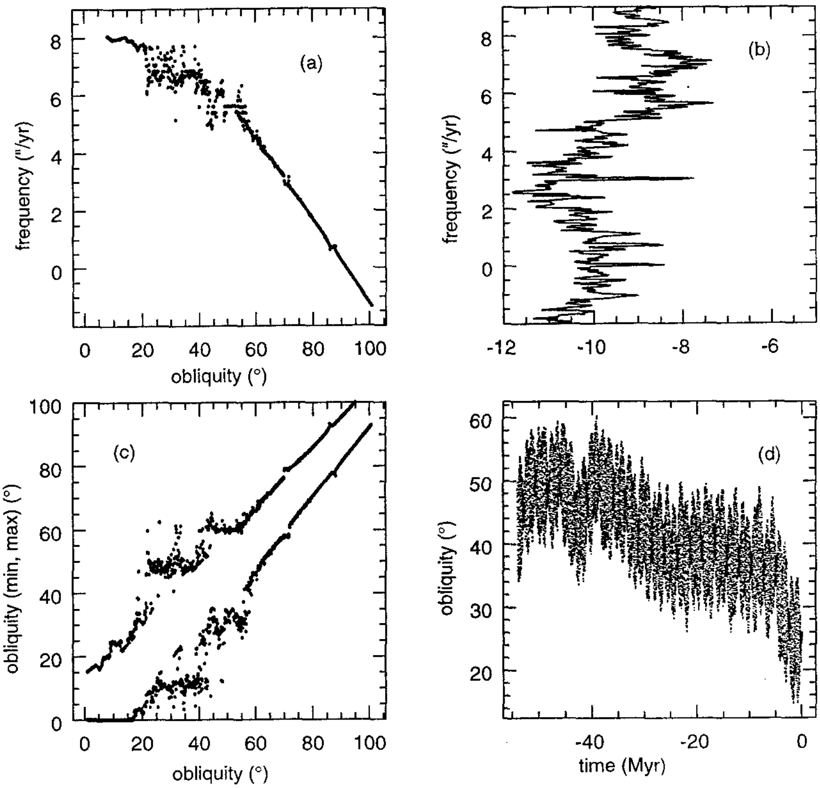


Figure 15. (a) Frequency map analysis of the obliquity of Mars over 56Myr. 1000 integrations of the obliquity of Mars have been conducted over 56 Myr for various initial conditions. A large chaotic zone is visible, ranging from 0° to 60°. In (b), the power spectrum of the orbital forcing term $\mathbf{A}(t) + i\mathbf{B}(t)$ is given in logarithmic scale, showing the correspondence of the chaotic zone with the main frequencies related to Venus and Mercury. (c) Maximum and minimum values of the obliquity reached over 56Myr. (d) Actual variations of the obliquity over 56 Myr for a selected orbit. (adapted from Laskar and Robutel, 1993).

It was generally assumed that Venus was formed upside down—or at least with its rotational axis in its orbital plane, since then dissipative effects arising from solar tides, core-mantle interactions, or from atmospheric tidal forces due to the Sun could bring it into an upside down position (Goldreich and Peale, 1970, Dobrovolski, 1980). Indeed, this was considered as a constraint on the models for the formation of the solar system, which would then require a "stochastic phase" at the end of the formation process, with a moderate number of large impacts by massive objects in order to obtain the desired orientation of this planet (e.g. Dones, and Tremaine, 1993a) We have shown instead that, even if Venus started with a rotational speed similar to the Earth's, and in the same direction, the presence of a large chaotic zone in its obliquity could subject it to severe tilting, bringing its rotational axis very nearly into its orbital plane. The dissipative effects just described could then bring it into its present position, where ultimately it might be stabilized as its rotation slowed further.

The situation for Mercury is slightly different. As is the case for Venus, we do not know Mercury's primordial rotational period, but it is enough to assume it was shorter than 300 hours to assure that, in the course of its history, Mercury underwent strongly chaotic variations in its obliquity, ranging from 0 to 90 degrees in the space of a few million years (Laskar and Robutel, 1993). As with Venus, the continued effects of tides could then slow its rotation, causing it to right itself and end up in its present position (Peale, 1974, 1976).

Mars is far from the Sun, and its satellites Phobos and Deimos have masses far too small to slow its rotation, so that its present rotational period of 24 hours 37 minutes is close to its primordial rotational period. Mars' equator is inclined 25 degrees with respect to its orbital plane, and its speed of precession, 7.26 seconds per year, is close to certain frequencies of motion of its orbit (Ward, 1974, Ward and Rudy, 1991). Moreover, variations in the inclination of Mars' orbit are considerably stronger than those of the Earth. It follows that variations in its obliquity over a period of one million years are also much stronger than the Earth's, and Ward has found obliquity variations on the order of ± 10 degrees about a mean value of 25 degrees. These variations bring about strong climatic changes on Mars' surface, and certain surface structures seem to bear witness to these changes.

Our computations (Laskar and Robutel, 1993), and numerical results obtained by Touma and Wisdom (1993), provide evidence that the motion of Mars' rotational axis is chaotic. This has two consequences. First, as it is also the case for the orbital motion of the inner planets, it is not possible to predict the orientation of Mars' axis for periods longer than a few million years.

But more important, the obliquity of Mars is subject to much larger variations than those predicted by Ward, ranging from about 0 to 60 degrees in less than 50 million years (fig. 15) (Laskar and Robutel, 1993). Models of the past climates of Mars need then to be reviewed in light of these new results. In particular, the large obliquity possibly reached for this planet will lead to higher temperature on its surface which may then allow the possibility of liquid water on its surface (Jakosky *et al.*, 1993).

On figure 15, obtained by frequency map analysis, it is clear that the size of the chaotic zone of the obliquity ranges from 0° to about 60°, and these values are actually reached during numerical integrations over less than 50 Myrs, but it can also be seen that the chaotic zone is divided into two main boxes: one is essentially related to secular resonances with Venus, and the second one with Mercury. The diffusion in each of these boxes is rapid, while the passage from one box to the other one is more difficult. This explains why Touma and Wisdom (1993), as they performed only a very limited numbers of integrations, were not able to see this transition and found only limited variations of Mars obliquity from 11° to 49°.

The existence of this large chaotic zone in the orientation motion of Mars also removes some constraints from models of solar system formation, since Mars' obliquity cannot be considered primordial, and its present orientation which is very similar to the Earth's, is purely due to chance.

On the other hand, our investigations showed that the obliquities of the outer planets are essentially stable. It is thus not possible to explain like that the very large obliquity of Uranus (98°), but it should be investigated if a chaotic behavior sufficient to lead to such an obliquity could have occurred during the formation of the solar system, at a time when it was supposed to be much more massive.

These results show that the situation of the Earth is very particular. The common status for all the terrestrial planets is to have experienced very large scale chaotic behaviour for their obliquity, which, in the case of the Earth and in absence of the Moon, may have prevented the appearance of evoluted forms of life. It is presently difficult to say exactly what would be the climate on the Earth with very large values of the obliquity, and even more with the possibility of drastic changes in the Earth orientation within a few million years, as no realistic models have yet been constructed taking into account these new results. But it is important to realize that up to now, it was generally assumed that in a planetary system similar to our, the planet located not too close to the sun, in which case runaway greenhouse effect may occur, and not to far from it in order to prevent runaway glaciation (Hart, 1978), would be very similar to the Earth. Our study demonstrated that this "reasonable hypothesis" is wrong, and that in the case of the Earth, we owe our relative present climate stability to an exceptional event: the presence of the Moon. While many results since the acceptance of heliocentrism have tendency to show that our Earth should be very common in the Universe, the present findings go in the opposite direction.

Moreover, the presence for the Earth of such a large satellite as the Moon, still puzzle astronomers, and the currently mostly accepted scenario for its origin relies on a non generic event: a large body, of the size of Mars, formed at the same time as the other planets enter in collision with the Earth, the subsequent accretion of the resulting debris forming the Moon (see the review of Stevenson, 1987). Indeed, if we accept that our presence on the Earth is correlated to the existence of the Moon, there is no problem for accepting an improbable scenario for the formation of the Moon, as soon as it does agree with all other physical and chemical constraints. Moreover, we may accept even more improbable models, if they better agree with the present observations. As a result, this may reduce the chances of finding extraterrestrial civilizations similar to ours around the nearby stars.

3.6. Planetary evolution on Gyr time scales

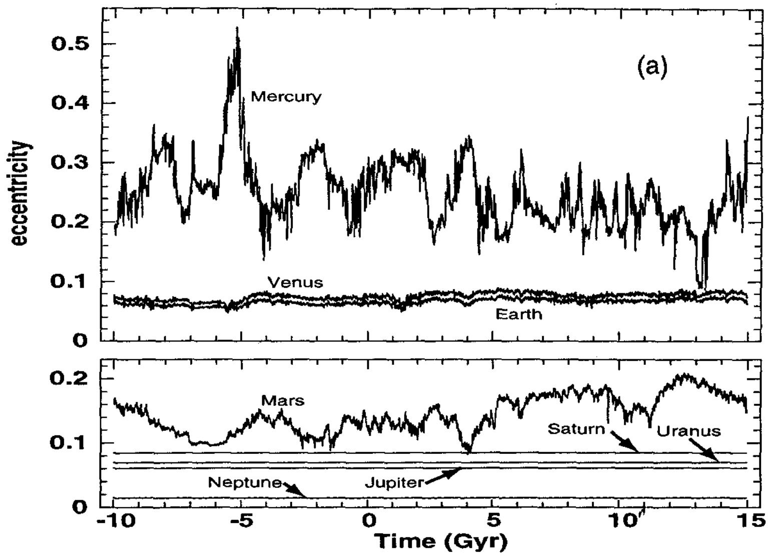
If the motion of the solar system were close to quasiperiodic, that is close to a KAM tori in the phase space, then it could be expected that some bound on the possible diffusion of the orbit over 5 Gyr would result from a Nekhoroshev like theorem (e.g. Niederman, 1994). In fact, as it was shown in (Laskar, 1990), although the system reduced to the outer planets may be considered as close to a KAM tori, the full solar system evolves far from a KAM tori, and diffusion of the action like variables (eccentricity and inclination) can occur. The natural question is thus to estimate this diffusion. Let us remind that in contrast to two degrees of freedom systems, where the diffusion may be bounded, in such a many degrees of freedom system (15 independent degrees of freedom for the secular system), there exist no results on the existence of invariant set which will bound the evolution of the system on infinite time span.

One may be tempted to integrate the motion of the solar system over 5 Gyr, that is over its expecting time life. For direct numerical integrations, this can be considered as an interesting challenge as it is still out of reach of present computer technology, but it should be stressed, that by no means it can be considered as the description of the evolution of the solar system over 5 Gyr. Indeed, because of the exponential divergence with a Lyapunov time of 5 Myr, after about 100 Myr the computed solution will be very different from the real solution followed by the actual solar system. Such a solution still present some interest, as it gives one of the possible behavior of the solar system, but it is much more important to obtain some description of the chaotic zone where the solar system evolves. In particular, it is more interesting to estimate the speed of diffusion in this chaotic zone. For such a goal, a single integration of the solar system over 5 Gyr will not be sufficient.

Quite surprisingly, we can use integrations over even longer time span, which will act as scouts exploring this chaotic zone. We can also send multiple of these explorers with very close initial conditions, in order to reach a larger portion of the phase space which can be attained by the solar system in 5 Gyr.

Doing this kind of search, it becomes obvious that we need to be able to integrate very rapidly the equations of motion for the solar system, and the present work analyzed the results of many such numerical integrations, totalling an integration time larger than 200 Gyr.

In order to achieve this task, the secular equations of the solar system were used, after some simplifications (laskar, 1994). Indeed, the initial secular system consisted into about 150000 polynomial terms, but many of them



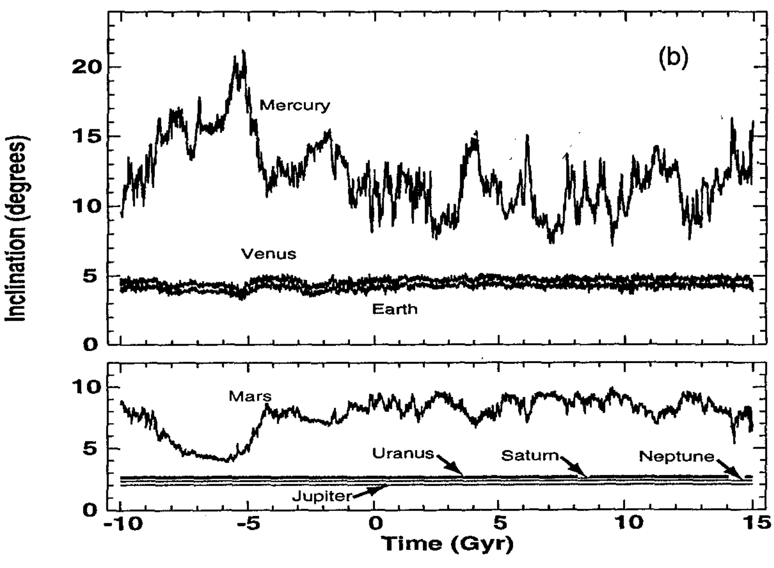


Figure 16. Numerical integration of the averaged equations of motion of the solar system 10 Gyr backward and 15 Gyr forward. For each planet, the maximum value obtained over intervals of 10 Myr for the eccentricity (a) and inclination (in degrees) from the fixed ecliptic J2000 (b) are plotted versus time. For clarity of the figures, Mercury, Venus and the Earth are plotted separately from Mars, Jupiter, Saturn, Uranus and Neptune. The large planets behavior is so regular that all the curves of maximum eccentricity and inclination appear as straight lines. On the contrary the corresponding curves of the inner planets show very large and irregular variations, which attest to their diffusion in the chaotic zone.(Laskar, 1994)

are of very small amplitude. It was thus possible to suppress them, and to reduce the system to only 50 000 terms, conserving the symmetries which were present in the equations. Doing that, only about 6000 terms need really to be computed during the evolution of the second hand member of the equations, and the computations could be achieved on an IBM RS6000/370 workstation at a rate of about 1 day of CPU time per Gyr, without any significant loss in the precision. Moreover, the numerical integration of the secular system has been improved, and the stepsize reduced to 250 years, which allowed the best precision. As we want to understand the dynamics of this secular system, it is actually necessary to make the integration with great accuracy. The secular system is an approximation of the real equations of motion, but the understanding of the global dynamical behavior of this system will provide a lot of information on the original system.

Some first integrations were conducted over 25 Gyr (-10 Gyr to + 15 Gyr) (fig. 16). It may seem strange to try to track the orbit of the solar system over such an extended time, longer than the age of the universe, but one should understand that it is done in order to explore the chaotic zone where the solar system evolves, and after 100 Myr, can give only an indication of what can happen. On the other hand, if there is a sudden increase of eccentricity for one planet after 10 Gyr, this still tells us that such an event could probably also occur over a much shorter time, for example in less than 5 Gyr. In the same way, what happens in negative time can happen as well in positive time.

In order to follow the diffusion of the orbits in the chaotic zone, one needs quantities which behave like action variables, that is quantities which will be almost constant for a regular (quasiperiodic) solution of the system. Such quantities are given here by the maximum eccentricity and inclination attained by each planet during intervals of 10 Myr (Fig. 16).

The behavior of the large planets is so regular that all the corresponding curves appear as straight lines (Fig. 16). On the contrary the maxima of eccentricity and inclination of the inner planets show very large and irregular variations, which attest to their diffusion in the chaotic zone. The diffusion of the eccentricity of the Earth and Venus is moderate, but still amounts to about 0.02 for both planets. The diffusion of the eccentricity of Mars is large and reaches more than 0.12, leading to values higher than 0.2 for the eccentricity of Mars. For Mercury, the chaotic zone is so large (more than 0.4) that it reaches values larger than 0.5 at some time. The behavior of the inclination is very similar.

Strong correlations between the different curves appear in figure 16. Indeed, as the solar system wanders in the chaotic zone, it is dominated by the linear coupling among the proper modes of the averaged equations (eq. 3), which induces a very similar behavior for the maximum eccentricity and inclination of Venus and the Earth. This coupling is also noticeable in the solution of Mars. On the other hand, an angular momentum integral exists in the averaged equations, and explains why when Mercury's maximum eccentricity and inclination increase, the similar quantities for Venus, the Earth and Mars decrease. Thus it appears that, despite the small values of the inner planets' masses, the conservation of angular momentum plays a decisive role in limiting their excursions in the chaotic zone. Thus the same argument which allowed Laplace to "prove" the stability of the solar system in the linear approximation (see section 3.1) appear to be indeed primordial for limiting the wandering of the Earth's orbit in the chaotic zone, and thus achieving practical stability over the age of the solar system.

3.7. Escaping planets

At some time, Mercury suffered a large increase in eccentricity (fig. 16) rising up to 0.5. But this is not sufficient to cross the orbit of Venus. The question then arises whether it is possible for Mercury to escape from the solar system in a time comparable to its age. A first attempt to answer this was made by slightly changing the initial conditions for the planets. Indeed, because of the chaotic behavior, very small changes in the initial conditions lead to completely different solutions after 100 Myr. Using this, only one coordinate in the position of the Earth was changed, amounting to a physical change of about 150 meters (10^{-9} in eccentricity). The full system was integrated with several of these modified solutions, but led to similar (although different) solutions. In fact, it should not be too easy to get rid of Mercury, otherwise it would be difficult to explain its presence in the solar system.

I thus decided to guide Mercury towards the exit. A first experiment was done for negative time: for 2 Gyr, the solution is left unchanged, then, 4 different solutions are computed for 500 Myr, in each of which the position of the Earth is shifted by 150 meters, in a different direction (due to the exponential divergence, this corresponds to a change smaller than Planck's length in the original initial conditions).

The solution which leads to the maximum value of Mercury's eccentricity is retained up to the nearest entire Myr, and is started again. In 18 of such steps, Mercury attains eccentricity values close to 1 at about -6 Gyr when the solution enters a zone of greater chaos, with Lyapunov time ≈ 1 Myr, giving rise to much stronger variations of the orbital elements of the inner planets. A second solution was also computed in positive time, with changes in initial condition of only 15 meters instead of 150 meters. As anticipated, this led to a similar increase in Mercury's eccentricity, this time in only 13 steps and about 3.5 Gyr (fig. 17).

While the eccentricity increases, the inclination of Mercury can change very much but the computation of the relative positions of the intersection

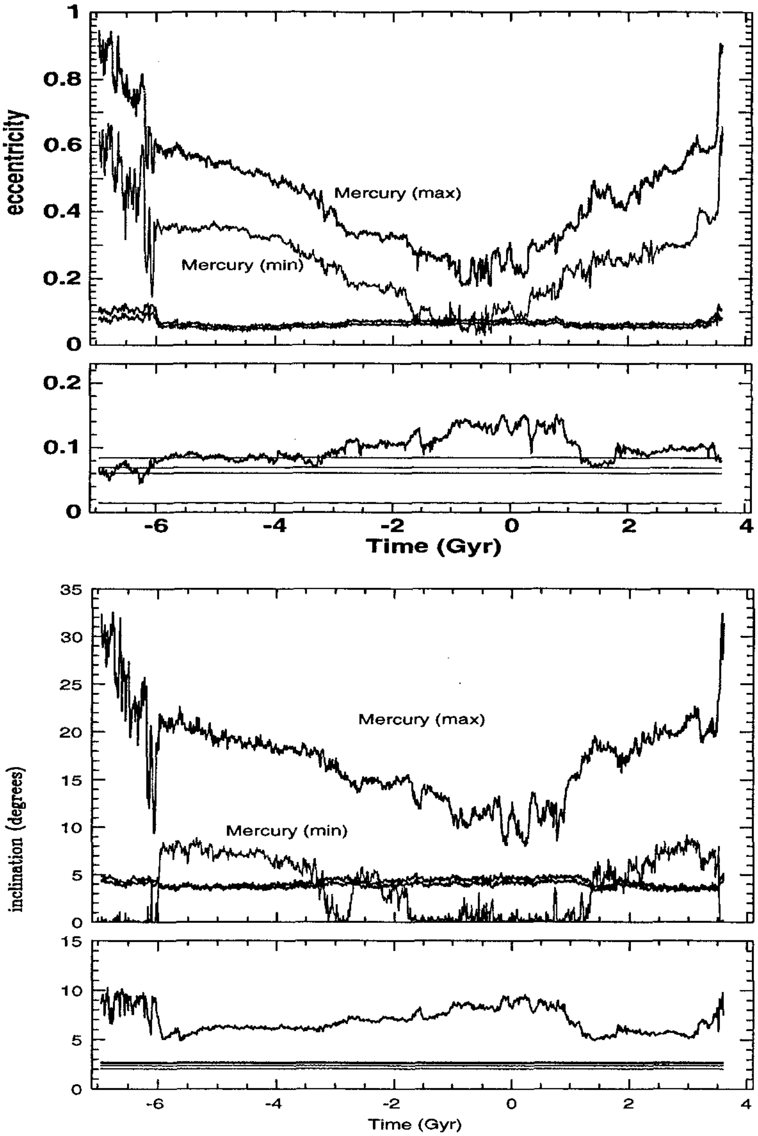


Figure 17. Orbit of the solar system leading to very large values for the eccentricity of Mercury, and possibility of escape at -6.6 Gyr and +3.5 Gyr. The plotted quantities are the same as in Fig. 16, except for Mercury, where minimum eccentricity and inclination over 10 Myr are also plotted. During all the integrations, the motion of the large planets is very regular (Laskar, 1994).

of the orbits of Mercury and Venus with their line of nodes demonstrated that the orbits effectively intersect at about 3.5 Gyr. At this time, the two planets can experience a close encounter which can lead to the escape of Mercury or to collision (Laskar, 1994).

For very high eccentricity of Mercury, the model used here no longer gives a very good approximation to the motion of Mercury, but in the real system the addition of the extra degrees of freedom related to semi-major axes and longitudes, will probably lead to even stronger chaotic behavior, as in general, addition of degrees of freedom increases the stochasticity of the motion. When one examines the results of direct numerical integrations of asteroids in secular resonances (see for example Farinella *et al.*, 1993), one can see very often that at the beginning, one can assist to a similar increase in eccentricity due to the secular resonance, but after some times, when the eccentricity is high enough, the perturbations related to mean motion resonances become very important, and large scale chaos related to mean motion resonances occurs, resulting from overlap of the mean motion resonances, which induces a large diffusion of the semi major axis. Most probably, the natural scenario for an escape of Mercury will be of this type and for high eccentricities the complete solar system should be much less stable than the secular system, where the semi-major axis are frozen.

Similar computations were made for Mars and the Earth, but did not lead up to now to an escaping solution. For the Earth, the maximum eccentricity reached after 5 Gyr is about 0.1, while for Mars, the eccentricity attained about 0.25 after 5 Gyr. With such a high eccentricity, Mars comes very close to the Earth, and it may be possible to find some escaping solution for Mars when considering the complete equations, but it should be noted that the search for the escaping solutions of Mars, in positive time uniquely, necessitated about 100 numerical integrations, each of them over 500 Myr. This is obviously out of reach of direct numerical integration with present computer technology, and a mixed solution should be envisaged, where most of the increase of eccentricity is made with the secular equations, and the final part with direct numerical integration.

3.8. Marginal stability of the solar system

The existence of an escaping orbit for Mercury does not mean that this escape is very likely to occur. In fact, the solution computed here which lead to an escape was very carefully tailored, by selecting at each step one solution among 4 or 5 equivalent ones. The result obtained here is a result of existence for an escaping orbit, but does not tell us the probability for this escape to occur. The computation of an estimate of this probability would require to follow more completely all the studied orbits, and also most probably to take into account the full equations in order to be accurate. From the present computation, it can be thought that this probability is small, but not null, which is compatible with the present existence of Mercury.

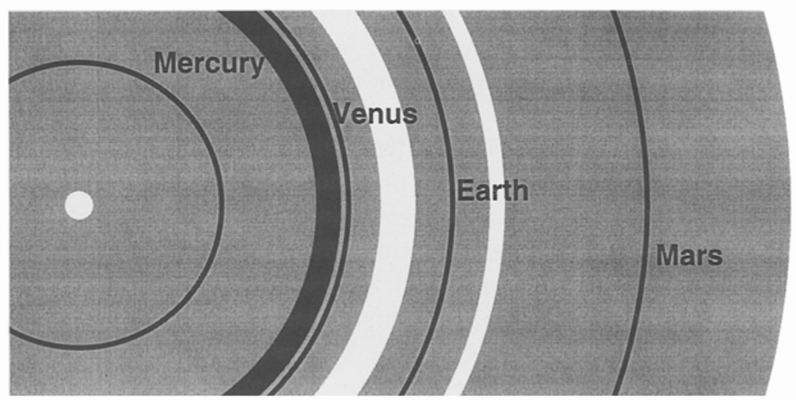


Figure 18. Estimates of the zones possibly occupied by the inner planets of the solar system over 5 Gyr. The circular orbits correspond to the bold lines, and the zones visited by each planet resulting from the possible increase of eccentricity are the shaded zones. In the case of Mercury and Venus, these shaded zones overlap. Mars can go as far as 1.9 AU, which roughly corresponds to the inner limit of the asteroid belt.

Without speaking of escaping orbits, the very large diffusion of the inner planets orbits is very striking. Even after the discovery of the chaotic behavior of the solar system, and despite the results of (Laskar, 1990) where estimates of the diffusion were already computed by means of frequency analysis, many people assumed that the chaotic diffusion in the solar system was very small. Here it is clearly demonstrated that for the inner planets, it is not the case. More, for the inner planets, the excursion of the eccentricity and inclination variables seems to be essentially constrained by the angular momentum conservation which explains that when the maximum eccentricity of Mercury increases, the maximum eccentricity of Venus, the Earth and Mars decreases. This is quite surprising, when considering that most of the angular momentum comes from the outer planets. In fact, the outer planets system is very regular, and practically no diffusion will take place among the degrees of freedom related to the outer planets.

On figure 16, it appears that the less massive planets are subject to the largest variation of eccentricity. This becomes obvious when considering that these variations are essentially bounded by the angular momentum conservation, which for each planets is proportional to $m\sqrt{a}$, where m is the mass of the planet, and a its semi major axis.

If for each planet, we consider the maximum diffusion of the eccentricity observed over 5 Gyr (fig. 18) during similar numerical experiments as for Mercury, we find that Mercury's eccentricity can go sufficiently high to allow Mercury's orbit to cross the orbit of Venus, Venus and the Earth's eccentricity can go up to 0.1, and Mars as high as 0.25. Apart from some small place in between Venus and the Earth, or the Earth and Mars, all the inner solar system is swept by the planetary orbits, and the small planets (Mercury and Mars) are the planets which present the largest excursions. Practically, we can conclude that the inner solar system is full. That is there is no room for any extra planet. Indeed, even if there are some place which seems not to be possibly reached in 5 Gyr, an additional planet orbit will most probably intersect one of the already existing ones. If we add a large planet, of the size of the Earth or Venus, its orbital elements will not vary much but it will induce strong short periods perturbations. On the contrary, a small object will suffer large orbital variations, as it will not be much constrained by the angular momentum conservation. In this case, encounters with the already existing planets is very probable.

The variations which are reported in fig. 17 are the maximum variations observed over 5 Gyr, and not the most probable variations, but the addition of an extra planet will most probably increase very much the diffusion by increasing the numbers of degrees of freedom, and these maximum possible variations can probably be considered as good estimates of the probable variations over 5 Gyr in the eventuality of this addition of an extra planet in the inner solar system. It thus becomes interesting to speak of marginal stability when considering the solar system. Maybe there were some extra planets at the early stage of formation of the solar system, and in particular in the inner solar system, but this led to so much instability that one of the planets (probably among the smallest ones, of the size of Mercury or Mars) suffered a close encounter, or a collision with the other ones. This lead eventually to the escape of this planet, and the remaining system gets more stable. Indeed, this is what was observed when Mercury was suppressed in the numerical simulation, after the crossing of Venus orbit. Quinlan (1993) also observed similar results on experiments conducted on examples of planetary systems with the full equations over shorter time scales. In this case, at each stage of its evolution, the system should have a time of stability comparable with its age, which is roughly what is achieved now, when ones finds that escape of one of the planets (Mercury) can occur within 5 Gyr.

4. Discussion

4.1. Stability of the solar system

The Lyapounov time of 5 Myr for the solar system (Laskar, 1989), as well as the existence of secular resonances of large amplitude in the inner solar system demonstrates that the motion of the solar system is not regular, and cannot be approximated by a quasiperiodic trajectory over more than 10 to 20 Myr. Moreover, it will be practically impossible to make any precise prediction for the evolution of the solar system beyond 100 Myr, due to the exponential divergence of the orbits. Thus, we are far from the regular solutions, whose existence was exhibited by Arnold.

Nevertheless, this result applies more specifically to the inner planets (Mercury, Venus, Earth and Mars). Although the outer planets (Jupiter, Saturn, Uranus and Neptune) are perturbed gravitationally by the inner planets, this perturbation is small, and the induced effect of their chaotic motion will only generate a small diffusion of their trajectories. For a planetary system restricted to the outer planets, and even more for the Jupiter -Saturn couple, it still should be possible to obtain rigorous (in the mathematical sense) stability results along the lines described by Arnold and Nekhoroshev, although this will necessitate specially adapted version of the theorems.

In their integrations of the outer planets system, Sussman and Wisdom (1992) have reported Lyapounov times ranging from 3 to 30 Myr. This result needs to be taken cautiously as it seems to be very dependent of the numerical procedure they used to integrate the equations. Moreover, as the Lyapounov time of the secular system seems to be much larger, these instabilities should be related to the fast orbital motion of the planets, and not to the slow precession of the orbits. They probably involve very high order mean motion resonances whose amplitude will be very small, and no physical consequence will result. The orbits of the outer planets should still be confined to very narrow regions over the age of the solar system.

For the secular system, the problem is very different. The main frequencies are of the order of 100 000 years. The Lyapounov time of 5 Myr (which is also of the same order as the libration period of the identified main secular resonance) is only equal to 50 times the fundamental periods of the motion, which explains why this can lead to large scale chaotic behavior. Indeed, we have seen that all the inner planets experience significant chaotic diffusion over billion years timescale, and the existence of an escaping orbit for Mercury demonstrates that the solar system in not stable, even when considering the strongest meaning of this word, that is the possibility of evasion or collision of the planets.

However, although the solar system is not stable, it can be considered as marginally stable, that is strong instabilities (collision or escape) can only occur on a time scale comparable to its age, that is about 5 Gyr. We have exhibited an escaping or collisional orbit for Mercury in less than 3.5 Gyr (Laskar, 1994). For Mars, the large diffusion of its orbit can drive the eccentricity to about 0.25, and it still should be checked, using the full equations of motion in order to add the possibility of instabilities related to the mean motion, whether this could also lead to a collisional orbit with the Earth.

On the other hand, the orbits of Venus and the Earth, because of their larger masses, their linear coupling, and because of the angular momentum conservation constraint, seem to be practically confined only to small deviations from their presents path. These two planets, although their orbits are not close to quasi periodic, can thus be considered as stable over the age of the solar system, without regard to the possibilities of collision with Mercury or Mars.

4.2. Constraints on the formation of the solar system

This new vision of the evolution of the solar system over its age also induces some changes in the dynamical constraints for the formation models of the solar system (Harris and Ward, 1982).

In particular, the recognition that the solar system is in a state of marginal stability suggests that the organization of the planets in the solar system (often quoted as the Titius-Bode law), and more particularly of its inner part, is most probably due to its long run orbital evolution, and not uniquely to its rapid (less than 100 million years) formation process. We have shown that the inner solar system is full from 0 AU to about 2 AU, which coincides with the inner edge of the asteroidal belt. Some extra inner planets may have existed, but their existence then gave rise to a much more instable system, leading to the escape or collision of one of the planets, the remaining part then being much more stabilized. Indeed, this is what was observed in our numerical computations, after the simulation of the escape of Mercury. In particular, these findings show that minor bodies in the inner solar system will probably not be able to survive for a very long time. This result is important for the understanding of the formation of the solar system, as it tells us that the solar system at the end of its formation process may have been significantly different from the present one, and has then evolved towards the present configuration because of the gravitationnal instabilities. It still should be very interesting to investigate this point further using simulations with the addition of an extra planet, but many features have already been deduced here from the present computations.

On the other hand, the outer system is very stable, but the long time recent numerical integrations (see section 2.5) also demonstrate that the outer solar system is full, that is most of the objects introduced in this system will escape on time scale much shorter than 5 Gyr. Apart from some special locations, like the Jupiter Lagrangian positions, stable zones only begin at about 40 AU, where several objects were recently founded.

Moreover, by showing that none of the obliquities of the inner planets are primordial (section 3.5), we have removed one of the constraint on the formation models for the solar system. We have also proven the stability of the obliquity of the outer planets, since the solar system is in its present state, but instabilities may have existed during the formation of the solar system, when the planetary disk was supposed to be more massive, and it is of great interest to study the possibility of such a scenario.

As was discussed in section 3.5, we have established the possibility of a strong correlation between our existence and the existence of the Moon, which should leave the possibility to accept an improbable scenario for its formation, if it properly accounts for the other physical and chemical observations. The models for the formations of the Moon thus need to be reevaluated in this scope.

4.3. Generic planetary systems

One may be now tempted to answer to the question of what will be a generic planetary system ?

Such a question is of course delicate to answer, after having only studied our solar system stability, but the observation that our solar system is in a state of marginal stability, that is practical stability on a time scale comparable to its age, can be a clue for answering this question.

Indeed, I would like to suggest that a planetary system will always be in this state of marginal stability, as a result of its gravitational interactions.

In particular, a planetary system with only one or two planets should be excluded, because it will then be much too stable*, or more precisely, if it does exist, it would be crowded with asteroids everywhere which would be the original remaining planetesimals, not ejected by planetary perturbations.

On the other hand, if the formation process is such that there exist some large outer planets, and some small inner planets, after 5 Gyr, the small inner planets will still be subject to instabilities similar to the present ones in the solar system, and thus so will be their obliquities.

It should indeed be noted that if a planet evolves at about 1 AU from a solar type star, that is in good condition to have liquid water on its surface, then its precession frequency will depend essentially on its rotation period and will thus be similar to the one of the Earth in absence of the Moon (fig. 14). Thus if the precessing frequencies of this planetary system are of the same order as those of our solar system, this planet will have a large probability to be subject to very large chaotic variations for its obliquity.

Moreover, in order to have an orbital stability comparable to the one of the Earth, a terrestrial planet probably needs to have a sufficiently large mass, otherwise it could be subject to orbital variations similar to the ones of Mercury or Mars, which would induce even larger variations of its obliquity.

These considerations show that it may not be so easy to find around a nearby star another planet with a similar orbital and rotational stability as the Earth, situated at a distance from the central star allowing the existence of liquid water.

* This results from some work in progress with P. Robutel

4.4. Epilogue

Many fundamental problems still remain in order to clarify the questions raised here on the genericity of our solar system and of the Earth orbital and rotational stability. Some concern the formation of the planetary system; in particular the understanding of the origin of the rotation of the planets (Dones and Tremaine, 1993b, Lissauer and Safronov, 1991) appears as a crucial point for the analysis of the stability of their orientations. As important will be possible improvements on the understanding of the response of the planets atmosphere behavior under insolation forcing. The direct observation of another planetary system, which may occur in the near future, should also provide important elements for answering these questions, but it should be stressed that improvements of the present theoretical knowledge of the global dynamics of planetary systems can also provide very important constraints on the possible organization of planetary systems.

Most of the results on the planetary orbits presented here rely on the analysis of the secular equations of the solar system, and not on the complete equations. This was the price to pay for allowing a more global approach on the problem of the stability and long time evolution of the solar system. Some integrations of the full equations are still welcome, but it is doubtful that these future integrations will change much the global landscape of the dynamics of the solar system portrayed here.

Acknowledgements

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