Peter J. Olver: *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1987.

Lie groups and Lie algebras have become not only an integral part of algebra, but also a common language for various fields of mathematics and mechanics, especially those where the global properties of geometric and topological objects should be derived from local ones. For example, nonlinear differential equations of field theory is only one of the areas where Lie group methods have been successfully applied and led to solutions of some key problems, both old and new.

The aim of this book is to present Lie groups as a language which is useful in applications to differential equations. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Within the classical framework of Lie, these groups consist of geometric transformations on the space of independent and dependent variables for the system and act on solutions by transforming their graphs.

All the examples of groups considered in geometry and theoretical physics are topological spaces on which the structure of a smooth manifold can be introduced in a natural way. As a general rule, we use the topology which arises on a transformation group after it is embedded in a matrix group whose topology is defined in the usual manner: matrices are assumed to be close if their elements are close. Thus, we arrive at a class of smooth manifolds such that the points on them can be 'multiplied', this multiplication satisfying all axioms of an algebraic group.

Chapter 1 contains the exposition of some fundamental ideas of Lie groups and Lie algebras. The first section gives a basic outline of the general concept of a manifold, the second doing the same for Lie groups, both local and global. Chapter 1 is addressed to those readers who are not very familiar with manifold language and can be considered as an introduction to the theory of Lie groups and algebras, in particular, the theory of transformation groups.

In theoretical physics, an important role is played by invariant functions. Let us consider a Lie group G acting on a smooth manifold M. The function $f \in C^{\infty}(M)$ is called the invariant of a given action if the following equality holds $f(g \cdot x) = f(x)$ for all $x \in M$, $g \in G$. Recall that the most efficient method for the construction of functions in the involution in Hamiltonian mechanics is the method of argument translation of the invariants of coadjoint representation. The great power of Lie group theory lies in the crucial observation that one can replace the complicated, nonlinear conditions for the invariance of a subset or function under the group transformations themselves, by an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action. This infinitesimal criterion is readily verifiable in practice and will thereby provide the key to the explicit determination of the symmetry groups of systems of differential equations. It is given as a useful, systematic, computational method that will explicitly determine the symmetry group of any given system of differential equations. A great number of problems are given in the book which are all of a practical character. Also well-presented is the classical theory of prolongation.

The significant question studied here is how to find group-invariant solutions of the differential equation. Also given is the construction of group-invariant solutions. As for the exposition, this is organized so that the applications-oriented reader can immediately learn the practical implementation of the method of constructing group-invariant solutions without having to delve into the theoretical foundations needed to justify the method. This general construction is illustrated by a number of interesting examples, including the heat equation, the Korteweg–de Vries equation, and the Euler equations of ideal fluid flow. Also given is the classification of group invariant solutions. From this point of view, the adjoint representation of the Lie group and the classification of subgroups and subalgebras is studied. The theory of group-invariant prolongation and reduction is also presented.

Variational problems constitute one of the most important classes of mathematical problems closely related to such fundamental physical and mechanical phenomena as motion and stability. For example, geodesic trajectories are solutions of the corresponding variational problem. The symmetry groups and conservation laws are considered within the framework of the calculus of variations. In particular, symmetries of Euler–Lagrange equations and the reduction of order are discussed. The general principle relating symmetry groups and conservation laws was first determined by E. Noether. The version presented in the book under review is the one most familiar to physicists and engineers, requiring only a knowledge of ordinary symmetry group theory. The result is still of great practical use and its effectiveness is illustrated by a number of examples of physical importance.

One of the themes presented in the book is that of the so-called generalized symmetries. The version of Noether's theorem appearing in previous considerations is only a special case of the more general theorem. The extension of Noether's methods to include divergence symmetries is presented within the framework of generalized symmetries. The basic theory of generalized vector fields and the associated group transformations are presented here and the connection between generalized symmetries of systems of differential equations and their prolongations is established. It is shown that many of the earlier applications of geometrical symmetries remain valid for generalized symmetries. In particular, Noether's theorem now provides a complete one-to-one correspondence between one-parameter groups of generalized variational symmetries of a function and the conservation laws of its associated Euler–Lagrange equations.

The final theme of this chapter is Helmholtz's version of the inverse problem of the calculus of variations which, when given a set of differential equations, forms the Euler–Lagrange equations for some variational problem. All of these results are manifestations of the exactness of the full variational complex. A selfcontained exposition of this complex is given, together with a much simplified proof of the exactness.

Mechanical systems are usually given as systems of ordinary differential equations on a Euclidean space. For example, the classical equations describing the motion of a three-dimensional rigid body with a fixed point are written in the three-dimensional Euclidean space $\mathbb{R}^3(x, y, z)$:

$$\dot{x} = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} yz, \qquad \dot{y} = \frac{\lambda_3 - \lambda_1}{\lambda_3 + \lambda_1} xz,$$

$$\dot{z} = \frac{\lambda_2 - \lambda_3}{\lambda_2 + \lambda_3} xy,$$

where $\lambda_1, \lambda_2, \lambda_3$ are real numbers. Some Hamiltonian systems given on $\mathbb{R}^n$ turn out to have a latent algebraic symmetry which, when revealed, allows them to be integrated. There are rather a lot of examples, but we shall single out a structure connected with Lie algebras. In its simplest form, this means the system we are studying preserves the orbits of the coadjoint representation action of a given Lie group whose Lie algebra is identified with the Euclidean space on which the system is given.

That is why the introduction to the theory of Poisson brackets, Hamiltonian vector fields, and Lie–Poisson structure on the manifolds are given in this book. In particular, the notion of the symplectic manifolds is discussed and the geometry of the moment map is studied. Also described is a simple and elegant construction which allows us to convert a Hamiltonian system which has a group of symmetries into a Hamiltonian system on a symplectic manifold of a lower dimension. This procedure is called the reduction of a Hamiltonian system and has applications in many interesting situations of classical mechanics.

The ideas of Lie group symmetry Hamiltonian systems of ordinary differential equations have been applied to the study of nonlinear partial differential equations. The principal innovations needed to convert a Hamiltonian system of ordinary differential equations to one of evolution equations are the replacing of the Hamiltonian function by a Hamiltonian functional, replacing the vector gradient operation by the variational derivative of the Hamiltonian functional, and replacing the skew-symmetric matrix by a skew-adjoint differential operator. That is why an introduction to the theory of Poisson brackets on functional spaces is given in this book and why the relation between symmetries and conservation laws is also discussed.

Finally, the recent theory of bi-Hamiltonian systems is also covered and we are interested in systems of evolution which, like the Korteweg–de Vries equation, can be written in Hamiltonian form in not just one but two different ways. Such systems have many remarkable properties, including soliton solutions, linearization by inverse scattering, and so on. A new proof of the basic theorem on bi-Hamiltonian systems is given here and a very interesting exposition of this theme is presented.

Applications of Lie Groups to Differential Equations is devoted to explaining a wide range of applications of continuous symmetry groups to physically important systems of differential equations. Emphasis is placed on significant applications of group-theoretic methods.

It is the first in the world literature that poses such a goal; it gives a systematic exposition of the idea of symmetry in the theory of differential equations. Therefore it will be very useful for mathematicians all over the world and can be used equally by undergraduates and those actively engaged in various fields. The exposition in the main part of the book is rather detailed but the book is supplemented by commentaries and exercises to sections where some results extending or complementing those in the main text are given.

This book should be an invaluable reference for mathematicians, physicists, and engineers.

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