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On the General Theory of Exact Controllability for Skew Symmetric Operators

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Abstract. We introduce a general formalism for linear evolution equations with skew adjoint operators. We make explicit the controllability operator as an expansion with respect to eigenfunctions. Using the fact that the eigenvalues are purely imaginary, we give sufficient controllability conditions. This approach is convenient for studying the asymptotic behaviour of the optimal control.

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0. Introduction

The problem of exact controllability for the wave equation has attracted considerable interest in recent years, see J. L. Lions [12, 13], I. A. Lasiecka and R. Triggiani [9], R. Triggiani [14] and references therein.

More recently, C. Bardos, G. Lebeau, and J. Rauch [1–4], using high frequency analysis, have shown the relation between the geometry of a ray and the exact controllability (or stabilization). Roughly, the result is that a necessary and sufficient condition of exact controllability is that any ray meets the control region before the time T. This result holds true both for Neumann and Dirichlet as well as for the Maxwell equation.

Here, we introduce a general formalism for linear evolution equations with skew adjoint operators. We make explicit the controllability operator as an expansion with respect to eigenfunctions. Using the fact that the eigenvalues are purely imaginary, we give sufficient controllability conditions. They are expressed in terms of eigenfunctions, which is natural since we are looking for the asymptotic behaviour or the behaviour for large time. In particular, this approach is convenient for studying the asymptotic behaviour of the optimal control, a problem considered by J. L. Lions [11], and Chin-Hsien-Li [6].

Let us briefly describe the main ideas of our approach in the context of *finite*-dimensional spaces.

Consider a linear system of the form

$$z' + \mathscr{A}z = \mathscr{B}v, \qquad z(0) = \varphi,$$

which is solved by the formula

$$z(t) = G(t)\varphi + \int_0^t G(t-s)\mathscr{B}v(s)\,\mathrm{d}s$$

where $G(t) = \mathrm{e}^{-\mathscr{A}t}$.

Controllability is linked with the *invertibility* of the controllability matrix

$$\Lambda(t) = \int_0^t G^*(s)\mathscr{B}\mathscr{B}^*G(s)\,\mathrm{d}s.$$

The state space (space of z) is denoted by $\mathscr{H}$, which is a *complex* finite-dimensional space of dimension $2N$. We assume that $\mathscr{A} = -\mathscr{A}^*$ and that the eigenvalues of $\mathscr{A}$ are $i\sqrt{\lambda_j}$, $-i\sqrt{\lambda_j}$, $j = 1, \ldots, N$. This feature will imply an interesting property leading to the invertibility of $\Lambda(t)$ for large t.

Let φ^j and $\bar{\varphi}^j$ be the eigenfunctions corresponding to $i\sqrt{\lambda_j}$, $-i\sqrt{\lambda_j}$, which form an orthonormal basis of $\mathscr{H}$. The operator $G(t)$ is expanded as

$$G(t)\varphi = \sum_{j=1}^N (\overline{c_j}\,\mathrm{e}^{-i\sqrt{\lambda_j}t}\varphi^j + c_j\,\mathrm{e}^{i\sqrt{\lambda_j}t}\bar{\varphi}^j),$$

where

$$\varphi = \sum_{j=1}^N (\overline{c_j}\varphi^j + c_j\bar{\varphi}^j).$$

Then the controllability matrix is expanded as

$$(\Lambda(t)\varphi, \tilde{\varphi}) = \int_0^t \sum_{j=1}^N (c_j\,\mathrm{e}^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j + \overline{c_j}\,\mathrm{e}^{-i\sqrt{\lambda_j}s}\mathscr{B}^*\varphi^j) \times$$

$$\times \sum_{j=1}^N (\tilde{c}_j\,\mathrm{e}^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j + \overline{\tilde{c}_j}\,\mathrm{e}^{-i\sqrt{\lambda_j}s}\mathscr{B}^*\varphi^j)\,\mathrm{d}s.$$

This form shows immediately that there is a principal term:

$$(\Lambda(t)\varphi, \tilde{\varphi}) = t\sum_{j=1}^N c_j\overline{\tilde{c}_j}|\mathscr{B}^*\varphi^j|^2 + (\tilde{\Lambda}(t)\varphi, \tilde{\varphi}),$$

where $\tilde{\Lambda}(t)$ involves all other terms. The important remark is that $|\tilde{\Lambda}(t)| \leqslant C$ independently of t. Therefore, if $|B^*\varphi^j|^2$ does not vanish, one can assert that

$$(\Lambda(t)\varphi, \varphi) \geqslant (c_0 t - c_1)|\varphi|^2$$

and, thus, for t sufficiently large, $\Lambda(t)$ is invertible.

This is the key property that we want to recover in infinite-dimensional spaces. The basic difficulty is the following. We may rely on an infinite expansion (i.e. $N \to \infty$), but this brings in a new parameter N, and terms must be controlled. Some additional features and assumptions are necessary, in particular flexibility of the state space.

After having developed the general theory, we see how it can be applied to recover the known results concerning the wave equation (Dirichlet and Neumann cases).

In a separate paper, we shall check that it also applies to the Maxwell equation. All these results have been derived in the literature, using the methodology of HUM, introduced by J. L. Lions.

Our approach has several advantages. It sheds the light on the fact that controllability for large time is a property to be expected for skew symmetric operators. The treatment for infinite-dimensional spaces is not as simple as the one sketched above.

Second, we have a general theorem which can be applied in specific cases by checking the assumptions. One does not need to redo all calculations for each case. Third, it 'probably' leads to the smallest time of controllability, since the calculations are exact in time (this has not been investigated, though). Besides, the asymptotic behaviour of the optimal control is easily obtained, whereas the problem was somewhat open for the wave equation.

Finally, our approach will fit nicely with a finite-dimensional approximation (Galerkin method) of the system, which will be the case in numerical approximations. It is well known that high frequencies are sources of difficulties in the numerical approximation of the wave equation.

1. A General Framework

1.1. NOTATION

Let V, H be two Hilbert spaces, with the usual set up $V \subset H = H' \subset V'$, each space being dense in the next one with a continuous injection.

We consider the operator A such that

$$\langle Av, \tilde{v} \rangle = ((v, \tilde{v})), \quad \forall v, \tilde{v} \in V, \tag{1.1}$$

with the usual notation $((\ .\ ))$, $(\ .\ )$ for the scalar product in V, H and $\langle\ .\ \rangle$ for the duality between V and V'.*

We introduce the domain of A in H defined by

$$D_A = \{v \in V \mid Av \in H\} \tag{1.2}$$

which is structured as a Hilbert space with the norm

$$\|v\|_{D_A} = |Av| \tag{1.3}$$

and A is an isometry from D_A into H, and V into V', since the norm of V' is

$$\|\xi\|_{V'} = \|A^{-1}\xi\|. \tag{1.4}$$

*For Hilbert spaces different from V, H we shall indicate explicitly the space in the norm notation; $|\ |_F$ will represent the norm of the Hilbert space F.

We thus have the following sequence

$$D_A \subset V \subset H = H' \subset V' \subset D'_A, \tag{1.5}$$

each space being dense in the next one with continuous injection. Note that A is also an isometry from H into D'_A.

1.2. THE CHANGE OF PIVOT SPACE

In the previous set-up, the space H is a pivot space. We shall need also a different set up where V' becomes the pivot space. Recall that V' is a Hilbert space for the scalar product

$$\begin{aligned}((\xi, \eta))_{V'} &= ((A^{-1}\xi, A^{-1}\eta)) \\ &= \langle A^{-1}\xi, \eta\rangle \end{aligned} \tag{1.6}$$

and $H \subset V'$, H being dense in V', with a continuous injection. We identify V' and its dual (which implies that this dual is now different from V). In this set-up, the dual of H becomes D'_A, since any linear continuous functional on H can be written as $h \to (h, A^{-1}\xi)$ where $\xi \in D'_A$, which coincides whenever $\xi \in V'$ with $((h, \xi))_{V'}$, hence we can write

$$H \subset V' \subset D'_A, \tag{1.7}$$

where it must be understood that V' coincides with its dual and D'_A is the dual of H.

To find the dual of V in this set up, we need to further introduce the space

$$\Delta_A = \{v \in V, Av \in V\}, \tag{1.8}$$

provided with the norm $\|v\|_{\Delta_A} = \|Av\|_V$. Then A is an isometry from Δ_A into V. Considering its dual Δ'_A when the pivot space is H, one can complete (1.5) as follows

$$\Delta_A \subset D_A \subset V \subset H = H' \subset V' \subset D'_A \subset \Delta'_A \tag{1.9}$$

and A is an isometry from V' into Δ'_A.

Now when we identify V' with its dual, Δ'_A appears as the dual of V, and we have

$$V \subset H \subset V' \subset D'_A \subset \Delta'_A. \tag{1.10}$$

The notation prime does not represent duality any more. To summarize if $h \in H$, $\xi \in D'_A$ we have

$$\langle h, \xi\rangle_{H,D'_A} = (h, A^{-1}\xi) \tag{1.11}$$

and if $v \in V$, $\zeta \in \Delta'_A$ we have

$$\langle v, \zeta\rangle_{V,\Delta'_A} = \langle v, A^{-1}\zeta\rangle. \tag{1.12}$$

1.3. THE OPERATOR $\mathscr{A}$

We shall now consider the product space

$$\mathscr{H} = H \times V' \tag{1.13}$$

which is identified with its *dual.* This means that in each component a different identification is made, H and its dual for the first component and V' and its dual for the second component. We thus have by construction $\mathscr{H} = \mathscr{H}'$.

Next we set

$$\mathscr{V} = V \times H, \qquad \mathscr{W} = D_A \times V$$

and according to Sections 1.1 and 1.2,

$$\mathscr{V}' = V' \times D'_A, \qquad \mathscr{W}' = D'_A \times \Delta'_A.$$

Moreover, A is an isometry from $\mathscr{W}$ into $\mathscr{H}$, $\mathscr{V}$ into $\mathscr{V}'$ and $\mathscr{H}$ into $\mathscr{W}'$. By A applied to a vector

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

we clearly mean

$$Az = \begin{pmatrix} Az_1 \\ Az_2 \end{pmatrix}.$$

We have the sequence

$$\mathscr{W} \subset \mathscr{V} \subset \mathscr{H} = \mathscr{H}' \subset \mathscr{V}' \subset \mathscr{W}'$$

each space being dense in the next one with continuous injection. We notice the duality products

$$\langle z, \eta \rangle_{\mathscr{V}, \mathscr{V}'} = \langle z_1, \eta_1 \rangle + (z_2, A^{-1}\eta_2), \tag{1.14}$$

$$\begin{aligned}\langle z, \eta \rangle_{\mathscr{W}, \mathscr{W}'} &= (Az, A^{-1}\eta)_{\mathscr{H}} \\ &= (Az_1, A^{-1}\eta_1) + \langle z_2, A^{-1}\eta_2 \rangle.\end{aligned} \tag{1.15}$$

We next define the operator $\mathscr{A}$ as follows

$$\mathscr{A} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix}$$

and we have the properties

$$\mathscr{A} \in \mathscr{L}(\mathscr{W}; \mathscr{V}) \cap \mathscr{L}(\mathscr{V}; \mathscr{H}) \cap \mathscr{L}(\mathscr{H}; \mathscr{V}') \cap \mathscr{L}(\mathscr{V}'; \mathscr{W}'). \tag{1.16}$$

We can check that $\mathscr{A}$ is skew symmetric, namely

$$\mathscr{A}^* = -\mathscr{A}. \tag{1.17}$$

For instance, let $\varphi \in \mathscr{W}$, $\psi \in \mathscr{V}'$, we must check that

$$\langle \mathscr{A}^*\varphi, \psi \rangle_{\mathscr{V}, \mathscr{V}'} = -\langle \mathscr{A}\varphi, \psi \rangle_{\mathscr{V}, \mathscr{V}'}$$

which amounts to checking that

$$\langle \varphi, \mathscr{A}\psi \rangle_{\mathscr{W}, \mathscr{W}'} = -\langle \mathscr{A}\varphi, \psi \rangle_{\mathscr{V}, \mathscr{V}'}$$

which is easy, using (1.13), (1.14).

We can also check that

$$((\mathscr{A}z, \tilde{z}))_F = -((z, \mathscr{A}\tilde{z}))_F, \quad \forall z, \tilde{z} \in \mathscr{Z} \tag{1.18}$$

where $(\mathscr{Z}, F)$ is, respectively, the pair $(\mathscr{W}, \mathscr{V})$; $(\mathscr{V}, \mathscr{H})$; $(\mathscr{H}, \mathscr{V}')$; $(\mathscr{V}', \mathscr{W}')$.

The operator $\mathscr{A}$ is monotone in $\mathscr{V}$, $\mathscr{H}$, $\mathscr{V}'$, $\mathscr{W}'$ and maximal monotone since the equation

$$\mathscr{A}z + z = f, \quad z \in \mathscr{Z}, \quad f \in F \tag{1.19}$$

has one and only one solution whenever the pair $(\mathscr{Z}, F)$ is as above.

By application of the Hille–Yosida theorem (see, for instance, H. Brezis [5]), one can solve the differential operational equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \mathscr{A}z = 0, \qquad z(0) = \varphi \tag{1.20}$$

with $\varphi \in \mathscr{Z} = \mathscr{W}$, $\mathscr{V}$, $\mathscr{H}$, $\mathscr{V}'$, respectively, and the solution belongs to $C^0([0, \infty); \mathscr{Z}) \cap C^1([0, \infty); F)$, where $F = \mathscr{V}$, $\mathscr{H}$, $\mathscr{V}'$, $\mathscr{W}'$, respectively.

1.4. THE GREEN OPERATOR

From (1.19) it follows that we can write $z(t) = G(t)\varphi$ and

$$G(t) \in \mathscr{L}(\mathscr{Z}; \mathscr{Z}), \quad \text{with } \mathscr{Z} = \mathscr{W}, \mathscr{V}, \mathscr{H}, \mathscr{H}'.$$

Moreover,

$$\|G(t)\varphi\|_F = \|\varphi\|_F, \quad \text{with } F = \mathscr{V}, \mathscr{H}, \mathscr{V}', \mathscr{W}',$$

which implies that $G(t)$ is a continuous semigroup of contractions on F. We may next check that $G(t)$ is a group and, more precisely,

$$G(-t) = G^{-1}(t) = JG(t)J,$$

where

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Moreover, consider the equation

$$-\frac{\mathrm{d}\zeta}{\mathrm{d}t} + \mathscr{A}^*\zeta = 0, \qquad \zeta(T) = \psi \tag{1.21}$$

and we can easily check by differential calculus that

$$(z(T), \psi)_{\mathscr{H}} = (\varphi, \zeta(0))_{\mathscr{H}}, \quad \forall \varphi, \psi \in \mathscr{V}. \tag{1.22}$$

This formula extends to φ, ψ in $\mathscr{H}$ and also

$$\langle z(T), \psi \rangle_{\mathscr{V}', \mathscr{V}''} = \langle \varphi, \zeta(0) \rangle_{\mathscr{V}', \mathscr{V}''} \tag{1.21'}$$

if $\psi \in \mathscr{V}$, $\varphi \in \mathscr{V}'$ or $\psi \in \mathscr{V}'$, $\varphi \in \mathscr{V}$.

Therefore, $\zeta(0) = G^*(T)\psi$, if $\psi \in \mathscr{V}, \mathscr{H}, \mathscr{V}'$. Because of (1.17) $\zeta(0) = G^{-1}(T)\psi$. Hence, we have

$$G^{-1}(t) = G^*(t) \in \mathscr{L}(\mathscr{H}; \mathscr{H}) \cap \mathscr{L}(\mathscr{V}; \mathscr{V}) \cap \mathscr{L}(\mathscr{V}'; \mathscr{V}'). \tag{1.23}$$

Since (1.21) extends to $\varphi \in \mathscr{W}'$, $\psi \in \mathscr{W}$ and $\varphi \in \mathscr{W}$, $\psi \in \mathscr{W}'$, respectively, and since from (1.17)

$$\|G(t)\varphi\|_{\mathscr{W}'} = \|\varphi\|_{\mathscr{W}'}$$

and similarly

$$\|G^*(t)\psi\|_{\mathscr{W}'} = \|\psi\|_{\mathscr{W}'}$$

we also deduce

$$\|G(t)\varphi\|_{\mathscr{W}} = \|\varphi\|_{\mathscr{W}}, \qquad \|G^*(t)\psi\|_{\mathscr{W}} = \|\psi\|_{\mathscr{W}} \tag{1.24}$$

and

$$G^{-1}(t) = G^*(t) \in \mathscr{L}(\mathscr{W}; \mathscr{W}) \cap \mathscr{L}(\mathscr{W}; \mathscr{W}'). \tag{1.25}$$

1.5. EIGENVALUES AND EXPANSION FORMULA

We consider the *complexified* versions of $\mathscr{W}, \mathscr{V}, \mathscr{H}, \mathscr{V}', \mathscr{W}'$, which is natural in the context of spectral analysis, and look for the eigenvalues of $\mathscr{A}$. At this stage, we need the additional assumption

$$\text{the injection of } V \text{ into } H \text{ is compact.} \tag{1.26}$$

This implies, together with the properties of A, that there exists a sequence $0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n < \cdots \lambda_n \uparrow +\infty$ of eigenvalues of A, such that

$$Aw_j = \lambda_j w_j, \quad w_j \in \Delta_A, \; |w_j|_H = 1 \tag{1.27}$$

and w_j is an orthonormal basis of H, whereas $w_j/\sqrt{\lambda_j}$, $\sqrt{\lambda_j}w_j$ are, respectively, orthonormal bases of V and V'. Note that (1.26) also implies

$$\text{the injection of } H \text{ into } V' \text{ is compact.} \tag{1.28}$$

Therefore

$$\text{the injection of } \mathscr{V} \text{ into } \mathscr{H} \text{ is compact} \tag{1.29}$$

It follows that the spectrum of $\mathscr{A}$ is discrete and, as easily checked, the eigenvalues are $i\sqrt{\lambda_j}$ and $-i\sqrt{\lambda_j}$ (the spectrum is purely imaginary).

The eigenvector corresponding to $i\sqrt{\lambda_j}$ is

$$\varphi^j = \frac{1}{\sqrt{2}}\begin{pmatrix} w_j \\ -i\sqrt{\lambda_j} \;\; w_j \end{pmatrix}$$

and that corresponding to $-i\sqrt{\lambda_j}$ is

$$\bar{\varphi}^j = \frac{1}{\sqrt{2}}\begin{pmatrix} w_j \\ i\sqrt{\lambda_j}\ w_j \end{pmatrix}$$

and it is easy to check that the sequence φ^j, $\bar{\varphi}^j$ forms an orthonormal basis of $\mathscr{H}$. Similarly, $\varphi^j/\sqrt{\lambda_j}$, $\bar{\varphi}^j/\sqrt{\lambda_j}$ forms an orthonormal basis of $\mathscr{V}$ and $\sqrt{\lambda_j}\varphi^j$, $\sqrt{\lambda_j}\bar{\varphi}^j$ forms an orthonormal basis of $\mathscr{V}'$.

The spectrum of $\mathscr{A}^* = -\mathscr{A}$ is also $i\sqrt{\lambda_j}$, $-i\sqrt{\lambda_j}$ and $\bar{\varphi}^j$ corresponds to $i\sqrt{\lambda_j}$, whereas φ^j corresponds to $-i\sqrt{\lambda_j}$.

We can give a representation formula for the solution of (1.20). If the initial condition is written as

$$\varphi = \sum_j (\bar{c}_j \varphi^j + c_j \bar{\varphi}^j), \tag{1.30}$$

where

$$c_j = (\varphi, \varphi^j)_{\mathscr{H}}. \tag{1.31}$$

The solutions of (1.19) is then

$$\begin{aligned} z(t) &= \sum_j (\bar{c}_j\, \mathrm{e}^{-i\sqrt{\lambda_j}t} \varphi^j + c_j\, \mathrm{e}^{i\sqrt{\lambda_j}t} \bar{\varphi}^j) \\ &= 2\,\mathrm{Re} \sum_j c_j\, \mathrm{e}^{i\sqrt{\lambda_j}t} \bar{\varphi}^j. \end{aligned} \tag{1.32}$$

2. Dynamic Systems – Controllability Operator

2.1. NONHOMOGENEOUS DIFFERENTIAL OPERATIONAL EQUATION

Let us consider the analogue of (1.20) with a right-hand side, namely

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \mathscr{A}z = f(t); \quad z(0) = \varphi, \tag{2.1}$$

where $\varphi \in \mathscr{Z} = \mathscr{W}, \mathscr{V}, \mathscr{H}, \mathscr{V}'$ and $f \in C^1([0, T]; F)$ with $F = \mathscr{V}, \mathscr{H}, \mathscr{V}', \mathscr{W}'$.

Then (2.1) has a unique solution $z \in C^1([0, T]; F) \cap C^0([0, T]; \mathscr{Z})$ given explicitly by

$$z(t) = G(t)\varphi + \int_0^t G(t-s) f(s)\, \mathrm{d}s. \tag{2.2}$$

Now (2.2) extends to $f \in L^1((0, T); F)$, and defines a function z in $C^0([0, T]; F)$ (note that $\mathscr{Z}$ is replaced for F). If F is, respectively, $\mathscr{V}, \mathscr{H}, \mathscr{V}'$, then $\mathrm{d}z/\mathrm{d}t \in L^1((0, T); E)$ with $E = \mathscr{H}, \mathscr{V}', \mathscr{W}'$.

2.2. CONTROLLED SYSTEM

Let $\mathscr{U}$ be a Hilbert space (the space of controls) which we *identify* with its dual. We consider an operator

$$\mathscr{B} \in \mathscr{L}(\mathscr{U}; \mathscr{H}) \tag{2.3}$$

and the dynamic system

$$\frac{dz}{dt} + \mathscr{A}z = \mathscr{B}v, \quad z(0) = \varphi \tag{2.4}$$

where $v(\,.\,) \in L^2_{loc}(0, \infty; \mathscr{U})$. Pick $\varphi \in \mathscr{H}$, then according to (2.2), Equation (2.4) has a unique solution $z(\,.\,)$ in $C^0([0, T]; \mathscr{H})$, $\forall T > 0$, given by the formula

$$z(t) = G(t)\varphi + \int_0^t G(t-s)\mathscr{B}v(s)\,ds. \tag{2.5}$$

2.3. EXACT CONTROLLABILITY

Consider the controls of the form

$$v(s) = \mathscr{B}^* G(s)\zeta, \tag{2.6}$$

where $\zeta \in \mathscr{H}$. Then the corresponding $z(t)$ can be written as

$$\begin{aligned} z(t) &= G(t)\left\{\varphi + \left(\int_0^t G^*(s)\mathscr{B}\mathscr{B}^* G(s)\,ds\right)\zeta\right\} \\ &= G(t)\{\varphi + \Lambda(t)\zeta\}, \end{aligned} \tag{2.7}$$

where

$$\Lambda(t) = \int_0^t G^*(s)\mathscr{B}\mathscr{B}^* G(s)\,ds \tag{2.8}$$

will be called the *controllability operator**. Clearly $\Lambda(t) \in \mathscr{L}(\mathscr{H}; \mathscr{H})$.

If for some t, we can solve the equation in ζ

$$\varphi + \Lambda(t)\zeta = 0, \tag{2.9}$$

then $z(t) = 0$ and the system is brought to a standstill.

2.4. BILINEAR FORM

Using the representation formula (1.32) we can give an explicit formula for $\Lambda(t)$. Let $\zeta, \tilde{\zeta} \in \mathscr{H}$, and

$$c_j = c_j(\zeta) = (\zeta, \varphi^j), \qquad \tilde{c}_j = c_j(\tilde{\zeta}) = (\tilde{\zeta}, \varphi^j),$$

then we associate to the controllability operator the bilinear form on $\mathscr{H}$ defined by

$$\begin{aligned} \lambda_t(\zeta, \tilde{\zeta}) &= (\Lambda(t)\zeta, \tilde{\zeta})_{\mathscr{H}} \\ &= 4\int_0^t \left(\sum_j \mathrm{Re}[c_j\, e^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j],\ \sum_k \mathrm{Re}[\tilde{c}^k\, e^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}} ds. \end{aligned} \tag{2.10}$$

The bilinear form λ_t is symmetric but not *coercive* in general in $\mathscr{H}$.

*It corresponds to the usual controllability operator of the pair $(\mathscr{A}^*, \mathscr{B}) = (-\mathscr{A}, \mathscr{B})$.

3. General Results on the Controllability Operator

3.1. EXTENSION OF $\Lambda(t)$

We are interested in further properties of $\Lambda(t)$ and of $\Gamma(t)=\Lambda(t)/t$ (in particular, its limit as $t\to\infty$), as well as invertibility properties. We shall first extend $\Lambda(t)$ and $\Gamma(t)$ as follows.

PROPOSITION 3.1. *$\Lambda(t)$ and $\Gamma(t)$ extend as operators $\in\mathscr{L}(\mathscr{V}';\mathscr{V}')$ and $\mathscr{L}(\mathscr{V};\mathscr{V})$.*

Proof. Let $\zeta\in\mathscr{V}'$ and $a_j=a_j(\zeta)=((\zeta,\sqrt{\lambda_j}\,\varphi^j))_{\mathscr{V}'}$. If $\zeta\in\mathscr{H}$ then $a_j=c_j/\sqrt{\lambda_j}$ and we have

$$\sum_j |a_j|^2=\tfrac{1}{2}\|\zeta\|^2_{\mathscr{V}'},\qquad \sum_j |c_j|^2=\tfrac{1}{2}|\zeta|^2_{\mathscr{H}}. \tag{3.1}$$

Similarly, we can state that

$$\sum_j \lambda_j|c_j|^2=\tfrac{1}{2}\|\zeta\|^2_{\mathscr{V}}. \tag{3.2}$$

We write (2.10) as

$$\begin{aligned}
\lambda_t(\zeta,\tilde{\zeta}) &= 4\int_0^t\left(\sum_j \operatorname{Re}[a_j\sqrt{\lambda_j}\,\mathrm{e}^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j],\ \sum_k \operatorname{Re}[\tilde{c}_k\,\mathrm{e}^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}}\mathrm{d}s\\
&= 4\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}s}\sum_j \operatorname{Im}[a_j\,\mathrm{e}^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j],\ \sum_k \operatorname{Re}[\tilde{c}_k\,\mathrm{e}^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}}\mathrm{d}s\\
&= 4\left(\sum_j \operatorname{Im}[a_j\,\mathrm{e}^{i\sqrt{\lambda_j}t}\mathscr{B}^*\bar{\varphi}^j],\ \sum_k \operatorname{Re}[\tilde{c}_k\,\mathrm{e}^{i\sqrt{\lambda_k}t}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}}-\\
&\quad -4\left(\sum_j \operatorname{Im}[a_j\mathscr{B}^*\bar{\varphi}^j],\ \sum_k \operatorname{Re}[\tilde{c}_k\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}}+\\
&\quad +4\int_0^t\left(\sum_j \operatorname{Im}[a_j\,\mathrm{e}^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j],\ \sum_k \operatorname{Im}[\sqrt{\lambda_k}\tilde{c}_k\,\mathrm{e}^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}}\mathrm{d}s,
\end{aligned}$$

hence

$$\begin{aligned}
|\lambda_t(\zeta,\tilde{\zeta})| &\leqslant C\left|\sum_j \operatorname{Im}[a_j\,\mathrm{e}^{i\sqrt{\lambda_j}t}\bar{\varphi}^j]\right|_{\mathscr{H}}\left|\sum_k \operatorname{Re}[\tilde{c}_k\,\mathrm{e}^{i\sqrt{\lambda_k}t}\bar{\varphi}^k]\right|_{\mathscr{H}}+\\
&\quad + C\left|\sum_j \operatorname{Im}[a_j\bar{\varphi}^j]\right|_{\mathscr{H}}\left|\sum_k \operatorname{Re}[\tilde{c}_k\bar{\varphi}^k]\right|_{\mathscr{H}}+\\
&\quad + C\int_0^t\left|\sum_j \operatorname{Im}[a_j\,\mathrm{e}^{i\sqrt{\lambda_k}s}\bar{\varphi}^j]\right|_{\mathscr{H}}\left|\sum_k \operatorname{Im}[\sqrt{\lambda_j}\tilde{c}_k\,\mathrm{e}^{i\sqrt{\lambda_k}s}\bar{\varphi}^k]\right|_{\mathscr{H}}\mathrm{d}s\\
&\leqslant C\left(\sum_j |a_j|^2\right)^{1/2}\left(\sum_k |\tilde{c}_k|^2\right)^{1/2}+Ct\left(\sum|a_j|^2\right)^{1/2}\left(\sum\lambda_k|\tilde{c}_k|^2\right)^{1/2}\\
&\leqslant C(t+1)\|\zeta\|_{\mathscr{V}'}\|\tilde{\zeta}\|_{\mathscr{V}}
\end{aligned}$$

and the desired result is obtained. □

We state the immediate corollary:

COROLLARY 3.1. *We have*

$$\begin{aligned} &\|\Lambda(t)\|_{\mathscr{L}(\mathscr{H};\mathscr{H})} \leqslant Ct, \\ &\|\Lambda(t)\|_{\mathscr{L}(\mathscr{V}';\mathscr{V}')} \leqslant C(t+1), \\ &\|\Lambda(t)\|_{\mathscr{L}(\mathscr{V};\mathscr{V})} \leqslant C(t+1). \end{aligned} \tag{3.3}$$

Remark 3.1. It is clear that if $\Lambda(t)$ is invertible for t_0, it is invertible for any $t \geqslant t_0$. For some type of problems (like plate and Schrödinger equations), $\Lambda(t)$ will be invertible for any time $t > 0$, or never invertible, E. Zuazua [15], G. Lebeau [10].

3.2. APPROXIMATION

Let P_N be the projector on the subspace of $\mathscr{H}$ generated by $\varphi^j, \bar{\varphi}^j, j = 1, \ldots, N$, hence

$$P_N\zeta = \sum_{j=1}^{N} (c_j\bar{\varphi}^j + \bar{c}_j\varphi^j).$$

Since also

$$P_N\zeta = \sum_{j=1}^{N} (a_j\sqrt{\lambda_j}\,\bar{\varphi}^j + \bar{a}_j\sqrt{\lambda_j}\,\varphi^j).$$

P_N is also a projector in $\mathscr{V}'$ and in $\mathscr{V}$. We shall denote

$$\gamma_t(\zeta, \tilde{\zeta}) = (\Gamma(t)\zeta, \tilde{\zeta}) = \frac{\lambda_t}{t}(\zeta, \tilde{\zeta}). \tag{3.4}$$

We have the

LEMMA 3.1. *On has the estimate*

$$|\gamma_t(\zeta, \tilde{\zeta}) - \gamma_t(P_N\zeta, P_N\tilde{\zeta})| \leqslant \frac{C}{\sqrt{\lambda_N}}\left(1 + \frac{1}{t}\right)|\zeta|_{\mathscr{H}}\,\|\tilde{\zeta}\|_{\mathscr{V}}. \tag{3.5}$$

Proof. We write

$$\begin{aligned} \gamma_t(\zeta, \tilde{\zeta}) - \gamma_t(P_N\zeta, P_N\tilde{\zeta}) &= \gamma_t(\zeta - P_N\zeta, \tilde{\zeta}) + \gamma_t(P_N\zeta, \tilde{\zeta} - P_N\tilde{\zeta}) \\ &= \frac{4}{t}\int_0^t \left(\sum_{j=N+1}^{\infty} \mathrm{Re}[c_j\, \mathrm{e}^{\mathrm{i}\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j],\ \sum_k \mathrm{Re}[\tilde{c}_k\, \mathrm{e}^{\mathrm{i}\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}} \mathrm{d}s + \\ &\quad + \frac{4}{t}\int_0^t \left(\sum_{j=1}^{N} \mathrm{Re}[c_j\, \mathrm{e}^{\mathrm{i}\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j],\ \sum_{k=N+1}^{\infty} \mathrm{Re}[\tilde{c}_k\, \mathrm{e}^{\mathrm{i}\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}} \mathrm{d}s = \mathrm{I} + \mathrm{II}. \end{aligned}$$

Operating as in the proof of Proposition 3.1, we have

$$\begin{aligned} \mathrm{I} &= \frac{4}{t}\left(\sum_{j=N+1}^{\infty} \mathrm{Im}\left[\frac{c_j}{\sqrt{\lambda_j}}\mathrm{e}^{\mathrm{i}\sqrt{\lambda_j}t}\mathscr{B}^*\bar{\varphi}^j\right],\ \sum_k \mathrm{Re}[\tilde{c}_k\, \mathrm{e}^{\mathrm{i}\sqrt{\lambda_k}t}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}} - \\ &\quad - \frac{4}{t}\left(\sum_{j=N+1}^{\infty} \mathrm{Im}\left[\frac{c_j}{\sqrt{\lambda_j}}\mathscr{B}^*\bar{\varphi}^j\right],\ \sum_k \mathrm{Re}[\tilde{c}_k\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}} + \\ &\quad + \frac{4}{t}\int_0^t \left(\sum_{j=N+1}^{\infty} \mathrm{Im}\left[\frac{c_j}{\sqrt{\lambda_j}}\,\mathrm{e}^{\mathrm{i}\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j\right],\ \sum_k \mathrm{Im}[\sqrt{\lambda_k}\,\tilde{c}_k\, \mathrm{e}^{\mathrm{i}\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]\right)_{\mathscr{U}} \mathrm{d}s \end{aligned}$$

and it follows that

$$|\mathrm{I}| \leqslant \frac{C}{\lambda_N}\left(1 + \frac{1}{t}\right) |\zeta|_{\mathscr{H}} \, \|\tilde{\zeta}\|_{\mathscr{V}'}.$$

We next write II as follows

$$\mathrm{II} = \frac{4}{t}\int_0^t \left(\sum_{j=1}^{N} \mathrm{Re}[c_j \, \mathrm{e}^{i\sqrt{\lambda_j}s} \mathscr{B}^* \bar{\varphi}^j], \sum_{k=N+1}^{\infty} \frac{\mathrm{Re}[\sqrt{\lambda_k}\,\tilde{c}_k \, \mathrm{e}^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k]}{\sqrt{\lambda_k}} \right)_{\mathscr{U}} \mathrm{d}s,$$

hence,

$$|\mathrm{II}| \leqslant \frac{C}{\lambda_N} |\zeta|_{\mathscr{H}} \, \|\tilde{\zeta}\|_{\mathscr{V}'}.$$

and the desired result has been proved. □

Similarly we prove the following lemma.

LEMMA 3.2. *We have the estimate*

$$|\gamma_t(\zeta, \tilde{\zeta}) - \gamma_t(P_N\zeta, P_N\tilde{\zeta})| \leqslant \frac{C}{\sqrt{\lambda_N}}\left(1 + \frac{1}{t}\right) \|\zeta\|_{\mathscr{W}'} \, \|\tilde{\zeta}\|_{\mathscr{V}'} + \frac{c}{t} |\zeta|_{\mathscr{H}} \, \|\tilde{\zeta}\|_{\mathscr{V}'}. \tag{3.6}$$

Proof. Since A is an isometry between $\mathscr{W}$ and $\mathscr{H}$, we have

$$\|\zeta\|^2_{\mathscr{W}'} = |A\zeta|^2_{\mathscr{H}} = 2\sum_j \lambda_j^2 |c_j|^2. \tag{3.7}$$

Considering the terms I and II defined in the previous lemma, we write

$$\begin{aligned}
\mathrm{I} &= \frac{4}{t}\int_0^t \left(\sum_{j=N+1}^{\infty} \mathrm{Re}[c_j \, \mathrm{e}^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j], \frac{\mathrm{d}}{\mathrm{d}s}\sum_k \mathrm{Im}\left[\frac{\tilde{c}_k}{\sqrt{\lambda_k}} \mathrm{e}^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k\right]\right)_{\mathscr{U}} \mathrm{d}s \\
&= \frac{4}{t}\left(\sum_{j=N+1}^{\infty} \mathrm{Re}[c_j \, \mathrm{e}^{i\sqrt{\lambda_j}t}\mathscr{B}^*\bar{\varphi}^j], \sum_k \mathrm{Im}\left[\frac{\tilde{c}_k}{\sqrt{\lambda_k}} \mathrm{e}^{i\sqrt{\lambda_k}t}\mathscr{B}^*\bar{\varphi}^k\right]\right)_{\mathscr{U}} - \\
&\quad - \frac{4}{t}\left(\sum_{j=N+1}^{\infty} \mathrm{Re}[c_j\mathscr{B}^*\bar{\varphi}^j], \sum_k \mathrm{Im}\left[\frac{\tilde{c}_k}{\sqrt{\lambda_k}}\mathscr{B}^*\bar{\varphi}^k\right]\right)_{\mathscr{U}} + \\
&\quad + \frac{4}{t}\int_0^t \left(\sum_{j=N+1}^{\infty} \mathrm{Im}[c_j\sqrt{\lambda_j}\, \mathrm{e}^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j], \sum_k \mathrm{Im}\left[\frac{\tilde{c}_k}{\sqrt{\lambda_k}} \mathrm{e}^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k\right]\right)_{\mathscr{U}} \mathrm{d}s
\end{aligned}$$

and from (3.7) we easily deduce

$$|\mathrm{I}| \leqslant \frac{c}{\sqrt{\lambda_N}}\left(1 + \frac{1}{t}\right) \|\zeta\|_{\mathscr{W}'} \, \|\tilde{\zeta}\|_{\mathscr{V}'}.$$

We then write, in a similar manner,

$$\begin{aligned}
\mathrm{II} = \frac{4}{t}\Bigg(\sum_{j=1}^{N} \operatorname{Re}[c_j\, e^{i\sqrt{\lambda_j}t}\mathscr{B}^*\bar{\varphi}^j], \sum_{k=N+1}^{\infty} \operatorname{Im}\left[\frac{\tilde{c}_k}{\sqrt{\lambda_k}}\, e^{i\sqrt{\lambda_k}t}\mathscr{B}^*\bar{\varphi}^k\right]\Bigg)_{\mathscr{U}} - \\
- \frac{4}{t}\Bigg(\sum_{j=1}^{N} \operatorname{Re}[c_j\mathscr{B}^*\bar{\varphi}^j], \sum_{k=N+1}^{\infty} \operatorname{Im}\left[\frac{\tilde{c}_k}{\sqrt{\lambda_k}}\mathscr{B}^*\bar{\varphi}^k\right]\Bigg)_{\mathscr{U}} - \\
- \frac{4}{t}\Bigg(\sum_{j=1}^{N} \operatorname{Im}[c_j\sqrt{\lambda_j}\, e^{i\sqrt{\lambda_j}t}\mathscr{B}^*\bar{\varphi}^j], \sum_{k=N+1}^{\infty} \operatorname{Re}\left[\frac{\tilde{c}_k}{\lambda_k}\, e^{i\sqrt{\lambda_k}t}\mathscr{B}^*\bar{\varphi}^k\right]\Bigg)_{\mathscr{U}} + \\
+ \frac{4}{t}\Bigg(\sum_{j=1}^{N} \operatorname{Im}[c_j\sqrt{\lambda_j}\mathscr{B}^*\bar{\varphi}^j], \sum_{k=N+1}^{\infty} \operatorname{Re}\left[\frac{\tilde{c}_k}{\lambda_k}\mathscr{B}^*\bar{\varphi}^k\right]\Bigg)_{\mathscr{U}} + \\
+ \frac{4}{t}\int_0^t\Bigg(\sum_{j=1}^{N} \operatorname{Re}[c_j\lambda_j\, e^{i\sqrt{\lambda_j}s}\mathscr{B}^*\bar{\varphi}^j], \sum_{k=N+1}^{\infty} \operatorname{Re}\left[\frac{\tilde{c}_k}{\lambda_k}\, e^{i\sqrt{\lambda_k}s}\mathscr{B}^*\bar{\varphi}^k\right]\Bigg)_{\mathscr{U}} \mathrm{d}s,
\end{aligned}$$

hence

$$|\mathrm{II}| \leqslant \frac{c}{t}|\zeta|_{\mathscr{H}}\,\|\tilde{\zeta}\|_{\mathscr{V}'} + \frac{c}{\lambda_N}\frac{1}{t}\|\zeta\|_{\mathscr{V}}\,\|\tilde{\zeta}\|_{\mathscr{H}} + \frac{c}{\lambda_N}\|\zeta\|_{\mathscr{W}}\,\|\tilde{\zeta}\|_{\mathscr{H}},$$

hence, (3.6) obtains. □

3.3. CONVERGENCE OF $\Gamma(t)$

Let us introduce the bilinear form on $\mathscr{H}$

$$\begin{aligned}
\gamma(\zeta, \tilde{\zeta}) &= 2\sum_j \operatorname{Re}\, c_j\bar{\tilde{c}}_j|\mathscr{B}^*\varphi^j|^2 \\
&= (\Gamma\zeta, \tilde{\zeta}),
\end{aligned} \tag{3.8}$$

then

$$\Gamma \in \mathscr{L}(\mathscr{H};\mathscr{H}) \cap \mathscr{L}(\mathscr{V};\mathscr{V}) \cap \mathscr{L}(\mathscr{V}';\mathscr{V}').$$

We clearly have

$$|\gamma(\gamma, \tilde{\zeta}) - \gamma(P_N\zeta, P_N\tilde{\zeta})| \leqslant \frac{c}{\sqrt{\lambda_N}}|\zeta|_{\mathscr{H}}\,\|\tilde{\zeta}\|_{\mathscr{V}'}, \tag{3.9}$$

$$|\gamma(\zeta, \tilde{\zeta}) - \gamma(P_N\zeta, P_N\tilde{\zeta})| \leqslant \frac{c}{\sqrt{\lambda_N}}\|\zeta\|_{\mathscr{W}}\,\|\tilde{\zeta}\|_{\mathscr{V}'}. \tag{3.10}$$

Moreover, we can state that

$$|\gamma_t(P_N\zeta, P_N\tilde{\zeta}) - \gamma(P_N\zeta, P_N\tilde{\zeta})| \leqslant \frac{\delta_N}{t}|\zeta|_{\mathscr{H}}\,\|\tilde{\zeta}\|_{\mathscr{V}'} \tag{3.11}$$

$$|\gamma_t(P_N\zeta, P_N\tilde{\zeta}) - \gamma(P_N\zeta, P_N\tilde{\zeta})| \leqslant \frac{\delta_N}{t}\|\zeta\|_{\mathscr{W}}\,\|\tilde{\zeta}\|_{\mathscr{V}'}. \tag{3.12}$$

These relations are easily obtained because we are now in the finite-dimensional case (see discussion of the Introduction). Note that the bound δ_N depends on N and tends to $+\infty$ as N tends to $+\infty$.

We next state the following convergence result.

THEOREM 3.1. *One has the property*

$$\Gamma(t)\zeta \to \Gamma\zeta \text{ in } \mathscr{V}', \quad \forall \zeta \text{ in } \mathscr{V}', \tag{3.13}$$

$$\Gamma(t)\zeta \to \Gamma\zeta \text{ in } \mathscr{V}, \quad \forall \zeta \text{ in } \mathscr{V}, \tag{3.14}$$

$$\Gamma(t)\zeta \to \Gamma\zeta \text{ in } \mathscr{H}, \quad \forall \zeta \text{ in } \mathscr{H}. \tag{3.15}$$

Proof. Let us first prove

$$\Gamma(t)\zeta \to \Gamma\zeta \text{ in } \mathscr{V}', \quad \forall \zeta \text{ in } \mathscr{H}. \tag{3.16}$$

Indeed, let $\tilde{\zeta} \in \mathscr{V}$, and consider

$$\gamma_t(\zeta, \tilde{\zeta}) - \gamma(\zeta, \tilde{\zeta}) = \gamma_t(\zeta, \tilde{\zeta}) - \gamma_t(P_N\zeta, P_N\tilde{\zeta}) + \\ + \gamma_t(P_N\zeta, P_N\tilde{\zeta}) - \gamma(P_N\zeta, P_N\tilde{\zeta}) + \gamma(P_N\zeta, P_N\tilde{\zeta}) - \gamma(\zeta, \tilde{\zeta}).$$

From (3.5), (3.9), and (3.11) we deduce

$$|\gamma_t(\zeta, \tilde{\zeta}) - \gamma(\zeta, \tilde{\zeta})| \leqslant \left(\frac{c}{\sqrt{\lambda_N}} + \frac{\delta_N}{t}\right)|\zeta|_{\mathscr{H}}\, \|\tilde{\zeta}\|_{\mathscr{V}}$$

and (3.16) follows easily.

To prove (3.13), note that for ζ in $\mathscr{V}'$, $P_N\zeta \in \mathscr{H}$ and tends to ζ in $\mathscr{V}'$, as $N \to \infty$. We can write

$$\Gamma(t)\zeta - \Gamma\zeta = \Gamma(t)(\zeta - P_N\zeta) + \Gamma(t)P_N\zeta - \Gamma P_N\zeta + \Gamma(P_N\zeta - \zeta).$$

Since $\|\Gamma(t)\|_{\mathscr{L}(\mathscr{V}';\mathscr{V}')} \leqslant K$, we deduce

$$\|\Gamma(t)\zeta - \Gamma\zeta\|_{\mathscr{V}'} \leqslant c\|\zeta - P_N\zeta\|_{\mathscr{V}'} + \|(\Gamma(t) - \Gamma)P_N\zeta\|_{\mathscr{V}'}$$

and the result follows from (3.16).

A similar proof holds for (3.14), using Lemma 3.2, (3.10) and (3.12), and proceeding as above. Now (3.15) is easily deduced from (3.14). □

3.4. SUFFICIENT CONDITION FOR EXACT CONTROLLABILITY

Instead of the equation of exact controllability (2.9), we shall consider the equivalent form

$$\Gamma(t)\zeta + \varphi = 0. \tag{3.17}$$

Clearly, if ζ is a solution of (2.9), then $t\zeta$ is a solution of (3.17), so it suffices to work with (3.17). We want to characterize the φ so that (3.17) has a solution. Naturally, the less regularity is required on ζ, the fewer conditions will be imposed on φ.

So we require ζ to be in $\mathcal{V}'$, and the problem amounts to characterizing the range of $\Gamma(t)$.

Without further assumptions, little can be said. We shall see that interesting results can be derived from the following assumptions

$$\gamma_t(\zeta, \zeta) \geq \beta \|\zeta\|^2_{\mathcal{V}'}, \quad \forall \zeta \in \mathcal{H}, \beta > 0, \tag{3.18}$$

at least for t large enough, and

$$|\mathcal{B}^* \varphi^j|^2 \geq \frac{c_0}{\lambda_j}. \tag{3.19}$$

Note that (3.19) implies at once from (3.8) that

$$\gamma(\zeta, \zeta) \geq 2c_0 \sum_j |a_j|^2 = c_0 \|\zeta\|^2_{\mathcal{V}'} \tag{3.20}$$

and, thus, the same property as (3.18) is true for $\gamma(\zeta, \zeta)$. From (3.18), it follows that $\gamma_t^{1/2}(\zeta, \zeta)$ is a *norm* in $\mathcal{H}$, not equivalent to that of $\mathcal{H}$, nor to that of $\mathcal{V}'$. We complete the space $\mathcal{H}$ with this norm and obtain a Hilbert space denoted $\mathcal{M}_t'$ (whose dual is denoted by $\mathcal{M}_t$).

We obtain the following inclusions

$$\mathcal{V} \subset \mathcal{M}_t \subset \mathcal{H} \subset \mathcal{M}_t' \subset \mathcal{V}'. \tag{3.21}$$

Therefore, $\Gamma(t) \in \mathcal{L}(\mathcal{M}_t'; \mathcal{M}_t)$ and has an inverse $\Gamma^{-1}(t) \in \mathcal{L}(\mathcal{M}_t; \mathcal{M}_t')$. It follows that if $\varphi \in \mathcal{M}_t$, there exists a unique solution of (3.17). From the inclusion (3.21), we deduce that it is sufficient to pick φ in $\mathcal{V}$, hence, we have the following proposition.

PROPOSITION 3.2. *If* (3.18) *holds, then there is exact controllability of the system* (2.4) *for* φ *in* $\mathcal{V}$. □

3.5. CONVERGENCE OF $\Gamma^{-1}(t)$

Let $\varphi \in \mathcal{V}$, and denote ρ_t the unique solution of

$$\Gamma(t)\rho_t + \varphi = 0, \quad \rho_t \in \mathcal{M}_t'. \tag{3.22}$$

In a similar way, we can solve (thanks to (3.19))

$$\Gamma\rho + \varphi = 0, \quad \rho \in \mathcal{M}', \tag{3.23}$$

where $\mathcal{M}'$ will be Hilbert space defined as $\mathcal{M}_t'$, with $\Gamma(t)$ replaced by Γ.

In fact, (3.23) has an explicit solution. Considering the formula (3.8), we deduce

$$c_j = (\rho, \varphi^j) = -\frac{(\varphi, \varphi^j)}{|\mathcal{B}^* \varphi^j|^2}. \tag{3.24}$$

We notice that

$$a_j = \frac{c_j}{\sqrt{\lambda_j}} = -\frac{1}{\lambda_j} \frac{((\varphi, \varphi^j/\sqrt{\lambda_j}))_{\mathcal{V}}}{|\mathcal{B}^* \varphi^j|^2}$$

and, therefore, by (3.19)

$$|a_j| \leqslant c_0 \left| \left(\left(\varphi, \frac{\varphi^j}{\sqrt{\lambda_j}} \right) \right)_{\mathscr{V}} \right|.$$

Since $\varphi \in \mathscr{V}$, we verify by this estimate that ρ is well defined in $\mathscr{V}'$ (in fact, we know a little bit more, $\rho \in \mathscr{M}'$). The explicit formula (3.24) allows us to check the following regularity result.

LEMMA 3.3. *If $\varphi \in \mathscr{W}$, then $\rho \in \mathscr{H}$, hence $\Gamma^{-1} \in \mathscr{L}(\mathscr{W}; \mathscr{H})$.*

Proof. If $\varphi \in \mathscr{W}$ then $\Sigma \lambda_j^2 |(\varphi, \varphi^j)|^2 < \infty$. But (3.24) implies $|c_j| \leqslant c_0 \lambda_j |(\varphi, \varphi^j)|$, hence $\Sigma |c_j|^2 < \infty$, which implies that $\rho \in \mathscr{H}$. □

We are now in a position to prove the following theorem.

THEOREM 3.2. *Assuming* (3.18), (3.19), *then we have*

$$\rho_t \to \rho \text{ in } \mathscr{V}' \quad \text{as } t \to \infty \tag{3.25}$$

Proof. We begin with the weak convergence. Since we have

$$\gamma_t(\rho_t, \rho_t) + \langle \varphi, \rho_t \rangle_{\mathscr{V}, \mathscr{V}'} = 0,$$

we deduce from (3.18) that $\|\rho_t\|_{\mathscr{V}'} \leqslant C$.

Therefore, we can extract a subsequence converging weakly to some σ in $\mathscr{V}'$. Now since $\Gamma(t)\rho_t + \varphi = 0$ for ζ in $\mathscr{V}$, we have

$$\langle \rho_t, \Gamma(t)\zeta \rangle + (\varphi, \zeta) = 0.$$

From (3.14) we can pass to the limit, as $t \to \infty$, to obtain $\langle \sigma, \Gamma\zeta \rangle + (\varphi, \zeta) = 0$, hence, $\Gamma\sigma + \varphi = 0$. Therefore

$$0 = \gamma(\sigma - \rho, \sigma - \rho) \geqslant c_0 \|\sigma - \rho\|_{\mathscr{V}'}^2,$$

hence, $\sigma = \rho$. By the uniqueness of the limit, the weak convergence of the whole sequence is obtained.

We next prove the strong convergence. It is done in two steps.

Strong convergence when $\varphi \in \mathscr{W}$. In that case we know that $\rho \in \mathscr{H}$, hence, from (3.15)

$$\Gamma(t)\rho \to \Gamma\rho \text{ in } \mathscr{H}. \tag{3.26}$$

Next we have

$$\gamma_t(\rho_t - \rho, \rho_t - \rho) = -\langle \varphi, \rho_t \rangle + 2(\varphi, \rho) + (\Gamma(t)\rho, \rho) \\ \to 0$$

hence, from (3.18), $\|\rho_t - \rho\|_{\mathscr{V}'} \to 0$.

General case. Let $\varphi \in \mathscr{V}$, and φ^μ in $\mathscr{W}$, such that $\varphi^\mu \to \varphi$ in $\mathscr{V}$, as $\mu \to 0$. Let ρ_t^μ be the solution of (3.22) with φ^μ instead of φ and, similarly, ρ^μ the solution of (3.23), for φ^μ.

We can write

$$\|\rho_t - \rho\|_{\mathscr{Y}'} \leqslant \|\rho_t - \rho_t^\mu\| + \|\rho_t^\mu - \rho^\mu\| + \|\rho^\mu - \rho\|. \tag{3.27}$$

But

$$\beta\|\rho_t - \rho_t^\mu\|^2 \leqslant \gamma_t(\rho_t - \rho_t^\mu, \rho_t - \rho_t^\mu) = -\langle \varphi - \varphi^\mu, \rho_t - \rho_t^\mu \rangle,$$

hence,

$$\|\rho_t - \rho_t^\mu\|_{\mathscr{Y}'} \leqslant c\|\varphi - \varphi^\mu\|_{\mathscr{Y}},$$

therefore from (3.27),

$$\|\rho_t - \rho\|_{\mathscr{Y}'} \leqslant c\|\varphi - \varphi^\mu\|_{\mathscr{Y}} + \|\rho_t^\mu - \rho^\mu\|_{\mathscr{Y}'}$$

and from the first part, the desired result easily follows. The proof has been completed. □

Remark 3.2. In practice, we are interested by ρ_t and not by ρ. But ρ_t will be computed through a numerical procedure whose purpose is to invert $\Gamma(t)$. Since ρ is an approximation of ρ_t for t large, and can be computed very easily, then it provides a way of checking that the numerical approximation used to compute ρ_t, whatever it is, is correct. It is enough to check it for large t and it should lead to an approximation of ρ.

4. Controllability for the Wave Equation with Dirichlet Conditions

4.1. NOTATIONS

Let Ω be a smooth bounded domain of R^n, and $\Gamma = \partial\Omega$. We shall take

$$H = L^2(\Omega), \qquad V = H_0^1(\Omega), \quad V' = H^{-1}(\Omega) \quad \text{and} \quad A = -\Delta.$$

Then $D_A = H^2 \cap H_0^1$.

Let $N = \partial/\partial\nu$, where ν denotes the outward unit normal. We have $N \in \mathscr{L}(D_A; L^2(\Gamma))$.

We pick $\mathscr{U} = L^2(\Gamma)$ and define

$$\mathscr{B}v = \begin{pmatrix} A^{-1}N^*v \\ 0 \end{pmatrix} \tag{4.1}$$

and $\mathscr{B} \in \mathscr{L}(\mathscr{U}; \mathscr{H})$. It is easy to check that

$$\mathscr{B}^*z = \frac{\partial}{\partial\nu} A^{-1}z_1, \quad \text{if } z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathscr{H}. \tag{4.2}$$

Considering the basis φ^j and $\bar{\varphi}^j$, we deduce

$$\mathscr{B}^*\varphi^j = \mathscr{B}^*\bar{\varphi}^j = \frac{1}{\sqrt{2}}\frac{1}{\lambda_j}\frac{\partial w_j}{\partial\nu}. \tag{4.3}$$

4.2. VERIFICATION OF THE ASSUMPTIONS OF THEOREM 3.2

Knowing that w_j satisfies

$$-\Delta w_j = \lambda_j w_j,\ w_j|_\Gamma = 0,\quad \int_\Omega w_j^2 = 1, \tag{4.4}$$

we first state the following lemma.

LEMMA 4.1. *Let* $q \in (C^1(R^n))^n$, *then we have*

$$\int_\Gamma \frac{\partial w_j}{\partial \nu}\frac{\partial w_k}{\partial \nu} q\cdot \nu \,\mathrm{d}\Gamma$$
$$= \int_\Omega \left[\frac{\partial w_j}{\partial x_\alpha}\frac{\partial w_k}{\partial x_\beta}\left(\frac{\partial q_\beta}{\partial x_\alpha} + \frac{\partial q_\alpha}{\partial x_\beta}\right) - \operatorname{div} q \cdot Dw_j, Dw_k - q_\alpha\left(\lambda_j w_j \frac{\partial w_k}{\partial x_\alpha} + \lambda_k w_k \frac{\partial w_j}{\partial x_\alpha}\right)\right]\mathrm{d}x. \tag{4.5}$$

Proof. We multiply (4.4) by $q_\alpha(\partial w_k/\partial x_\alpha)$ and perform integration by parts. Inverting j and k and adding up, the desired result follows. □

We apply (4.5) with $q = m$ and with $m(x) = x - x_0$. We deduce

$$\int_\Gamma \frac{\partial w_j}{\partial \nu}\frac{\partial w_k}{\partial \nu} m\cdot \nu \,\mathrm{d}\Gamma$$
$$= (2-n)\sqrt{\lambda_j}\sqrt{\lambda_k}\,\delta_{jk} - \int_\Omega m_\alpha\left(\lambda_j w_j \frac{\partial w_k}{\partial x_\alpha} + \lambda_k w_k \frac{\partial w_j}{\partial x_\alpha}\right)\mathrm{d}x. \tag{4.6}$$

In particular,

$$\int_\Gamma \left(\frac{\partial w_j}{\partial \nu}\right)^2 m\cdot \nu \,\mathrm{d}\Gamma = 2\lambda_j. \tag{4.7}$$

We have used the fact deduced from (4.4)

$$\int_\Omega Dw_j \cdot Dw_k \,\mathrm{d}x = \sqrt{\lambda_j}\sqrt{\lambda_k}\,\delta_{jk}.$$

Let $R(x_0) = \sup_{x\in\Gamma} |m(x)|$, then $m, \nu \leqslant R(x_0)$, hence, from (4.7),

$$\int_\Gamma \left(\frac{\partial w_j}{\partial \nu}\right)^2 \mathrm{d}\Gamma \geqslant \frac{2\lambda_j}{R(x_0)}.$$

Together with (4.3), it follows immediately that (3.19) holds with $c_0 = \frac{1}{2}R(x_0)$.

We next check (3.18). First, from (2.10) and (4.3), we have

$$\Lambda_t(\zeta, \zeta) = 2\int_0^t \int_\Gamma \left|\sum_j \operatorname{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s})\frac{\partial \hat{w}_j}{\partial \nu}\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma, \tag{4.8}$$

where we have set $\hat{w}_j = w_j/\sqrt{\lambda_j}$. We then state the following proposition

PROPOSITION 4.1. *The property* (3.18) *holds.*

Proof. We have from (4.8)

$$\lambda_t(\zeta, \zeta) \geqslant \frac{2}{R(x_0)} \int_0^t \int_\Gamma m\nu \left| \sum_j \operatorname{Re}(a_j\, e^{i\sqrt{\lambda_j}s}) \frac{\partial \hat{w}_j}{\partial \nu} \right|^2 ds\, d\Gamma. \tag{4.9}$$

Now from (4.6)

$$\begin{aligned} X &= \int_0^t \int_\Gamma m\nu \left| \sum_j \operatorname{Re}(a_j\, e^{i\sqrt{\lambda_j}s}) \frac{\partial \hat{w}_j}{\partial \nu} \right|^2 ds\, d\Gamma \\ &= (2-n) \int_0^t \sum_j |\operatorname{Re}(a_j\, e^{i\sqrt{\lambda_j}s})|^2\, ds - \\ &\quad - 2 \int_0^t \int_\Omega m_\alpha \left[\sum_j \operatorname{Re}(a_j\, e^{i\sqrt{\lambda_j}s}) \sqrt{\lambda_j}\, w_j \right] \left[\sum_k \operatorname{Re}(a_k\, e^{i\sqrt{\lambda_k}s}) \frac{\partial \hat{w}_k}{\partial x_\alpha} \right] dx\, ds \\ &= (2-n) \int_0^t \sum_j |\operatorname{Re}(a_j\, e^{i\sqrt{\lambda_j}s})|^2\, ds + Y. \end{aligned}$$

Then

$$\begin{aligned} Y &= 2 \int_0^t \int_\Omega m_\alpha \operatorname{Re}\left[i \frac{d}{ds} \left(\sum_j a_j\, e^{i\sqrt{\lambda_j}s} w_j \right) \right] \operatorname{Re}\left[\sum_k a_k\, e^{i\sqrt{\lambda_k}s} \frac{\partial \hat{w}_k}{\partial x_\alpha} \right] dx\, ds \\ &= - \int_\Omega 2m_\alpha \left\{ \operatorname{Im}\left[\sum_j a_j\, e^{i\sqrt{\lambda_j}t} w_j \right] \operatorname{Re}\left[\sum_k a_k\, e^{i\sqrt{\lambda_k}t} \frac{\partial \hat{w}_k}{\partial x_\alpha} \right] - \right. \\ &\quad \left. - \operatorname{Im}\left[\sum_j a_j w_j \right] \operatorname{Re}\left[\sum_k a_k \frac{\partial \hat{w}_k}{\partial x_\alpha} \right] \right\} ds - \\ &\quad - 2 \int_0^t \int_\Omega m_\alpha \operatorname{Im}\left[\sum_j a_j\, e^{i\sqrt{\lambda_j}s} w_j \right] \operatorname{Im}\left[\sum_k a_k\, e^{i\sqrt{\lambda_k}s} \frac{\partial w_k}{\partial x_\alpha} \right] dx\, ds. \end{aligned}$$

Therefore

$$\begin{aligned} X &= (2-n) \int_0^t \sum_j |\operatorname{Re}(a_j\, e^{i\sqrt{\lambda_j}s})|^2\, ds + n \int_0^t \sum_j |\operatorname{Im}(a_j\, e^{i\sqrt{\lambda_j}s})|^2\, ds - \\ &\quad - 2 \int_\Omega m_\alpha \left\{ \left(\sum_j \operatorname{Im}(a_j\, e^{i\sqrt{\lambda_j}t}) w_j \right) \left(\sum_k \operatorname{Re}(a_k\, e^{i\sqrt{\lambda_k}t}) \frac{\partial \hat{w}_k}{\partial x_\alpha} \right) - \right. \\ &\quad \left. - \left(\sum_j \operatorname{Im} a_j w_j \right) \left(\sum_k \operatorname{Re}(a_k) \frac{\partial \hat{w}_k}{\partial x_\alpha} \right) \right\} dx. \\ &= t \sum_j |a_j|^2 + \frac{n-1}{2} \sum_j \frac{1}{\sqrt{\lambda_j}} (\operatorname{Im} a_j^2 - \operatorname{Im} a_j^2\, e^{2i\sqrt{\lambda_j}t}) - \\ &\quad - 2 \int_\Omega m_\alpha \left\{ \left(\sum_j \operatorname{Im}(a_j\, e^{i\sqrt{\lambda_j}t}) w_j \right) \left(\sum_k \operatorname{Re}(a_k\, e^{i\sqrt{\lambda_k}t}) \frac{\partial \hat{w}_k}{\partial x_\alpha} \right) - \right. \\ &\quad \left. - \left(\sum_j \operatorname{Im} a_j w_j \right) \left(\sum_k \operatorname{Re}(a_k) \frac{\partial \hat{w}_k}{\partial x_\alpha} \right) \right\} dx. \end{aligned}$$

Collecting the results, it follows that

$$\lambda_t(\zeta, \zeta) \geqslant \frac{1}{R(x_0)}\{t - c_0 - c_1\mu(x_0)\}\|\zeta\|^2_{\mathcal{V}'}, \tag{4.10}$$

where $\mu(x_0) = \sup_{x\in\Omega}|x - x_0|$, and the desired result follows. □

4.3. ADDITIONAL RESULTS

In fact, in the present context, one can even say that

$$\mathcal{M}'_t = \mathcal{V}'. \tag{4.11}$$

This follows from the following proposition.

PROPOSITION 4.2. *We have the estimate*

$$\lambda_t(\zeta, \zeta) \leqslant c(t + 1)\|\zeta\|^2_{\mathcal{V}'}. \tag{4.12}$$

Proof. We shall use (4.5) with $q = h$, such that $h, \nu = 1$, which can be found. Hence, we can state

$$\begin{aligned}
&\int_\Gamma \frac{\partial \hat{w}_j}{\partial \nu}\frac{\partial \hat{w}_k}{\partial \nu}\,\mathrm{d}\Gamma \\
&= \int_\Omega \left[\frac{\partial \hat{w}_j}{\partial x_\alpha}\frac{\partial \hat{w}_k}{\partial x_\beta}\left(\frac{\partial h_\beta}{\partial x_\alpha} + \frac{\partial h_\alpha}{\partial x_\beta}\right) - \operatorname{div} h D\hat{w}_j \cdot D\hat{w}_k - \right.\\
&\quad \left. - h_\alpha\left(\sqrt{\lambda_j}\, w_j \frac{\partial \hat{w}_k}{\partial x_\alpha} + \sqrt{\lambda_k}\, w_k \frac{\partial \hat{w}_j}{\partial x_\alpha}\right)\right]\mathrm{d}x.
\end{aligned} \tag{4.13}$$

We deduce

$$\begin{aligned}
\lambda_t(\zeta,\zeta) &= 2\int_0^t\int_\Omega \left(\frac{\partial h_\alpha}{\partial x_\beta} + \frac{\partial h_\beta}{\partial x_\alpha}\right)\left(\sum_j \operatorname{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s})\frac{\partial \hat{w}_j}{\partial x_\alpha}\right) \times \\
&\quad \times \left(\sum_k \operatorname{Re}(a_k\, \mathrm{e}^{i\sqrt{\lambda_k}s})\frac{\partial \hat{w}_k}{\partial x_\alpha}\right)\mathrm{d}x\,\mathrm{d}s \\
&\quad - 2\int_0^t\int_\Omega \operatorname{div} h\left|\sum_j \operatorname{Re}(a_j \mathrm{e}^{i\sqrt{\lambda_j}s})D\hat{w}_j\right|^2 \mathrm{d}x\,\mathrm{d}s - \\
&\quad - 2\int_0^t\int_\Omega h_\alpha\left(\sum_j \operatorname{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s})\sqrt{\lambda_j}\, w_j\right)\left(\sum_k \operatorname{Re}(a_k\, \mathrm{e}^{i\sqrt{\lambda_k}s})\frac{\partial \hat{w}_k}{\partial x_\alpha}\right)\mathrm{d}x\,\mathrm{d}s \\
&= \lambda_t^1 + \lambda_t^2 + \lambda_t^3.
\end{aligned} \tag{4.14}$$

We check easily that

$$|\lambda_t^2|,\ |\lambda_t^2| \leqslant ct\sum_j |a_j|^2.$$

For λ_t^3 we operate as in the proof of Proposition 4.1 (see the treatment of the last integral) to derive the desired result. □

4.4. INTERPRETATION

Let us interpret the dynamic system (2.4) with the operators $A = -\Delta$ and $\mathscr{B}$ given by (4.1). We write

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and obtain

$$\begin{aligned} &z_1' - z_2 = (-\Delta)^{-1}N^*v, \qquad z_2' - \Delta z_1 = 0, \\ &z_1(0) = \varphi_1, \qquad z_2(0) = \varphi_2, \\ &z_1 \in C([0, T]; L^2), \qquad z_2 \in C(0, T; H^{-1}). \end{aligned} \tag{4.15}$$

If we set $z_2 = \eta$, then $z_1 = -(-\Delta)^{-1}\eta'$, then we have

$$\begin{aligned} &(-\Delta)^{-1}\eta'' + \eta = -(-\Delta)^{-1}N^*v, \\ &\eta(0) = y_0 = \varphi_2; \qquad \eta'(0) = y_1 = \Delta\varphi_1, \\ &\eta \in C([0, T]; H^{-1}); \qquad (-\Delta)^{-1}\eta' \in C(0, T; L^2); \\ &(-\Delta)^{-1}\eta'' \in L^2(0, T; H^{-1}) \end{aligned} \tag{4.16}$$

and η is 'formally' the solution of

$$\eta'' - \Delta\eta = 0, \quad \eta|_\Sigma = v, \qquad \eta(0) = y_0, \quad \eta'(0) = y_1. \tag{4.17}$$

Indeed we use the fact that the functions in L^2, $\psi = -(-\Delta)^{-1}N^*v$, are formally the solution of $-\Delta\psi = 0$, $\psi|_\Gamma = v$.

From Proposition 3.2, it follows that there exists exact controllability if $\varphi_1, \varphi_2 \in \mathscr{V}'$, hence, $y_0 \in L^2$ and $y_1 \in H^{-1}$, with $v(\,.\,) \in L^2(0, T; L^2(\Gamma))$, for some convenient T. The control is explicitly given by (2.6), hence, from 4.2, we easily check that

$$v = \frac{\partial\theta}{\partial\nu}, \tag{4.18}$$

where θ is the solution of

$$\begin{aligned} &\theta'' - \Delta\theta = 0, \quad \theta|_\Sigma = 0, \\ &\theta(0) = (-\Delta)^{-1}\zeta_1, \qquad \theta'(0) = (-\Delta)^{-1}\zeta_2, \end{aligned} \tag{4.19}$$

where

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in H^{-1} \times (H^2 \cap H_0^1)'$$

is the solution of $\Lambda(T)\zeta + \varphi = 0$.

This is exactly the result given by 'HUM' (cf. J. L. Lions [12]).

Applying formulas (3.24), we get for the limit ρ

$$a_j = \sqrt{2}\lambda_j \frac{[\langle w_j/\sqrt{\lambda_j}, y_1\rangle + i(w_j, y_0)]}{|\partial w_j/\partial v|^2},$$

hence,

$$\rho = \begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}$$

is given by

$$\rho_1 = 2\sum_j \lambda_j \frac{\langle w_j, y_1\rangle}{|\partial w_j/\partial v|^2} w_j,$$

$$\rho_2 = -2\sum_j \lambda_j^2 \frac{(w_j, y_0)}{|\partial w_j/\partial v|^2} w_j, \tag{4.20}$$

hence, also,

$$(-\Delta)^{-1}\rho_1 = 2\sum_j \lambda_j \frac{\langle w_j, y_1\rangle}{|\partial w_j/\partial v|^2} w_j \in H_0^1,$$

$$(-\Delta)^{-1}\rho_2 = -2\sum_j \lambda_j \frac{(w_j, y_0)}{|\partial w_j/\partial v|^2} w_j \in L^2. \tag{4.21}$$

4.5. AN ADDITIONAL REMARK

Consider the case when the domain Ω is star-shaped with respect to x_0, hence, we have

$$m.v \geqslant 0. \tag{4.22}$$

We modify the operator $\mathscr{B}$, defined in (4.1), and then follows

$$\mathscr{B}v = \begin{pmatrix}(-\Delta)^{-1}N^*[v(mv)^{1/2}]\\ 0\end{pmatrix}, \tag{4.23}$$

hence, v has first been multiplied by $(mv)^{1/2}$. The operator $\mathscr{B}^* \in \mathscr{L}(\mathscr{U}; \mathscr{H})$ reads

$$\mathscr{B}^* z = (mv)^{1/2}\frac{\partial}{\partial v}[(-\Delta)^{-1}z_1]. \tag{4.24}$$

The bilinear form (2.10) becomes

$$\lambda_t(\zeta, \tilde{\zeta}) = 2\int_0^t \int_\Gamma mv \operatorname{Re}\left[\sum_j a_j\, e^{i\sqrt{\lambda_j}s}\frac{\partial \hat{w}_j}{\partial v}\right]\operatorname{Re}\left[\sum_j \tilde{a}_j\, e^{i\sqrt{\lambda_j}s}\frac{\partial \hat{w}_j}{\partial v}\right] ds\, d\Gamma. \tag{4.25}$$

Note that (4.10) is changed into

$$\lambda_t(\zeta, \zeta) \geqslant (t - c_0 - c_1\mu(x_0))\|\zeta\|_{\mathscr{Y}'}^2. \tag{4.26}$$

Note that from (4.7)

$$|\mathscr{B}^*\varphi^j|^2 = \frac{1}{2\lambda_j^2}\int_\Gamma mv\left(\frac{\partial w_j}{\partial v}\right)^2 \mathrm{d}\Gamma = \frac{1}{\lambda_j}.$$

Therefore, the bilinear form $\gamma(\zeta, \tilde{\zeta})$ on $\mathscr{V}'$ defined by (3.8), becomes extremely simple, namely

$$\gamma(\zeta, \tilde{\zeta}) = 2\sum_j \operatorname{Re} a_j\bar{\tilde{a}}_j = ((\zeta, \tilde{\zeta}))_{\mathscr{V}'}. \tag{4.27}$$

and, thus, $\Gamma = (-\Delta)^{-1}$.

Formulas (4.21) are then replaced by

$$(-\Delta)^{-1}\rho_1 = (-\Delta)^{-1}y_1, \qquad (-\Delta)^{-1}\rho_2 = -y_0. \tag{4.28}$$

5. Controllability for the Wave Equation with Neumann Conditions

5.1. NOTATION

We consider here the situation

$$H = L^2(\Omega), \qquad V = H^1(\Omega), \qquad A = -\Delta + I,$$

associated with the Neumann boundary condition. Hence,

$$D_A = \left\{v \in H^2 \left|\frac{\partial v}{\partial v}\right|_\Gamma = 0\right\},$$

$$\Delta_A = \left\{v \in H^3 \left|\frac{\partial v}{\partial v}\right|_\Gamma = 0\right\}.$$

The eigenvectors w_j are defined by the equation

$$-\Delta w_j + w_j = \lambda_j w_j, \qquad \left.\frac{\partial w_j}{\partial v}\right|_\Gamma = 0. \tag{5.1}$$

Let

$$\Gamma(x_0) = \{x \in \Gamma \mid m, v > 0\}, \quad m(x) = x - x_0,$$

$$\Gamma^*(x_0) = \Gamma - \Gamma(x_0).$$

We consider

$$\mathscr{U} = (L^2(\Gamma^*(x_0)))^{n+1} \times L^2(\Gamma(x_0))$$

and define

$$\gamma_0 = \text{trace on } \Gamma(x_0) \in \mathscr{L}(V;\, L^2(\Gamma(x_0))) \cap \mathscr{L}(D_A;\, H^1(\Gamma(x_0)))$$

and $\bar{\gamma}_0 =$ trace on $\Gamma^*(x_0)$ defined similarly with $\Gamma^*(x_0)$ instead of $\Gamma(x_0)$, $\sigma_j \in \mathscr{L}(H^1(\Gamma^*(x_0)); L^2(\Gamma^*(x_0)))$ such that

$$\bar{\gamma}_0 \frac{\partial \varphi}{\partial xj} = v_j \frac{\partial \varphi}{\partial v} \mathbf{1}_{\Gamma^*(x_0)} + \sigma_j \bar{\gamma}_0 \varphi, \quad \forall \varphi \in H^2.$$

Let $\bar{\gamma} \in \mathscr{L}(\mathscr{W}; \mathscr{U})$ be defined by

$$\bar{\gamma}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{\gamma}_0 z_1 \\ \sigma_1 \bar{\gamma}_0 z_1 \\ \vdots \\ \sigma_n \bar{\gamma}_0 z_1 \\ \gamma_0 z_2 \end{pmatrix}$$

then

$$\bar{\gamma}^* \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \bar{\gamma}_0^*(v_0 + \sigma_1^* v_1 + \cdots + \sigma_n^* v_n) \\ A \gamma_0^* v_{n+1} \end{pmatrix} \in \mathscr{W}'.$$

Next define

$$\mathscr{B} = -A^{-1}\bar{\gamma}^* \in \mathscr{L}(\mathscr{U}; \mathscr{H}),$$

hence,

$$\mathscr{B}v = \begin{pmatrix} -\bar{N}_0(v_0 + \sigma_1^* v_1 + \cdots + \sigma_n^* v_n) \\ -\gamma_0^* v_{n+1} \end{pmatrix}, \tag{5.2}$$

where we have set

$$\bar{N}_0 = A^{-1}\bar{\gamma}_0^*$$

and

$$\bar{N}_0 \in \mathscr{L}(L^2(\Gamma^*(x_0)); V) \cap \mathscr{L}((H^1(\Gamma^*(x_0)))'; L^2).$$

'Formally', $\varphi = \bar{N}_0 g$ is the solution of

$$-\Delta\varphi + \varphi = 0, \qquad \left.\frac{\partial \varphi}{\partial v}\right|_{\Gamma^*(x_0)} = g, \qquad \left.\frac{\partial \varphi}{\partial v}\right|_{\Gamma(x_0)} = 0. \tag{5.3}$$

Next we can check that

$$\mathscr{B}^*\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\begin{pmatrix} \bar{N}_0^* z_1 \\ \sigma_1 \bar{N}_0^* z_1 \\ \vdots \\ \sigma_n \bar{N}_0^* z_1 \\ \gamma_0 A^{-1} z_2 \end{pmatrix} = -\begin{pmatrix} \bar{\gamma}_0 A^{-1} z_1 \\ \sigma_1 \bar{\gamma}_0 A^{-1} z_1 \\ \vdots \\ \sigma_n \bar{\gamma}_0 A^{-1} z_1 \\ \gamma_0 A^{-1} z_2 \end{pmatrix} \tag{5.4}$$

for

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathscr{H} \quad \text{or} \quad \mathscr{B}^* z = \gamma A^{-1} z.$$

Considering the base φ^j and $\bar{\varphi}^j$, we deduce

$$\mathscr{B}^* \bar{\varphi}^j = \frac{1}{\sqrt{2}} \begin{pmatrix} -\bar{\gamma}_0 \dfrac{w_j}{\lambda_j} \\ -\sigma_1 \bar{\gamma}_0 \dfrac{w_j}{\lambda_j} \\ \vdots \\ -\sigma_n \bar{\gamma}_0 \dfrac{w_j}{\lambda_j} \\ -i\gamma_0 \dfrac{w_j}{\sqrt{\lambda_j}} \end{pmatrix}$$

and $\mathscr{B}^* \varphi^j$ is the conjugate.

Denote

$$D_\sigma \text{ the tangential gradient} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix};$$

and $\gamma = \gamma_0 + \bar{\gamma}_0$, the trace on Γ. Note that

$$\gamma D\varphi = D_\sigma \gamma \varphi, \quad \text{if } \frac{\partial \varphi}{\partial v} = 0. \tag{5.5}$$

The bilinear form λ_t becomes (cf. (2.10)).

$$\begin{aligned} \lambda_t(\zeta, \tilde{\zeta}) = 2 \int_0^t \int_{\Gamma^*(x_0)} & \left[\left(\sum_j \operatorname{Re}(c_j \, e^{i\sqrt{\lambda_j} s}) D_\sigma \frac{w_j}{\lambda_j} \right) \cdot \left(\sum_k \operatorname{Re}(\tilde{c}_k \, e^{i\sqrt{\lambda_k} s}) D_\sigma \frac{w_k}{\lambda_k} \right) \mathrm{d}s \, \mathrm{d}\Gamma \, + \right. \\ & + 2 \int_0^t \int_{\Gamma^*(x_0)} \left. \left(\sum_j \operatorname{Re}(c_j \, e^{i\sqrt{\lambda_j} s}) \frac{w_j}{\lambda_j} \right) \left(\sum_k \operatorname{Re}(\tilde{c}_k \, e^{i\sqrt{\lambda_k} s}) \frac{w_k}{\lambda_k} \right) \right] \mathrm{d}s \, \mathrm{d}\Gamma \, + \\ & + 2 \int_0^t \int_{\Gamma(x_0)} \left(\sum_j \operatorname{Im}(c_j \, e^{i\sqrt{\lambda_j} s}) \frac{w_j}{\sqrt{\lambda_j}} \right) \left(\sum_k \operatorname{Im}(\tilde{c}_k \, e^{i\sqrt{\lambda_k} s}) \frac{w_k}{\sqrt{\lambda_k}} \right) \mathrm{d}s \, \mathrm{d}\Gamma. \end{aligned} \tag{5.6}$$

To simplify the writing, we have written w_j instead of $\bar{\gamma}_0 w_j$ or $\gamma_0 w_j$ in the boundary integrals. It is implicit.

We shall use the following result.

LEMMA 5.1. *We have the formula*

$$\begin{aligned} & \int_\Gamma m v (D_\sigma w_j \cdot D_\sigma w_k + w_j w_k) \, \mathrm{d}\Gamma \\ & \quad = (n-2)\sqrt{\lambda_k \lambda_j}\, \delta_{jk} + 2\delta_{jk} + \int_\Omega m_\alpha \left(\lambda_j w_j \frac{\partial w_k}{\partial x_\alpha} + \lambda_k w_k \frac{\partial w_j}{\partial x_\alpha} \right) \mathrm{d}x. \end{aligned} \tag{5.7}$$

We deduce from (5.7) that considering $\hat{w}_j = w_j/\sqrt{\lambda_j}$; we have

$$\int_\Gamma mv(D_\sigma \hat{w}_j \cdot D_\sigma \hat{w}_k + \hat{w}_j \hat{w}_k)\,\mathrm{d}\Gamma$$
$$= (n-2)\delta_{jk} + \frac{2}{\sqrt{\lambda_j \lambda_k}}\delta_{jk} + \int_\Omega m_\alpha\left(\sqrt{\lambda_j}\, w_j \frac{\partial \hat{w}_k}{\partial x_\alpha} + \lambda_k w_k \frac{\partial \hat{w}_j}{\partial x_\alpha}\right)\mathrm{d}x \tag{5.8}$$

and

$$\int_\Gamma mv(|D_\sigma \hat{w}_j|^2 + (\hat{w}_j)^2)\,\mathrm{d}\Gamma = -2 + \frac{2}{\lambda_j} + \int_\Gamma mv w_j^2\,\mathrm{d}\Gamma. \tag{5.9}$$

5.2. VERIFICATION OF THE ASSUMPTIONS OF THEOREM 3.2

We can write

$$\lambda_t(\zeta, \zeta) = 2\int_0^t \int_{\Gamma^*(x_0)} \left|\sum_j \mathrm{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) D_\sigma \hat{w}_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma +$$
$$+ 2\int_0^t \int_{\Gamma^*(x_0)} \left|\sum_j \mathrm{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) \hat{w}_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma +$$
$$+ 2\int_0^t \int_{\Gamma(x_0)} \left|\sum_j \mathrm{Im}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) w_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma. \tag{5.10}$$

We are going to prove the following proposition.

PROPOSITION 5.1. *The following estimate holds*

$$\lambda_t(\zeta, \zeta) \geqslant \left\{t \min\left(\frac{|\Gamma|}{2|\Omega|}, \frac{1}{R(x_0)}\left(1 - \frac{1}{\lambda_2}\right)\right) - \left(\frac{C_0 + C_1\mu(x_0)}{R(x_0)} + \frac{C|\Gamma|}{|\Omega|}\right)\right\} \|\zeta\|^2_{\mathscr{V}'}{}^{\star} \tag{5.11}$$

Proof. There is a special treatment for the first eigenvalue since $\lambda_1 = 1$, $w_1 = 1/|\Omega|^{1/2}$, hence, we write

$$\lambda_t(\zeta, \zeta) = 2\int_0^t \int_{\Gamma^*(x_0)} \left|\sum_{j \geqslant 2} \mathrm{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) D_\sigma \hat{w}_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma +$$
$$+2\int_0^t \int_{\Gamma^*(x_0)} \left|\frac{\mathrm{Re}(a_1\, \mathrm{e}^{is})}{|\Omega|^{1/2}} + \sum_{j \geqslant 2} \mathrm{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) \hat{w}_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma +$$
$$+ 2\int_0^t \int_{\Gamma(x_0)} \left|\frac{\mathrm{Im}(a_1\, \mathrm{e}^{is})}{|\Omega|^{1/2}} + \sum_{j \geqslant 2} \mathrm{Im}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) w_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma$$
$$= 2\int_0^t \int_{\Gamma^*(x_0)} \left|\sum_{j \geqslant 2} \mathrm{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) D_\sigma \hat{w}_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma +$$
$$+2\int_0^t \int_{\Gamma^*(x_0)} \left|\sum_{j \geqslant 2} \mathrm{Re}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) \hat{w}_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma +$$
$$+2\int_0^t \int_{\Gamma(x_0)} \left|\sum_{j \geqslant 2} \mathrm{Im}(a_j\, \mathrm{e}^{i\sqrt{\lambda_j}s}) w_j\right|^2 \mathrm{d}s\,\mathrm{d}\Gamma +$$

⋆We recall that $R(x_0) = \sup_{x \in \Gamma}|x - x_0|$, $\mu(x_0) = \sup_{x \in \Omega}|x - x_0|$.

$$+\frac{t|\Gamma|}{|\Omega|}|a_1|^2+\frac{|\Gamma^*(x_0)|-|\Gamma(x_0)|}{2|\Omega|}\operatorname{Im}[a_1^2(e^{2it}-1)]+$$

$$+\frac{2}{|\Omega|^{1/2}}\sum_{j\geqslant 2}\operatorname{Im}\left\{a_1\left[a_j\frac{(e^{i(1+\sqrt{\lambda_j})t}-1}{1+\sqrt{\lambda_j}}+\bar{a}_j\frac{(e^{i(1-\sqrt{\lambda_j})t}-1)}{1-\sqrt{\lambda_j}}\right]\right\}\int_{\Gamma^*(x_0)}\hat{w}_j\,d\Gamma-$$

$$-\frac{2}{|\Omega|^{1/2}}\sum_{j\geqslant 2}\operatorname{Im}\left\{a_1\left[a_j\frac{(e^{i(1+\sqrt{\lambda_j})t}-1)}{1+\sqrt{\lambda_j}}-\bar{a}_j\frac{(e^{i(1-\sqrt{\lambda_j})t}-1)}{1-\sqrt{\lambda_j}}\right]\right\}\int_{\Gamma(x_0)}w_j\,d\Gamma.$$

We then treat the different terms as follows

$$2\int_0^t\int_{\Gamma^*(x_0)}\left(\left|\sum_{j\geqslant 2}\operatorname{Re}(a_j\,e^{i\sqrt{\lambda_j}s})D_\sigma\hat{w}_j\right|^2+\sum_{j\geqslant 2}\operatorname{Re}(a_j\,e^{i\sqrt{\lambda_j}s})\hat{w}_j\Bigg|^2\right)ds\,d\Gamma+$$

$$+2\int_0^t\int_{\Gamma(x_0)}\left|\sum_{j\geqslant 2}\operatorname{Im}(a_j\,e^{i\sqrt{\lambda_j}s})w_j\right|^2 ds\,d\Gamma$$

$$\geqslant-\frac{2}{R(x_0)}\int_0^t\int_\Gamma mv\left[\left|\sum_{j\geqslant 2}\operatorname{Re}(a_j\,e^{i\sqrt{\lambda_j}s})D_\sigma\hat{w}_j\right|^2+\left|\sum_{j\geqslant 2}\operatorname{Re}(a_j\,e^{i\sqrt{\lambda_j}s})\hat{w}_j\right|^2\right]ds\,d\Gamma+$$

$$+\frac{2}{R(x_0)}\int_0^t\int_\Gamma mv\left|\sum_{j\geqslant 2}\operatorname{Im}(a_j\,e^{i\sqrt{\lambda_j}s})w_j\right|^2 ds\,d\Gamma=\frac{2}{R(x_0)}X.$$

From (5.8) we have

$$X=\int_0^t\sum_{j\geqslant 2}\left(2-n-\frac{2}{\lambda_j}\right)|\operatorname{Re}(a_j\,e^{i\sqrt{\lambda_j}s})|^2\,ds-$$

$$-2\int_0^t\int_\Omega m_\alpha\left(\sum_{j\geqslant 2}\sqrt{\lambda_j}\,w_j\operatorname{Re}(a_j\,e^{i\sqrt{\lambda_j}s})\right)\left(\sum_{k\geqslant 2}\frac{\partial\hat{w}_k}{\partial x_\alpha}\operatorname{Re}(a_k\,e^{i\sqrt{\lambda_k}s})\right)dx+$$

$$+\int_0^t\int_\Gamma mv\left|\sum_{j\geqslant 2}\operatorname{Im}(a_j e^{i\sqrt{\lambda_j}s})w_j\right|^2 ds\,d\Gamma.$$

Operating as in the proof of Proposition 4.1, we can write

$$X=\int_0^t\sum_{j\geqslant 2}\left(2-n-\frac{2}{\lambda_j}\right)|\operatorname{Re}(a_j\,e^{i\sqrt{\lambda_j}s})|^2\,ds+n\int_0^t\sum_{j\geqslant 2}|\operatorname{Im}(a_j\,e^{i\sqrt{\lambda_j}s})|^2\,ds-$$

$$-2\int_\Omega m_\alpha\left[\left(\sum_{j\geqslant 2}w_j\operatorname{Im}(a_j\,e^{i\sqrt{\lambda_j}t})\right)\left(\sum_{k\geqslant 2}\frac{\partial\hat{w}_k}{\partial x_\alpha}\operatorname{Re}(a_k\,e^{i\sqrt{\lambda_k}t})\right)-\right.$$

$$\left.-\left(\sum_{j\geqslant 2}w_j\operatorname{Im}a_j\right)\left(\sum_{k\geqslant 2}\frac{\partial\hat{w}_k}{\partial x_\alpha}\operatorname{Re}(a_k)\right)\right]dx$$

$$\geqslant\left(1-\frac{1}{\lambda_2}\right)t\sum_{j\geqslant 2}|a_j|^2+\left(1-\frac{1}{\lambda_2}-n\right)\sum_{j\geqslant 2}\frac{\operatorname{Im}[a_j^2(e^{2i\sqrt{\lambda_j}t}-1)]}{2\sqrt{\lambda_j}}-$$

$$-2\int_\Omega m_\alpha\left[\left(\sum_{j\geqslant 2}w_j\operatorname{Im}(a_j\,e^{i\sqrt{\lambda_j}t})\right)\left(\sum_{k\geqslant 2}\frac{\partial\hat{w}_k}{\partial x_\alpha}\operatorname{Re}(a_k\,e^{i\sqrt{\lambda_k}t})\right)-\right.$$

$$\left.-\left(\sum_{j\geqslant 2}w_j\operatorname{Im}(a_j)\right)\left(\sum_{k\geqslant 2}\frac{\partial\hat{w}_k}{\partial x_\alpha}\operatorname{Re}(a_k)\right)\right]dx$$

$$\geqslant\left(1-\frac{1}{\lambda_2}\right)t\sum_{j\geqslant 2}|a_j|^2-(c_0+c_1\mu(x_0))\sum_{j\geqslant 2}|a_j|^2.$$

Next consider the term

$$Z_1 = -\frac{2}{|\Omega|^{1/2}} \sum_{j\geqslant 2} \operatorname{Im}\left\{a_1\left[a_j \frac{((e^{i(1+\sqrt{\lambda_j})t}-1)}{1+\sqrt{\lambda_j}} - \bar{a}_j \frac{(e^{i(1-\sqrt{\lambda_j})t}-1)}{1-\sqrt{\lambda_j}}\right]\right\} \int_{\Gamma(x_0)} w_j \, d\Gamma$$

$$= \int_{\Gamma(x_0)} \left(\sum_j g_j(t)\hat{w}_j\right) d\Gamma,$$

where

$$g_j(t) = -\frac{2\sqrt{\lambda_j}}{|\Omega|^{1/2}} \operatorname{Im}\left\{a_1\left[a_j \frac{((e^{i(1+\sqrt{\lambda_j})t}-1)}{1+\sqrt{\lambda_j}} - \bar{a}_j \frac{(e^{i(1-\sqrt{\lambda_j})t}-1)}{1-\sqrt{\lambda_j}}\right]\right\},$$

hence

$$|Z_1| \leqslant \int_{\Gamma(x_0)} \left|\sum_{j\geqslant 2} g_j(t)\hat{w}_j\right| d\Gamma \leqslant |\Gamma(x_0)|^{1/2} \left(\int_{\Gamma(x_0)} \left|\sum_{j\geqslant 2} g_j(t)\hat{w}_j\right|^2 d\Gamma\right)^{1/2}$$

$$\leqslant C \frac{|\Gamma|}{|\Omega|^{1/2}} \left\|\sum_{j\geqslant 2} g_j(t)\hat{w}_j\right\|_{H^1}$$

$$= C \frac{|\Gamma|}{|\Omega|^{1/2}} \left(\sum_{j\geqslant 2} |g_j(t)|^2\right)^{1/2}$$

$$\leqslant C \frac{|\Gamma|}{|\Omega|} |a_1| \left(\sum_{j\geqslant 2} |a_j|^2\right)^{1/2}.$$

Similarly, considering

$$Z_2 = \frac{2}{|\Omega|^{1/2}} \sum_{j\geqslant 2} \operatorname{Im}\left\{a_1\left[a_j \frac{((e^{i(1+\sqrt{\lambda_j})t}-1)}{1+\sqrt{\lambda_j}} - \bar{a}_j \frac{(e^{i(1-\sqrt{\lambda_j})t}-1)}{1-\sqrt{\lambda_j}}\right]\right\} \int_{\Gamma^*(x_0)} \hat{w}_j \, d\Gamma.$$

An estimate similar to that holding for $|Z_1|$, holds for $|Z_2|$.

Collecting the results, one obtains

$$\lambda_t(\zeta,\zeta) \geqslant t\frac{|\Gamma|}{|\Omega|}|a_1|^2 + \frac{2}{R(x_0)} t\left(1-\frac{1}{\lambda_2}\right) \sum_{j\geqslant 2} |a_j|^2 -$$

$$- \frac{2}{R(x_0)} (c_0 + c_1\mu(x_0)) \sum_{j\geqslant 2} |a_j|^2 - c\frac{|\Gamma|}{|\Omega|}\left(|a_1|^2 + \sum_{j\geqslant 2} |a_j|^2\right) \tag{5.12}$$

and the desired result (5.11) is obtained. □

Clearly (3.18) follows from Proposition 5.1. Let us check (3.19). But

$$|\mathscr{B}^*\varphi^j|^2 = \frac{1}{2}\int_{\Gamma^*(x_0)} \left(\frac{w_j^2}{\lambda_j^2} + \frac{|D_\sigma w_j|^2}{\lambda_j^2}\right) d\Gamma + \frac{1}{2}\int_{\Gamma(x_0)} \frac{w_j^2}{\lambda_j} d\Gamma,$$

hence

$$|\mathscr{B}^*\varphi^{1/2} = \frac{1}{2}\frac{|\Gamma|}{|\Omega|}$$

and for $j \geqslant 2$

$$|\mathscr{B}^*\varphi^j|^2 \geqslant -\frac{1}{2R(x_0)}\int_\Gamma \frac{mv}{\lambda_j^2}(w_j^2 + |D_\sigma w_j|^2)\,d\Gamma + \frac{1}{2R(x_0)}\int_\Gamma mv\frac{w_j^2}{\lambda_j}\,d\Gamma$$

and from (5.9)

$$= \frac{1}{R(x_0)}\frac{1}{\lambda_j}\left(1 - \frac{1}{\lambda_j}\right) \geqslant \frac{1}{R(x_0)}\frac{1}{\lambda_j}\left(1 - \frac{1}{\lambda_2}\right)$$

and, thus, (3.19) is also established with

$$c_0 = \min\left(\frac{1}{2}\frac{|\Gamma|}{|\Omega|}, \frac{1}{R(x_0)}\left(1 - \frac{1}{\lambda_2}\right)\right).$$

The assumptions of Theorem 3.2 are thus verified.

5.3. INTERPRETATION

We now interpret the dynamic system (2.4) with the operators $A = -\Delta + J$, and $\mathscr{B}$ given by (5.2). We write $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and obtain

$$\begin{aligned}
&z_1' - z_2 = -\bar{N}_0(v_0 + \sigma_1^* v_1 + \cdots + \sigma_n^* v_n), \qquad z_2' + Az_1 = -\gamma_0^* v_{n+1}\\
&z_1(0) = \varphi_1, \qquad z_2(0) = \varphi_2,\\
&z_1 \in C([0, T]; L^2), \qquad z_2 \in C([0, T]; (H^1)'),\\
&z_1' \in L^2 \in L^2(0, T; (H^1)'), \qquad z_2' \in L^2([0, T]; D_A').
\end{aligned} \tag{5.13}$$

Set $\eta = z_2$, then $z_1 = -A^{-1}\eta' - N_0 v_{n+1}$ It follows that

$$\begin{aligned}
&A^{-1}\eta'' + \eta = -N_0 v_{n+1}' + \bar{N}_0(v_0 + \sigma_1^* v_1 + \cdots + \sigma_n^* v_n),\\
&\eta(0) = \varphi_2 = y_0,\\
&\eta'(0) = -A\varphi_1 - \gamma_0^* v_{n+1}(0) = y_1 - \gamma_0^* v_{n+1}(0).
\end{aligned} \tag{5.14}$$

From the interpretation of N_0, $\bar{N}_0$, we can write (5.14) as

$$\begin{aligned}
&\eta'' - \Delta n + \eta = 0,\\
&\left.\frac{\partial \eta}{\partial v}\right|_{\Gamma^*(x_0)} = v_0 + \sigma_1^* v_1 + \cdots + \sigma_n^* v_n,\\
&\left.\frac{\partial \eta}{\partial v}\right|_{\Gamma(x_0)} = -v_{n+1}',\\
&\eta(0) = y_0 \;.\; \eta'(0) = y_1 - \gamma_0^* v_{n+1}(0).
\end{aligned} \tag{5.15}$$

Note that $\eta'(0)$ is defined only when v_{n+1} is continuous. It is also very important to notice that, unlike the Dirichlet case (see (4.16)), we cannot impose $\eta'(0) = y_1$. We must allow some control to influence the initial condition.

We can relate (5.14) to the transposition method (cf. J. L. Lions [12]).

Let us consider the equation

$$\theta'' - \Delta\theta + \theta = f, \qquad \left.\frac{\partial\theta}{\partial\nu}\right|_{\Gamma} = 0, \qquad \theta(T) = 0, \qquad \theta'(T) = 0. \tag{5.16}$$

If we proceed with a formal integration by parts between (5.14) and (5.15), we obtain

$$\begin{aligned}
&\int_0^T \langle f, \eta\rangle \, \mathrm{d}t \\
&= -(\theta'(0), y_0) + \langle\theta(0), y_1\rangle + \int_0^T \int_{\Gamma^*(x_0)} \times \\
&\times \left[\theta v_0 + \frac{\partial\theta}{\partial x_1} v_1 + \cdots + \frac{\partial\theta}{\partial x_n} v_n\right] \mathrm{d}s\,\mathrm{d}\Gamma + \\
&+ \int_0^T \int_{\Gamma(x_0)} \theta' v_{n+1}\, \mathrm{d}s\, \mathrm{d}\Gamma,
\end{aligned} \tag{5.17}$$

which provides a rigorous definition of η, for (5.14), which coincides with the second component of z in (5.13).

From Proposition 3.2, it follows that there exists exact controllability if $\varphi \in \mathscr{V}'$, hence, $\varphi_1 \in H^1$, $\varphi_2 \in L^2$, which implies $y_0 \in L^2$, $y_1 \in (H^1)'$.

Exact controllability here means $z_1(T) = 0$; $z_2(T) = 0$ for some convenient T. In terms of the function η, this means

$$\eta(T) = 0, \qquad \eta'(T) = -\gamma_0^* v_{n+1}(T) \tag{5.18}$$

and, thus, $\eta'(T)$ is different from 0.

Let us make explicit the control obtained from formula (2.6), which yields exact controllability. From (5.4) we see that

$$\begin{aligned}
v_0(s) &= -\bar{\gamma}_0 \psi, \\
v_1(s) &= -\sigma_1 \bar{\gamma}_0 \psi, \\
&\ \vdots \\
v_n(s) &= -\sigma_n \bar{\gamma}_0 \psi, \\
v_{n+1}(s) &= -\gamma_0 \psi',
\end{aligned} \tag{5.19}$$

where ψ is the solution of

$$\psi'' - \Delta\psi + \psi = 0, \qquad \left.\frac{\partial\psi}{\partial\nu}\right|_{\Sigma} = 0, \qquad \psi(0) = A^{-1}\zeta_1, \qquad \psi'(0) = A^{-1}\zeta_2, \tag{5.20}$$

where $\zeta_1 \in (H^1)'$, $\zeta_2 \in D_A'$ is the solution of $\Lambda(T)\zeta + \varphi = 0$.

Inserting (5.19) in (5.15) yields

$$\eta'' - \Delta\eta + \eta = 0,$$
$$\left.\frac{\partial\eta}{\partial v}\right|_{\Gamma^*(x_0)} = -(-\Delta_{\Gamma^*(x_0)} + I)\bar{\gamma}_0\psi,$$
$$\left.\frac{\partial\eta}{\partial v}\right|_{\Gamma(x_0)} = \gamma_0\psi'', \qquad \eta(0) = y_0, \qquad \eta'(0) = y_1 + \gamma_0^*\gamma_0 A^{-1}\zeta_2, \tag{5.21}$$

where we have set $-\Delta_{\Gamma^*(x_0)} = \sigma_1^*\sigma_1 + \cdots + \sigma_n^*\sigma_n$.

The precise definition of the solution η of (5.20) can be seen by the transposition method, see (5.17), namely considering θ given by (5.16) and ψ given by (5.20), then from (5.17)

$$\int_0^T \langle f, \eta\rangle \,\mathrm{d}t$$
$$= -(\theta'(0), y_0) + \langle\theta(0), y_1\rangle -$$
$$- \int_0^T \int_{\Gamma^*(x_0)} [\psi + D\theta . D\psi] \,\mathrm{d}s \,\mathrm{d}\Gamma - \int_0^T \int_{\Gamma(x_0)} \theta'\psi' \,\mathrm{d}s \,\mathrm{d}\Gamma. \tag{5.22}$$

We find the formula given by J. L. Lions [12], from 'HUM'. However, we must insert the control at the initial value of $\eta'(0)$, otherwise there is some contradiction (see J. L. Lions [12], Chap. III, Section 1.5).

There is another way to represent the control on $\Gamma(x_0)$ in (5.15). Consider the operator $J_0 \in \mathscr{L}(H^1(0, T; L^2(\Gamma(x_0))); L^2(0, T; L^2(\Gamma(x_0))))$ defined by $J_0 g = g'$, then we can write (5.15) as follows

$$\eta'' - \Delta\eta + \eta = 0,$$
$$\left.\frac{\partial\eta}{\partial v}\right|_{\Gamma^*(x_0)} = v_0 + \sigma_1^* v_1 + \cdots + \sigma_n^* v_n,$$
$$\left.\frac{\partial\eta}{\partial v}\right|_{\Gamma(x_0)} = J_0^* v_{n+1}, \qquad \eta(0) = y_0, \qquad \eta'(0) = y_1. \tag{5.23}$$

Let us justify (5.23). First note that $J_0^* v_{n+1} \in (H^1(0, T; L^2(\Gamma(x_0))))'$ and that if we write the following duality between (5.15) and (5.21)

$$\int_0^T \langle f, \eta\rangle \,\mathrm{d}t = -(\theta'(0), y(0)) + \langle\theta(0), \eta'(0)\rangle + \int_0^T \int_\Gamma \theta \frac{\partial\eta}{\partial v} \,\mathrm{d}s \,\mathrm{d}\Gamma \tag{5.24}$$

then we must interpret the boundary integral as

$$\int_0^T \langle\bar{\gamma}_0\theta, \bar{\gamma}_0 v_0 + \sigma_1^* v_1 + \cdots + \sigma_n^* v_n\rangle_{H^1(\Gamma^*(x_0)),(H^1(\Gamma^*(x_0)))'} \,\mathrm{d}s +$$
$$+ \langle\gamma_0\theta, J_0^* v_{n+1}\rangle_{(H^1(0,T;L^2(\Gamma(x_0))),(H^1(0,T;L^2(\Gamma(x_0))))'} \,\mathrm{d}s$$

and we recover (5.17).

Writing (5.23) has some advantages, since it has a meaning for $v_{n+1} \in L^2(0, T; L^2(\Gamma(x_0)))$ and allows us to write $\eta'(0) = y_1$, although this is misleading, since the quantity $J_0^* v_{n+1}$ influences the initial value.

Another merit of (5.23) is that it emphasizes the fact that the boundary control is an element of $L^2(0, T; (H^1(\Gamma^*(x_0)))') \times (H^1(0, T; L^2(\Gamma(x_0))))'$.

Using (5.18) we obtain

$$\begin{aligned} &\eta'' - \Delta\eta + \eta = 0, \\ &\quad \left.\frac{\partial\eta}{\partial v}\right|_{\Gamma^*(x_0)} = -(-\Delta_{\Gamma^*(x_0)} + I)\bar{\gamma}_0\psi, \\ &\quad \left.\frac{\partial\eta}{\partial v}\right|_{\Gamma(x_0)} = -J_0^*\gamma_0\psi', \\ &\qquad \eta(0) = y_0, \qquad \eta'(0) = y_1. \end{aligned} \tag{5.25}$$

Besides, instead of (5.18), we can write

$$\eta(T) = 0, \qquad \eta'(T) = 0 \tag{5.26}$$

for reasons similar to those justifying the writing of the initial condition. Indeed, consider (5.16) with a nonzero condition at T, namely

$$\theta'' - \Delta\theta + \theta = f, \qquad \left.\frac{\partial\theta}{\partial v}\right|_{\Gamma} = 0, \qquad \theta(T) = \theta_0, \qquad \theta'(T) = \theta_1, \tag{5.27}$$

then (5.22) is still valid, which justifies (5.26). The fact that (5.22) holds can be seen by proceeding as for (5.17) and taking account of (5.18). In other words, the initial conditions and the final conditions (5.26) are met, provided we leave aside the Dirac measures at 0 and T arising in J_0^*.

Let us finally describe what is the limit value ρ. We again apply (3.23) and get

$$\begin{aligned} a_j &= \frac{1}{\sqrt{2}} \frac{[\langle y_1, w_j/\sqrt{\lambda_j}\rangle + i(y_0, w_j)]}{\lambda_j |\mathscr{B}^*\varphi^j|^2} \\ &= \frac{\sqrt{2}[\langle y_1, w_j/\sqrt{\lambda_j}\rangle + i(y_0, wj)]}{\int_{\Gamma^*(x_0)} (\hat{w}_j^2 + |D_\sigma \hat{w}_j|^2)\, d\Gamma + \int_{\Gamma(x_0)} w_j^2\, d\Gamma} \end{aligned} \tag{5.28}$$

$$\hat{w}_j = \frac{w_j}{\sqrt{\lambda_j}}. \tag{5.29}$$

Therefore $\rho = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is defined by

$$\begin{aligned} \rho_1 &= 2 \sum_j \frac{\langle y_1, w_j\rangle w_j}{\int_{\Gamma^*(x_0)} (\hat{w}_j^2 + |D_\sigma \hat{w}_j|^2)\, d\Gamma + \int_{\Gamma(x_0)} w_j^2\, d\Gamma}, \\ \rho_2 &= -2 \sum_j \frac{\lambda_j (y_0, w_j) w_j}{\int_{\Gamma^*(x_0)} (\hat{w}_j^2 + |D_\sigma \hat{w}_j|^2)\, d\Gamma + \int_{\Gamma(x_0)} w_j^2\, d\Gamma} \end{aligned} \tag{5.30}$$

and also

$$A^{-1}\rho_1 = 2\sum_j \frac{((y_1, \sqrt{\lambda_j}\, w_j))_{V'} w_j/\sqrt{\lambda_j}}{\int_{\Gamma^*(x_0)}(\hat{w}_j^2 + |D_\sigma \hat{w}_j|^2)\,\mathrm{d}\Gamma + \int_{\Gamma(x_0)} w_j^2\,\mathrm{d}\Gamma} \in H^1,$$

$$A^{-1}\rho_2 = -2\sum_j \frac{(y_0, w_j)w_j}{\int_{\Gamma^*(x_0)}(\hat{w}_j^2 + |D_\sigma \hat{w}_j|^2)\,\mathrm{d}\Gamma + \int_{\Gamma(x_0)} w_j^2\,\mathrm{d}\Gamma} \in L^2. \tag{5.31}$$

Remark 5.1. From the control point of view, the main difference between the Dirichlet and Neumann cases stems from the nonexistence of a sharp theorem for the trace of the Neumann problem (see Bardos *et al.* [4], formula (3.15) and Corollary 3.9).

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