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SEQUENTIAL ORDER STATISTICS AND K-OUT-OF-N SYSTEMS WITH SEQUENTIALLY ADJUSTED FAILURE RATES

ERHARD CRAMER AND UDO KAMPS

Institute of Statistics, Aachen University of Technology,
Wüllnerstr. 3, D-52056 Aachen, Germany

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Abstract. k-out-of-n systems frequently appear in applications. They consist of n components of the same kind with independent and identically distributed life-lengths. The life-length of such a system is described by the $(n-k+1)$-th order statistic in a sample of size n when assuming that remaining components are not affected by failures. Sequential order statistics are introduced as a more flexible model to describe 'sequential k-out-of-n systems' in which the failure of any component possibly influences the other components such that their underlying failure rate is parametrically adjusted with respect to the number of preceding failures. Useful properties of the maximum likelihood estimators of the model parameters are shown, and several tests are proposed to decide whether the new model is the more appropriate one in a given situation. Moreover, for specific distributions, e.g. Weibull distributions, simultaneous maximum likelihood estimation of the model parameters and distribution parameters is considered.

Key words and phrases: Sequential k-out-of-n-system, sequential order statistics, generalized order statistics, type II censoring, maximum likelihood estimators, extremal quotient, Weibull distributions.

1. Introduction

In many applications, technical systems or sub-systems have k-out-of-n structure which has been extensively investigated in the literature. We consider a $(n - r + 1)$-out-of-n system which consists of n components of the same kind with independent and identically distributed life-lengths. All components start working simultaneously and the system fails, if r or more components fail. A $(n-r+1)$-out-of-n system of this type is also called $(n-r+1)$-out-of-n:G as well as r-out-of-n:F system. Obviously, the life-length of the system is described by the r-th order statistic in a sample of size n. Results on statistical inference in this model can also be found in the analysis of type II censoring which is described by order statistics (see e.g. Lawless (1982)). However, in some systems the failure of a component may more or less strongly influence the remaining components. This can be thought of as damage caused by the i-th component failure, $1 \leq i \leq r$, in the system. Thus, a more flexible model, which is therefore more applicable to practical situations, must take some dependence among the components into account. We suppose that, after each failure, the remaining components possess a possibly different failure rate than before; i.e., the underlying failure rate of the remaining components is adjusted according to the number of preceding failures. For this purpose, the model of sequential order statistics is proposed in Kamps (1995*a*). Therein, a concept of generalized order statistics is introduced containing order statistics, sequential order statistics and several other models of ordered random variables as particular cases. Moreover, the distribution theory for generalized order statistics is developed and results with respect to moments are shown. For further properties and a detailed discussion of generalized order statistics, including reliability properties, we refer to Kamps (1995*b*). Well known results for order statistics, and by this for corresponding ordinary k-out-of-n systems, are contained as special cases.

In the present paper, we are concerned with statistical inference for $(n - r+1)$-out-of-n systems with component damages as described above by means of sequential order statistics. Thus we term this structure 'sequential $(n-r+1)$-out-of-n system'.

In Section 2 we introduce sequential order statistics and summarize some distribution theoretical results. Maximum likelihood estimators for the parameters in our model are derived in Section 3 and several of their properties are shown. Based on these estimators, we present short-cut tests and the likelihood ratio test in Section 4 to decide whether or not our proposed model is appropriate for describing some $(n - r + 1)$-out-of-n system. For illustration, we show the results of a simulation study. In the last section we are concerned with simultaneous maximum likelihood estimation of the model parameters and distribution parameters for specific underlying distributions such as one- and two-parameter Weibull distributions.

2. Sequential order statistics

We consider a sequential $(n - r + 1)$-out-of-n system where the life-length distribution of the remaining components in the system may change after each failure of the components. If we observe the i-th failure at time x, the remaining components are now supposed to have a possibly different life-length distribution. This one is truncated on the left at x to ensure realizations arranged in ascending order of magnitude. In the definition of sequential order statistics we start with some triangular scheme of random variables where the i-th line contains $n-i+1$ random variables with distribution function F_i, $1 \leq i \leq n$.

DEFINITION 2.1. Let $(Y_j^{(i)})_{1\leq i\leq n, 1\leq j\leq n-i+1}$ be independent random variables with $(Y_j^{(i)})_{1\leq j\leq n-i+1} \sim F_i$, $1 \leq i \leq n$, where $F_1, \ldots, F_n$ are strictly increasing and continuous distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$.

Let $X_j^{(1)} = Y_j^{(1)}$, $1 \leq j \leq n$, $X_*^{(1)} = \min\{X_1^{(1)}, \ldots, X_n^{(1)}\}$, and for $2 \leq i \leq n$ let $X_j^{(i)} = F_i^{-1}(F_i(Y_j^{(i)})(1 - F_i(X_*^{(i-1)})) + F_i(X_*^{(i-1)}))$, $X_*^{(i)} = \min\{X_j^{(i)}, 1 \leq j \leq$

$n-i+1\}$. Then the random variables $X_*^{(1)}, \dots, X_*^{(n)}$ are called sequential order statistics (based on $F_1, \dots, F_n$).

Given the realization $z_{1,n-i+2}^{(i-1)}$ of the minimum in line $i-1$, we obtain the conditional distribution of the random variables $X_1^{(i)}$ for $1 \le i \le n$. Since $Y_1^{(i)}$ and $X_*^{(i-1)}$ are independent, we have

$$P(X_1^{(i)} \le t \mid X_*^{(i-1)} = s) = P\left(F_i(Y_1^{(i)}) \le \frac{F_i(t) - F_i(s)}{1 - F_i(s)}\right)$$
$$= \frac{F_i(t) - F_i(s)}{1 - F_i(s)} = G_i(t \mid s), \qquad \text{say.}$$

Thus we describe our modified $(n-r+1)$-out-of-n-system as follows:

Consider a triangular scheme $(Z_j^{(i)})_{1 \le i \le n, 1 \le j \le n-i+1}$ of random variables where the $(Z_j^{(i)})_{1 \le j \le n-i+1}$ are iid according to $G_i(\cdot \mid z_{1,n-i+2}^{(i-1)})$, $1 \le i \le n$, $z_{1,n+1}^{(0)} = -\infty$.

After the occurrence of the i-th failure in the system at time $z_{1,n-i+1}^{(i)}$ (i.e. the realization of the sample minimum in line i), the next failure time is modelled as the minimum in the sample $Z_1^{(i+1)}, \dots, Z_{n-i}^{(i+1)}$ of iid r.v.'s with distribution function $G_{i+1}(\cdot \mid z_{1,n-i+1}^{(i)})$.

At this point the question arises as to whether we can obtain the distribution theory of sequential order statistics and their properties by analogy with ordinary order statistics which have been extensively investigated in the literature (see e.g. David (1981)). In the general setting of Definition 2.1, the model of sequential order statistics turns out to be too extensive to establish analogous properties as found in the case of ordinary order statistics.

ASSUMPTION. In the following we restrict ourselves to a particular choice of the distribution functions $F_1, \dots, F_n$, namely

$$F_i(t) = 1 - (1 - F(t))^{\alpha_i}, \qquad 1 \le i \le n, \tag{2.1}$$

with some absolutely continuous and strictly increasing distribution function F and positive real numbers $\alpha_1, \dots, \alpha_n$. Let f be the corresponding density function.

From (2.1) we have $\alpha_i \cdot \frac{f}{1-F}$ as the failure rate of the underlying distribution in line i of the above triangular scheme.

LEMMA 2.1. (cf. Kamps (1995*b*), p. 62) *Let $X_*^{(1)}, \dots, X_*^{(n)}$ be sequential order statistics based on (2.1). Then the joint density function of the first r sequential order statistics $X_*^{(1)}, \dots, X_*^{(r)}$ is given by*

(2.2) $f^{X_*^{(1)}, \dots, X_*^{(r)}}(x_1, \dots, x_r)$

$$= \frac{n!}{(n-r)!}\left(\prod_{j=1}^{r}\alpha_j\right)\left(\prod_{j=1}^{r-1}(1-F(x_j))^{m_j}f(x_j)\right)$$
$$\cdot (1-F(x_r))^{\alpha_r(n-r+1)-1}f(x_r),$$
$$x_1 < \cdots < x_r, \quad r \le n,$$
$$\textit{with} \quad m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1} - 1, \quad 1 \le j \le n-1.$$

Ordinary order statistics (for the description of ordinary $(n-r+1)$-out-of-n systems) are contained in the model of sequential order statistics in the distribution theoretical sense. Choosing $r = n$ and $\alpha_1 = \cdots = \alpha_n$ in (2.2), we obtain the joint density function of the order statistics $X_{1,n} \le \cdots \le X_{n,n}$ based on iid random variables $X_1, \dots, X_n$ with distribution function $1-(1-F)^{\alpha_1}$ (see e.g. David (1981)).

3. Maximum likelihood estimation for arbitrary distributions

Let us consider $s \ge 1$ independent and identically distributed observations of some sequential $(n-r+1)$-out-of-n system leading to the set of data

$$(x_{ij})_{1\le i\le s, 1\le j\le r}, \qquad x_{i1} \le \cdots \le x_{ir}, \quad 1 \le i \le s.$$

That is, we suppose knowledge of all the times of failures of components during the life-length of the system which we describe by sequential order statistics based on (2.1). Let

$$(X_{*i}^{(j)})_{1\le i\le s, 1\le j\le r}$$

be the corresponding random variables.

Then the likelihood function is given by

(3.1) $L(\alpha_1, \dots, \alpha_r; x_{ij}, 1 \le i \le s, 1 \le j \le r)$
$$= \left(\frac{n!}{(n-r)!}\right)^s \left(\prod_{j=1}^{r}\alpha_j\right)^s \left(\prod_{i=1}^{s}\prod_{j=1}^{r-1}(1-F(x_{ij}))^{m_j}f(x_{ij})\right)$$
$$\cdot \prod_{i=1}^{s}(1-F(x_{ir}))^{\alpha_r(n-r+1)-1}f(x_{ir}) = L(\alpha_1, \dots, \alpha_r, F), \qquad \text{say},$$

and we obtain

LEMMA 3.1. *The maximum likelihood estimators of* $\alpha_1, \dots, \alpha_r$ *are*

$$\alpha_1^* = -\frac{s}{n}\left(\log\prod_{i=1}^{s}(1-F(x_{i1}))\right)^{-1} \qquad \textit{and}$$
$$\alpha_j^* = -\frac{s}{n-j+1}\left(\log\prod_{i=1}^{s}\frac{1-F(x_{ij})}{1-F(x_{i,j-1})}\right)^{-1}, \qquad 2 \le j \le r.$$

PROOF. We express the logarithm of the likelihood function in (3.1) in terms of the hazard function

$$h(t) = f(t)/(1 - F(t))$$

and the cumulative hazard function

$$H(t) = -\log(1 - F(t)):$$

$$\begin{aligned}\log L(\alpha_1, \dots, \alpha_r, F) = s \log \frac{n!}{(n-r)!} &+ s \sum_{j=1}^{r} \log \alpha_j - n\alpha_1 \sum_{i=1}^{s} H(x_{i1}) \\ &- \sum_{j=2}^{r} (n-j+1)\alpha_j \sum_{i=1}^{s} (H(x_{ij}) - H(x_{i,j-1})) \\ &+ \sum_{j=1}^{r} \sum_{i=1}^{s} \log h(x_{ij}).\end{aligned}$$

For brevity, let $\sum_{i=1}^{s} H(x_{i1}) = A_1$ and $\sum_{i=1}^{s}(H(x_{ij}) - H(x_{i,j-1})) = A_j$, $2 \le j \le r$. Since

$$\log \alpha_j = \log \alpha_j/\alpha_j^* + \log \alpha_j^* \le \alpha_j/\alpha_j^* - 1 + \log \alpha_j^*,$$

$$s \sum_{j=1}^{r} \alpha_j/\alpha_j^* = \sum_{j=1}^{r} (n-j+1)\alpha_j A_j$$

and

$$\sum_{j=1}^{r} \log \alpha_j^* = -\log \frac{n!}{(n-r)!} - \sum_{j=1}^{r} \log \left(\frac{1}{s} A_j\right)$$

we obtain

$$\begin{aligned}\log L(\alpha_1, \dots, \alpha_r, F) &\le -sr - s \sum_{j=1}^{r} \log \left(\frac{1}{s} A_j\right) + \sum_{j=1}^{r} \sum_{i=1}^{s} \log h(x_{ij}) \\ &= \Psi(x_{ij}, F), \qquad \text{say,}\end{aligned}$$

with equality iff $\alpha_j = \alpha_j^*$, $1 \le j \le r$.

In the case of ordinary order statistics, similar MLE's result in the following situation (see Sarkar (1971)): The components of a series system are independent and exponentially distributed with possibly different parameters and the MLE's of these parameters are based on independent copies of the respective components under type II censoring.

In order to prove helpful properties of the MLE's α_j^*, $1 \le j \le r$, we make use of the following theorem (see Kamps (1995*b*), p. 81).

THEOREM 3.1. *Let $X_*^{(1)}, \ldots, X_*^{(n)}$ be sequential order statistics based on (2.1) with $F(x) = 1 - e^{-x}$, $x \geq 0$. Then the random variables*

$$n\alpha_1 X_*^{(1)} \quad \text{and} \quad (n-j+1)\alpha_j(X_*^{(j)} - X_*^{(j-1)}), \qquad 2 \leq j \leq n,$$

are iid according to F.

THEOREM 3.2. *The MLE's $\alpha_1^*, \ldots, \alpha_r^*$ given in Lemma 3.1 have the following properties:*

(i) *$\alpha_1^*, \ldots, \alpha_r^*$ are independent and $\alpha_j^* \sim \alpha_j(\frac{1}{s}\sum_{i=1}^s V_{ij})^{-1}$, $1 \leq j \leq r$, where the $(V_{ij})_{i,j}$ are iid with $P(V_{11} \leq x) = 1 - e^{-x}$, $x \geq 0$.*

(ii) *$E(\alpha_j^*)^k = \frac{(s-k-1)!}{(s-1)!}(s\alpha_j)^k$, if $k \leq s-1$; hence $E(\alpha_j^*) = \frac{s}{s-1}\alpha_j$, $s > 1$ (i.e. α_j^* is asymptotically unbiased), $\mathrm{Var}(\alpha_j^*) = \frac{s^2}{(s-1)^2(s-2)}\alpha_j^2$, $s > 2$, $MSE(\alpha_j^*) = \frac{s+2}{(s-1)(s-2)}\alpha_j^2$, $s > 2$, $1 \leq j \leq r$.*

(iii) *The statistic $(\alpha_1^*, \ldots, \alpha_r^*)$ is sufficient for $(\alpha_1, \ldots, \alpha_r)$.*

(iv) *The sequences of estimators $(\alpha_j^*)_{s\in\mathbb{N}}$ are strongly consistent $(1 \leq j \leq r)$.*

(v) *α_j^* is asymptotically normal (AN), $1 \leq j \leq r$; i.e. $\sqrt{s}(\alpha_j^*/\alpha_j - 1) \xrightarrow{d} \mathcal{N}(0,1)$. Moreover, we have $\sqrt{s}\alpha_j/\alpha_j^*(1 - \alpha_j/\alpha_j^*) \xrightarrow{d} \mathcal{N}(0,1)$.*

PROOF.

(i) Let $X_*^{(1)}, \ldots, X_*^{(n)}$ be sequential order statistics based on $F_i = 1 - (1 - F)^{\alpha_i}$, $1 \leq i \leq n$. Then $(-\log(1 - F(X_*^{(1)})), \ldots, -\log(1 - F(X_*^{(n)})))$ has the same distribution as sequential order statistics $Z_*^{(1)}, \ldots, Z_*^{(n)}$ based on $G = 1 - (1-G)^{\alpha_i}$ where $G(x) = 1 - e^{-x}$, $x \geq 0$.

Applying Theorem 3.1, we obtain the assertions since $Z_*^{(1)}, Z_*^{(2)} - Z_*^{(1)}, \ldots, Z_*^{(n)} - Z_*^{(n-1)}$ are independent and $n\alpha_1 Z_*^{(1)} \sim (n-j+1)\alpha_j(Z_*^{(j)} - Z_*^{(j-1)}) \sim G$, $2 \leq j \leq n$.

(v) Since $\frac{1}{s}\sum_{i=1}^s V_{ij}$ is AN$(1, \frac{1}{s})$, we use the transformation $g(x) = \alpha_j/x$ to obtain that α_j^* is AN$(\alpha_j, \alpha_j^2/s)$ (cf. Serfling (1980), p. 118).

In Theorem 3.2(i) it is shown that α_j^* is distributed as $1/T_j$ where the random variable T_j is gamma distributed with parameters s and $s \cdot \alpha_j$; i.e., its density is given by

$$f^{T_j}(t) = \frac{(s\alpha_j)^s}{(s-1)!} t^{s-1} e^{-s\alpha_j t}, \qquad t \geq 0.$$

The distribution of α_j^* is known as inverted gamma distribution which is used as a prior density in Bayesian analysis (cf. Bain (1983)).

4. Short-cut tests

Given some sequential $(n-r+1)$-out-of-n system and a set of data as in Section 3, we have to choose an appropriate model. Following, we present two short-cut tests in Subsections 4.1 and 4.2 and we consider the likelihood ratio test in Subsection 4.3. These tests provide procedures for testing the hypothesis

$$H : \alpha_1 = \cdots = \alpha_r$$

against the alternative

$$A : \exists i \neq j, \quad i, j \in \{1, \ldots, r\}, \quad \text{such that} \quad \alpha_i \neq \alpha_j$$

at the level of significance α.

Hence, if the hypothesis is rejected, then the model of sequential order statistics is adequate and ordinary order statistics should not be used to describe the system. If there is some prior information about the possibly common value α_0 of $\alpha_1, \ldots, \alpha_r$, then we may be interested to consider the hypothesis

$$\tilde{H} : \alpha_1 = \cdots = \alpha_r = \alpha_0.$$

In Subsection 4.4 we propose a suitable level α test.

The MLE's $\alpha_1^*, \ldots, \alpha_r^*$ of $\alpha_1, \ldots, \alpha_r$ are independent and inverted gamma distributed with parameters s and $s \cdot \alpha_j$, respectively (see Theorem 3.2(i)). In the following tests we use the random variables $\beta_1^*, \ldots, \beta_r^*$ defined by

$$\beta_j^* = s/\alpha_j^*, \qquad 1 \leq j \leq r,$$

which are independently gamma distributed with parameters s and α_j, respectively. Under the hypothesis H (or $\tilde{H}$), $\beta_1^*, \ldots, \beta_r^*$ are identically distributed.

4.1 *Test* A

Let $\beta_{1,r}^* \leq \cdots \leq \beta_{r,r}^*$ be the order statistics corresponding to $\beta_1^*, \ldots, \beta_r^*$. Reject hypothesis H, if the ratio $\beta_{1,r}^*/\beta_{r,r}^*$ is too small, i.e., if

$$\beta_{1,r}^*/\beta_{r,r}^* \leq c$$

where c is determined by

$$P_H(\beta_{1,r}^*/\beta_{r,r}^* \leq c) = \alpha.$$

Now we have to compute the critical value c. We may use a representation of the joint density of the minimum $X_{1,r}$ and the maximum $X_{r,r}$ in the sample $X_1, \ldots, X_r$ of nonnegative iid random variables with distribution function F and density function f (cf. David (1981), p. 10)

$$f^{X_{1,r}, X_{r,r}}(x_1, x_r) = r(r-1)(F(x_r) - F(x_1))^{r-2} f(x_1) f(x_r), \qquad x_1 < x_r,$$

to obtain the density of the ratio $X_{1,r}/X_{r,r}$

$$f^{X_{1,r}/X_{r,r}}(z) = r(r-1) \int_0^\infty (F(y) - F(yz))^{r-2} f(y) f(yz) y dy, \qquad z \in (0,1).$$

If $F(0) = 0$, then interchanging the order of integration yields

$$P(X_{1,r}/X_{r,r} \le x) = \int_0^x f^{X_{1,r}/X_{r,r}}(z)dz$$
$$= 1 - r\int_0^\infty f(y)(F(y) - F(yx))^{r-1}dy.$$

Applying this formula to $\beta^*_{1,r}$ and $\beta^*_{r,r}$, we find

$$P_H(\beta^*_{1,r}/\beta^*_{r,r} \le c) = 1 - \frac{r}{(s-1)!}\int_0^\infty \left(\sum_{i=0}^{s-1} \frac{z^i}{i!}(c^i e^{-cz} - e^{-z})\right)^{r-1} z^{s-1}e^{-z}dz.$$

The ratio of the smallest and the largest order statistic from a sample of size r has been considered earlier in the literature. Gumbel and Keeney (1950) call it the extremal quotient and study its asymptotic distribution with respect to r for symmetrical, continuous and unlimited parent distributions. They use extreme value theory along with the fact that the minimum and the maximum are asymptotically independent. We do not use this method here since the number of components is usually small. Corresponding tables are given in Gumbel and Pickands (1967).

Muenz and Green (1977) show representations of the distribution functions of arbitrary ratios of order statistics based on an absolutely continuous and strictly increasing distribution function F. In particular, we find

$$P(X_{1,r}/X_{r,r} \le x) = 1 - r\int_0^1 (t - F(xF^{-1}(t)))^{r-1}dt.$$

In terms of $\beta^*_{1,r}$ and $\beta^*_{r,r}$ we again obtain the above expression.

Table A shows critical values for $\alpha = 0.01, 0.05, 0.1$ and several values of r and s.

Table A. Values of α-quantiles c in Test A.

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
s	$r = 2$	$r = 3$	$r = 2$	$r = 3$	$r = 2$	$r = 3$
1	.0050	.0022	.0256	.0114	.0526	.0235
2	.0432	.0273	.1041	.0647	.1565	.0964
3	.0903	.0641	.1718	.1196	.2334	.1606
4	.1334	.1006	.2256	.1666	.2909	.2124
5	.1710	.1340	.2690	.2064	.3358	.2547
6	.2038	.1639	.3051	.2404	.3722	.2900
7	.2326	.1906	.3357	.2698	.4026	.3202
8	.2581	.2147	.3621	.2957	.4285	.3464
9	.2809	.2365	.3853	.3186	.4510	.3694
10	.3014	.2564	.4058	.3391	.4708	.3898

4.2 *Test* B

Since, under the hypothesis H, $\beta_1^*, \ldots, \beta_r^*$ are iid gamma random variables, it is near at hand to use

$$\mathcal{B} = \beta_1^* / \sum_{j=1}^{r} \beta_j^*$$

as a test statistic which has a beta distribution on $(0,1)$ with parameters s and $(r-1)s$. The density of $\mathcal{B}$ is given by

$$f^{\mathcal{B}}(z) = \frac{1}{B(s,(r-1)s)} z^{s-1}(1-z)^{(r-1)s-1}, \qquad z \in (0,1).$$

Thus, reject hypothesis H, if either $\mathcal{B}$ is too small or too large, i.e., if

$$\mathcal{B} \le c_1 \quad \text{or} \quad \mathcal{B} > c_2$$

where c_1 and c_2 are determined by

$$P_H(\mathcal{B} \le c_1) = \alpha/2 \quad \text{and} \quad P_H(\mathcal{B} > c_2) = \alpha/2.$$

4.3 *Likelihood ratio test*

The likelihood ratio procedure for testing H against A is based on the statistic

$$Q = \sup_{\alpha_1=\cdots=\alpha_r} L(\alpha_1, \ldots, \alpha_r, F) / \sup_{\alpha_1,\ldots,\alpha_r} L(\alpha_1, \ldots, \alpha_r, F).$$

With the notations and the representation of $\Psi(x_{ij}, F)$ in the proof of Lemma 3.1 and noticing that the supremum in the numerator of Q is attained at

$$\alpha_1 = \cdots = \alpha_r = sr / \sum_{j=1}^{r} (n-j+1)A_j,$$

we obtain

$$Q = \left(\prod_{j=1}^{r} (n-j+1)A_j \right)^{s} \cdot \left(\frac{1}{r} \sum_{j=1}^{r} (n-j+1)A_j \right)^{-rs}.$$

Under hypothesis H, $Q^{1/(rs)}$ is distributed as the ratio of geometric and arithmetic mean of iid gamma variables $W_1, \ldots, W_r$ with density function

$$f^{W_1}(w) = \frac{1}{(s-1)!} w^{s-1} e^{-w}, \qquad w > 0:$$

$$Q^{1/(rs)} \sim \left(\prod_{j=1}^{r} W_j \right)^{1/r} \Bigg/ \left(\frac{1}{r} \sum_{j=1}^{r} W_j \right).$$

The statistic $-\frac{1}{rs}\log Q$ appears in the computation of maximum likelihood estimators of the parameters of gamma distributions. Tests and confidence intervals for these parameters are also based on Q. However, the exact distribution of Q is very complicated and thus chi-square approximations are used for large sample sizes (see Bain (1983)). In our situation, the role of the sample size is taken up by the parameter r which is usually small (cf. Subsection 4.1). Since numerical calculations using the approximations shown in Bain (1983) lead to inaccurate and therefore unsatisfactory results, we do not show a table of critical values here.

4.4 *Test* C

If we consider hypothesis $\tilde{H}$ with some fixed $\alpha_0 > 0$, then we propose to accept the alternative, if the range $\beta^*_{r,r} - \beta^*_{1,r}$ of the random variables $\beta^*_1, \ldots, \beta^*_r$ is too large, i.e.,

$$\beta^*_{r,r} - \beta^*_{1,r} > c/\alpha_0$$

where c is determined by

$$P_{\tilde{H}}(\beta^*_{r,r} - \beta^*_{1,r} > c/\alpha_0) = \alpha.$$

Applying a formula for the distribution function of the range (cf. David (1981), p. 12) we obtain

$$\begin{aligned} &P_{\tilde{H}}(\beta^*_{r,r} - \beta^*_{1,r} \leq c/\alpha_0) \\ &= \frac{r}{(s-1)!}\int_0^\infty \left(e^{-z}\sum_{i=0}^{s-1}\frac{z^i}{i!} - e^{-(z+c)}\sum_{i=0}^{s-1}\frac{(z+c)^i}{i!}\right)^{r-1} z^{s-1}e^{-z}dz \\ &= \frac{r}{(s-1)!}e^{-c(r-1)}\int_0^\infty \left(\sum_{i=0}^{s-1}\frac{z^i}{i!}\sum_{j=s-i}^{\infty}\frac{c^j}{j!}\right)^{r-1} z^{s-1}e^{-zr}dz. \end{aligned}$$

The distribution of the range obviously depends on α_0. Thus, if we consider the distribution of the ratio $(\beta^*_{r,r} - \beta^*_{1,r})/\beta^*_{1,r}$, e.g., α_0 drops out which amounts to using the extremal quotient as test statistic (see Subsection 4.1).

Table C shows critical values for $\alpha = 0.01, 0.05, 0.1$ and several values of r and s.

4.5 *Simulation study*

The following simulation study illustrates the use of our model and of the above tests. We generated 20000 experiments with $s = 50$ iid copies each of a sequential 2-out-of-3 system [1-out-of-3 system] based on the standard exponential distribution ($F(x) = 1 - e^{-x}$, $x \geq 0$) and $\alpha_1 = 1$, $\alpha_2 = 2$ [$\alpha_3 = 2$] (i.e. $F_1(x) = 1 - e^{-x}$ and $F_2(x) = F_3(x) = 1 - e^{-2x}$). α^*_1, α^*_2 [and α^*_3] are the MLE's of α_1, α_2 [and α_3]. If we assume, for comparison, that we observe outcomes of ordinary 2-out-of-3 systems [1-out-of-3 systems] based on an exponential distribution with parameter λ, then $\lambda^* = 2/(1/\alpha^*_1 + 1/\alpha^*_2)$ [$\lambda^* = 3/(1/\alpha^*_1 + 1/\alpha^*_2 + 1/\alpha^*_3)$] is the corresponding MLE.

Table C. Values of α-quantiles c/α_0 in Test C.

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
s	$r = 2$	$r = 3$	$r = 2$	$r = 3$	$r = 2$	$r = 3$
1	4.6052	5.2958	2.9957	3.6761	2.3026	2.9697
2	5.9902	6.8675	4.1130	5.0055	3.2718	4.1672
3	7.0598	8.0774	4.9686	6.0224	4.0104	5.0798
4	7.9704	9.1057	5.6926	6.8826	4.6329	5.8495
5	8.7793	10.0185	6.3327	7.6434	5.1816	6.5287
6	9.5155	10.8488	6.9132	8.3335	5.6780	7.1435
7	10.1963	11.6165	7.4483	8.9699	6.1346	7.7095
8	10.8329	12.3341	7.9472	9.5635	6.5597	8.2368
9	11.4330	13.0107	8.4166	10.1221	6.9591	8.7324
10	12.0025	13.6527	8.8612	10.6513	7.3369	9.2014

Table 1.

MLE	2-out-of-3 system		1-out-of-3 system	
	emp. mean	emp. variance	emp. mean	emp. variance
α_1^*	1.0193	0.0214	1.0195	0.0212
α_2^*	2.0416	0.0890	2.0385	0.0864
α_3^*	—	—	2.0407	0.0864
λ^*	1.3475	0.0206	1.5104	0.0170

Table 1 shows the empirical means and the empirical variances of the MLE's α_1^*, α_2^* $[\alpha_3^*]$ and λ^* based on the data of 20000 simulation steps.

For the sequential 2-out-of-3 system the hypothesis H (ordinary model) was rejected in favor of the alternative A (new model) in 93.4% of the experiments by Test A as well as by Test B ($\alpha = 0.05$) which are equivalent tests in the case $r = 2$. For the sequential 1-out-of-3 system H was rejected in 94.8% and 98.0% of the experiments by Test A and Test B, respectively ($\alpha = 0.05$).

5. Maximum likelihood estimation for specific distributions

In the situation described in Section 3 we are now concerned with specific distribution functions F. We consider simultaneous MLE's of $\alpha_1, \ldots, \alpha_r$ and the distribution parameters. We start with a simple exponential family.

Let F be given by

$$F(t) = 1 - e^{-\lambda g(t)}, \qquad t \geq 0, \tag{5.1}$$

with $\lambda > 0$ unknown and some increasing and differentiable function g on $[0, \infty)$ satisfying $g(0) = 0$ and $\lim_{t \to \infty} g(t) = \infty$.

The particular cases $g(t) = t^\beta$, $\beta > 0$, and $g(t) = \log(t^a)$, $a > 0$, correspond to standard Weibull (exponential, $\beta = 1$) and Pareto distributions, respectively.

Putting the above distribution function F in Ψ (see the proof of Lemma 3.1), we obtain that the upper bound Ψ is independent of λ. Hence there is no finite MLE of λ and we find

LEMMA 5.1. *In the case of* (5.1) *the MLE's of* $\tilde{\alpha}_j = \lambda\alpha_j$, $1 \le j \le r$, *are given by* $\tilde{\alpha}_j^* = \lambda\alpha_j^*$ *with* α_j^* *as in Lemma* 3.1; *i.e.*

$$\tilde{\alpha}_1^* = \frac{s}{n}\left(\sum_{i=1}^{s} g(x_{i1})\right)^{-1} \quad \text{and}$$

$$\tilde{\alpha}_j^* = \frac{s}{n-j+1}\left(\sum_{i=1}^{s}(g(x_{ij}) - g(x_{i,j-1}))\right)^{-1}, \quad 2 \le j \le r.$$

Following, we consider (5.1) with an additional shift parameter.

Let F be given by

$$F(t) = 1 - e^{-\lambda(g(t)-\eta)}, \quad t \ge g^{-1}(\eta), \tag{5.2}$$

with $\lambda > 0$, $\eta \in \mathbb{R}$ unknown and some increasing and differentiable function g on $[g^{-1}(\eta), \infty)$ satisfying $\lim_{t\to\infty} g(t) = \infty$.

The particular cases $g(t) = t$ and $g(t) = \log(t^a)$, $a > 0$, correspond to two-parameter exponential and Pareto distributions, respectively (cf. Varde (1970) for $g(t) = t$ and $r = 2$).

LEMMA 5.2. *In the case of* (5.2) *the MLE's of* η *and* $\tilde{\alpha}_j = \lambda\alpha_j$, $1 \le j \le r$, *are given by*

$$\eta^* = \min_{1\le i\le s} g(x_{i1}),$$

$$\tilde{\alpha}_1^* = \frac{s}{n}\left(\sum_{i=1}^{s} g(x_{i1}) - \eta^*\right)^{-1} \quad \text{and}$$

$$\tilde{\alpha}_j^* = \frac{s}{n-j+1}\left(\sum_{i=1}^{s}(g(x_{ij}) - g(x_{i,j-1}))\right)^{-1}, \quad 2 \le j \le r.$$

Moreover, the statistic $(\eta^*, \tilde{\alpha}_1^*, \dots, \tilde{\alpha}_r^*)$ *is sufficient for* $(\eta, \lambda\alpha_1, \dots, \lambda\alpha_r)$.

PROOF. Since $\eta \le \min_{1\le i\le s} g(x_{i1})$ and $L(\alpha_1, \dots, \alpha_r, F)$ increases with respect to η for all fixed $\alpha_1, \dots, \alpha_r$, we conclude that $\eta^* = \min_{1\le i\le s} g(x_{i1})$ following the argument of Lawless ((1982), p. 127) and Epstein ((1957), p. 20) in the case of type II censoring described by ordinary order statistics.

Putting η^* in the likelihood function $L(\alpha_1, \dots, \alpha_r, F)$, the assertion follows by analogy with Lemma 5.1.

Finally, we consider two-parameter Weibull distributions with $g(t) = t^\beta$ in (5.1) and we aim at determining simultaneous MLE's of $\lambda\alpha_1, \dots, \lambda\alpha_r$ and β. The Weibull distribution is frequently used in statistical models, and a variety of papers can be found in the literature dealing with parameter estimation. For MLE in type II censoring, i.e. based on ordinary order statistics, we refer to, e.g., Cohen (1965), Harter and Moore (1965), Pike (1966), McCool (1970), Rockette *et al.* (1974) and Lawless ((1982), Ch. 4.1). Usually, the case $s = 1$ is considered in the literature.

Let F be given by

$$F(t) = 1 - e^{-\lambda t^\beta}, \qquad t \geq 0, \tag{5.3}$$

with $\lambda, \beta > 0$ unknown.

However, in turns out that in general there is no explicit solution for the MLE of β. This problem also arises in the classical case (cf. Lawless (1982), p. 143 and the references above). To show the existence and the uniqueness of a MLE of β we use the following auxiliary result which is of interest in its own.

LEMMA 5.3. *Let $a_1, \dots, a_s, b_1, \dots, b_s \in \mathbb{R}$ with $a_i \geq b_i$, $1 \leq i \leq s$, and let*

$$S(k) = \sum_{i=1}^{s} (a_i^k e^{a_i} - b_i^k e^{b_i}), \qquad k = 0, 1, 2.$$

Then we have

$$S(2) \cdot S(0) - S^2(1) + S^2(0) \geq 0$$

with equality iff $a_i = b_i$ for all $1 \leq i \leq s$.

PROOF. With

$$A_i = ((a_i^2 + 1)e^{a_i} - (b_i^2 + 1)e^{b_i})(e^{a_i} - e^{b_i})$$

and

$$B_i = (a_i e^{a_i} - b_i e^{b_i})^2, \qquad 1 \leq i \leq s,$$

we find that

$$A_i \geq B_i \quad \text{iff} \quad (e^{a_i - b_i} - 1)^2 \geq (a_i - b_i)^2 e^{a_i - b_i}.$$

The latter is seen to hold true for all $a_i \geq b_i$ with equality iff $a_i = b_i$.

Then the Cauchy-Schwarz inequality yields

$$S(2) \cdot S(0) + S^2(0) \geq \left(\sum_{i=1}^{s} A_i^{1/2} \right)^2$$

with equality iff $a_i = b_i$ for all $1 \leq i \leq s$ and hence the assertion.

THEOREM 5.1. *For a distribution function F as in (5.3) we obtain:*

(i) *For $s = 1$ there is no MLE of β.*

(ii) *If* $s \geq 2$ *and* $\max_{1\leq k\leq s} x_{kr} > \min_{1\leq k\leq s} x_{k1}$, *then a unique MLE* β^* *of* β *exists, and it is given by the solution of the likelihood equation*

$$(5.4)\quad r/\beta + \sum_{i=1}^{s}\sum_{j=1}^{r}\log x_{ij} - \sum_{j=2}^{r}\left(\sum_{i=1}^{s}(x_{ij}^{\beta} - x_{i,j-1}^{\beta})\right)^{-1}\sum_{i=1}^{s}(x_{ij}^{\beta}\log x_{ij} - x_{i,j-1}^{\beta}\log x_{i,j-1}) - \left(\sum_{i=1}^{s}x_{i1}^{\beta}\right)^{-1}\sum_{i=1}^{s}x_{i1}^{\beta}\log x_{i1} = 0.$$

The MLE's $\tilde{\alpha}_j^*$ *of* $\tilde{\alpha}_j = \lambda\alpha_j$, $1 \leq j \leq r$, *are determined according to Lemma* 5.1 *with* $g(t) = t^{\beta^*}$.

PROOF. Differentiating $\Psi(x_{ij}, F)$ (see the proof of Lemma 3.1) with respect to β leads to the necessary condition (5.4).

(i) In the case $s = 1$, the condition simplifies to

$$\frac{\partial}{\partial\beta}\Psi(x_{ij}, F) = r/\beta - \sum_{j=2}^{r}(\log x_{1j} - \log x_{1,j-1})/((x_{1j}/x_{1,j-1})^{\beta} - 1) = 0.$$

Since $\frac{\partial}{\partial\beta}\Psi(x_{ij}, F) \geq \frac{1}{\beta}$ for $\beta \in (0, \infty)$, there is no solution of this equation.

(ii) Let $s \geq 2$. Then we have

$$\lim_{\beta\to 0}\frac{\partial}{\partial\beta}\Psi(x_{ij}, F) = \infty \quad \text{and}$$
$$\lim_{\beta\to\infty}\frac{\partial}{\partial\beta}\Psi(x_{ij}, F) = \sum_{j=1}^{r}\sum_{i=1}^{s}\left(\log x_{ij} - \max_{1\leq k\leq s}\log x_{kj}\right) < 0$$

(cf. Pike (1966) and McCool (1970) in the case of type II censoring).

Moreover, $\Psi(x_{ij}, F)$ is a convex function with respect to β, since

$$\frac{\partial^2}{\partial\beta^2}\Psi(x_{ij}, F) = -rs/\beta^2 - s\sum_{j=2}^{r}\left(\sum_{i=1}^{s}(x_{ij}^{\beta} - x_{i,j-1}^{\beta})\right)^{-2} \cdot \left(\left(\sum_{i=1}^{s}(x_{ij}^{\beta}\log^2 x_{ij} - x_{i,j-1}^{\beta}\log^2 x_{i,j-1})\right)\left(\sum_{i=1}^{s}(x_{ij}^{\beta} - x_{i,j-1}^{\beta})\right) - \left(\sum_{i=1}^{s}(x_{ij}^{\beta}\log x_{ij} - x_{i,j-1}^{\beta}\log x_{i,j-1})\right)^2\right)$$

$$- s\left(\sum_{i=1}^{s} x_{i1}^{\beta}\right)^{-2}$$

$$\cdot\left(\left(\sum_{i=1}^{s} x_{i1}^{\beta}\log^2 x_{i1}\right)\left(\sum_{i=1}^{s} x_{i1}^{\beta}\right) - \left(\sum_{i=1}^{s} x_{i1}^{\beta}\log x_{i1}\right)^2\right) \le 0$$

applying Lemma 5.3. Equality holds iff $x_{ij} = x_{11}$ for all $1 \le i \le s$ and $1 \le j \le r$. Thus the assertion follows.

Evidently there is no MLE of β in the case $s = 1$, since then $r + 1$ parameters are to be estimated based on r observations.

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