ON THE NUMBER OF CENTRAL CONFIGURATIONS IN THE N-BODY PROBLEM

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Abstract. Central configurations are critical points of the potential function of the n-body problem restricted to the topological sphere where the moment of inertia is equal to constant. For a given set of positive masses $m_1, \ldots, m_n$ we denote by $N(m_1, \ldots, m_n, k)$ the number of central configurations of the n-body problem in $\mathbf{R}^k$ modulus dilatations and rotations. If $N(m_1, \ldots, m_n, k)$ is finite, then we give a bound of $N(m_1, \ldots, m_n, k)$ which only depends of n and k.

Key words: N-body problem, central configuration

1. Introduction and Statement of the Results

A very old problem in Celestial Mechanics is the study of the central configurations of the n-body problem. Central configurations are initial positions of the bodies that lead to particular solutions of the n-body problem for which the ratios of the mutual distances between the bodies remain constant. There is an extensive literature concerning these solutions. For a classical background, see the sections on central configurations in (Wintner 1941) and (Hagihara 1970). For a modern background one can see (Smale 1970a, 1970b) and (Saari 1980). More recent work can be found in (Buck 1989, 1991; Cedó and Llibre 1989; Elmabsout 1988; Meyer 1987; Meyer and Schmidt 1988a, 1988b; Moeckel 1985, 1989; Palmore 1973, 1975a, 1975b; Pacella 1987; Perko and Walter 1985; Schmidt 1988; Shub 1970 and Simó 1977.

If $r_i = (x_i, y_i, z_i)$ is the position vector of the ith positive mass m_i relative to the center of mass of the system, then the particles form a *central configuration* at time t if and only if there exists some scalar λ such that $\ddot{r}_i = -\lambda r_i$ for $i = 1, 2, \ldots, n$. By replacing the acceleration vector $\ddot{r}_i$ by the force vector this equation becomes

$$\lambda r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{r_i - r_j}{r_{ij}^3} \quad \text{for} \quad i = 1, \ldots, n ,$$

which is an equation which is independent of the dynamics. Here r_{ij} is the mutual distance between the ith and jth particles. It is well known that the constant λ in the above system is positive.

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The central configurations are called *collinear* when the n bodies are contained in a straight line, and *planar* when the n bodies are contained in a plane.

Let $N(m_1, \ldots, m_n, k)$ be the number of central configurations, modulus dilatations and rotations, of the n-body problem in $\mathbb{R}^k$ with positive masses $m_1, \ldots, m_n$. Of course from a physical point of view the more interesting cases are for $k = 1, 2, 3$.

Moulton (1910) characterized the number of collinear central configurations $N(m_1, \ldots, m_n, 1)$ by showing that there exist exactly $n!/2$ collinear central configurations of the n-body problem for a given set of positive masses. Considering non collinear central configurations a natural question is: *is $N(m_1, \ldots, m_n, k)$ finite for $k \geq 2$?*

Smale (1970a) stated that to solve the above question for the planar central configurations seems to be a major open problem, and in (Smale 1970b) he conjectured that for almost all positive values of $m_1, \ldots, m_n$, the number $N(m_1, \ldots, m_n, 2)$ is finite. Here by assuming that $N(m_1, \ldots, m_n, k)$ is finite we show how to compute a bound of $N(m_1, \ldots, m_n, k)$ which only depends on n and k. For $k = 2, 3$ the bounds are given in the next theorem, for $k > 3$ see Remark 3.

THEOREM 1. *If $N(m_1, \ldots, m_n, k)$ is finite, then*

- $N(m_1, \ldots, m_n, 2) \leq 8^{\binom{n}{2}} \cdot 2 \cdot 6^{\binom{n-2}{2}}$ *for $n \geq 4$, and*
- $N(m_1, \ldots, m_n, 3) \leq 10^{\binom{n}{2}} \cdot 2 \cdot 8^{\binom{n-3}{2}}$ *for $n \geq 5$.*

Theorem 1 will be proved in Section 2.

It is well known that the number of planar central configurations modulus dilatations and rotations for n equal to 2 and 3 are 1 and 5 respectively, and that the number of spatial (non-planar) central configurations modulus dilatations and rotations for $n = 4$ is 1 (see Proposition 2).

2. Proof of the Results

We start this section by summarizing some well-known results on central configurations (for more details see Dziobek 1900, Chapter 3 of Hagihara 1970 and Meyer and Schmidt 1988a).

Consider the center of mass of the n-bodies at the origin and the function

$$V = U + \delta(I - I_0) ,$$

where

$$U = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{r_{ij}}$$

is the *potential function* of the n-body problem,

$$I = \frac{1}{2} \sum_{i,j=1}^{n} m_i m_j r_{ij}^2$$

is the *moment of inertia* multiplied by the total mass $M = \sum_{i=1}^{n} m_i$, I_0 is a positive constant, and $\delta = \lambda/M$ is positive. Then the central configurations are extrema of the function V, i.e. central configurations are critical points of the potential U restricted to the topological sphere where the moment of inertia is equal to a constant. So to find central configurations we must solve the system

$$\begin{aligned} I - I_0 &= 0, \\ \frac{\partial U}{\partial x_i} &= 0, \\ \frac{\partial U}{\partial y_i} &= 0, \\ \frac{\partial U}{\partial z_i} &= 0, \end{aligned}$$

where $i = 1, \ldots, n$.

Of course we must subtract from the topological sphere where the moment of inertia is constant the set of all collisions Δ where the potential U is not defined, i.e. Δ is the set of elements of the form $(r_{12}, \ldots, r_{n-1,n}) \in \mathbb{R}^{\binom{n}{2}}$ such that $\sum_{i,j=1}^{n} m_i m_j r_{ij}^2 = I_0$ and $r_{ij} = 0$ for some $1 \leq i < j \leq n$.

If we want to find the extrema of V for the planar central configurations (i.e. $z_i = 0$ for $1, 2, \ldots, n$) by using the variables r_{ij} rather than the variables (x_i, y_i) we must take into account the fact that the number of necessary relations among the $n(n-1)/2$ mutual distances r_{ij} in order that the n bodies lie all in a plane is $(n-2)(n-3)/2$. To prove this assertion we note that in the case $n = 4$ only one condition is needed to specify that the tetrahedron formed by these four points has volume zero. Suppose now that the number of conditions to specify that n points lie in a plane are $(n-2)(n-3)/2$. Then we shall prove by induction that $(n-1)(n-2)/2$ conditions are required in order that $n+1$ points lie in a plane. Let A and B be two of the first n points. Clearly to assure that the $n+1$ points lie all on a plane we have to add the conditions obtained by setting equal to zero the volume of the tetrahedrons formed by A, B, the $(n+1)$th point and each of the remaining $n-2$ points among the first n ones. These $n-2$ conditions are new because they involve the $(n+1)$th point. Hence the number of necessary relations is $n - 2 + [(n-2)(n-3)/2] = (n-1)(n-2)/2$.

The condition to assure that four points r_i $(i = 1, 2, 3, 4)$ lie all on a plane in terms of the mutual distances r_{ij} is that the function

$$F(r_1, r_2, r_3, r_4) = det \begin{pmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 & 1 \\ r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

is zero because F is a positive constant by the square of the volume of the tetrahedron formed by the four points (for more details, see Dziobek 1900 and Hagihara 1970).

In short the problem of finding the planar central configurations is equivalent to find the extrema of V under $(n-2)(n-3)/2$ conditions of type $F = 0$. More precisely, it is sufficient to find the extrema of the function

$$V_P = U + \delta(I - I_0) + \rho_1 F_1 + \ldots + \rho_l F_l, \quad l = (n-2)(n-3)/2,$$

with Lagrange's indeterminate multipliers δ and ρ_k for $k = 1, \ldots, l$.

Proof of Theorem 1 in the planar case. After differentiating V_P with respect to r_{ij}; δ, ρ_k and the extrema of V are found from the following set of $\binom{n}{2} + 1 + \binom{n-2}{2}$ polynomial equations in the variables r_{ij}, δ and ρ_k:

$$\begin{aligned} m_i m_j(\delta r_{ij}^3 - 1) + 2r_{ij}^3 \sum_{k=1}^{l} \rho_k \frac{\partial F_k}{\partial r_{ij}^2} &= 0, \\ I - I_0 &= 0, \\ F_k &= 0, \end{aligned}$$

where $1 \leq i < j \leq n$ and $k = 1, \ldots, l$.

Since $N(m_1, \ldots, m_n, 2)$ is finite by hypothesis, from Bezout's Theorem it follows that the number of real solutions of the above polynomial system is at most the product of the degrees of the above $\binom{n}{2} + 1 + \binom{n-2}{2}$ polynomials, i. e. $8^{\binom{n}{2}} \cdot 2 \cdot 6^{\binom{n-2}{2}}$. ∎

Now we want to find the extrema of V for the spatial central configurations by using the variables r_{ij} instead of the variables $r_i = (x_i, y_i, z_i)$. Then we must take account of the fact that the number of necessary relations in the space of $n(n-1)/2$ mutual distances r_{ij} in order that the n bodies lie all in $\mathbb{R}^3$ is $(n-3)(n-4)/2$. As for the planar central configuration we shall prove this assertion inductively on n. In the case $n = 5$ only one condition is needed to specify that the hypertetrahedron in $\mathbb{R}^4$ formed by five points has volume zero. Suppose now that the number of conditions to specify that n points lie in $\mathbb{R}^3$ are $(n-3)(n-4)/2$. Then we shall show that the number of conditions in order that $n+1$ points lie in $\mathbb{R}^3$ are $(n-2)(n-3)/2$. Let A, B and C be three of the first n points. Clearly to assure that the $n+1$ points all lie on $\mathbb{R}^3$ we have to add the conditions obtained setting equal to zero the volume of the hypertetrahedrons in $\mathbb{R}^4$ formed by A, B, C, the $(n+1)$th point and each of the remaining $n-3$ points among the first n ones. These $n-3$ conditions are new because they involve the $(n+1)$th point. Hence the number of necessary relations is $n-3+[(n-3)(n-4)/2] = (n-2)(n-3)/2$.

The conditions to ensure that the ten mutual distances between the five points $r_1, \ldots, r_5$ allow for a geometric realization in $\mathbb{R}^3$ is that the function

$$G(r_1, r_2, r_3, r_4, r_5) = det \begin{pmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & r_{15}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 & r_{25}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 & r_{35}^2 & 1 \\ r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 & r_{45}^2 & 1 \\ r_{15}^2 & r_{25}^2 & r_{35}^2 & r_{45}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

is zero because G is a constant by the square of the volume of the hypertetrahedron in $\mathbb{R}^4$ formed by the five points $r_1, \ldots, r_5$ (for more details, see Meyer and Schmidt 1988a and Schmidt 1988).

In short the problem of finding the spatial central configurations is equivalent to finding the extrema of V under $(n-3)(n-4)/2$ conditions of type $G = 0$. More precisely, it is sufficient to find the extrema of

$$V_S = U + \delta(I - I_0) + \rho_1 G_1 + \ldots + \rho_h G_h, \quad h = (n-3)(n-4)/2,$$

with Lagrange's indeterminate multipliers δ and ρ_k for $k = 1, \ldots, h$.

Proof of Theorem 1 in the spatial case. After differentiating V_S with respect to r_{ij}; δ, ρ_k and the extrema of V are found from the following set of $\binom{n}{2} + 1 + \binom{n-3}{2}$ polynomial equations in the variables r_{ij}, δ and ρ_k:

$$\begin{aligned} m_i m_j(\delta r_{ij}^3 - 1) + 2r_{ij}^3 \sum_{k=1}^{h} \rho_k \frac{\partial G_k}{\partial r_{ij}^2} &= 0, \\ I - I_0 &= 0, \\ G_k &= 0, \end{aligned}$$

where $1 \le i < j \le n$ and $k = 1, \ldots, h$.

Since $N(m_1, \ldots, m_n, 3)$ is finite by hypothesis, from Bezout's Theorem it follows that the number of real solutions of the above polynomial system is at most the product of the degrees of the above polynomials, i.e. $10^{\binom{n}{2}} \cdot 2 \cdot 8^{\binom{n-3}{2}}$. ∎

PROPOSITION 2. *Given* 4 *positive masses there is a unique spatial central configuration of the* 4*-body problem modulus dilatations and rotations.*

Proof: The system of polynomial equations in the proof of Theorem 1 for the spatial case becomes

$$\begin{aligned} \delta r_{ij}^3 - 1 &= 0, \\ I - I_0 &= 0. \end{aligned}$$

So the proposition follows. ∎

The proof of Theorem 1 can be extended in a similar way to the case $k > 3$. Then we obtain:

REMARK 3. *If $N(m_1, \ldots, m_n, k)$ is finite for $k > 3$ then we can obtain that*

$$N(m_1, \ldots, m_n, k) \leq [2(k+2)]^{\binom{n}{2}} \cdot 2 \cdot [2(k+1)]^{\binom{n-k}{2}},$$

if $n \geq k + 2$.

In what follows we analyse the bound given by this method for the collinear n-body problem.

The condition to assure that three masses m_i, m_j and m_k lie on a straight line in terms of the mutual distances r_{lh} is that the function $H = r_{ij} - r_{ik} - r_{kj}$ is zero. Since the number of necessary relations of type H among the $n(n-1)/2$ mutual distances in order that the n bodies all lie in a straight line is $(n-1)(n-2)/2$, to find the collinear central configurations we must find the extrema of the function

$$V_C = U + \delta(I - I_0) + \rho_1 H_1 + \ldots + \rho_l H_l, \quad m = (n-1)(n-2)/2,$$

with Lagrange's indeterminate multipliers δ and ρ_k for $k = 1, \ldots, m$.

After differentiating V_C with respect to r_{ij}; δ, ρ_k and the extrema of V are found from the following set of $\binom{n}{2} + 1 + \binom{n-1}{2}$ polynomial equations:

$$\begin{aligned} m_i m_j(\delta r_{ij}^3 - 1) + r_{ij}^2 \sum_{k=1}^{m} \rho_k \frac{\partial H_k}{\partial r_{ij}} &= 0, \\ I - I_0 &= 0, \\ H_k &= 0, \end{aligned}$$

where $1 \leq i < j \leq n$ and $k = 1, \ldots, m$.

From Moulton's results we know that $N(m_1, \ldots, m_n, 1)$ is finite, so from Bezout's Theorem it follows that the number of real solutions of the above polynomial system is at most the product of the degrees of the above polynomials in the variables r_{ij}, δ and ρ_k, i. e. $4^{\binom{n}{2}} \cdot 2 \cdot 1^{\binom{n-1}{2}} = 2^{n^2-n+1}$.

Of course, the exact number of collinear central configurations for the n-body problem $n!/2$ satisfies that $n!/2 < 2^{n^2-n+1}$ for $n \geq 2$. Clearly, this inequality is due to the fact of the existence of complex roots which are not real, to the multiplicities of the different roots, and that some roots can be at infinity.

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