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ESTIMATION OF PARAMETERS IN THE DISCRETE DISTRIBUTIONS OF ORDER k

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Abstract. This paper considers estimating parameters in the discrete distributions of order k such as the binomial, the geometric, the Poisson and the logarithmic series distributions of order k. It is discussed how to calculate maximum likelihood estimates of parameters of the distributions based on independent observations. Further, asymptotic properties of estimators by the method of moments are investigated. In some cases, it is found that the values of asymptotic efficiency of the moment estimators are surprisingly close to one.

Key words and phrases: Binomial distribution of order k, geometric distribution of order k, Poisson distribution of order k, logarithmic series distribution of order k, parametric estimation, method of moments, maximum likelihood.

1. Introduction

Let k be a positive integer. Suppose we are given independent trials with success probability p. The distribution of the number of occurrences of consecutive k successes until the n-th trial is called the binomial distribution of order k and is denoted by $B_k(n,p)$. The distribution of the number of trials until the first occurrence of the k-th consecutive success is called the geometric distribution of order k and is denoted by $G_k(p)$. In addition to these distributions, there are some important distributions of order k such as the negative binomial ($NB_k(r,p)$), the Poisson ($P_k(\lambda)$), the logarithmic series distributions of order k ($LS_k(p)$), etc. The properties of each distribution and relationships among them have often been investigated in the literature (cf. e.g., Philippou *et al.* (1983), Aki *et al.* (1984), Hirano (1986), Philippou (1986), Hirano and Aki (1987) and Aki and Hirano (1988)). However, there are not many papers which treat estimation of the parameters in the distributions of order k, since the probability functions are too complicated. To the best of our knowledge, moment estimation of the parameters of $G_k(p)$, $NB_k(p)$ and $P_k(\lambda)$ was considered only in the remarks by Philippou *et al.* (1983) and Philippou (1983, 1984). Considering that $P_k(\lambda)$ is one of the generalized Poisson distributions, we have to mention Douglas (1955) and Shumway and Gurland (1960*a*, 1960*b*). They discussed the problem of calculating the MLE of the parameter of certain generalized Poisson distributions such as the Neyman type A, the Poisson binomial and the Poisson Pascal, respectively.

In the present paper we discuss how to calculate the maximum likelihood estimates (MLE's) of the parameters in the distributions based on independent observations. Let $X_1, X_2, \dots, X_n$ be independent discrete random variables with common probability function $f(x, \theta)$. In order to calculate the MLE $\hat{\theta}$, we have to solve the likelihood equation iteratively

$$F(\theta) = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \theta} f(X_i, \theta)}{f(X_i, \theta)} = 0 . \tag{1.1}$$

Since the sequence of the iteration θ_m is determined by the equation

$$\theta_{m+1} = \theta_m - \frac{F(\theta_m)}{F'(\theta_m)} , \tag{1.2}$$

where

$$F'(\theta) = \sum_{i=1}^{n} \frac{\left(\frac{\partial^2}{\partial \theta^2} f(X_i, \theta)\right) \cdot f(X_i, \theta) - \left(\frac{\partial}{\partial \theta} f(X_i, \theta)\right)^2}{(f(X_i, \theta))^2} ,$$

the solution of (1.1) can be obtained if $f(x, \theta)$, $(\partial / \partial \theta) f(x, \theta)$ and $(\partial^2 / \partial \theta^2) f(x, \theta)$ are given. Therefore, the most important problem for maximum likelihood estimation of the distributions of order k is how quickly the probability function, the first and the second derivatives of the probability function with respect to the parameter can be calculated.

Besides the MLE's we investigate moment method estimators (MME's) of parameters in the distributions of order k. MME's are relatively simpler than MLE's in these distributions. In some cases, it is found that the values of asymptotic efficiency of the MME's are surprisingly close to one. In Section 2 estimation of the parameter p in the binomial distribution of order k is discussed. We study, in Section 3, estimation of the parameters in the geometric, the Poisson and the logarithmic series distributions of order k.

2. The binomial distribution of order k

2.1 *The maximum likelihood estimation*

Some properties of the binomial distribution of order k are given by Feller (1968), Hirano (1986), Philippou and Makri (1986) and Aki and Hirano (1988). As we stated in Section 1, the calculation of the probability function, the first and the second derivatives of the probability function with respect to the parameter is necessary for getting the MLE based on independent observations.

Though Hirano (1986) and Philippou and Makri (1986) gave exactly the probability function of $B_k(n,p)$ as

$$(2.1)\quad \sum_{m=0}^{k-1} \sum_{x_1+2x_2+\cdots+kx_k=n-m-kx} \times \begin{bmatrix} x_1+\cdots+x_k+x \\ x_1,\ldots,x_k,x \end{bmatrix} p^n \left(\frac{q}{p}\right)^{x_1+\cdots+x_k}, \quad \text{for} \quad x=0,1,\ldots,\left[\frac{n}{k}\right],$$

where $q = 1-p$, $[a]$ means the largest integer not exceeding a and the inner summation is over all nonnegative integers $x_1,\ldots,x_k$ such that $x_1+2x_2+\cdots+kx_k = n-m-kx$, the formula (2.1) is not suitable for calculation.

Aki and Hirano (1988) proved that the following recurrence relation for the probability function $B_k(n,p;x)$ holds

$$(2.2)\quad B_k(n,p;x) = B_k(n-1,p;x) + p^k(B_k(n-k,p;x-1) - qB_k(n-k-1,p;x) - pB_k(n-k-1,p;x-1)) \quad \text{if} \quad n>k \text{ and } x=1,2,\ldots,\left[\frac{n}{k}\right],$$

$$(2.3)\quad B_k(n,p;0) = B_k(n-1,p;0) - p^k qB_k(n-k-1,p;0) \quad \text{if} \quad n>k,$$

$$(2.4)\quad B_k(k,p;0) = 1-p^k, \quad B_k(k,p;1) = p^k,$$

$$(2.5)\quad B_k(n,p;0) = 1 \quad \text{if} \quad 0 \le n < k.$$

By differentiating both sides of (2.2)–(2.5), we have the following recurrence relation for $(\partial/\partial p)B_k(n,p;x)$ ($\equiv B_k'(n,p;x)$) and $(\partial^2/\partial p^2)B_k(n,p;x)$ ($\equiv B_k''(n,p;x)$).

$$(2.6)\quad B_k'(n,p;x) = B_k'(n-1,p;x) + kp^{k-1}\{B_k(n-k,p;x-1) - (1-p)B_k(n-k-1,p;x) - pB_k(n-k-1,p;x-1)\}$$

$$+ p^k\{B_k'(n-k,p;x-1) + B_k(n-k-1,p;x) - (1-p)B_k'(n-k-1,p;x) - B_k(n-k-1,p;x-1) - pB_k'(n-k-1,p;x-1)\}$$

$$\text{if} \quad n > k \text{ and } x = 1,2,\ldots,\left[\frac{n}{k}\right],$$

(2.7) $$B_k'(n,p;0) = B_k'(n-1,p;0) - p^{k-1}(kq-p)B_k(n-k-1,p;0) - p^k q B_k'(n-k-1,p;0) \quad \text{if} \quad n > k\,,$$

(2.8) $$B_k'(k,p;0) = -kp^{k-1}, \qquad B_k'(k,p;1) = kp^{k-1}\,,$$

(2.9) $$B_k'(n,p;0) = 0 \quad \text{if} \quad 0 \le n < k\,,$$

(2.10) $$\begin{aligned} B_k''(n,p;x) = {} & B_k''(n-1,p;x) \\ & + k(k-1)p^{k-2}\{B_k(n-k,p;x-1) - (1-p)B_k(n-k-1,p;x) - pB_k(n-k-1,p;x-1)\} \\ & + 2kp^{k-1}\{B_k'(n-k,p;x-1) + B_k(n-k-1,p;x) - (1-p)B_k'(n-k-1,p;x) - B_k(n-k-1,p;x-1) - pB_k'(n-k-1,p;x-1)\} \\ & + p^k\{B_k''(n-k,p;x-1) + 2B_k'(n-k-1,p;x) - (1-p)B_k''(n-k-1,p;x) - 2B_k'(n-k-1,p;x-1) - pB_k''(n-k-1,p;x-1)\} \end{aligned}$$

$$\text{if} \quad n > k \text{ and } x = 1,2,\ldots,\left[\frac{n}{k}\right],$$

(2.11) $$\begin{aligned} B_k''(n,p;0) = {} & B_k''(n-1,p;0) \\ & - \{(k-1)p^{k-2}(k-(k+1)p) - (k+1)p^{k-1}\}B_k'(n-k-1,p;0) \\ & - 2p^{k-1}(k-(k+1)p)B_k'(n-k-1,p;0) \\ & - p^k(1-p)B_k''(n-k-1,p;0) \quad \text{if} \quad n > k\,, \end{aligned}$$

(2.12) $B_k''(k,p;0) = -k(k-1)p^{k-2}$,

(2.13) $B_k''(k,p;1) = k(k-1)p^{k-2}$,

(2.14) $B_k''(n,p;0) = 0 \quad \text{if} \quad 0 \le n < k$.

2.2 *The moment estimation of p*

Let $X_1, X_2, \ldots, X_m$ be independent identically distributed random variables with probability function $f(x;p)$. If we set $\bar{X} = (1/m)\sum_{i=1}^{m} X_i$, then, from the central limit theorem, it holds that $\sqrt{m}(\bar{X} - f(p))$ converges in distribution to the Gaussian distribution with mean zero and variance $\sigma^2(p)$ as $m \to \infty$, where $f(p) = EX_i$ and $\sigma^2(p) = \mathrm{Var}(X_i)$. Now we define a moment estimator $\hat{p}$ of p by the solution of the equation $f(p) = \bar{X}$. For simplicity we assume that f is strictly monotone in p. Then, $\hat{p}$ is written as $f^{-1}(\bar{X})$ and hence it holds that $\sqrt{m}(\hat{p} - p)$ converges in distribution to the Gaussian distribution with mean zero and variance $\sigma^2(p)/(f'(p))^2$ as $m \to \infty$.

When the X's are distributed as $B_k(n,p)$, the moment estimator is much simpler than the MLE. In fact, from Proposition 2.4 of Aki and Hirano (1988), $f(p)$ can be written as a very simple polynomial

$$\sum_{j=1}^{[n/k]} \{(n - jk + 1)p^{jk} - (n - jk)p^{jk+1}\}. \tag{2.15}$$

It is very easy to solve the equation $f(p) = \bar{X}$ numerically. Now we shall see that the function (2.15) is monotonously increasing with respect to $p \in (0, 1)$, which implies that the moment estimate can be determined uniquely. Though we have not yet succeeded in showing this analytically for all integers n, we can give an algorithm for proving this for each given integer n. Since the function (2.15) is a polynomial with integral coefficients, it can be proved exactly by using classical results of algebra like Sturm's theorem that the polynomial is monotonously increasing in $p \in (0, 1)$ (see Aki (1987)).

Here we give an algorithm for checking a sufficient condition for the problem. This algorithm is simpler than that of Aki (1987) and may be more suitable in particular for proving that the function (2.15) is monotonously increasing in $p \in (0, 1)$. To prove the problem, it is sufficient to show that the derivative of (2.15)

$$f'(p) = \sum_{j=1}^{[n/k]} p^{jk-1}\{jk(n - jk + 1) - (jk + 1)(n - jk)p\}$$

is positive for all $0 < p < 1$. Since $f'(0) = 0$, we consider the polynomial which is obtained by dividing $f'(p)$ by p^{k-1}

$$g(p)=\sum_{j=1}^{[n/k]}p^{(j-1)k}\{jk(n-jk+1)-(jk+1)(n-jk)p\}\,.$$

Then it suffices to prove that $p=1$ is a lower bound for positive roots of $g(p)=0$. Setting $r=[n/k]$, consider the equation which is obtained by transforming the equation $g(p)=0$ by $p=1/x$

$$h(x)=\sum_{j=1}^{r}\{jk(n-jk+1)x^{(r-j)k+1}-(jk+1)(n-jk)x^{(r-j)k}\}=0\,.$$

Then it suffices to show that $x=1$ is an upper bound for positive roots of $h(x)=0$. From Newton's theorem, it is easily checked by showing that $h(1)$, $h^{(1)}(1),\dots,h^{((r-1)k)}(1)>0$. We have proved at present by using the computer algebra system REDUCE (see Hearn (1984)) that $f(p)$ is monotonously increasing for $k=2,3,\dots,10$ and $n=k,k+1,\dots,100$.

2.3 *Estimation of p based on a censored sample*

The binomial distribution of order k is closely related to the reliability of the system called a consecutive-k-out-of-n:F system (cf. e.g., Aki (1985), Hirano (1986) and Philippou (1986)). The system, which was introduced by Chiang and Niu (1981) and was further studied by many authors (cf. e.g., Derman *et al.* (1982)), consists of n components in sequence and fails whenever k consecutive components are failed. On the assumption that all components fail independently with identical probability p, the probability $P\{X_i=0\}$ ($P\{X_i\geq 1\}$) means the probability of the event that the system is functioning (resp. failed). Assume that it is observable only whether the system fails or not. Then it corresponds to considering the following censoring.

Let $X_1,X_2,\dots,X_m$ be independent identically distributed random variables with probability function $B_k(n,p;x)$. Let Y_1, $Y_2,\dots$, Y_m be the random variables defined by

$$Y_i=\begin{cases}0 & \text{if}\quad X_i\geq 1\,,\\ 1 & \text{if}\quad X_i=0,\quad i=1,2,\dots,m\,.\end{cases}$$

Assuming that only Y's are observable, we shall consider estimation of p based on Y_1, $Y_2,\dots$, Y_m. Since Y_i is distributed as the binomial distribution $B(1,c(n))$, where $c(n)=B_k(n,p;0)$, the likelihood equation can be written as $\bar{Y}=c(n)$, where $\bar{Y}=\sum_{i=1}^{m}Y_i$. From Proposition 2.3 of Aki and Hirano (1988), we have the following recurrence relations for $c(n)$ and $c'(n)$

$$c(n) = \begin{cases} 1 & \text{if} \quad 0 \le n < k\,, \\ 1 - p^k & \text{if} \quad n = k\,, \\ c(n-1) - p^k(1-p)c(n-k-1) & \text{if} \quad n > k\,, \end{cases}$$

$$c'(n) = \begin{cases} 0 & \text{if} \quad 0 \le n < k\,, \\ -kp^{k-1} & \text{if} \quad n = k\,, \\ c'(n-1) - (kp^{k-1} - (k+1)p^k)c(n-k-1) & \\ \qquad - p^k(1-p)c'(n-k-1) & \text{if} \quad n > k\,, \end{cases}$$

where $c'(n) = (\partial/\partial p)c(n)$. Then the estimation procedure is feasible. As the first part of Subsection 2.2, it is easy to see that $\sqrt{m}(\hat{p}_c - p)$ converges in distribution to the Gaussian distribution with mean zero and variance $AV(p)$ as $m \to \infty$, where $\hat{p}_c$ is the MLE based on Y's and $AV(p) = c(n)(1 - c(n))/c'(n)^2$.

2.4 *Estimation of p when n is large*

In this subsection, we shall consider estimation of p based on one observation X_n which is assumed to be distributed as $B_k(n,p)$. Since the sample size is one, the asymptotic theory which we have stated in the previous subsections does not hold. Another type of asymptotic result, however, will be expected when n is large. If $k = 1$, X_n can be written in distribution as a sum of n independent identically distributed random variables which are distributed as $B(1,p)$ and hence the law of large numbers and the central limit theorem can be applied directly.

Feller ((1968), Chapter XIII) proved the next theorem.

THEOREM 2.1. (*Feller* (1968)) *If X_n is a random variable distributed as $B_k(n,p)$, then $\sqrt{n}(X_n/n - 1/\mu)$ converges weakly to the Gaussian distribution with mean zero and variance σ^2/μ^3, where $\mu = (1-p^k)/(1-p)p^k$ and $\sigma^2 = \{1 - (2k+1)qp^k - p^{2k+1}\}/q^2p^{2k}$ are the mean and variance of $G_k(p)$, respectively.*

Now we define an estimator of p by the solution of $n/X_n = (1-p^k)/(1-p)p^k$. Then the estimator is consistent and asymptotically normal from the above theorem.

3. Some other distributions of order k

3.1 *The geometric distribution of order k*

As Philippou *et al.* (1983) indicated, the mean of $G_k(p)$ is monotonously decreasing and hence the moment estimate which was defined in the first part of Subsection 2.2 can be determined uniquely. Since $f(p)$ and

$\sigma^2(p)$ are written as $(1-p^k)/(1-p)p^k$ and $\{1-(2k+1)qp^k-p^{2k+1}\}/q^2p^{2k}$, respectively, the asymptotic variance of the MME can be written as

$$\frac{p^2q^2\{1-(2k+1)(1-p)p^k-p^{2k+1}\}}{(-p^{k+1}+p+kp-k)^2}\,.$$

Next we shall consider how to calculate the MLE based on independent observations. As we stated in Section 1, the calculation of the probability function, the first and the second derivatives of the probability function with respect to the parameter is crucial. There are some recurrence relations to be used for the calculation of the probability function (cf. Aki *et al.* (1984), Aki (1985) and Philippou and Makri (1985)). Among them the next formula given by Philippou and Makri (1985) will be suitable for the calculation since it can be used satisfactorily even if k is large.

We denote by $G_k(p;x)$ the probability function of $G_k(p)$. Let $a_1=p^k$ and $a_2=p^k(1-p)$. Then the following recurrence relation holds

$$\text{(3.1)}\qquad G_k(p;x)=\begin{cases}0 & \text{if } 0\le x<k\,,\\ a_1 & \text{if } x=k\,,\\ a_2 & \text{if } k+1\le x\le 2k\,,\\ G_k(p;x-1)-a_2G_k(p;x-k-1) & \text{if } x\ge 2k+1\,.\end{cases}$$

By differentiating both sides of (3.1) we have the following recurrence relation for $(\partial/\partial p)G_k(p;x)$ $(\equiv G_k'(p;x))$ and $(\partial^2/\partial p^2)G_k(p;x)$ $(\equiv G_k''(p;x))$

$$\text{(3.2)}\qquad G_k'(p;x)=\begin{cases}0 & \text{if } 0\le x<k\,,\\ a_3 & \text{if } x=k\,,\\ a_4 & \text{if } k+1\le x\le 2k\,,\\ G_k'(p;x-1)-a_4G_k(p;x-k-1)-a_2G_k'(p;x-k-1) & \\ & \text{if } x\ge 2k+1\,,\end{cases}$$

where $a_3=kp^{k-1}$ and $a_4=a_3-(k+1)p^k$,

$$\text{(3.3)}\qquad G_k''(p;x)=\begin{cases}0 & \text{if } 0\le x<k\,,\\ a_5 & \text{if } x=k\,,\\ a_6 & \text{if } k+1\le x\le 2k\,,\\ G_k''(p;x-1)-a_6G_k(p;x-k-1) & \\ \quad -2a_4G_k'(p;x-k-1)-a_2G_k''(p;x-k-1) & \\ & \text{if } x\ge 2k+1\,,\end{cases}$$

where $a_5 = k(k-1)p^{k-2}$ and $a_6 = k(k-1)p^{k-2} - (k+1)kp^{k-1}$.

3.2 *The Poisson distribution of order k*

Since the mean and variance of $P_k(\lambda)$ are $(k(k+1)/2)\lambda$ and $(k(k+1)(2k+1)/6)\lambda$, respectively (cf. e.g., Aki *et al.* (1984)), the MME of λ is given by $(2/k(k+1))\bar{X}$ and the asymptotic variance of the MME can be written as $(2(2k+1)/3k(k+1))\lambda$. Moreover, we can easily see that the MME is unbiased and that the exact variance of the MME is given by $2(2k+1)\lambda/3k(k+1)n$.

A recurrence relation which is necessary for calculation of the MLE was given by Adelson (1966) (cf. also Aki *et al.* (1984)) as follows:

$$(3.4)\qquad \begin{cases} P_k(\lambda; 0) = e^{-k\lambda}, \\ P_k(\lambda; x) = \dfrac{\lambda}{x}\displaystyle\sum_{j=1}^{k\wedge x} jP_k(\lambda; x-j), \end{cases}$$

where $P_k(\lambda; x)$ is the probability function of $P_k(\lambda)$ and $a \wedge b$ means the minimum of a and b. Differentiating both sides of (3.4), we have

$$(3.5)\qquad \begin{cases} P_k'(\lambda; 0) = -ke^{-k\lambda}, \\ P_k'(\lambda; x) = \dfrac{1}{x}\displaystyle\sum_{j=1}^{k\wedge x} jP_k(\lambda; x-j) + \dfrac{\lambda}{x}\displaystyle\sum_{j=1}^{k\wedge x} jP_k'(\lambda; x-j), \end{cases}$$

and

$$(3.6)\qquad \begin{cases} P_k''(\lambda; 0) = k^2e^{-k\lambda}, \\ P_k''(\lambda; x) = \dfrac{2}{x}\displaystyle\sum_{j=1}^{k\wedge x} jP_k'(\lambda; x-j) + \dfrac{\lambda}{x}\displaystyle\sum_{j=1}^{k\wedge x} jP_k''(\lambda; x-j), \end{cases}$$

where $P_k'(\lambda; x) = (\partial/\partial\lambda)P_k(\lambda; x)$ and $P_k''(\lambda; x) = (\partial^2/\partial\lambda^2)P_k(\lambda; x)$.

3.3 *The logarithmic series distribution of order k*

First we study the MME of the parameter p of $LS_k(p)$, which has been discussed generally in the first part of Subsection 2.2. Aki *et al.* (1984) gave the mean of $LS_k(p)$ as

$$(3.7)\qquad f(p) = \frac{-1}{k\log p}\left\{\frac{1-p^k}{(1-p)p^k} - k\right\}.$$

The next proposition implies that the MME is determined uniquely if the sample mean exceeds $(k+1)/2$.

PROPOSITION 3.1. *The mean of* $LS_k(p)$ (3.7) *is monotonously decreasing in* $p \in (0,1)$ *and* $\lim_{p \to 0} f(p) = \infty$ *and* $\lim_{p \to 1} f(p) = (k+1)/2$ *hold.*

PROOF. The derivative of (3.7) is written as

$$\frac{1}{kp^k \log p}\left\{ \frac{1 - k\log p}{-p\log p}\{1 + p + p^2 + \cdots + p^{k-1} - kp^k\} - \frac{1 + p + p^2 + \cdots + p^{k-1} - kp^{k-1}}{1-p} + k^2p^{k-1} \right\}.$$

Since $p^k < p^{k-1}$, we have

$$\begin{aligned}
&\frac{1 - k\log p}{-p\log p}\{1 + p + p^2 + \cdots + p^{k-1} - kp^k\} \\
&\quad - \frac{1 + p + p^2 + \cdots + p^{k-1} - kp^{k-1}}{1-p} + k^2p^{k-1} \\
&\qquad > \frac{1 - k\log p}{-p\log p}\{1 + p + p^2 + \cdots + p^{k-1} - kp^k\} \\
&\qquad\quad - \frac{1 + p + p^2 + \cdots + p^{k-1} - kp^{k}}{1-p} + k^2p^{k-1} \\
&\qquad = (1 + p + p^2 + \cdots + p^{k-1} - kp^k) \\
&\qquad\quad \cdot \left\{ \frac{1 - k\log p}{-p\log p} - \frac{1}{1-p} \right\} + k^2p^{k-1}.
\end{aligned}$$

Noting that $1 + p + p^2 + \cdots + p^{k-1} - kp^k > 0$, it is sufficient to show that $(1 - k\log p)/(-p\log p) - 1/(1-p)$ is positive in order to prove that $f(p)$ is monotonously decreasing in $p \in (0,1)$. If we write

$$\frac{1 - k\log p}{-p\log p} - \frac{1}{1-p} = \frac{h(p)}{(1-p)(-p\log p)},$$

where $h(p) = (1 - k\log p)(1-p) + p\log p$, then the denominator is positive. Hence it suffices to show that $h(p)$ is positive in $p \in (0,1)$. But it is easily seen from the fact that $h(1) = 0$ and $h'(p)$ is negative in $p \in (0,1)$. It is easy to see that $\lim_{p \to 0} f(p) = \infty$ and $\lim_{p \to 1} f(p) = (k+1)/2$. This completes the proof.

As for recurrence relations of the probability function, which is necessary for getting the MLE, two different methods exist given by

Proposition 3.3 of Aki *et al.* (1984) and by Charalambides ((1986), Section 4). The next recurrence relation was obtained by Charalambides.

$$
\begin{aligned}
&LS_k(p;1) = \frac{1-p}{-k\log p}, \\
&LS_k(p;x) = \frac{1-p}{x}\sum_{j=1}^{x-1}(x-j)p^{j-1}LS_k(p;x-j) - \frac{1-p}{k\log p}p^{x-1} \\
&\qquad\qquad \text{if} \quad 1 < x \le k, \\
&LS_k(p;x) = \frac{1-p}{x}\sum_{j=1}^{k}(x-j)p^{j-1}LS_k(p;x-j) \quad \text{if} \quad x > k,
\end{aligned} \tag{3.8}
$$

where $LS_k(p;x)$ denotes the probability function of $LS_k(p)$. Differentiating both sides of (3.8), we have

$$
\begin{aligned}
&LS_k'(p;1) = \frac{1}{-k}\frac{-\log p - (1-p)/p}{(\log p)^2}, \\
&LS_k'(p;x) = \frac{1}{x}\sum_{j=1}^{x-1}(x-j)p^{j-2}\{(1-p)(j-1)-p\}LS_k(p;x-j) \\
&\qquad + \frac{1-p}{x}\sum_{j=1}^{x-1}(x-j)p^{j-1}LS_k'(p;x-j) \\
&\qquad - \frac{p^{x-2}\{((1-p)(x-1)-p)\log p - (1-p)\}}{k(\log p)^2} \\
&\qquad\qquad \text{if} \quad 1 < x \le k, \\
&LS_k'(p;x) = \frac{1}{x}\sum_{j=1}^{k}(x-j)p^{j-2}\{(1-p)(j-1)-p\}LS_k(p;x-j) \\
&\qquad + \frac{1-p}{x}\sum_{j=1}^{k}(x-j)p^{j-1}LS_k'(p;x=j) \\
&\qquad\qquad \text{if} \quad x > k,
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&LS_k''(p;1) = \frac{-1}{k}\left\{\frac{2}{p(\log p)^2} + \frac{1-p}{p^2(\log p)^2} + \frac{2(1-p)}{p^2(\log p)^3}\right\}, \\
&LS_k''(p;x) = \frac{1}{x}\sum_{j=1}^{x-1}(x-j)p^{j-3}\{(1-p)(j-1)(j-2) \\
&\qquad\qquad - (j-2)p - jp\}LS_k(p;x-j)
\end{aligned}
$$

$$+\frac{2}{x}\sum_{j=1}^{x-1}(x-j)p^{j-2}\{(1-p)(j-1)-p\}LS_k'(p;x-j)$$

$$+\frac{1}{x}\sum_{j=1}^{x-1}(x-j)(1-p)p^{j-1}LS_k''(p;x-j)$$

$$-\frac{1}{k}\frac{p^{x-3}}{(\log p)^4}[(\log p)^3\{(x-2)(x-xp-1)-xp\}$$

$$+(\log p)^2\{(1-p)-2(x-xp-1)\} \tag{3.10}$$

$$+2(\log p)(1-p)]$$

$$\text{if} \quad 1<x\le k\,,$$

$$LS_k''(p;x)=\frac{1}{x}\sum_{j=1}^{k}(x-j)p^{j-3}\{(1-p)(j-1)(j-2)$$

$$-(j-2)p-jp\}LS_k(p;x-j)$$

$$+\frac{2}{x}\sum_{j=1}^{k}(x-j)p^{j-2}\{(1-p)(j-1)-p\}LS_k'(p;x-j)$$

$$+\frac{1}{x}\sum_{j=1}^{k}(x-j)(1-p)p^{j-1}LS_k''(p;x-j)$$

$$\text{if} \quad x>k\,,$$

where $LS_p'(p;x)=(\partial/\partial p)\,LS_k(p;x)$ and $LS_p''(p;x)=(\partial^2/\partial p^2)LS_k(p;x)$.

4. Numerical results

In the previous sections we have proposed some estimators besides MLE's. Values of asymptotic efficiencies of those estimators at some fixed points of the parameters are given in Tables 1–5.

Tables 1 and 2 treat the case of $B_k(n,p)$. Of course, the MME coincides with the MLE when $k=1$. It is surprising that the values of asymptotic efficiency of the MME are very near to 1 (which are between 0.99 and 1), for all values of the parameter listed in Table 1. From a practical viewpoint the MME may be better than the MLE, because the former can be calculated very easily and the round-off error which occurs in the calculation of the MLE cannot be ignored when n is large. Table 2 shows that the values of asymptotic efficiency of the MLE based on the censored sample are small when the values of the parameter p are close to one.

In the cases of the geometric and the Poisson distributions of order k, it can be seen from Tables 3 and 4 that the MME's are fairly good. In the

Table 1. Values of asymptotic efficiency of the MME in the case of $B_k(n,p)$.

p	(k,n)					
	(2, 10)	(2, 30)	(3, 10)	(3, 30)	(4, 10)	(4, 30)
.1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.2	.99998	.99999	.99999	1.00000	1.00000	1.00000
.3	.99992	.99998	.99995	.99998	.99999	.99999
.4	.99984	.99995	.99984	.99995	.99990	.99997
.5	.99973	.99991	.99969	.99989	.99977	.99992
.6	.99959	.99985	.99958	.99980	.99970	.99982
.7	.99944	.99977	.99964	.99966	.99979	.99965
.8	.99948	.99964	.99988	.99942	.99996	.99944
.9	.99983	.99948	1.00000	.99899	.99999	.99970

Table 2. Values of asymptotic efficiency of the MLE based on the censored sample in the case of $B_k(n,p)$.

p	(k,n)					
	(2, 10)	(2, 30)	(3, 10)	(3, 30)	(4, 10)	(4, 30)
.1	.96993	.88265	.99815	.98935	.99991	.99906
.2	.88413	.60080	.98545	.92282	.99854	.98630
.3	.75072	.29392	.95164	.76927	.99234	.93730
.4	.58211	.09156	.88725	.53501	.97497	.82358
.5	.39785	.01568	.78409	.27795	.93724	.62777
.6	.22535	.00122	.63720	.08995	.86760	.37018
.7	.09446	.00003	.44984	.01349	.75246	.13501
.8	.02304	.00000	.24309	.00053	.57602	.01907
.9	.00160	.00000	.06817	.00000	.32090	.00029

Table 3. Values of asymptotic efficiency of the MME in the case of $G_k(p)$.

p	$k=2$	$k=3$	$k=4$	$k=5$
.50	.99395	.99635	.99821	.99919
.55	.99108	.99404	.99675	.99834
.60	.98719	.99059	.99433	.99680
.65	.98204	.98553	.99044	.99407
.70	.97538	.97826	.98430	.98936
.75	.96697	.96809	.97485	.98144
.80	.95665	.95424	.96066	.96842
.85	.94441	.93603	.94010	.94769
.90	.93044	.91313	.91167	.91623
.95	.91532	.88610	.87508	.87205

case of the logarithmic series distribution of order k, we can see from Table 5 that the MME may not be good when the parameter p is very close to one.

Table 4. Values of asymptotic efficiency of the MME in the case of $P_k(\lambda)$.

λ	$k=2$	$k=3$	$k=4$	$k=5$
.1	.92049	.89523	.88745	.88710
.2	.93721	.92427	.92571	.93204
.3	.95064	.94568	.95139	.95938
.4	.96131	.96112	.96813	.97548
.5	.96970	.97210	.97889	.98484
.6	.97626	.97986	.98577	.99029
.7	.98136	.98531	.99017	.99349
.8	.98532	.98915	.99300	.99540
.9	.98839	.99185	.99485	.99657
1.0	.99076	.99376	.99607	.99731
1.5	.99667	.99774	.99835	.99869
2.0	.99842	.99870	.99892	.99908
2.5	.99902	.99905	.99917	.99929
3.0	.99928	.99924	.99933	.99942
3.5	.99941	.99936	.99943	.99950
4.0	.99950	.99945	.99951	.99957
4.5	.99956	.99952	.99957	.99962
5.0	.99961	.99957	.99961	.99966

Table 5. Values of asymptotic efficiency of the MME in the case of $LS_k(p)$.

p	$k=2$	$k=3$	$k=4$
.40	.96076	.97108	.98000
.45	.94770	.96031	.97168
.50	.93149	.94646	.96091
.55	.91138	.92867	.94666
.60	.88636	.90575	.92774
.65	.85510	.87612	.90245
.70	.81568	.83757	.86838
.75	.76537	.78694	.82197
.80	.69996	.71951	.75786
.85	.61252	.62787	.66751
.90	.49050	.49931	.53625
.95	.30794	.30924	.33589

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