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Dislocation layers applied to moving cracks in orthotropic crystals

G. E. TUPHOLME
School of Mathematics, University of Bradford, Bradford, England
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ABSTRACT

In this paper, we consider loaded Griffith-type strip cracks moving in orthotropic crystals using the powerful method of dislocation layers. Expressions for the components of the stress fields created are obtained in closed forms and some representative numerical results are given. The applications of the method to the BCS model of moving cracks with plastic flow are briefly discussed.

RÉSUMÉ

Dans cet article, nous traitons des fissures sur ruban chargé de type Griffith se déplacant en cristaux orthotropiques utilisant la puissante méthode de couches de dislocation.

Des formules pour les composantes des domaines chargés qui sont crés, sont obtenues en formes fermées et quelques résultats numériques représentatifs sont donnés. Les applications, au modèle BCS, de cette méthode de fissures mobiles avec écoulement plastique sont brièvement discutées.

ZUSAMMENFASSUNG

In diesem Referat handelt es sich um geladene Bänderrisse des Griffith-Typus, die sich unter Anwendung der kraftvollen Methode der Schichtenverschiebung in orthotropischen Kristallen bewegen. Ausdrücke für die Bestandteile der erzeugten Kraftfelder werden in geschlossenen Formen ermittelt und einige typische, zahlenmässige Ergebnisse angegeben. Die Anwendungen der Methode auf das BSC-Modell der sich bewegenden Risse mit plastischem Fluss werden auch kurz erörtert.

1. Introduction

The technique of simulating strip-type cracks in linearly elastic media by equivalent continuous distributions of dislocations has been discussed and applied extensively (see, for example, Bilby and Eshelby [1]) since the early work of Zener [2] and Friedel [3] and is now well-established. Recently, in fact, it has been demonstrated by Guidera and Lardner [4] that corresponding methods can be advantageously applied to investigations of the analogous penny-shaped cracks in isotropic media. This so-called dislocation layer method is found to be extremely useful for studying situations in which the traditional techniques of integral transform and complex potential function theories are rather unwieldy.

The present paper demonstrates that it is especially suited to providing details of the stress fields created by particular orientations of loaded straight cracks moving through orthotropic crystals. Corresponding analyses have recently been presented for moving cracks in isotropic media by Lardner and Tupholme [5] and for stationary cracks in orthotropic crystals by Tupholme [6], whilst a comprehensive discussion of a more general nature of interfacial cracks between bonded anisotropic half-spaces has been given by Willis [7]. In order to study cracks moving in orthotropic crystals in this way it is essential to have detailed information of the stress fields around uniformly moving straight dislocations in such media. For this, we appeal to the fundamental work of Bullough and Bilby [8]. Their results have provided the foundations for the subsequent general discussions of Teutonico [9, 10] and the analyses of dislocations moving in various particular cubic and hexagonal materials undertaken by Teutonico [11, 12] and Weertman [13, 14]. The model of a crack which we employ here to gain some insight into the fundamental problems of a growing crack is based on that used by Yoffe [15]. An interesting comparison of this with the one used by Craggs [16] has recently been given by Atkinson [17].

The basic situations with which we are dealing are formulated in section 2, whilst sections 3 and 4 are concerned with deriving and analysing the stress fields' components for mode II and mode III cracks, respectively. In particular, the results enable approximations valid near the crack tips to be obtained and typical numerical results are presented graphically. Finally, in section 5, ways of treating the corresponding elastoplastic cracks using the BCS model are indicated.

2. Basic formulation

We consider a plane strip crack of Griffith type moving parallel to its axis with uniform velocity in its own plane through a homogeneous crystal which is orthotropically symmetrical in its elastic response. We suppose that the material is initially everywhere at rest and stress-free in a natural reference state and situated so that its three mutually perpendicular planes of symmetry are the coordinate planes of a system of rectangular Cartesian coordinates x, y, z.

At time t, the crack is assumed to occupy the region $y = 0, vt-c < x < vt+c, -\infty < z < \infty$ of the $x-z$ plane, so that $2c$ is the width of the crack and v its speed of propagation. Defining a moving coordinate ξ given by

$$\xi = x - vt,$$

we suppose that a traction, $T(\xi)$, is applied symmetrically to the two faces of the crack and translates with the crack. It has become standard practice in fracture mechanics to consider separately three fundamental modes of loading. Letting $\sigma_{xy}, \sigma_{xx}, \sigma_{yy}, \sigma_{yz}$ denote the components of the stress-tensor referred to the x, y, z system of coordinates, these modes are represented by the following boundary conditions holding for $|\xi| < c$:

$$\begin{aligned} &\text{Mode I:} && \sigma_{yy}(\xi, 0) = T(\xi), && \sigma_{xy}(\xi, 0) = 0, \\ &\text{Mode II:} && \sigma_{yy}(\xi, 0) = 0, && \sigma_{xy}(\xi, 0) = T(\xi), \\ &\text{Mode III:} && \sigma_{yz}(\xi, 0) = T(\xi), && \end{aligned} \tag{1}$$

with the medium remaining stress free at infinity. Plane strain deformations are created by the first two modes, whilst the third is antiplane strain. With respect to the x, y, z coordinate system, the relationship connecting the components of the stress and strain tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ respectively, for an orthotropic material can be written in the form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}, \tag{2}$$

where the c_{ij} denote the elastic constants referred to the chosen coordinate system.

Hexagonal and cubic crystals are important special classes of orthotropic crystals and a fuller discussion of the way in which the expression (2) can be modified to provide an analysis applicable to them is given by Tupholme [6]. It is sufficient to simply state here that if the basal plane of a hexagonal crystal is chosen to be the $x-z$ plane then our results describe a crack moving in this basal plane when we make the substitutions

$$\begin{aligned} &c_{13} = c^h_{12}, \quad c_{12} = c_{23} = c^h_{13}, \quad c_{22} = c^h_{33}, \quad c_{11} = c_{33} = c^h_{11}, \\ &c_{55} = \tfrac{1}{2}(c^h_{11} - c^h_{12}), \quad c_{44} = c_{66} = c^h_{44} \end{aligned} \tag{3}$$

throughout. Here the superfix h is used to indicate that the five elastic constants are those of a hexagonal crystal referred to the more standard hexagonal system of coordinates in which the z-axis (rather than the y-axis which we find more convenient to use here) is parallel to the six-fold axis. Similarly for discussing a crack in the $x-z$ plane of a cubic crystal situated with its three cubic edges coinciding with the x, y, z axes we make, with an obvious notation, the replacements

$$c_{13} = c_{23} = c^c_{12}, \quad c_{22} = c_{33} = c^c_{11}, \quad c_{55} = c_{66} = c^c_{44}. \tag{4}$$

The mathematical studies and properties of the mode I and mode II situations are very similar. For brevity, we only discuss here a shear crack subjected to mode II surface tractions, the extension to the normally loaded crack being tedious but straightforward.

3. Moving inplane shear crack

Firstly, consider a straight edge dislocation with line in the z-direction and Burgers vector in the x-direction which is gliding through the crystal with constant speed v in the positive x-direction. We suppose it corresponds to a displacement discontinuity given by

$$\boldsymbol{u}^{\mathrm{II}}(\xi, 0+) - \boldsymbol{u}^{\mathrm{II}}(\xi, 0-) = (-b, 0, 0) \qquad \text{for } \xi > 0$$

where b is a constant. Throughout this section, a superfix II is attached to the displacement vector $\boldsymbol{u}$ and the components of the corresponding stress tensor. The displacement and stress fields of such a dislocation situated at the origin can be deduced from the general discussions of Bullough and Bilby [8] and the subsequent investigations undertaken by Teutonico [9]. The analysis is found to depend crucially upon the solutions of the quartic equation

$$K_4\zeta^4 - K_2\zeta^2 + K_0 = 0 \tag{5}$$

with

$$\begin{aligned} K_4 &= c_{22}c_{66}, \\ K_2 &= (c_{11}c_{22} - c^2_{12} - 2c_{12}c_{66}) - (c_{22} + c_{66})\rho v^2, \\ K_0 &= (c_{11} - \rho v^2)(c_{66} - \rho v^2), \end{aligned} \tag{6}$$

ρ being the density of the material in the reference state. This, in fact, is simply a quadratic in ζ^2 and has solutions λ^2_1, λ^2_2 given by

$$\lambda^2_1 = \{q + (q^2 - 4s)^{\frac{1}{2}}\}/2, \qquad \lambda^2_2 = \{q - (q^2 - 4s)^{\frac{1}{2}}\}/2, \tag{7}$$

where

$$\left.\begin{aligned} q &= K_2/K_4 = \{c_{11}c_{22}\beta_1^2 + c_{66}^2\beta_6^2 - (c_{12}+c_{66})^2\}/c_{22}c_{66} \\ s &= K_0/K_4 = c_{11}\beta_1^2\beta_6^2/c_{22} \\ &\text{with} \\ \beta_6^2 &= 1-\delta_6^2, \qquad \beta_1^2 = 1 - c_{66}\delta_6^2/c_{11}, \qquad \delta_6^2 = \rho v^2/c_{66}. \end{aligned}\right\} \tag{8}$$

If $q^2 - 4s > 0$, then clearly λ_1^2 and λ_2^2 are both real, whilst they are complex if $q^2 - 4s < 0$. The value of $q^2 - 4s$ depends upon the elastic constants of the medium together with the speed of propagation v. Suppose we let

$$\lambda_n = p_n + iq_n, \qquad n = 1, 2. \tag{9}$$

For convenience and brevity we restrict our attention in the main text to situations for which $q^2 - 4s > 0$ and briefly outline the corresponding results for $q^2 - 4s < 0$ in Appendix II.

After many cumbersome algebraic manipulations it is found that when $q^2 - 4s > 0$ the stress field of this dislocation has components given by

$$\left.\begin{aligned} \sigma_{xy}^{\mathrm{II}}(\xi, y) &= \frac{bc_{66}\xi}{2\pi}\sum_{n=1}^{2}\frac{\phi_n}{\xi^2 + p_n^2 y^2}, \\ \sigma_{xx}^{\mathrm{II}}(\xi, y) &= \frac{by}{2\pi}\sum_{n=1}^{2}\frac{c_{11}k_n + c_{12}w_n}{\xi^2 + p_n^2 y^2}, \\ \sigma_{yy}^{\mathrm{II}}(\xi, y) &= \frac{by}{2\pi}\sum_{n=1}^{2}\frac{c_{12}k_n + c_{22}w_n}{\xi^2 + p_n^2 y^2}, \end{aligned}\right. \tag{10}$$

where, for $n = 1, 2$,

$$k_n = p_n A_{ni}(c_{22}p_n^2 - \beta_6^2 c_{66}), \qquad w_n = -p_n^3 A_{ni}(c_{12}+c_{66}), \qquad \phi_n = (w_n/p_n^2) - k_n, \tag{11}$$

with

$$\left.\begin{aligned} A_{1i} &= -(p_2^2 c_{22} + c_{12}\beta_6^2)/c_{22}\beta_6^2(c_{12}+c_{66})(p_1^2 - p_2^2), \\ A_{2i} &= (p_1^2 c_{22} + c_{12}\beta_6^2)/c_{22}\beta_6^2(c_{12}+c_{66})(p_1^2 - p_2^2) \end{aligned}\right. \tag{12}$$

and $p_n = \lambda_n$ $(n = 1, 2)$ given by equations (7) and (8).

It is known, from the general techniques of the dislocation layer method, that a loaded crack can be studied by replacing it by an equivalent continuous planar distribution of dislocations. For this moving mode II shear crack we utilize straight edge dislocations with line in the z-direction having Burgers vectors and velocity of magnitude v in the x-direction. From equation (10_1) we observe that on $y = 0$, where the boundary conditions (1_2) are to be satisfied, $\sigma_{xy}^{\mathrm{II}}$ is given by

$$\sigma_{xy}^{\mathrm{II}}(\xi, 0) = bc_{66}\Omega/2\pi, \tag{13}$$

where

$$\Omega = \phi_1 + \phi_2. \tag{14}$$

If the number of dislocations in the interval $(\xi, \xi + d\xi)$ is $f(\xi)d\xi$ for all ξ lying between $-c$ and $+c$, then, recalling equation (13), the corresponding stress component at a point on the x-axis is given by

$$\sigma_{xy}(\xi, 0) = \int_{-c}^{c} \sigma_{xy}^{\mathrm{II}}(\xi-\xi', 0) f(\xi')\mathrm{d}\xi'$$

$$= \frac{bc_{66}\Omega}{2\pi} \int_{-c}^{c} \frac{f(\xi')}{\xi-\xi'}\mathrm{d}\xi'. \tag{15}$$

When

$$\sigma_{xy}(\xi, 0) = \lim_{y\to 0} \int_{-c}^{c} \sigma_{xy}^{\mathrm{II}}(\xi-\xi', y) f(\xi')\mathrm{d}\xi'$$

is evaluated rigorously using equation (10_1) and the Plemelj formulae it becomes clear that the integral in (15) must be interpreted as a Cauchy principal value integral. To satisfy the second of the conditions (1_2) we equate the expression (15) to the prescribed function $T(\xi)$. The solution of the resulting integral equation for the density function $f(\xi)$ is deducible from the results of Muskhelishvili [18] and Gakhov [19] and can be written in the form

$$f(\xi) = \frac{2}{\pi bc_{66}\Omega} \frac{1}{(c^2-\xi^2)^{\frac{1}{2}}} \int_{-c}^{c} \frac{(c^2-\xi'^2)^{\frac{1}{2}}}{\xi'-\xi} T(\xi')\mathrm{d}\xi', \tag{16}$$

if the relative displacement of the two crack faces is assumed to vanish at $\xi = \pm c$. Having derived an expression for $f(\xi)$, any of the stress components produced by the crack can now be calculated by direct substitution into the formula

$$\sigma_{ij}(\xi, y) = \int_{-c}^{c} \sigma_{ij}^{\mathrm{II}}(\xi-\xi', y) f(\xi')\mathrm{d}\xi', \tag{17}$$

with the necessary $\sigma_{ij}^{\mathrm{II}}$ given by the corresponding expressions (10).

It is convenient at this stage to define the functions $\mathscr{F}_n(\theta_n)$ for $n = 1, 2$, which subsequently occur, by

$$\mathscr{F}_n(\theta_n) = \frac{1}{\pi\Omega} \int_{-c}^{c} \frac{\{p_n y \cos\theta_n + (\xi-\xi')\sin\theta_n\}}{\mathscr{R}_n\{(\xi-\xi')^2+p_n^2y^2\}} (c^2-\xi'^2)^{\frac{1}{2}} T(\xi')\mathrm{d}\xi' \tag{18}$$

where the functions $\mathscr{R}_n(\xi, y)$ and $\theta_n(\xi, y)$ for $n = 1, 2$ are given by

$$\mathscr{R}_n e^{i\theta_n} = \{c^2-(\xi+ip_n y)^2\}^{\frac{1}{2}}. \tag{19}$$

The branches of the square root functions are specified by choosing θ_n to be zero for $-c < \xi < c$, $y = 0+$ and extending it by analytic continuation elsewhere. Using the equations (A.1) and (A.2) of Appendix I, it can be shown that the stress components given by equations (17) have the simplified forms

$$\sigma_{xy}(\xi, y) = \sum_{n=1}^{2} \phi_n \mathscr{F}_n(\theta_n),$$

$$\sigma_{xx}(\xi, y) = \frac{1}{c_{66}} \sum_{n=1}^{2} \frac{c_{11}k_n + c_{12}w_n}{p_n} \mathscr{F}_n(\theta_n - \pi/2), \tag{20}$$

$$\sigma_{yy}(\xi, y) = \frac{1}{c_{66}} \sum_{n=1}^{2} \frac{c_{12}k_n + c_{22}w_n}{p_n} \mathscr{F}_n(\theta_n - \pi/2).$$

These agree with the results of Lardner and Tupholme [5] in the isotropic limit in which $c_{11} = c_{22} = c_{33} = \lambda+2\mu$, $c_{12} = c_{13} = c_{23} = \lambda$, $c_{44} = c_{55} = c_{66} = \mu$, where λ and μ are the Lamé elastic constants of the isotropic material evaluated in the reference state. Using equation (20_3), a further application of the Plemelj formulae verifies that $\sigma_{yy}(\xi, y)$ does vanish as y approaches zero as is required to fulfil the first of the two traction conditions (1_2) on the crack.

From the expression (16) for the density function $f(\xi)$ it is clear that the analysis breaks down when $\Omega = 0$. For specified values of the elastic constants, this first occurs at a particular value v_R of the speed of propagation v. Recalling equation (14) and combining the expressions (11) and (12) it can be shown that $\delta_R = (\rho v_R^2/c_{66})^{\frac{1}{2}}$ is a root of the equation

$$c_{22}c_{66}^2(c_{22}-c_{66})V^6 - c_{22}c_{66}\{2(c_{11}c_{22}-c_{12}^2)+c_{66}(c_{22}-c_{11})\}V^4 + (c_{11}c_{22}-c_{12}^2)\{(c_{11}c_{22}-c_{12}^2)+2c_{22}c_{66}\}V^2 - (c_{11}c_{22}-c_{12}^2)^2 = 0 \quad (21)$$

where

$$V^2 = \rho v^2/c_{66}.$$

This cubic equation in V^2 corresponds to the usual Rayleigh surface wave equation in isotropic media and has been previously shown to govern the speed of a surface wave of plane strain in an orthotropic crystal by Hearmon [20, p. 86]. It can be shown that there always exists a root such that $0 < \delta_R < 1$. This equation (21) can easily be solved numerically for various media and using the data of Huntingdon [21] and Baker, Chou and Kelly [22] it is found, in fact, that for most common crystals $\delta_R > 0.83$, so that the analysis is not severely restricted by this limitation.

The interesting distribution of the stress field close to a tip of the crack can be shown to be qualitatively similar to that found by Lardner and Tupholme [5] and Tupholme [6] in corresponding situations. This can be illustrated by putting

$$\xi = c + r\cos\alpha, \qquad y = r\sin\alpha$$

into equations (20) and considering situations in which $r \ll c$. It can easily be shown from equation (19) that the quantities $\mathscr{R}_n$ and θ_n are approximately given by

$$\begin{aligned} \mathscr{R}_n &\sim \{2cr(\cos^2\alpha + p_n^2\sin^2\alpha)^{\frac{1}{2}}\}^{\frac{1}{2}}, \\ \theta_n &\sim -(\pi - \Phi_n)/2, \end{aligned} \quad (22)$$

as $r \to 0$. Here Φ_n is defined for $n = 1$ and 2 by

$$\Phi_n = \tan^{-1}(p_n \tan\alpha) \quad (23)$$

and $\tan^{-1}(\ldots)$ is used to denote the principal value of the inverse tangent for $0 < \alpha < \pi/2$ and π plus the principal value for $\pi/2 \leqq \alpha \leqq \pi$. When the expressions (22) and (23) are substituted into the representations (18) and (20) it is found that the stress components are approximately given by

$$\begin{aligned} \sigma_{xy}(r,\alpha) &\sim \frac{K}{\Omega r^{\frac{1}{2}}}\sum_{n=1}^{2}\frac{\phi_n}{\Delta_n}\cos\left(\frac{\Phi_n}{2}\right), \\ \sigma_{xx}(r,\alpha) &\sim \frac{K}{c_{66}\Omega r^{\frac{1}{2}}}\sum_{n=1}^{2}\frac{c_{11}k_n + c_{12}w_n}{p_n\Delta_n}\sin\left(\frac{\Phi_n}{2}\right), \\ \sigma_{yy}(r,\alpha) &\sim \frac{K}{c_{66}\Omega r^{\frac{1}{2}}}\sum_{n=1}^{2}\frac{c_{12}k_n + c_{22}w_n}{p_n\Delta_n}\sin\left(\frac{\Phi_n}{2}\right), \end{aligned} \quad (24)$$

as $r \to 0$, where we have put

$$K = -\frac{1}{\pi(2c)^{\frac{1}{2}}}\int_{-c}^{c}\left(\frac{c+\xi'}{c-\xi'}\right)T(\xi')\mathrm{d}\xi',$$
$$\Delta_n = (\cos^2\alpha + p_n^2\sin^2\alpha)^{\frac{1}{4}}. \tag{25}$$

We observe from equations (24) and (25) that near the tip $\xi = c$ of this mode II crack the stress components depend on the loading $T(\xi)$ only through K which is the corresponding stress intensity factor at the end of a stationary or moving crack in an isotropic medium. The anisotropy of the material clearly does not affect the stress components on the ξ-axis ahead of the crack, since we see that

$$\sigma_{xy}(r, 0) \sim K/r^{\frac{1}{2}}, \qquad \sigma_{yy}(r, 0) \sim 0$$

as $r \to 0$. It is found that similar observations can be made for mode I and mode III cracks also.

The asymptotic behaviour as $r \to 0$ of the physically interesting radial shear stress component $\sigma_{r\alpha}$ can be deduced from the expressions (24) using the identity

$$\sigma_{r\alpha} = \tfrac{1}{2}(\sigma_{yy}-\sigma_{xx})\sin 2\alpha + \sigma_{xy}\cos 2\alpha. \tag{26}$$

whilst the properties of the corresponding tangential stress component $\sigma_{\alpha\alpha}$ can be calculated from

$$\sigma_{\alpha\alpha} = \sigma_{xx}\sin^2\alpha + \sigma_{yy}\cos^2\alpha - \sigma_{xy}\sin 2\alpha. \tag{27}$$

From these, for a fixed value of r, the approximate variations of $\sigma_{r\alpha}$ and $\sigma_{\alpha\alpha}$ with α in any orthotropically symmetry crystal for which $q^2 - 4s > 0$ can easily be calculated for various speeds of propagation. Appropriate values for the required elastic constants for many orthotropic, and in particular cubic and hexagonal, crystals have been given by Huntingdon [21] and Baker, Chou and Kelly [22]. Typical numerical results are presented graphically in Figures 1 and 2.

Figure 1 illustrates the angular variation of the scaled stress component, $r^{\frac{1}{2}}\sigma_{r\alpha}/K$, around the tip, $\xi = c$, of a crack in the basal planes of magnesium and graphite for a range of values of $\delta_6 = (\rho v^2/c_{66})^{\frac{1}{2}}$. For magnesium the features are very similar to those found by Lardner and Tupholme [5] for a crack moving in an isotropic material in the sense that the maximum stress occurs in a non-forward direction (i.e. off the x-axis) for a sufficiently high speed of propagation. This first occurs at the critical speed of about $\delta_6 = 0.78$ here. For speeds higher than this the ratio of the maximum stress to its value in the forward direction rapidly increases. By solving equation (21) for magnesium the limiting speed at which the analysis fails is given by approximately $\delta_R = 0.94$. By contrast, for graphite which exhibits a far greater anisotropy we find that the effect of the speed of propagation upon the value of the non-forward maximum stress which Tupholme [6] showed occurs even in the static case ($\delta_6 = 0$) is very small for speeds less than the limiting speed given by $\delta_R = 0.99$.

Figure 2 exhibits the distribution around the tip $\xi = c$ of $r^{\frac{1}{2}}\sigma_{\alpha\alpha}/K$ again for magnesium and graphite with various values of δ_6. It is well-known that for a stationary crack in an isotropic material $\sigma_{\alpha\alpha}$ has a maximum at about $\alpha = -70°$ from which it is frequently concluded that a crack in a brittle material grows in a tensile mode at an angle of $-70°$ to its initial direction when shear forces are applied to it. We see from our results that this maximum becomes sharper and also moves round towards an angle of $-90°$ with the crack as the speed or strength of anisotropy (typified by the curves for graphite) increases. Again the behaviour for graphite is found to have very little dependence upon speed within the range under consideration, whilst that of magnesium is close to that of an isotropic material.

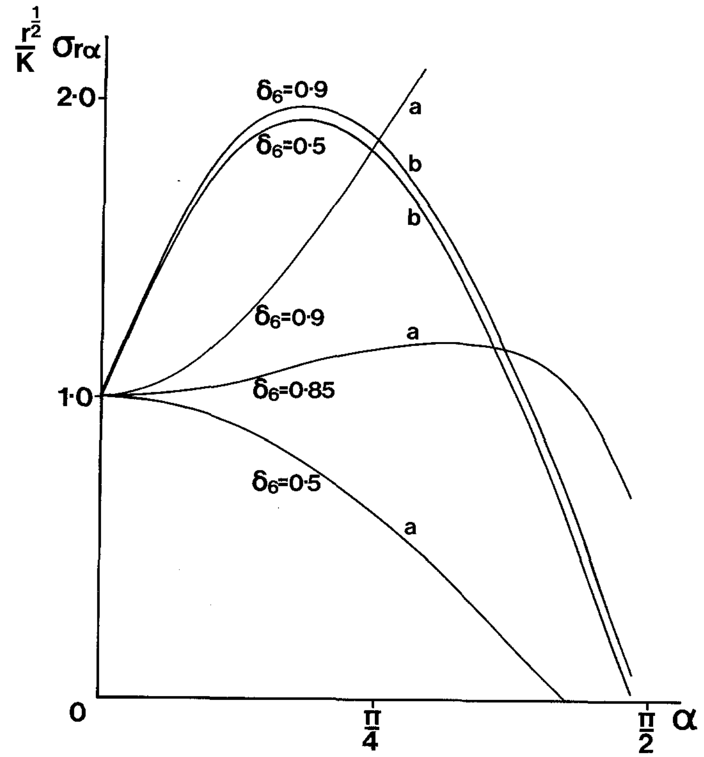

Figure 1. Distribution of the stress component $\sigma_{r\alpha}$ around the tip of a moving mode II crack for a range of speeds in (a) magnesium (b) graphite.

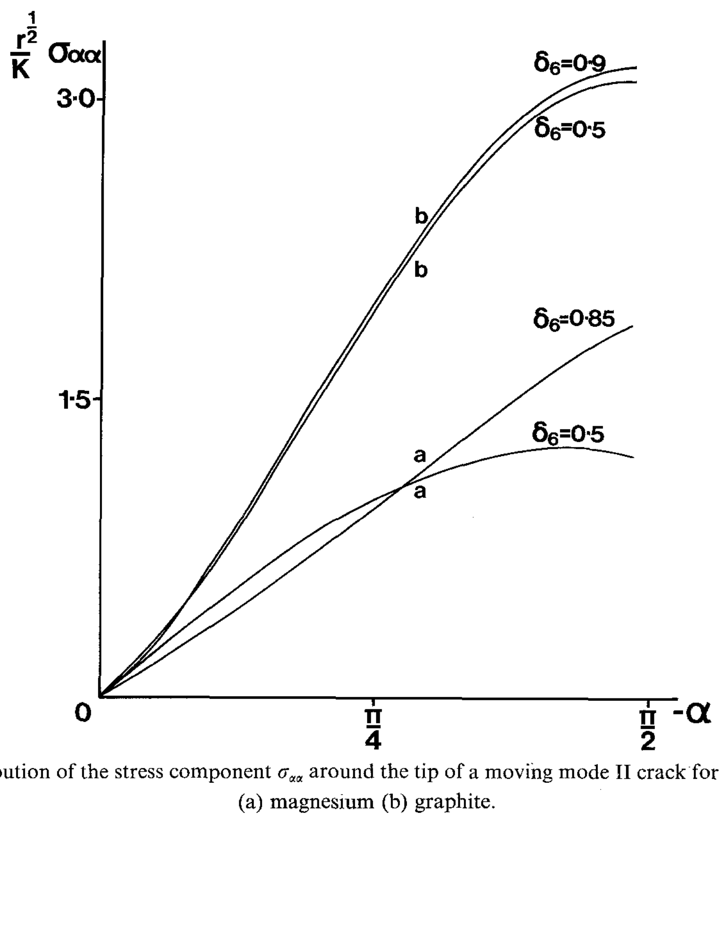

Figure 2. Distribution of the stress component $\sigma_{\alpha\alpha}$ around the tip of a moving mode II crack for a range of speeds in (a) magnesium (b) graphite.

4. Moving antiplane shear crack

As a preliminary to studying a moving mode III crack, we consider a moving screw dislocation whose displacement discontinuity is given by

$$\boldsymbol{u}^{\mathrm{III}}(\xi, 0+)-\boldsymbol{u}^{\mathrm{III}}(\xi, 0-) = (0, 0, -b) \qquad \text{for } \xi > 0.$$

The stress field of such a dislocation situated at the origin can be extracted from the analyses of Bullough and Bilby [8] and Teutonico [9] and is found to have non-zero components given by

$$\left.\begin{aligned} \sigma_{xz}^{\mathrm{III}}(\xi, y) &= -\frac{bK_s}{2\pi}\frac{\eta^2\beta_5 y}{\xi^2+\eta^2\beta_5^2y^2}, \\ \sigma_{yz}^{\mathrm{III}}(\xi, y) &= \frac{bK_s}{2\pi}\frac{\beta_5\xi}{\xi^2+\eta^2\beta_5^2y^2} \end{aligned}\right\} \tag{28}$$

where

$$K_s = (c_{44}c_{55})^{\frac{1}{2}}, \qquad \eta = (c_{55}/c_{44})^{\frac{1}{2}}, \qquad \beta_5^2 = 1-\delta_5^2, \qquad \delta_5^2 = \rho v^2/c_{55}. \tag{29}$$

A distribution of such screw dislocations with density function $f(\xi)$ can be used to replace the moving mode III crack. From equation (28_2) we see that

$$\sigma_{yz}^{\mathrm{III}}(\xi, 0) = bK_s\beta_5/2\pi\xi$$

and the boundary condition (1_3) therefore yields the integral equation

$$\frac{bK_s\beta_5}{2\pi}\int_{-c}^{c}\frac{f(\xi')\mathrm{d}\xi'}{\xi-\xi'} = T(\xi).$$

The solution of this equation which is appropriate is

$$f(\xi) = \frac{2}{\pi bK_s\beta_5}\frac{1}{(c^2-\xi^2)^{\frac{1}{2}}}\int_{-c}^{c}\frac{(c^2-\xi'^2)^{\frac{1}{2}}}{\xi'-\xi}T(\xi')\mathrm{d}\xi'.$$

After simplification using equations (28) and (A.1), (A.2) it then follows from the formula

$$\sigma_{ij}(\xi, y) = \int_{-c}^{c}\sigma_{ij}^{\mathrm{III}}(\xi-\xi', y)f(\xi')\mathrm{d}\xi'$$

that the non-zero stress components can be written in the forms

$$\left.\begin{aligned} \sigma_{xz}(\xi, y) &= -(\eta/\beta_5)\mathscr{F}(\theta-\pi/2), \\ \sigma_{yz}(\xi, y) &= \mathscr{F}(\theta). \end{aligned}\right\} \tag{30}$$

Here the quantities $\mathscr{F}(\theta)$, $\mathscr{R}(\xi, y)$ and $\theta(\xi, y)$ are given by

$$\left.\begin{aligned} \mathscr{F}(\theta) &= \frac{1}{\pi}\int_{-c}^{c}\frac{\eta\beta_5 y\cos\theta+(\xi-\xi')\sin\theta}{\mathscr{R}\{(\xi-\xi')^2+\eta^2\beta_5^2y^2\}}(c^2-\xi'^2)^{\frac{1}{2}}T(\xi')\mathrm{d}\xi', \\ \mathscr{R}e^{i\theta} &= \{c^2-(\xi+i\eta\beta_5 y)^2\}^{\frac{1}{2}} \end{aligned}\right\} \tag{31}$$

with θ chosen to be zero on $y = 0+$ for $|\xi| < c$ and continued analytically elsewhere.

Near the crack tip we find that

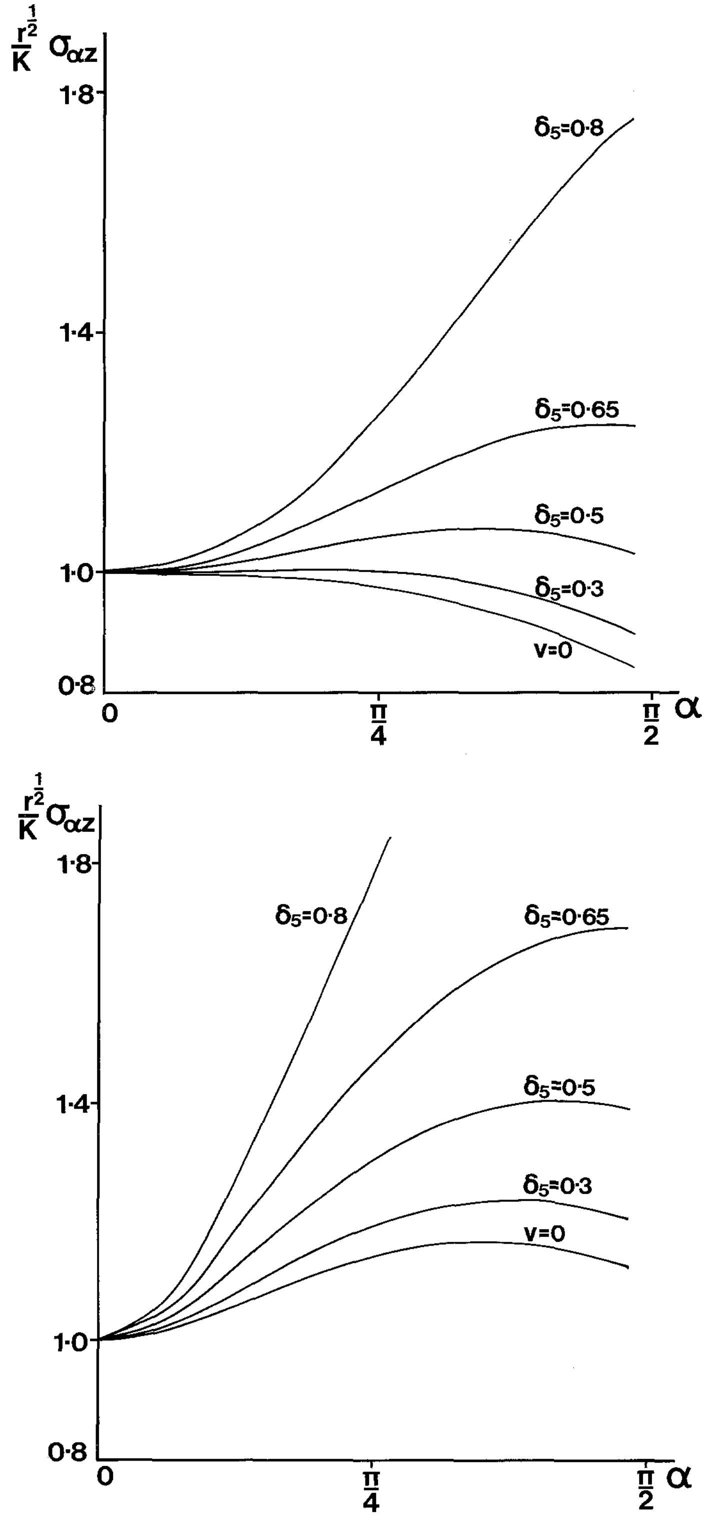

Figure 3. (i) Distribution of the stress component $\sigma_{\alpha z}$ around the tip of a moving mode III crack for a range of speeds for $\eta = 1.36$ (graphite).
(ii) Distribution of the stress component $\sigma_{\alpha z}$ around the tip of a moving mode III crack for a range of speeds for $\eta = 2.5$.

$$\left.\begin{aligned}\sigma_{xz}(r,\alpha) &\sim -\frac{\eta K}{\beta_5 r^{\frac{1}{2}}}\frac{\sin(\phi/2)}{\Delta},\\ \sigma_{yz}(r,\alpha) &\sim \frac{K}{r^{\frac{1}{2}}}\frac{\cos(\phi/2)}{\Delta}\end{aligned}\right\} \tag{32}$$

as $r \to 0$, with

$$\phi = \tan^{-1}(\eta\beta_5 \tan\alpha),$$
$$\Delta = (\cos^2\alpha + \eta^2\beta_5^2 \sin^2\alpha)^{\frac{1}{4}} \tag{33}$$

and K given by equation (25_1). $\sigma_{xz}(r, 0)$ and $\sigma_{yz}(r, 0)$ near the crack tip are easily seen to be unaffected by the anisotropy of the material.

The behaviour of the stress component σ_{xz} near the crack tip is deduced from the expressions (32) to be governed by

$$\sigma_{\alpha z}(r,\alpha) \sim \left\{\frac{\eta\beta_5^{-1}\sin(\phi/2)\sin\alpha + \cos(\phi/2)\cos\alpha}{\Delta}\right\}\frac{K}{r^{\frac{1}{2}}} \tag{34}$$

as $r \to 0$. This component has a non-forward maximum for large enough values of η or δ_5. The curves in Figure 3 depict the variation with α of the scaled stress component, $r^{\frac{1}{2}}\sigma_{\alpha z}/K$, for a range of values of δ_5 for graphite ($\eta = 1.36$) and for the case $\eta = 2.5$.

5. Moving crack with plastic flow

The properties of a mode II crack when the material in the vicinity of the tips of the crack behaves plastically are often discussed using the BCS model suggested by Bilby, Cottrell and Swinden [23]. This model replaces the plastic zones by plane distributions of dislocations coplanar with the crack spread over the regions $-a < \xi < -c$, $c < \xi < a$ in which the constant yield stress, σ_1, of the material is such that $\sigma_{xy}(\xi, 0) = -\sigma_1$. The loading $T(\xi)$ for $|\xi| < c$ is supposed to be an even function of ξ so that the plastic zones are symmetrical about $\xi = 0$. The value of a is determined by the requirement that there should be no stress singularities at the ends $\xi = \pm a$ of the plastic zones. The stress intensity factor must therefore vanish there. Hence, recalling equation (25_1), it can be shown that

$$\sigma_1 \int_{c<|\xi'|<a} \frac{d\xi'}{(a^2-\xi'^2)^{\frac{1}{2}}} = \int_{-c}^{c} \frac{T(\xi')}{(a^2-\xi'^2)^{\frac{1}{2}}}\, d\xi'$$

which determines the length $(a-c)$ of the plastic zones in terms of the applied load. This is observed to be independent of both the anisotropy of the crystal and the speed of propagation of the crack.

The density of dislocations in the region $|\xi| < a$ can be deduced from equation (16) to be given by

$$f(\xi) = \frac{2}{\pi b c_{66}\Omega}\frac{1}{(a^2-\xi^2)^{\frac{1}{2}}}\left\{\int_{-c}^{c}\frac{(a^2-\xi'^2)^{\frac{1}{2}}}{\xi'-\xi}T(\xi')d\xi' - \sigma_1\int_{c<|\xi'|<a}\frac{(a^2-\xi'^2)^{\frac{1}{2}}}{\xi'-\xi}d\xi'\right\}.$$

This is the corresponding density for the stationary case (see Tupholme [6]) multiplied by a factor of $K_e/c_{66}\Omega$ where

$$K_e = \{(c_{11}c_{22})^{\frac{1}{2}} + c_{12}\}\left[\frac{c_{66}\{(c_{11}c_{22})^{\frac{1}{2}} - c_{12}\}}{c_{22}\{(c_{11}c_{22})^{\frac{1}{2}} + c_{12} + 2c_{66}\}}\right]^{\frac{1}{2}}.$$

It therefore follows that the ratio, $\Phi^{\mathrm{II}}(v)/\Phi^{\mathrm{II}}(0)$, of the plastic displacement at the moving crack tip defined by

$$\Phi^{\mathrm{II}}(v) = \int_c^a bf(\xi)\mathrm{d}\xi$$

to its stationary value is given by

$$\Phi^{\mathrm{II}}(v)/\Phi^{\mathrm{II}}(0) = K_e/c_{66}\Omega. \tag{35}$$

This ratio is 1 when $v = 0$ and becomes infinite as the limiting speed given by equation (21) is approached. If the same model is applied to a mode III crack then, with an obvious notation, it is found that

$$\Phi^{\mathrm{III}}(v)/\Phi^{\mathrm{III}}(0) = 1/\beta_5, \tag{36}$$

β_5 being given by equation (29_3) and the length of the plastic zones is again unaffected.

The possible implications of these results to a discussion of the growth of ductile and brittle cracks are similar to those given by Lardner and Tupholme [5] for cracks in isotropic media.

Appendix I

Contour integrations can be used to verify that

$$\int_{-c}^{c} \frac{\mathrm{d}\xi''}{(c^2-\xi''^2)^{\frac{1}{2}}(\xi'-\xi'')\{(\xi-\xi'')^2+\kappa^2y^2\}} = \frac{\pi\{\kappa y \sin\Theta-(\xi-\xi')\cos\Theta\}}{y\kappa\mathscr{R}\{(\xi-\xi')^2+\kappa^2y^2\}}, \tag{A.1}$$

$$\int_{-c}^{c} \frac{(\xi-\xi'')\mathrm{d}\xi''}{(c^2-\xi''^2)^{\frac{1}{2}}(\xi'-\xi'')\{(\xi-\xi'')^2+\kappa^2y^2\}} = \frac{\pi\{\kappa y \cos\Theta+(\xi-\xi')\sin\Theta\}}{\mathscr{R}\{(\xi-\xi')^2+\kappa^2y^2\}} \tag{A.2}$$

for constant κ, where the branches of

$$\mathscr{R}e^{i\Theta} = \{c^2-(\xi+i\kappa y)^2\}^{\frac{1}{2}}$$

are chosen similarly to those in equation (19).

Appendix II

When the straight edge dislocation described in section 3 is moving through an orthotropic crystal for which $q^2-4s<0$ it can be shown that the λ_n ($n = 1, 2$) defined in equation (9) are complex. It is found, in fact, that their real and imaginary parts are given by

$$\left.\begin{aligned} p_1 &= p_2 = \{(4s)^{\frac{1}{2}}+q\}^{\frac{1}{2}}/2, \\ q_2 &= q_1 = \{(4s)^{\frac{1}{2}}-q\}^{\frac{1}{2}}/2. \end{aligned}\right\} \tag{A.3}$$

If we define the functions

$$\left.\begin{aligned} A_{1i} &= A_{2i} = 1/2\beta_6^2(c_{12}+c_{66}), \\ A_{1r} &= -A_{2r} = \{c_{12}\beta_6^2+c_{22}(p_1^2-q_2^2)\}/4p_1q_2c_{22}\beta_6^2(c_{12}+c_{66}), \end{aligned}\right\} \tag{A.4}$$

and

$$\left.\begin{aligned} S_{nr} &= c_{22}(p_n^2-q_n^2)-c_{66}\beta_6^2, \\ S_{ni} &= 2p_nq_nc_{22}, \\ R_{nr} &= -q_n(c_{12}+c_{66}), \\ R_{ni} &= p_n(c_{12}+c_{66}), \end{aligned}\right\} \tag{A.5}$$

for $n = 1, 2$, the stress components of such a dislocation situated at the origin can be shown, after lengthy manipulations, to be given by

$$\left.\begin{aligned} \sigma_{xy}^{\mathrm{II}}(\xi, y) &= \frac{bc_{66}}{2\pi}\sum_{n=1}^{2}\frac{\xi\phi_n+yv_n}{\xi^2+p_n^2y^2} \\ \sigma_{xx}^{\mathrm{II}}(\xi, y) &= \frac{b}{2\pi}\sum_{n=1}^{2}\frac{\xi(c_{11}\alpha_n+c_{12}l_n)+y(c_{11}k_n+c_{12}w_n)}{\xi^2+p_n^2y^2}, \\ \sigma_{yy}^{\mathrm{II}}(\xi, y) &= \frac{b}{2\pi}\sum_{n=1}^{2}\frac{\xi(c_{12}\alpha_n+c_{22}l_n)+y(c_{12}k_n+c_{22}w_n)}{\xi^2+p_n^2y^2}, \end{aligned}\right\} \tag{A.6}$$

where here, for $n = 1, 2$,

$$\left.\begin{aligned} \alpha_n &= A_{nr}S_{nr}-A_{ni}S_{ni}, & \gamma_n &= A_{nr}S_{ni}+A_{ni}S_{nr}, \\ \delta_n &= A_{nr}R_{nr}-A_{ni}R_{ni}, & \varepsilon_n &= A_{nr}R_{ni}+A_{ni}R_{nr}, \\ k_n &= \gamma_np_n-\alpha_nq_n, & l_n &= -(\delta_nq_n+\varepsilon_np_n), \\ w_n &= \delta_n(p_n^2+q_n^2), & \phi_n &= \delta_n-\alpha_nq_n-\gamma_np_n, \\ v_n &= \varepsilon_np_n-\delta_nq_n+\alpha_n(p_n^2+q_n^2). \end{aligned}\right\} \tag{A.7}$$

It follows from equations (A.3) to (A.5) and (A.7) that

$$\sum_{n=1}^{2}\alpha_n = \sum_{n=1}^{2}l_n = 0.$$

In particular therefore, we observe from the expressions (A.6) that

$$\left.\begin{aligned} \sigma_{xx}^{\mathrm{II}}(\xi, 0) &= \sigma_{yy}^{\mathrm{II}}(\xi, 0) = 0, \\ \sigma_{xy}^{\mathrm{II}}(\xi, 0) &= bc_{66}\Omega/2\pi\xi \end{aligned}\right\} \tag{A.8}$$

Equation ($A.8_2$) is identical to equation (13) and hence the density function of the system of such dislocations equivalent to a mode II crack will again be given by equation (16). Results corresponding to those of section 3 can then, if required, be calculated by substituting this with the expressions (A.6) into equation (17).

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