NEKHOROSHEV ESTIMATE FOR ISOCHRONOUS NON RESONANT SYMPLECTIC MAPS

ARMANDO BAZZANI, STEFANO MARMI AND GIORGIO TURCHETTI
Dipartimento di Fisica, Universitá di Bologna
INFN-Sezione di Bologna
Via Irnerio 46, I-40126 Bologna, Italy
E-mail BAZZANI@MVXBO2.INFN.IT

(Received : 22 May, 1989; accepted : 26 January, 1990)

Abstract. We prove that non resonant isochronous symplectic maps in a neighborhood of an elliptic fixed point are stable for exponentially long times with the inverse of the distance from the fixed point. In the proof we make use of the majorant series method together with an idea for optimizing remainder estimates first applied to Hamiltonian problems by Nekhoroshev.

Keywords : symplectic maps, stability, normal forms

1. Introduction

The stability analysis of motions in a neighborhood of an equilibrium point is very relevant in many branches of physics such as celestial mechanics, accelerators theory and classical statistical mechanics.

The first positive general result for quasi integrable Hamiltonian systems is the celebrated Kolmogorov-Arnold-Moser (KAM) theorem [1,2,3] which, in the case of systems with $n = 2$ degrees of freedom insures stability for all times in any region bounded by a preserved two-dimensional torus. However for systems with $n \geq 3$ degrees of freedom the existence of preserved $\mathbf{T}^n$ tori does not imply any stability result. In fact, as the resonant set is connected and dense in phase space, arbitrarily close to any initial condition there are orbits which can get very close to any point on the constant energy manifold by moving along resonances (*topological instability*).

This complex diffusive phenomenon was first pointed out by Arnold [4] who gave an explicit example and found that the time needed for this diffusion was exponentially long with the small perturbative parameter. Later a general theorem [5] was proved by Nekhoroshev who analized the perturbation theory for quasi integrable Hamiltonians of the form $H(j,\varphi,\varepsilon) = H_0(j) + \varepsilon V(j,\varphi)$, where $(j,\varphi) \in \mathbf{R}^n \times \mathbf{T}^n$ are action-angle variables, ε is the (small) perturbation parameter and H_0 is a non isochronous integrable Hamiltonian. By partitioning the phase space into resonant and non-resonant blocks and combining resonant and non-resonant finite order perturbation theory with suitable remainder estimates, Nekhoroshev proved stability for finite but exponentially long times on an open set of initial conditions. More precisely he showed that $| I(t) - I(0) | \leq \varepsilon^a$, for times $| t | \leq \exp(\varepsilon_0/\varepsilon)^b$, where a, ε_0 and b are positive real numbers.

A similar result had already been obtained 20 years before by J.E. Littlewood [6] in the study of the equilateral configuration in the restricted three body problem. About this exponentially long stability time his comment is : "while not eternity, this is a considerable slice of it".

Results of this form can be relevant for physics and celestial mechanics [7] : clearly the relevant time scale for the stability of the Solar system is not longer than the estimated age of the universe.

Nekhoroshev analysis simplifies when restricted to perturbed isochronous systems [8,9,10,11]. In this case one deals with Birkhoff perturbation series and the remainder estimates are $A^N(N!)^a\varepsilon^N$, where A and a are positive constants and N denotes the truncation order. The asymptotic character of the series is evident and at the optimal truncation order $N \approx \varepsilon^{-1/a}/e$ one obtains an exponentially small remainder which implies an exponentially long stability time.

In the present work we extend this result to isochronous symplectic maps of $\mathbf{R}^{2d}$. The motivation is twofold : on one side there are important physical systems which are naturally modelled by maps, on the other side maps are slightly more general than flows and there are relevant technical differences, so that something new can be learned analyzing them. As J. Palis and J.C. Yoccoz have proved [12] in full generality for smooth diffeomorphisms of compact boundaryless manifolds, there exists a large class of maps with trivial "*centralizer*", that is these maps do not embed in smooth flows as their time one map. Therefore the transposition of results from Hamiltonian flows to symplectic maps is not immediate.

Symplectic maps frequently appear in theoretical physics, applied physics and celestial mechanics [13,14,15,16]. They are obviously more convenient than flows for any numerical experiment and occur as the natural models to describe the magnetic lattice of a particle accelerator [17,18,19,20,21] or the magnetic surfaces in the toroidal machines for the controlled thermonuclear fusion [22,23,24,25].

In the present work we consider a symplectic map of $\mathbf{R}^{2d}$ in the neighborhood of an elliptic fixed point and analyze the asymptotic properties of the Birkhoff series [26,27], using the majorant series method of Cauchy. Such a method was first succesfully used by Moser [28] to prove the convergence of the Birkhoff series which bring to normal form an area preserving map in the neighborhood of a hyperbolic fixed point.

In the elliptic case we consider, the precise mechanism which determines the divergence of the series is not precisely known, even though some heuristic arguments have been given for area preserving maps [29,30,31]. Here we find that our majorant series satisfies a functional equation, which seems to be at the origin of the $N!$ growth of the power series coefficients, even if the small divisors contribution is absent. Indeed the same majorant series can be used also to study the normal forms of analytic maps of $\mathbf{C}^n$ in the resonant case [32] where no divisors occur if $n = 1$.

The generating function $g(x)$ of the majorant series satisfies the following functional equation

$$3g(x) = g(g(x)) + x^2 + 2x \qquad g(0) = 0,\ g'(0) = g''(0) = 1 \qquad x \in \mathbf{R} \tag{1.1}$$

whose analyticity properties are presently investigated and seem to imply the presence of a fractal natural boundary [33]. A complete understanding of the analyticity of $g(x)$ could be the first step towards understanding the analyticity structure of the true symplectic transformation which leads to the Birkhoff normal forms.

Future work will consist in sharpening the estimates, possibly with a computer assisted proof, in order to compare them with numerical experiments for maps of $\mathbf{R}^4$ where the diffusion can be observed. The extension to the resonant case will also be considered : the extension of the present estimates to the resonant normal forms is straightforward, while the construction of the interpolating hamiltonian and the corresponding remainder estimates are needed to obtain the stability estimate and require a further non-negligible technical effort.
The present work is organized as follows :

In section 2 we introduce our notations and the equation which formally conjugates a symplectic, isochronous map with its normal form. In section 3 we estimate the growth of the Birkhoff series. In section 4 we estimate the remainder and in the last section we estimate the iterated remainder, obtaining a bound to the stability time.

2. Notations, statement of the main result. Formal solution of the conjugacy equations

2. 1. THE MAP

Consider a map in $\mathbf{R}^{2d}$, $d \geq 1$, defined by

$$\begin{aligned} p' &= f(q,p) \\ q' &= g(q,p), \end{aligned} \tag{2.1}$$

where $(q,p) \in \mathbf{R}^d \times \mathbf{R}^d$ and f, g are real analytic functions with values in $\mathbf{R}^d$. We assume that the origin is a fixed point $f(0,0) = g(0,0) = 0$ and that the map is symplectic, namely the following condition is satisfied

$$\mathcal{F}(q,p)\, J\, \tilde{\mathcal{F}}(q,p) = J \tag{2.2}$$

where the jacobian matrix $\mathcal{F}(q,p)$ and the matrix J are defined by

$$\mathcal{F}(q,p) = \frac{\partial(f,g)}{\partial(q,p)} \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.3}$$

where I_n is the $n \times n$ identity matrix.

2. 2. COMPLEX COORDINATES

According to Birkhoff we introduce the complex coordinates $z_k = q_k + ip_k$, $k = 1, \ldots, d$ and their canonical conjugates $w_k = z_k^* = q_k - ip_k$ and define $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$.
We consider also the complex functions $F_k = f_k + ig_k$ and their conjugates $G_k = f_k - ig_k$ for $k = 1, \ldots, d$. Since the coordinates q, p are real one is interested only in the $w = z^*$ subspace of $\mathbf{C}^{2d}$ where one has $G = F^*$, namely the map (F, G) as a map in $\mathbf{C}^{2d}$ leaves this subspace invariant.
Assuming that the origin is an elliptic fixed point, the map F can be written as

$$z' = F(z, z^*) = e^{i\omega}(z + \sum_{n=2}^{+\infty}[F]_n(z, z^*)) \tag{2.4}$$

The following notation has been used : $e^{i\omega} = \mathrm{diag}\,(e^{i\omega_1}, \ldots, e^{i\omega_d})$, $e^{i\omega}z = (e^{i\omega_1}z_1, \ldots, e^{i\omega_d}z_d)$ and $[F]_n(z, z^*)$ denotes the projection of F on the subspace of homogenuous polynomials of degree n.

In the space $\mathbf{C}^d$ we use the following norm

$$\|z\| = \max_{1 \le j \le d} |z_j| \qquad z \in \mathbf{C}^d \tag{2.5}$$

and the corresponding sphere $\|z\| \le r$ is indeed a polydisk.
Given any map $G \ : \ \mathbf{C}^{2d} \to \mathbf{C}^d$ holomorphic on some closed polydisk $D_r \ := \{(z, w) \in \mathbf{C}^{2d} \mid \ \|z\| \le r, \|w\| \le r \ \}$ with $G = (G_1, \ldots, G_d)$, $G(0) = 0$, we write its Taylor series as

$$G(z, w) = \sum_{\mathbf{k} \in \mathbf{N}^{2d}} G_{\mathbf{k}} \ z_1^{k_1} w_1^{k_2} \cdots z_d^{k_{2d-1}} w_d^{k_{2d}} \ , \tag{2.6}$$

where $G_{\mathbf{k}} \in \mathbf{C}^d$, $G_{\mathbf{k}} = (G_{\mathbf{k}}^1, \ldots, G_{\mathbf{k}}^d)$.
Its projection on the subsapce of homogeneous polynomials of order n and of order $\le N$ respectively are denoted with

$$\begin{aligned} [G]_n(z, w) &= \sum_{k_1 + \cdots + k_{2d} = n} G_{\mathbf{k}} z_1^{k_1} w_1^{k_2} \cdots z_d^{k_{2d-1}} w_d^{k_{2d}} \ , \ n \ge 1 \, , \\ [G]_{\le N} &= \sum_{n=1}^{N} [G]_n \ , \ N \ge 1 \, . \end{aligned} \tag{2.7}$$

The same notation is used for the projections of holomorphic maps of $\mathbf{C}^{2d}$ with values in $\mathbf{C}$.

2. 3. DIOPHANTINE CONDITION

We assume the linear part of the map (2.4) to be non-resonant ($\mathbf{k} \cdot \omega = 0$ and $\mathbf{k} \in \mathbf{Z}^d \Rightarrow \mathbf{k} = 0$) and furthermore we require that the frequencies ω satisfy a Diophantine condition, i.e. for some constants $\gamma > 0$ and $\eta \geq d$ one has

$$|e^{i\mathbf{k}\cdot\omega} - 1| \geq \gamma^{-1}|\mathbf{k}|^{-\eta} \quad \text{for all } \mathbf{k} \in \mathbf{Z}^d \setminus \{0\} \tag{2.8}$$

where $|\mathbf{k}| = \sum_{j=1}^{d} |k_j|$. Without loss of generality, in the sequel we will assume that $\gamma > 1$.

2. 4. NORMAL FORMS

We say that the map G is a normal form of degree 1 if

$$G(e^{i\omega}z, e^{-i\omega}z^*) = e^{i\omega}G(z, z^*) \,. \tag{2.9}$$

If ω is *non resonant* and the map is *symplectic*, (2.9) implies that G is the direct product of rotations in the planes $z_1, \ldots, z_d$ with variable real frequencies $\mathbf{\Omega} = \mathbf{\Omega}(zz^*) \equiv \Omega(|z_1|^2, \ldots, |z_d|^2)$ such that $\partial\Omega^j/\partial\rho_k = \partial\Omega^k/\partial\rho_j$ with $\rho_j = z_j z_j^*$ for $j, k = 1, \ldots, d$. As a consequence

$$G(z, z^*) = e^{i\mathbf{\Omega}(zz^*)}z \,, \tag{2.10}$$

We shall denote with Π_1 the projector on non-resonant normal forms of degree 1. If G is not in normal form then if $\mathbf{k} \in \mathbf{N}^{2d}$, $|\mathbf{k}| = n$, and $G^j_{\mathbf{k}}$ are the Taylor coefficients of its j-th component, $1 \leq j \leq d$, we have

$$(\Pi_1 G)^j_{\mathbf{k}} = \begin{cases} G^j_{\mathbf{k}}, & \text{if } k_{2m-1} = k_{2m} \text{ for } m \neq j \text{ and } k_{2j-1} = k_{2j} + 1 \,; \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

We say that a map G is *in normal form of degree* 0 if

$$G(e^{i\omega}z, e^{-i\omega}z^*) = G(z, z^*) \,. \tag{2.12}$$

If ω is non resonant, the projector Π_0 on the non-resonant normal form of degree 0, acts on a generic map G according to :

$$(\Pi_0 G)^j_{\mathbf{k}} = \begin{cases} G^j_{\mathbf{k}} & \text{if } k_{2m-1} = k_{2m} \text{ for all } 1 \leq m \leq d; \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

2. 5. NORMS

For any map G analytic in the polydisk D_r with a Taylor expansion $G = \sum_{n=1}^{\infty} [G]_n$ we consider the following norms [34,35,36]

$$
\begin{aligned}
\|[G^j]_n\|_r \; &:= r^n \sum_{k_1+\cdots+k_{2d}=n\,,\,k_i\geq 0} |G^j_{\mathbf{k}}|\,, \\
\|G^j\|_r \; &:= \sum_{n=1}^{+\infty} \|[G^j]_n\|_1 r^n = \sum_{n=1}^{+\infty} \|[G^j]_n\|_r\,, \\
\|G\|_r \; &:= \max_{1\leq j\leq d} \|G^j\|_r\,.
\end{aligned}
\tag{2.14}
$$

The given symplectic map (2.4) is assumed to be holomorphic in a polydisk D_{r_F} for some r_F. With a simple scaling of the variables $z \to z/r_F$ we obtain a new map analytic in the unit polydisk. Scaling with some $r \leq r_F$ the following conditions can always be fulfilled

$$
\begin{aligned}
&\|[F]_1\|_1 = 1 \qquad \|[F]_2\|_1 \leq 3^{-\eta}\gamma^{-1} < 1 \\
&\|[F]_n\|_1 \leq 1, \quad n \geq 2.
\end{aligned}
\tag{2.15}
$$

2. 6. MAIN RESULT

Under assumptions (2.8) and (2.15) the following exponential estimate for the stability of the orbits under iteration of the map F is obtained.

THEOREM . *Let $\rho_\star = 2/(49\gamma)$, and $\rho \leq \rho_\star(3e)^{-(9+\eta)}$. If $\|z_0\| \leq \rho/2$ and $z_t = F(z_{t-1}, z^*_{t-1})$, $t \in \mathbf{Z}, t \geq 1$, then $\|z_t\| \leq (1+\frac{1}{24})\rho$ for all $t \leq T$, where*

$$
T = \frac{7}{8}\sqrt{\frac{\rho}{6}} \exp\left[\frac{9+\eta}{2e}\left(\frac{\rho_\star}{\rho}\right)^{\frac{1}{9+\eta}}\right] . \tag{2.16}
$$

2. 7. OUTLINE OF THE PROOF

The proof of the theorem consists of four steps.
First (section 2) one considers the formal solution of the functional equation connecting the given map with its normal form. The existence of a symplectic solution was proved in reference [27].
In the second step (section 3) the norms of the homogeneous polynomials defining the conjugation function and the normal forms are estimated up to a given truncation order N using Cauchy's method.
In the third step (section 4) a bound is given for the norm of remainder of the functional equation in a disc D_r.
In the fourth and last step (section 5) the iterations of the map are considered and related to the iterations of the normal form through an "iterated remainder" whose norm is estimated. The optimization in N of the remainder norm in $D_{r/2}$ and the requirement that it is also smaller than $r/2$ gives the final Nekhoroshev estimate once the result is tranformed back from the normal coordinates to the initial ones.

2. 8. THE FUNCTIONAL EQUATION

The key point of the proof is the transformation through a change of coordinates close to the identity

$$z = \Phi(\zeta, \zeta^*) = \zeta + \sum_{n \geq 2} [\Phi]_n(\zeta, \zeta^*) \tag{2.17}$$

of the given map F to a new map which is still symplectic but in normal form

$$\zeta' = U(\zeta, \zeta^*) \equiv e^{i\Omega(\zeta\zeta')}\, \zeta. \tag{2.18}$$

The obvious reason is that the iterations of (2.18) are trivial. Formally the problem is solved by the functional equation

$$F(\Phi(\zeta, \zeta^*), \Phi^*(\zeta, \zeta^*)) = \Phi(e^{i\boldsymbol{\Omega}(\zeta \cdot \zeta^*)}\zeta, e^{-i\boldsymbol{\Omega}(\zeta \cdot \zeta^*)}\zeta^*)\,. \tag{2.19}$$

Since no analytic solutions are known to exist in a neighbourhood of the origin we replace Φ and Ω with polynomials of order $\leq N$, namely with $[\Phi]_{\leq N}$ and $[\Omega]_{\leq N}$, which are determined by requiring that the functional equation is still satisfied up to order N. As a consequence we replace (2.19) with

$$[\ F \circ [\Phi]_{\leq N} - [\Phi]_{\leq N} \circ e^{i[\Omega]_{\leq N}}\ ]_n = 0 \qquad 2 \leq n \leq N\,, \tag{2.20}$$

where we have adopted the compact notation $(F \circ \Phi)(\zeta, \zeta^*) = F(\Phi(\zeta, \zeta^*), \Phi^*(\zeta, \zeta^*))$ and $(\Phi \circ e^{i\boldsymbol{\Omega}})(\zeta, \zeta^*) = \Phi(\zeta e^{i\boldsymbol{\Omega}(\zeta \cdot \zeta^*)}, \zeta^* e^{-i\boldsymbol{\Omega}(\zeta \cdot \zeta^*)})$. For $n > N$ the functional equation is no longer satisfied and one is left with a remainder R_N which is a sum of polynomials of degree $\geq N+1$

$$R_N \ := F \circ [\Phi]_{\leq N} - [\Phi]_{\leq N} \circ e^{i[\boldsymbol{\Omega}]_{\leq N}}\,. \tag{2.21}$$

For $n \leq N$, let us introduce the linear operator Δ defined by

$$\Delta[\Phi]_n = [\Phi]_n \circ e^{i\omega} - e^{i\omega} \circ [\Phi]_n \tag{2.22}$$

and notice that the relation between U and $[\Omega]_{\leq N}$ is

$$[U]_n = [e^{i[\boldsymbol{\Omega}]_{\leq N}}\zeta]_n\,. \tag{2.23}$$

It is convenient to remark that the null space of Δ is the space of normal form of degree 1 and for later convenience we write explicitly that

$$\Pi_1[U]_n = [U]_n\,, \tag{2.24}$$

$$\Pi_1\Delta[\Phi]_n = 0\,. \tag{2.25}$$

2. 9. THE RECURRENCE

We can extract an explicit recurrence from equation (2.20) according to

$$[U]_n + \Delta[\Phi]_n = [S]_n\,, \tag{2.26}$$

where $[S]_n$ is defined by

$$[S]_n = [F \circ [\Phi]_{\leq N} - [\Phi]_{\leq N} \circ e^{i[\Omega]_{\leq N}}]_n + \Delta[\Phi]_n + [U]_n\,, \tag{2.27}$$

and manifestly depends only on $[F]_k$, $[\Phi]_k$ and $[U]_k$ with $1 \leq k \leq n-1$.

The solution of the recurrence is obtained by projecting the equation (2.26) on the subspace of normal forms of degree 1 and its complement according to

$$[U]_n = \Pi_1[S]_n \tag{2.28}$$

$$\Delta(1-\Pi_1)[\Phi]_n = (1-\Pi_1)[S]_n\,, \tag{2.29}$$

The linear operator Δ, (which commutes with Π_1) can now be inverted since its null space is the space of normal forms and we write

$$(1-\Pi_1)[\Phi]_n = \Delta^{-1}(1-\Pi_1)[S]_n\,. \tag{2.30}$$

The explicit expression of the r.h.s. reads

$$(\Delta^{-1}(1-\Pi_1)[S^j]_n)(\zeta,\zeta^*) = \sum_{k_1+\cdots+k_{2d}=n}' [e^{i(\omega_1(k_1-k_2)+\cdots+\omega_d(k_{2d-1}-k_{2d}))} - e^{i\omega_j}]^{-1}$$
$$S^j_{\mathbf{k}} z_1^{k_1} z_1^{*k_2} \cdots z_d^{k_{2d-1}} z_d^{*k_{2d}} \quad 1 \leq j \leq d \tag{2.31}$$

where in the sum $\sum'$ the coefficients for which the square bracket vanishes, namely $k_1 = k_2, \ldots, k_{2j-1} = k_{2j}+1, \ldots, k_{2d-1} = k_{2d}$, are excluded.

2. 10. SIMPLECTICITY

We notice that the normal form components of the transformation $\Pi_1[\Phi]_n$ are undetermined and at the same time the recurrence does not garantee that $[\Phi]_{\leq N}$ and $[U]_{\leq N}$ are symplectic, namely that equation (2.2) is satisfied up to order N.

For area preserving maps $d=1$, one can determine $\Pi_1[\Phi]_{\leq N}$ by imposing that the Π_0 projection of the jacobian of Φ vanishes at any order. It is then proved that $[\Phi]_{\leq N}$ and $[U]_{\leq N}$ are symplectic up to order N so that $[\Omega]_{\leq N}$ turns out to be real [27].

For $d > 1$ the $\Pi_1[\Phi]_{\leq N}$ is overconstrained and a direct construction is not workable. However using a symplectic generator, the existence of a solution $\Pi_1[\Phi]_{\leq N}$, symplectic up to order N was proved [27]; as a consequence the existence of a normal form $[U]_{\leq N}$ symplectic up to order N is also proved, and the reality of $[\Omega]_{\leq N}$ follows.

Moreover the functional equation (2.19) has a solution (Φ, U) with U in normal form according to (2.18) also if we impose that Φ verifies a condition weaker than symplecticity. Actually it is straightforward to see that if the couple (Φ, U) satisfies the equation (2.19), the couple (Φ', U) where we have defined

$$\Phi' = e^{i\Xi(\zeta\zeta^*)} \circ \Phi \tag{2.32}$$

with $\Xi(\zeta\zeta^*)$ real, is also a solution. This is a trivial consequence of the fact that the maps in normal form commute with all the rotations in the coordinate planes. It is also possible to prove that in this way we obtain all the possible perturbative solutions of eq. (2.19).

If the initial transformation Φ is symplectic, then one can easily see that the new trasformation Φ' satisfies the Poisson brackets

$$\{\Phi'^j, \Phi'^{*j}\} = 1 \qquad j = 1...d \tag{2.33}$$

where

$$\{\ ,\ \} = \sum_{s=1}^{d} \frac{\partial}{\partial\zeta_s}\frac{\partial}{\partial\zeta_s^*} - \frac{\partial}{\partial\zeta_s^*}\frac{\partial}{\partial\zeta_s}\ , \tag{2.34}$$

We recall that if the functions Ξ satisfy the relations $\partial\Xi^j/\partial\rho_k = \partial\Xi^k/\partial\rho_j$ with $\rho_j = \zeta_j\zeta_j^*$ for $j, k = 1, \dots, d$ then the transformation Φ' is a real symplectic map.

When we replace Φ' with its expansion up to order N in eq. (2.32) and (2.33) and project the resulting equations respectively with Π_1 and Π_0, observing that $\Pi_0\partial\Phi/\partial\zeta = \partial\Pi_1\Phi/\partial\zeta$, we obtain the relations

$$\Pi_1[\Phi'^j]_N = i\zeta_j[\Xi^j]_{N-1} + \Pi_1[\Phi^j]_N + \dots. \tag{2.35}$$

$$\begin{aligned}\frac{\partial}{\partial\zeta_i}\Pi_1[\Phi^i]_n + \frac{\partial}{\partial\zeta_i^*}\Pi_1[\Phi^{*i}]_n = \\ = \Pi_0\sum_{l=2}^{n-1}\sum_{s=1}^{d} -\frac{\partial}{\partial\zeta_s}[\Phi^i]_l\frac{\partial}{\partial\zeta_s^*}[\Phi^{*i}]_{n+1-l} + \frac{\partial}{\partial\zeta_s^*}[\Phi^i]_l\frac{\partial}{\partial\zeta_s}[\Phi^{*i}]_{n+1-l}\end{aligned} \tag{2.36}$$

where $[\Xi^j]_{N-1}$ is an arbitrary real function. The equation (2.35) means that the immaginary part of $\Pi_1[\Phi^j]_N$ can be choosen arbitrarily. Then we estimate the remaining coefficients $\Pi_1[\Phi^j]_N$ by means of eq. (2.36) together with the constraints

$$\frac{\partial}{\partial\zeta_i}\Pi_1[\Phi^i]_n = \frac{\partial}{\partial\zeta_i^*}\Pi_1[\Phi^{*i}]_n \tag{2.37}$$.

which remove completely the freedom left in the choice of the symplectic transformation Φ according to (2.35). This completes the construction of the final perturbative solution of the conjugation equation.

3. Estimate of the norm of the Birkhoff series coefficients

3. 1. PRELIMINARIES

In order to obtain an estimate of the norms $\|(1-\Pi_1)[\Phi]_n\|_1$ and $\|[U]_n\|_1$ for $n \le N$ we use the conjugation equation, namely (2.27), (2.28) and (2.30); in order to estimate $\|\Pi_1[\Phi]_n\|$ the simplecticity conditions (2.36) and (2.37).
Since the composition of holomorphic functions or polynomials frequently occurs, we state the following result whose proof is straightforward and is left to the reader. Let P and Q be two maps : $\mathbf{C}^{2d} \mapsto \mathbf{C}^d$ holomorphic in a neighborhood of the origin and such that $P(0) = Q(0) = 0$; for the composed map $Q \circ P$ the following estimate holds

$$\|[Q \circ P]_n\|_1 \le \sum_{s=1}^{n} \|[Q]_s\|_1 \sum_{\substack{m_1+\ldots+m_s=n \\ m_j \ge 1}} \|[P]_{m_1}\|_1 \cdots \|[P]_{m_s}\|_1 \,. \tag{3.1}$$

In order to bound the r.h.s. of equation (2.31) using the diophantine condition (2.8) it is convenient to introduce the notation

$$\sigma_n = 2\gamma(n+1)^\eta \tag{3.2}$$

so that

$$\max_{k_1+\ldots+k_{2d}=n,k_i\ge 0} \left|e^{i(\omega_1(k_1-k_2)+\ldots+\omega_d(k_{2d-1}-k_{2d}))} - e^{i\omega_j}\right|^{-1} \le \frac{\sigma_n}{2} \tag{3.3}$$

and consequently from (2.28) and (2.30) we have

$$\|[U]_n\|_1 \le \|[S]_n\|_1 \le \sigma_n \|[S]_n\|_1 \tag{3.4}$$

$$\|(1-\Pi_1)[\Phi]_n\|_1 \le \frac{\sigma_n}{2} \|[S]_n\|_1 . \tag{3.5}$$

since $\sigma_n > 1$.

3. 2. THE INDUCTION

The estimates on the norms of $[U]_n$ and $[\Phi]_n$ are obtained with an inductive procedure. A central role is played by a numeric sequence $\{\mu_n\}$ which satisfies the following recursion

$$\mu_n = \sum_{s=2}^{n-1} \mu_s \sum_{\substack{m_1+\ldots+m_s=n \\ m_j \ge 1}} \mu_{m_1} \cdots \mu_{m_s} \,. \tag{3.6}$$

initialized by

$$\mu_1 = \mu_2 = 1 \tag{3.7}$$

Lemma. *With the above definitions and assuming that (2.8) and (2.15) are satisfied, we have*

$$\|[U]_1\|_1 = 1 \qquad \|[U]_n\|_1 \le (3\sigma_N)^{n-2}\mu_n \qquad 2 \le n \le N \tag{3.8}$$

$$\|[\Phi]_n\|_1 = 1 \qquad \|[\Phi]_n\|_1 \le (3\sigma_N)^{n-2}\mu_n \qquad 2 \le n \le N. \tag{3.9}$$

Proof. The proof is by induction.
The statement is true for $n = 2$ since $[S]_2 = [F]_2$ and the subspace of normal forms is empty for polynomials of even degree so that $\|[U]_2\|_1 = 0$ and from (2.16) and (3.5) one has $\|[\Phi]_2\|_1 \le 1$.
If $n > 2$ we suppose (3.8) to be satisfied at any lower order namely $\|[\Phi]_l\|_1 \le (3\sigma_l)^{l-2}\mu_l$ and $\|[U]_l\|_1 \le (3\sigma_N)^{l-2}\mu_l$ for $2 \le l < n \le N$. Starting from the definition of $[S]_n$ (2.27) and using (3.1), we find

$$\begin{aligned} \chi_n \equiv \sigma_n\|[S]_n\|_1 \le \sigma_n\|[F]_n\|_1 \\ + \sigma_n\sum_{s=2}^{n-1}\|[F]_s\|_1 \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} \|[\Phi]_{m_1}\|_1\cdots\|[\Phi]_{m_s}\|_1 \\ + \sum_{s=2}^{n-1}\|[\Phi]_s\|_1 \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} \|[U]_{m_1}\|_1\cdots\|[U]_{m_s}\|_1 \,. \end{aligned} \tag{3.10}$$

By the inductive hypotesis and using (2.15) we obtain

$$\begin{aligned} \chi_n \le \sigma_n + \sigma_n\sum_{s=2}^{n-1} \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} (3\sigma_N)^{n-2s+k(\mathbf{m})}\mu_{m_1}\cdots\mu_{m_s} + \\ + \sigma_n\sum_{s=2}^{n-1}(3\sigma_N)^{s-2}\mu_s \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} (3\sigma_N)^{n-2s+k(\mathbf{m})}\mu_{m_1}\cdots\mu_{m_s} \le \\ \le \sigma_n\left(1 + 2(3\sigma_N)^{(n-3)}\sum_{s=2}^{n-1}\mu_s \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} \mu_{m_1}\cdots\mu_{m_s}\right). \end{aligned} \tag{3.11}$$

In the above equations $k(\mathbf{m})$ denotes the number of indices m_j which are equal to 1, since the estimates (3.8) at order 1 and higher orders are different. Taking into account that $0 \leq k(\mathbf{m}) \leq s-1$ because of the costraint $m_1 + \ldots + m_s = n$, and using the recurrence (3.6) for the μ_s we have

$$\begin{aligned} \chi_n &\leq \sigma_n(1 + 2(3\sigma_N)^{(n-3)}\mu_n) \\ &\leq (3\sigma_N)^{(n-2)}\mu_n \left[\frac{2}{3} + \frac{1}{3\mu_n(3\sigma_N)^{(n-3)}}\right] \leq \mu_n(3\sigma_N)^{(n-2)} \end{aligned} \tag{3.12}$$

since $\mu_n(3\sigma_N)^{(n-2)} > 1$.

From (3.4) and (3.12) the inequality (3.8) follows.

From (3.5) and (3.12) we obtain $\|(1-\Pi_1)[\Phi]_n\|_1 \leq \frac{1}{2}\mu_n(3\sigma_N)^{(n-2)}$. As a consequence in order to show that (3.9) is also satisfied we have to prove that $\|\Pi_1[\Phi]_n\|_1 \leq \frac{1}{2}\mu_n(3\sigma_N)^{(n-2)}$. This is achieved using the symplecticity conditions (2.34) and (2.35).

3. 3. SIMPLECTICITY CONDITIONS

Taking the norm of (2.36) and taking into account (2.37) we obtain

$$\|\frac{\partial}{\partial\zeta_i}\Pi_1[\Phi^i]_n\|_1 \leq \sum_{l=2}^{n-1}\sum_{s=1}^{d} \|\frac{\partial}{\partial\zeta_s}[\Phi^i]_l\|_1 \|\frac{\partial}{\partial\zeta_s^*}[\Phi^{*i}]_{n+1-l}\|_1 \quad 1 \leq i \leq d. \tag{3.13}$$

Since $\Pi_1[\Phi^i]_n(\zeta = 0, \zeta^*) = 0$ it is not difficult to see that $\|\frac{\partial}{\partial\zeta_i}\Pi_1[\Phi^i]_n\|_1 \geq \|\Pi_1[\Phi^i]_n\|_1$, and eq. (3.13) then reads

$$\begin{aligned} \|\Pi_1[\Phi]_n\|_1 &\leq \sum_{l=2}^{n-1}\sum_{s=1}^{d} l(n+1-l)\|[\Phi]_l\|_1\|[\Phi]_{n+1-l}\|_1 \\ &= d\sum_{l=2}^{n-1} l(n+1-l)\|[\Phi]_l\|_1\|[\Phi]_{n+1-l}\|_1 \,. \end{aligned} \tag{3.14}$$

Using our induction hypothesis $\|[\Phi]_l\|_1 \leq (3\sigma_N)^{l-2}\mu_l$ for $2 \leq l \leq n-1$ we obtain

$$\|\Pi_1[\Phi]_n\|_1 \leq d(3\sigma_N)^{n-2}\sum_{l=2}^{n-1}\frac{l(n+1-l)}{3\sigma_N}\mu_l\mu_{n+1-l} \tag{3.15}$$

In order to obtained the desired result, all that we need to prove is

$$l\mu_{n+1-l} \leq \sum_{\substack{m_1+\ldots+m_l=n \\ m_j\geq 1}} \mu_{m_1}\cdots\mu_{m_l} \tag{3.16}$$

which follows immediatly recalling that $\mu_1 = 1$, and

$$\frac{d(n+1-l)}{3\sigma_N} = \frac{1}{2}\,\frac{d(n+1-l)}{3\gamma(N+1)^\eta} \leq \frac{1}{2}\,\frac{d(n-1)}{3(N+1)^d} \leq \frac{1}{2}\,\frac{d}{3(N+1)^{d-1}} < \frac{1}{2} \tag{3.17}$$

where (3.2) has been used and the constraints $\gamma > 1$, $\eta \geq d$ have been taken into account. As a consequence (3.13) and (3.14) yield

$$\|\Pi_1[\Phi]_n\|_1 \leq (3\sigma_N)^{n-2}\frac{1}{2}\sum_{l=2}^{n-1}\mu_l \sum_{\substack{m_1+\ldots+m_l=n \\ m_j\geq 1}} \mu_{m_1}\cdots\mu_{m_l} = (3\sigma_N)^{n-2}\frac{\mu_n}{2} \tag{3.18}$$

and finally one has

$$\|[\Phi]_n\|_1 \leq \|\Pi_1[\Phi]_n\|_1 + \|(1-\Pi_1)[\Phi]_n\|_1 \leq (3\sigma_N)^{n-2}\mu_n\ .$$

This completes the proof.

3. 4. FINAL ESTIMATES

The bounds which will be used in the next sections depend on estimates on the μ_n. These are considered in the Appendix A. Here we simply report the result

$$\mu_n \leq (n!)^3. \tag{3.19}$$

We point out that the result (3.19) could be sharpened becouse a numerical analysis of the sequence shows that $\mu_n \leq C^n(n!)$. By (3.2) we can write

$$(3\sigma_N)^{(n-2)}(n!)^3 \leq [6\gamma(N+1)^\eta]^{(n-2)}\,N^{(3n)} \leq \left[6\gamma(N+1)^{(9+\eta)}\right]^{n-2} \tag{3.20}$$

The final estimates can therefore be written as

$$\|[U]_1\|_1 = 1 \qquad \|[U]_n\|_1 \leq A^{n-2} \qquad 2\leq n\leq N \tag{3.21}$$

$$\|[\Phi]_n\|_1 = 1 \qquad \|[\Phi]_n\|_1 \leq A^{n-2} \qquad 2\leq n\leq N. \tag{3.22}$$

where

$$A = 6\gamma(N+1)^{9+\eta} \qquad A > 2\cdot 3^5 \tag{3.23}$$

The lower bound to A holds also when $9+\eta$ is replaced with $3+\eta$; this exponent would appear in (3.20) if the $n!$ estimate to μ_n, suggested by numerical evidence replaces the proved $(n!)^3$ bound. The $2\cdot 3^5$ lower bound to A will systematically be used, so that the final results will hold under the substitution $9+\eta \to 3+\eta$ once the $n!$ bound to μ_n is proved.

3. 5. INVERSE TRANSFORMATION

Together with the transformation $[\Phi]_{\leq N}$ one has to consider the inverse transformation Ψ, which exists in a neighbourhood of the origin due to the Inverse Function Theorem, and is defined by the functional equation

$$([\Phi]_{\leq N} \circ \Psi)(z, z^*) = z. \tag{3.24}$$

From (3.24) we obtain the series solution for Ψ which at order $n \leq 2$ reads

$$[\Psi]_1 = \zeta \quad [\Psi]_2 = -[\Phi]_2 \implies \|[\Psi]_1\|_1 = 1 \quad \|[\Psi]_2\|_1 \leq 1 \tag{3.25}$$

and at higher orders gives a recurrence, which, through (3.1) leads to recurrent inequalities for the norms

$$\|[\Psi]_n\|_1 \leq \|[\Phi]_n\|_1 + \sum_{s=2}^{n-1} \|[\Phi]_s\|_1 \sum_{\substack{m_1+\ldots+m_s=n \\ m_j \geq 1}} \|[\Psi]_{m_1}\|_1 \cdots\cdots \|[\Psi]_{m_s}\|_1 \tag{3.26}$$

setting $\|[\Phi]_n\|_1 = 0$ if $n > N$. Therefore using (3.22) we obtain

$$\|[\Psi]_n\|_1 \leq A^{n-2} + \sum_{s=2}^{n-1} A^{s-2} \sum_{\substack{m_1+\ldots+m_s=n \\ m_j \geq 1}} \|[\Psi]_{m_1}\|_1 \cdots\cdots \|[\Psi]_{m_s}\|_1 \, . \tag{3.27}$$

It is convenient to scale the norms $\|[\Psi]_n\|_1$ introducing the sequence y_n defined by

$$\|[\Psi]_1\|_1 = y_1 = 1, \qquad \|[\Psi]_n\|_1 = A^{n-2} y_n \qquad \text{for} \quad n \geq 2 \tag{3.28}$$

By the same arguments used to prove (3.11) we find that that the sequence y_n satisfies the recurrent inequalities

$$y_n \leq 1 + \frac{1}{A} \sum_{s=2}^{n-1} \sum_{\substack{m_1+\ldots+m_s=n \\ m_j \geq 1}} y_{m_1} \cdots\cdots y_{m_s} \qquad \text{with} \quad n > 2 \tag{3.29}$$

initialized by $y_1 = 1$, $y_2 \leq 1$.
We introduce a new sequence $\{\hat{y}_n\}$ bounding from above the $\{y_n\}$ and defined by the recursion

$$\begin{aligned} \hat{y}_1 &= \hat{y}_2 = 1, \\ \hat{y}_n &= \sum_{s=2}^{n-1} \sum_{\substack{m_1+\ldots+m_s=n \\ m_j \geq 1}} \hat{y}_{m_1} \cdots \hat{y}_{m_s} \qquad \text{for} n > 2 \end{aligned} \tag{3.30}$$

The sequence $\{\hat{y}_n\}$ is analyzed in Appendix A where we obtain the result

$$y_n \leq \hat{y}_n \leq \frac{147}{4}(7^{n-2}) \qquad \text{for all} \quad n > 2\,. \tag{3.31}$$

Finally from (3.28) and (3.31) we obtain the desired estimate for $\|[\Psi]_n\|_1$ which reads

$$\begin{cases} \|[\Psi]_n\|_1 = 1, & \text{if } n = 1,2 \\ \|[\Psi]_n\|_1 \leq \frac{147}{4}\,(7A)^{n-2}\,, & \text{with } n > 2. \end{cases} \tag{3.32}$$

4. Estimate of the remainder norm

According to (2.21) the remainder R_N gives a nonvanishing contribution at orders $n \geq N+1$ and we can write

$$[R_N]_n = \left[F \circ [\Phi]_{\leq N} - [\Phi]_{\leq N} \circ e^{i[\Omega]_{\leq N}}\zeta\right]_n \qquad \text{for} \quad n > N\,. \tag{4.1}$$

In order to bound the norm of the remainder we need first to estimate the norms $\|[\Omega]_n\|_1$ with $n \leq N$ using the estimates of $\|[U]_n\|_1$ obtained for $n \leq N$ in the previous section and then bound $\|[e^{-[\Omega]_{\leq N}}\zeta]_n\|_1$ for any value of n.

4. 1. BOUNDS TO $\boldsymbol{\Omega}$

The definition (2.23) implies that $[\Omega]_{\leq N}$ is determined by $[U]_{\leq N}$ namely

$$\zeta_j e^{i[\Omega^j]_{\leq N}(\zeta\cdot\zeta^*)} = [U^j]_{\leq N}(\zeta,\zeta^*) \;+\; O(|\zeta|^N) \qquad \text{where} \quad j = 1,\ldots,d\,, \tag{4.2}$$

and the relation can be inverted according to

$$i\left([\Omega^j]_{\leq N}(\zeta\cdot\zeta^*) - \omega^j\right) = \ln\left[1 + \left(\frac{[U^j]_{\leq N}(\zeta,\zeta^*)e^{-i\omega^j}}{\zeta_j} - 1 + O(|\zeta|^N)\right)\right] \tag{4.3}$$

where the logarithm is evaluated according to the expansion $\ln(1+x) = -\sum_{s=1}^{\infty}\frac{(-x)^s}{s}$.
The following inequality is obtained by (3.1)

$$\|[\Omega^j]_n\|_1 \leq \sum_{s=1}^{n}\frac{1}{s} \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\geq 1}} \|[U]_{m_1+1}\|_1\cdots\|[U]_{m_s+1}\|_1\,, \tag{4.4}$$

where we have taken into account that

$$\left\|\left[\frac{[U^j]_{\leq N}(\zeta,\zeta^*)e^{-i\omega^j}}{\zeta_j} - 1\right]_n\right\|_1 = \|[U^j]_{n+1}\|_1 \leq \|[U]_{n+1}\|_1\,. \tag{4.5}$$

Using the estimate (3.21) for $\|[U]_n\|_1$, equation (4.4) becomes

$$\begin{aligned} \|[\Omega]_n\|_1 &\le \sum_{s=1}^{n} \frac{1}{s} A^{n-s} \binom{n-1}{s-1} \\ &\le A^{n-1} \sum_{s=1}^{\infty} \frac{A^{-(s-1)}}{(s-1)!} (n-1)^{s-1} = \left(Ae^{\frac{1}{A}}\right)^{n-1}, \end{aligned} \tag{4.6}$$

where the binomial coefficient $\binom{n-1}{s-1}$ is the number of terms of the sum $\sum_{m_1+\ldots+m_s=n}$ (see Appendix A).

By a similar procedure, using (4.6) we find

$$\begin{aligned} \left\|[e^{i[\Omega]_{\le N}}]_n\right\|_1 &\le \sum_{s=1}^{n} \frac{1}{s!} \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} \|[\Omega]_{m_1}\|_1 \ldots \|[\Omega]_{m_s}\|_1 \le \\ &\le \sum_{s=1}^{n} \frac{1}{s!} \left(Ae^{\frac{1}{A}}\right)^{n-s} \binom{n-1}{s-1} \\ &\le \left(Ae^{\frac{1}{A}}\right)^{n-1} \sum_{s=1}^{\infty} \frac{1}{(s-1)!} \left(Ae^{\frac{1}{A}}\right)^{-(s-1)} (n-1)^{s-1} = B^{n-1} \end{aligned} \tag{4.7}$$

where

$$B = A \exp\left\{1/(Ae^{\frac{1}{A}}) + 1/A\right\}. \tag{4.8}$$

From (3.23) it follows that

$$2 \cdot 3^5 < A \le B \le 2A \tag{4.9}$$

4. 2. REMAINDER ESTIMATES

In order to estimate the remainder $\|R_N\|_r$ we first notice that

$$\left\|\left[e^{i[\Omega]_{\le N}}\zeta\right]_1\right\|_1 = 1, \quad \left\|\left[e^{i[\Omega]_{\le N}}\zeta\right]_n\right\|_1 = \left\|\left[e^{i[\Omega]_{\le N}}\right]_{n-1}\right\|_1 \le B^{n-2} \quad \text{for } n \ge 2 \tag{4.10}$$

and recall that $\|[\Phi]_n\|_1 = 0$ for $n > N$, so that the following inequality is obtained

$$\begin{aligned} \|[R_N]_n\|_1 \le \sum_{s=2}^{n-1} \|[F]_s\|_1 & \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} \|[\Phi]_{m_1}\|_1 \ldots \|[\Phi]_{m_s}\|_1 + \|[F]_n\|_1 \\ + \sum_{s=1}^{N} \|[\Phi]_s\|_1 & \sum_{\substack{m_1+\ldots+m_s=n \\ m_j\ge 1}} \left\|[e^{i[\Omega]_{\le N}}\zeta]_{m_1}\right\|_1 \ldots \left\|[e^{i[\Omega]_{\le N}}\zeta]_{m_s}\right\|_1 \end{aligned} \tag{4.11}$$

Using the estimates(2.15), (3.22) and (4.10) we obtain, with the same manipulations used in (3.11)

$$\|[R_N]_n\|_1 \le A^{n-3}\sum_{s=2}^{n-1}\binom{n-1}{s-1} + 1 + B^{n-3}\sum_{s=2}^{N}\binom{n-1}{s-1} + B^{n-2}. \tag{4.12}$$

Recalling that $\sum_{s=0}^{n-1}\binom{n-1}{s} = 2^{n-1}$, taking (4.9) into account, we find for $n-1 \ge N \ge 2$

$$\begin{aligned}\|[R_N]_n\|_1 &\le 2^{n-1}\left(A^{n-3}+B^{n-3}\right)+1+B^{n-2}\\ &\le (2B)^{n-2}\left(\frac{2}{B}+\frac{2}{A}+\frac{1}{2B}+\frac{1}{2}\right) \le (2B)^{n-2}\end{aligned} \tag{4.13}$$

Then we have immediately

$$\|R_N\|_r \le \frac{1}{(2B)^2}\sum_{n=N+1}^{\infty}(2Br)^n = \frac{1}{(2B)^2}\frac{(2Br)^{N+1}}{1-2Br}, \qquad \text{for } r < \frac{1}{2B}. \tag{4.14}$$

4. 3. A NEW REMAINDER

In order to compare the normal form dynamics and the true dynamics the transformation $\Psi = [\Phi]_{\le N}^{-1}$ must be applied to both members of the equation

$$F \circ [\Phi]_{\le N} = [\Phi]_{\le N} \circ e^{i[\Omega]_{\le N}}\zeta + R_N\,, \tag{4.15}$$

obtaining

$$G\ := \Psi \circ F \circ [\Phi]_{\le N} = e^{i[\Omega]_{\le N}} + \hat{R}_N\,. \tag{4.16}$$

where $\hat{R}_N$ is the new remainder defined by

$$\hat{R}_N\ := \Psi \circ \left([\Phi]_{\le N} \circ e^{i[\Omega]_{\le N}}\zeta + R_N\right) - \Psi \circ [\Phi]_{\le N} \circ e^{i[\Omega]_{\le N}}\zeta. \tag{4.17}$$

In order to estimate the new remainder we first evaluate the norms of the functions F, Φ and Ψ respectively.

4. 4. SOME ESTIMATES

Using (2.15) we immediatly have

$$\|F\|_r \le \frac{r}{1-r} \qquad r < r_F = 1, \tag{4.18}$$

using (3.22) we obtain

$$\|[\Phi]_{\le N}\|_r \le r + \frac{r^2}{1-Ar} \qquad r < r_\Phi = 1/A \tag{4.19}$$

and finally using (3.32) we get

$$\|\Psi\|_r \leq r + r^2 + \frac{147}{4} r^2 \frac{7Ar}{1-7Ar} \qquad r < r_\Psi = 1/(7A). \tag{4.20}$$

One can notice now that the composition $\Psi \circ F \circ [\Phi]_{\leq N}$ is well defined if we choose

$$r \leq \frac{1}{9B} \tag{4.21}$$

Indeed from (4.19) it follows

$$\|[\Phi]_{\leq N}\|_r \leq \|[\dot{\Phi}]_{\leq N}\|_{\frac{1}{9B}} \leq \frac{1}{8B}\left(\frac{8}{9} + \frac{1}{9B}\right) < \frac{1}{8B} \tag{4.22}$$

and from (4.18)

$$\|F \circ [\Phi]_{\leq N}\|_r \leq \|F\|_{\|[\Phi]_{\leq N}\|_r} \leq \|F\|_{\frac{1}{8B}} \leq \frac{1}{8B-1} < \frac{1}{7A}\,. \tag{4.23}$$

We wish also to check that

$$\left\|[\Phi]_{\leq N} \circ e^{i[\Omega]_{\leq N}}\zeta + R_N\right\|_r < r_\Psi \qquad \text{with} \quad r \leq \frac{1}{9B}. \tag{4.24}$$

In fact using (4.14) and (4.22) we have

$$\begin{aligned} &\left\|[\Phi]_{\leq N} \circ e^{i[\Omega]_{\leq N}}\zeta + R_N\right\|_r \leq \|[\Phi]_{\leq N}\|_{\frac{1}{9B}} + \|R_N\|_{\frac{1}{9B}} \\ &\frac{1}{8B}\left(\frac{8}{9} + \frac{1}{9B}\right) + \frac{1}{4B^2}\frac{9}{7}\left(\frac{2}{9}\right)^{N+1} \leq \frac{1}{8B} \end{aligned} \tag{4.25}$$

where we have used $N \geq 2$ and $\left\|e^{i[\Omega]_{\leq N}}\zeta\right\|_r = 1$, which is a consequence of the reality of the frequency Ω.

4. 5. NORM OF THE NEW REMAINDER

In order to bound $\hat{R}_N$ we have to use a Cauchy estimate. If f is analytic in D_ρ and g, h are analytic in D_r with $r < \rho$ then

$$\|f \circ (g+h) - f \circ g\|_r \leq \frac{\|f\|_{\|g\|_r + \|h\|_r + \varepsilon}\|h\|_r}{\varepsilon} \tag{4.26}$$

provided that $\|g\|_r + \|h\|_r + \varepsilon \leq \rho$. The proof is given in appendix B.
As a consequence letting $\rho = \|[\Phi]_{\leq N}\|_r + \|R_N\|_r$ and choosing still $r \leq 1/(9B)$ we have $\rho \leq 1/(8B)$, see (4.25), and consequently

$$\|\hat{R}_N\|_r \leq \frac{1}{\varepsilon}\|\Psi\|_{\rho+\varepsilon}\|R_r\|_r. \tag{4.27}$$

Since we must have $\rho + \varepsilon < r_\Psi = 1/(7A)$ we choose

$$\varepsilon = \frac{1}{112B} \Longrightarrow \rho + \varepsilon \leq \frac{1}{112B} + \frac{1}{8B} = \frac{15}{112B} < \frac{1}{7B} < \frac{1}{7A} = r_\Psi \tag{4.28}$$

A straightforward calculation using (4.20) gives

$$\|\Psi\|_{\rho+\epsilon} \leq \|\Psi\|_{\frac{15}{112B}} \leq \frac{18}{112B}. \tag{4.29}$$

Finally from (4.27), (4.29) and (4.9),(4.13) we obtain the estimate of $\|\hat{R}_N\|_r$

$$\|\hat{R}_N\|_r \leq 18\|R_N\|_r \leq \frac{18}{4B^2}\,\frac{9}{7}(2Br)^{N+1} \;\leq (4Ar)^{N+1}. \tag{4.30}$$

5. The Nekhoroshev's Estimate

5. 1. ITERATED REMAINDERS

Here we first write a recurrence for the remainders of the map iterated $t \leq T$ times. Iterating once the function G defined by (4.16) we obtain

$$\begin{aligned} G \circ G = \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right) = e^{2i[\Omega]_{\leq N}}\zeta \\ + \left[\left(e^{i[\Omega]_{\leq N}}\zeta\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right) - \left(e^{i[\Omega]_{\leq N}}\zeta\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta\right)\right] + \\ + \left[\hat{R}_N \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right)\right] = e^{2i[\Omega]_{\leq N}}\zeta + \hat{R}_N(2) \end{aligned} \tag{5.1}$$

where $\hat{R}_N(2)$ is the sum of the two terms between square brackets. By induction we easily have

$$\begin{aligned} G^{\circ(t+1)} = G^{\circ(t)} \circ G = \left(e^{it[\Omega]_{\leq N}}\zeta + \hat{R}_N(t)\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right) = \\ = e^{i(t+1)[\Omega]_{\leq N}}\zeta + \hat{R}_N(t+1) \end{aligned} \tag{5.2}$$

where we define the remainder after $t+1$ iterations as

$$\begin{aligned} \hat{R}_N(t+1) = \left[\left(e^{ti[\Omega]_{\leq N}}\zeta\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right) - \left(e^{it[\Omega]_{\leq N}}\zeta\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta\right)\right] \\ + \left[\hat{R}_N(t) \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right)\right]. \end{aligned} \tag{5.3}$$

Since $\hat{R}_N$ is defined in a disc D_r with $r \leq 1/(9B)$ and a fortiori for

$$r \leq \frac{1}{18A} \leq \frac{1}{9B}, \tag{5.4}$$

the next remainders will be defined in smaller domains, namely $\hat{R}_N(t)$ will be defined in $D_{r_{t-1}}$ where r_t is a decreasing sequence on which we require that, for instance, $r_T = r/2$.
As a consequence we can choose

$$r_0 = r,\ \ r_1 = r - \frac{r}{2T},\ \ldots.\ \ r_t = r_{t-1} - \frac{r}{2T} = r - \frac{rt}{2T},\ \ldots\ \ r_T = \frac{r}{2} \tag{5.5}$$

We use now the Cauchy estimates (4.26) in order to evaluate the norm of $\hat{R}_N(t)$. The norm of the first term in (5.3) is bounded by

$$\begin{aligned}&\left\|\left(e^{it[\Omega]_{\leq N}}\zeta\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right) - \left(e^{it[\Omega]_{\leq N}}\zeta\right) \circ \left(e^{i[\Omega]_{\leq N}}\zeta\right)\right\|_{r_t} \\ &\leq \frac{1}{\varepsilon}\ \|e^{it[\Omega]_{\leq N}}\zeta\|_{r_t+\|\hat{R}_N\|+\varepsilon}\ \|\hat{R}_N\|_{r_t} = \frac{r_t + \varepsilon + \|\hat{R}_N\|_{r_t}}{\varepsilon}\ \|\hat{R}_N\|_{r_t}\end{aligned} \tag{5.6}$$

Of course we must impose that $r_t + \|\hat{R}_N\|_{r_t} + \varepsilon \leq r_{t-1}$. A possibility is to choose, taking into account (4.30)

$$\varepsilon = \frac{r_{t-1} - r_t}{2} \qquad \|\hat{R}_N\|_{r_t} \leq \|\hat{R}_N\|_r \leq (4Ar)^{N+1} \leq \frac{r_{t-1} - r_t}{2} \tag{5.7}$$

With this condition the l.h.s. of equation (5.6) is bounded by $2r_{t-1}/(r_{t-1} - r_t)\|\hat{R}_N\|_r$. By (5.5) we have also

$$\left\|\hat{R}_N(t) \circ \left(e^{i[\Omega]_{\leq N}}\zeta + \hat{R}_N\right)\right\|_{r_t} \leq \|\hat{R}_N(t)\|_{r_t+\|\hat{R}_N\|_{r_t}} \leq \|\hat{R}_N(t)\|_{r_{t-1}} \tag{5.8}$$

Combining the last results finally we obtain

$$\begin{aligned}\|\hat{R}_N(t+1)\|_{r_t} &\leq 2\frac{r_{t-1}}{r_{t-1} - r_t}\|\hat{R}_N\|_r + \|\hat{R}_{\tilde{N}}(t)\|_{r_{t-1}} = \\ &= 2(2T - t + 1)\|\hat{R}_N\|_r + \|\hat{R}_{\tilde{N}}(t)\|_{r_{t-1}}\end{aligned} \tag{5.9}$$

The recurrent inequalities are immediatly solved and one obtains

$$\|\hat{R}_N(t)\|_{r_{t-1}} \leq \left[\frac{1}{2} + (2T+1)(t-1) - \frac{t(t-1)}{2}\right] 2\|\hat{R}_N\|_r \leq 4T^2\|\hat{R}_N\|_r \tag{5.10}$$

for $2 \leq t \leq T$.

5. 2. MINIMIZATION OF THE REMAINDERS

The optimization is obtained by choosing for any fixed r satisfying (5.4) a truncation order $N = \tilde{N}$ which makes the remaninder $\|\hat{R}_N\|_r$ minimum. For this purpose it is convenient to rewrite $\|\hat{R}_N\|_r$ as follows; recalling (4.30) and (3.23)

$$\|\hat{R}_N\|_r \leq f(N,r) \equiv (4Ar)^{N+1} = \left[\frac{r}{r_*}(N+1)^{9+\eta}\right]^{N+1} \qquad N \geq 2 \tag{5.11}$$

where we have defined

$$r_* = \frac{1}{24\gamma}. \tag{5.12}$$

The inequality (5.4) now reads

$$\frac{r}{r_*} \leq \frac{2}{9}(N+1)^{-9-\eta}. \tag{5.13}$$

The minimum of $f(N,r)$ is obtained for $N = \bar{N}$ where

$$\bar{N} = \frac{1}{e}\left(\frac{r^*}{r}\right)^{\frac{1}{9+\eta}} - 1 \tag{5.14}$$

and the corresponding value of f is

$$f(\bar{N}, r) = \exp\left[-(9+\eta)\frac{1}{e}\left(\frac{r_*}{r}\right)^{\frac{1}{9+\eta}}\right] \tag{5.15}$$

We observe that writing (5.14) as

$$\frac{r}{r_*} = \left(\frac{1}{e}\right)^{9+\eta} (\bar{N}+1)^{-9-\eta} \tag{5.16}$$

condition (5.13) is trivially satisfyed for any $\bar{N} \geq 2$. Indeed we have only to insure that the minimum effectively occurs for for $\bar{N} \geq 2$. This gives the only true constraint on r which reads from (5.14)

$$\bar{N} \geq 2 \quad \Longrightarrow \quad r \leq \frac{r_*}{(3e)^{9+\eta}} \tag{5.17}$$

5. 3. ADDITIONAL CONSTRAINTS

Let us observe now that equation (5.7), which bounds $\|\hat{R}_N\|_r$ in order that the compositions occuring in the remainders $\hat{R}_N(t)$ are well defined, now reads

$$f(\bar{N}, r) \leq \frac{r}{4T} \tag{5.18}$$

We should also notice that the image by the map $e^{it[\Omega]\leq N}\zeta + \hat{R}_N(t)$ of a point in $D_r/2$ is well defined for any $t \leq T$ and will remain confined in D_r if we require that the norm of the remainder $\hat{R}_N(t)$ does not exceed $r/2$. This is essentially the stability condition which, according to (5.10) can be written

$$\|\hat{R}_N(t)\|_{r_{t-1}} \leq 4T^2 f(\bar{N}, r) \leq \frac{r}{2} \quad \Longrightarrow \quad f(\bar{N}, r) \leq \frac{r}{8T^2} \quad 1 \leq t \leq T \tag{5.19}$$

Condition (5.19) fixes the optimal stability time T : $\zeta \in D_{r/2} \Rightarrow \zeta_t \in D_r \;\; 1 \leq t \leq T$

$$T = \sqrt{\frac{r}{8f(\bar{N}, r)}} = \sqrt{\frac{r}{8}}\exp\left[\frac{9+\eta}{2e}\left(\frac{r_*}{r}\right)^{\frac{1}{9+\eta}}\right] \tag{5.20}$$

5. 4. FINAL STABILITY CONDITION

Now for any fixed N the stability condition is fulfilled if any orbit with initial point $z \in D_{\rho/2}$ remains up to time T in a disc $D_{\rho'}$, with $\rho' > \rho$, namely that $z_t \in D_{\rho'}$ for $t \leq T$. If we consider the image $\zeta, \ldots, \zeta_t, \ldots, \zeta_T$ of the orbit $z, \ldots, z_t, \ldots, z_T$ by the tranformation Ψ, then the previous statement becomes : that $\zeta \in D_{r/2}$ implies $\zeta_t \in D_{r'}$ for $t \leq T$.
We have seen above how to fix T so that $r' = r$. Let us now recall that

$$\Psi \circ F^{o(t)} \circ [\Phi]_{\leq N} = e^{-t[\Omega]_{\leq N}} \zeta + \hat{R}_N(t) \tag{5.21}$$

and that

$$\|\zeta\| = \|\Psi\|_{\frac{\rho}{2}} \leq \frac{49}{48} \frac{\rho}{2} \qquad \text{for} \quad \rho \leq \frac{1}{7A} = r_\Psi \tag{5.22}$$

and also that

$$\|z\| = \|[\Phi]_{\leq N}\|_{r'} \leq \frac{145}{144} r' \qquad \text{for} \quad r' \leq \frac{1}{2A} = \frac{r_\Phi}{2} \tag{5.23}$$

Le us choose now

$$\frac{r}{2} = \frac{49}{48} \frac{\rho}{2} \qquad r' = r \tag{5.24}$$

which implies

$$\rho' = \frac{145}{144} r' \leq \left(1 + \frac{1}{24}\right) \rho \tag{5.25}$$

Introducing the variable

$$\rho_* = \frac{48}{49} r_* = \frac{2}{49\gamma} \tag{5.26}$$

the optimal stability time T : $z \in D_{\rho/2} \Rightarrow z_t \in D_{(1+\frac{1}{24})\rho} \quad 1 \leq t \leq T$ is obviously given by

$$T = \frac{7}{8} \sqrt{\frac{\rho}{6}} \exp\left[\frac{9+\eta}{2e} \left(\frac{\rho_*}{\rho}\right)^{\frac{1}{9+\eta}}\right] \tag{5.27}$$

provided that

$$\rho \leq \frac{\rho_*}{(3e)^{9+\eta}} \tag{5.28}$$

According to (5.4) the choice (5.24) implies $\rho \leq \frac{48}{49} \frac{1}{9A} \leq \frac{1}{7A}$ so that the conditions to apply (5.22) and (5.23) are obviously satisfyed.
This concludes the proof of the main theorem.

Appendix A : Proof of (3.19) and (3.31)

Let $\mu_1 = \mu_2 = 1$, and μ_n , $n \geq 2$ be recursively defined by

$$\mu_n = \sum_{s=2}^{n-1} \mu_s \sum_{k_1+\ldots+k_s=n\, k_i\geq 1} \mu_{k_1} \ldots \mu_{k_s} \,. \tag{A.1}$$

Lemma A1 *For all* $n \geq 1$

$$\mu_n \leq (n!)^3 \tag{3.19}$$

Proof. By induction we assume that (3.19) holds for $1 \leq j \leq n$. First of all we remark that

$$\max_{k_1+\ldots+k_s=n, 1\leq k_1\leq \ldots k_s} k_1! \ldots k_s! = (n-s+1)! \,. \tag{A2}$$

In fact this is true if $s = 1$ and by induction on s

$$\begin{aligned}\max_{\substack{k_1+\ldots+k_{s+1}=n \\ 1\leq k_1\leq\ldots\leq k_{s+1}}} k_1! \ldots k_{s+1}! &\leq \max_{1\leq j\leq n-s} (j! \max_{\substack{k_1+\ldots+k_s=n-j \\ 1\leq k_1\leq\ldots\leq k_s}} k_1! \ldots k_s!) \\ &\leq \max_{1\leq j\leq n-s} j!(n-j-s+1)! \leq (n-s)! \,.\end{aligned}$$

On the other hand the number card(s, n) of terms in the sum $\sum_{k_1+\ldots+k_s=n}$ is given by the binomial coefficient

$$\binom{n-1}{s-1} = \frac{(n-1)!}{(s-1)!(n-s)!} \tag{A3}$$

as follows immediately by induction on $1 \leq s \leq n$:

$$\text{card}(s+1, n) = \sum_{j=1}^{n-s} \text{card}(s, n-j) = \sum_{j=1}^{n-s} \binom{n-j-1}{s-1} = \binom{n-1}{s} \,.$$

From (A2) and (A3) follows that

$$\begin{aligned}\mu_n &\leq \sum_{s=2}^{n-1} (s!)^3 \sum_{k_1+\ldots+k_s=n\, k_i\geq 1} (k_1!)^3 \ldots (k_s!)^3 \leq \sum_{s=2}^{n-1} (s!)^3 \binom{n-1}{s-1} ((n-s+1)!)^3 \\ &= (n-1)! \sum_{s=2}^{n-1} (s!(n-s+1)!)^2 s(n-s+1)\end{aligned}$$

so that (3.19) is proved if one shows that for all $n \geq 3$

$$\sum_{s=2}^{n-1} (s!(n-s+1)!)^2 s(n-s+1) \leq (n!)^2 n \,. \tag{A4}$$

As $s!(n-s+1)!$,$2 \le s \le n-1$ has its maximum for $s=2$ one has

$$s!(n-s+1)! \le 2(n-1)! \tag{A5}$$

while the sum $\sum_{s=2}^{n-1} s(n-s+1)$ can be explicitly computed

$$\sum_{s=2}^{n-1} s(n-s+1) = \frac{n}{6}[(n-1)(n+4)-6]\,. \tag{A6}$$

By (A5) and (A6) we finally obtain

$$\sum_{s=2}^{n-1}(s!(n-s+1)!)^2 s(n-s+1) \le (2(n-1)!)^2 \frac{n}{6}[(n-1)(n+4)-6] \le (n!)^2 n$$

for all $n \ge 3$, i.e. (A4). This completes the proof of the lemma.

We now consider the sequence recursively defined by

$$\hat{y}_1 = 1\ ,\ \hat{y}_n = \sum_{s=2}^{n} \sum_{k_1+\ldots+k_s=n\,,\,k_i\ge1} \hat{y}_{k_1}\cdots\hat{y}_{k_s} \quad \text{for } n \ge 2 \tag{A7}$$

Lemma A2 *For all* $n \ge 1$

$$\hat{y}_n \le \frac{21}{4}7^{n-1}\,. \tag{3.31}$$

Proof. The formal power series

$$g(z) = \sum_{n=1}^{+\infty} \hat{y}_n z^n\,, \tag{A8}$$

where $z \in \mathbf{C}$, solves the functional equation

$$g(z) = \frac{(g(z))^2}{1-g(z)} + z\,. \tag{A9}$$

Let $w = g(z)$; w is a root of the quadratic equation $w^2 + z(1-w) = (1-w)w$ from which one has immediately

$$w = g(z) = \frac{1+z-\sqrt{1+(z^2-6z)}}{4}\,.$$

The first singularity of $g(z)$ is at $z = 3-2\sqrt{2}$ and the series will converge for instance for $|z| \le \frac{1}{7} < 3-2\sqrt{2}$ and using Cauchy's theorem we have

$$\hat{y}_n \le 7^n \max_{|z|\le 1/7} |g(z)| \le \frac{3}{4}7^n\,.$$

Appendix B : Proof of (4.26)

Lemma B3 *Let f be analytic in the polydisk D_ρ and g, h be analytic in D_r with $r < \rho$. Then*

$$\|f \circ (g+h) - f \circ g\|_r \leq \frac{\|f\|_{\|g\|_r + \|h\|_r + \varepsilon} \, \|h\|_r}{\varepsilon} \tag{4.26}$$

provided that $\|g\|_r + \|h\|_r + \varepsilon < \rho$.

Proof. Let $z \in D_r$ be fixed and let $G(z,t) \ : = f(t(g(z)+h(z)) + (1-t)g(z))$, so that

$$f(g(z)+h(z)) - f(g(z)) = \int_0^1 \frac{d}{dt} G(z,t) dt \,.$$

By the standard Cauchy theorem for one complex variable we can evaluate the derivative of G with respect to t by integrating over a small circle around t. Following [36] we introduce

$$\hat{G}(z,t,s) \ : = f((t+se^{ix})(g(z)+h(z)) + (1-t-se^{ix})g(z))$$

so that

$$f(g(z)+h(z)) - f(g(z)) = \int_0^1 \int_0^{2\pi} \frac{\hat{G}(z,t,s)}{2\pi s e^{ix}} dx dt \,. \tag{$B1$}$$

If

$$s = \frac{\varepsilon}{\|h\|_r} \tag{$B2$}$$

then $\hat{G}(z,t,s)$ is well defined for all $z \in D_r$ and its norm as an analytic function of z is bounded by the norm of f uniformly as (t,s) vary in $[0,1] \times [0,2\pi]$:

$$\|\hat{G}(.,t,s)\|_r \leq \|f\|_{\|g\|_r + \|h\|_r + \varepsilon} \,. \tag{$B3$}$$

By taking norms on both sides of (B1) and using (B3) we have

$$\|f \circ (g+h) - f \circ g\|_r \leq \int_0^1 \int_0^{2\pi} \frac{\|\hat{G}(.,t,s)\|_r}{2\pi s} dx dt \leq \|f\|_{\|g\|_r + \|h\|_r + \varepsilon} \int_0^1 \frac{dt}{s}$$

From the definition (B2) of s one finally obtains

$$\|f \circ (g+h) - f \circ g\|_r \leq \frac{\|f\|_{\|g\|_r + \|h\|_r + \varepsilon} \, \|h\|_r}{\varepsilon} \,.$$

Note added in proof. The estimate (3.19) can be actually replaced by $\mu_n \leq 8^{n-1}(n-1)!$ as proved recently by one of us (A.B.) in an unpublished work. Therefore the final stability estimate (2.16) can be improved by replacing $9+\eta$ with $3+\eta$.

References

[1] Kolmogorov, A.N. : 1954, *Dokl. Akad. Nauk SSSR*, **98**, 527-530.

[2] Arnol'd, V.I. : 1963, *Russ. Math. Surv.*, **18**, 9-36, and 85-191.

[3] Moser, J. : 1962, *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl.* **1**, 1.

[4] Arnol'd, V.I. : 1964, *Sov. Math. Dokl.*, **5**, 581-585.

[5] Nekhoroshev, N.N. : 1977, *Russ. Math. Surv.*, **32**, 1-65.

[6] Littlewood, J.E. : 1959, *Proc. London Math. Soc.*, **9**, 343-372.

[7] Benettin, G. : 1988, "Nekhoroshev-like results for Hamiltonian Dynamical Systems". In *Nonlinear Evolution and Chaotic Phenomena* (G. Gallavotti and P.F Zweifel editors), Plenum Press, New York.

[8] Gallavotti, G. : 1984, "Quasi-Integrable Mechanical Systems". In *Critical Phenomena, Random Systems, Gauge Theories* (K. Osterwalder and R. Stora editors), Les Houches, Session XLIII, North Holland, Amsterdam.

[9] Giorgilli, A. and Galgani, L. : 1985, *Cel. Mech.*, **37**, 95-112.

[10] Benettin, G. and Gallavotti, G. : 1986, *J. Stat. Phys.*, **44**, 293-338.

[11] Giorgilli, A., Delshams, A., Fontich, E., Galgani L. and Simò, C. : 1989, "Effective Stability for a Hamiltonian System near an Elliptic Equilibrium Point, with an Application to the Restricted Three-Body Problem." *J. Diff. Eq.*, **77**, 167-198.

[12] Palis, J. and Yoccoz, J.C. : 1988, "Rigidity of Centalizers of Diffeomorphisms." *Ann. Ec. Norm. Sup.*, in press.

[13] Roels, J. and Henon, M. : 1967, *Bull. Astr. (Ser. 3)*, **2**, 267-285.

[14] Rannou, F. : 1974, *Astron. and Astrophys.*, **31**, 289-301.

[15] Wisdom, J. : 1982, *Astron. J.*, **87**, 577.

[16] Wisdom, J. : 1983, *Icarus*, **56**, 51.

[17] Herrera, J.C. : 1979, "Nonlinear Dynamics and the beam-beam interaction." *Am. Inst. of Phys.*, **232**.

[18] Dragt, A.J. : 1982, "Lectures in nonlinear orbit dynamics." In *A.I.P. Conf. Proc.*, **87**.

[19] Dragt, A.J. : 1986, *Nonlinear Dynamics and particle accelerators*, Lecture Notes in Physics, **247**.

[20] Bazzani, A., Mazzanti, P., Servizi, G. and Turchetti, G. : 1988, *Nuovo Cimento*, **102B**, 51-80.

[21] Bazzani, A., Leemann, T., Scandale, W., Servizi, G. and Turchetti, G. : 1989, "Description of nonlinear beam dynamics in the CERN large hadron collider by using normal form algorithms." In *EPAC Conference proceedings* (S. Tazzari Ed.), World Scientific.

[22] Liechtenberg, A. J. and Lieberman, M. A. : 1983, *Regular and stochastic motion*, Chap 6, Springer Verlag.

[23] Ortolani, A. : 1987, "Physics of mirrors, reversed field pinches and compact tori." International School of Plasma Physics. Ed. Compositori.

[24] Bazzani, A. and Turchetti, G. : 1989, "Analisi di superfici magnetiche in geometria toroidale." *Atti Sem. Mat. Fis. Un. Modena*, **XXXVII**, 157.

[25] Bazzani, A., Malavasi, M., Siboni, S. and Turchetti, G. : 1989, "Perturbative methods in the analysis of magnetic fields structure." *Dip. di Fisica della Università di Bologna*, preprint.

[26] Birkhoff, G.D. : 1920, *Acta Math.*, **43**, 1-119.

[27] Bazzani, A. : 1988, "Normal Forms for symplectic maps in $\mathbf{R}^{2n}$." *Cel. Mech.*, **42**, 107-128.

[28] Moser, J. : 1956, *Comm. Pure Appl. Math.*, **9**, 673-692.

[29] Servizi, G., Turchetti, G., Benettin, G. and Giorgilli, A. : 1983, *Phys. Lett.*, **A95**, 11.

[30] Servizi, G. and Turchetti, G. : 1986, *Nuovo Cimento*, **B95**, 121.

[31] Turchetti, G. : 1988, "Perturbative Methods for Hamiltonian Maps." In *Methods and Applications of Nonlinear Dynamics* (A.W. Sáenz Ed.), 95-154, World Scientific, Singapore.

[32] Malavasi, M., Marmi, S., Servizi, G. and Turchetti, G. : 1990, in preparation.

[33] Bessis, D., Marmi, S. and Turchetti, G. : 1989, "On the Singularities of Divergent Majorant Series arising from Normal Form Theory." *Rendiconti di Matematica*, to appear.
[34] Meyer, K.R. : 1974, "The Implicit Function Theorem and Analytic Differential Equations." *Lect. Notes in Math.*, **468**, 191-208.
[35] Liverani, C. and Turchetti, G. : 1986, *J. Stat. Phys.*, **45**, 1071-1086.
[36] Marmi, S. : 1988, *Meccanica*, **23**, 139-146.