ON THE EFFECT OF THE MANTLE ELASTICITY ON THE EARTH'S ROTATION

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Abstract. Starting from the Hamiltonian model for a solid Earth with an elastic mantle previously developped by the authors, analytical expressions are derived which give the nutation series corresponding to the plane perpendicular to the angular momentum vector, to the plane perpendicular to the rotational axis and to the equator of figure, as well as the series that give the polar motion. The effects of the different perturbations – solid Earth, centrifugal and tidal potentials – are calculated separately. The corrections due to the elasticity of the mantle, which mostly correspond to the Oppolzer terms, are calculated with an accuracy of 10^{-6} arc sec., given that the intrinsic observational accuracy has reached 0.01 mas.

Key words: Elastic Earth, Earth's rotation, nutations, tides, Chandler's period.

1. Introduction

The aim of this paper is to derive analytical expressions and the corresponding numerical series for the perturbations in the rotation of the Earth when the mantle is assumed to be elastic instead of rigid, and is deformed by the very rotation itself and by the lunisolar attraction. The periodic perturbations are offered in the form of the nutations series in longitude and obliquity of the fundamental planes: Andoyer plane – or perpendicular to the angular momentum vector -, equatorial one and the plane perpendicular to the rotation axis. We can add the polar motion, which is the displacement of the rotation axis with respect to the figure axis.

In the present approach we follow essentially the same lines as those described in our recent Hamiltonian theory for an elastic Earth, published in a set of four papers (Getino and Ferrándiz, 1990, 1991a, 1991b, 1992), that is: the introduction of a system of elastic Andoyer variables to express the Hamiltonian of the problem and the later application of a canonical perturbation method based on the Lie series for eliminating the periodic terms. The resulting secular Hamiltonian is studied in a way analogous to the classic one of Kinoshita's theory (1977), obtaining the values of some parameters necessary for numerically evaluating the expressions giving the perturbations, and, in our case, including the Chandler's period.

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The advantage of this procedure is that it allows us, clearly and conveniently, to separate the influence of each kind of the perturbations taken into account (mainly rigid Earth potential, and centrifugal and tidal variation of the inertia tensor) on the different variables considered, as well as the secular and periodic effects. It is also susceptible to an easy adaptation to include new perturbations, such as those of the second order derived by Kinoshita and Souchay (1990), without having to reconstruct the theory.

The paper is organized in the following way: In Section 2, a brief description of the canonical variables used is made, and the principal terms which are considered in the Hamiltonian are reviewed, without forgetting the important distinction between the *perturbing* and *perturbed* roles played for a body when deformations are considered. In Section 3 the analytical expressions of the relevant Hamiltonian are given.

The tidal and rotational potentials are obtained directly using the same procedure, that is, starting from the solutions of the displacement vector of the deformation. The difference with the classical expressions is that both potentials do not depend on a sole global k_2, but they turn out to be functions of respective coefficients dependent on integrals of functions of the radius and the rheological parameters.

We then summarize the first order integration for the rigid Earth, presenting expressions for the nutation of the Andoyer plane, as well as the Oppolzer terms for the equator and for the plane perpendicular to the rotation axis, and finally the solutions for the polar motion. Naturally, the results are similar to those of Kinoshita's theory. Nevertheless, it seems opportune to devote a few pages on this basic reference with the aim of making later developments corresponding to the elastic case both clearer and briefer, also facilitating the interpretation and comparison of results.

In Section 5 the effect of the centrifugal deformation is studied. The elimination of periodic terms follows a process analogous to that of the rigid case, and concludes with the expressions of the perturbations of the fundamental planes, the polar motion and the interpretation of the results. For the motion of the Andoyer plane and the Oppolzer terms of the figure axis, the principal effect can be considered as a *coupling* with the results of a rigid Earth, so that the coefficient responsible for the amplitude of the periodic perturbations solely has to be re-examined. On the other hand, for the Oppolzer terms corresponding to the motion of the rotation axis or the polar perturbations, new terms appear which cannot be paired with those of a rigid Earth. A similar remark can be done for the secular Hamiltonian.

In Section 6 the same procedure is followed but for the tidal deformation, which acts through a perturbation induced in the inertia tensor. The effect on the Andoyer plane is negligible, while this effect diminishes the Oppolzer terms corresponding to a rigid Earth, which coincides with the results of Sasao *et al.* (1980) and Kubo (1991).

In the last section we proceed to the numerical representation of the previous results. First of all, the basic constants are calculated in the way above mentioned. Those corresponding to the deformation which come from geophysical data, are taken from a previous paper (Getino and Ferrándiz, 1991a). With them the four pairs of perturbation series considered – nutations of the three fundamental planes and polar motion – can be numerically evaluated.

The results, which are shown in tables, are consistent with those published by other authors [Kubo (1991) and Sasao et al. (1980)]. Taking into account the fact that some observations reach the internal accuracy level of a few 0.01 mas, as pointed out by Kinoshita and Souchay (1990), we have retained all the contributions of the elasticity to the nutation series up to 0.001 mas. This does not mean that the final series have such an accuracy, since we should also include the second order corrections of the rigid Earth theory for this to be true. On the other hand, we must note that some of our values do not coincide with the corresponding previous ones given by us, since, when obtaining certain perturbing series some errors slipped in, which mistake was discovered by the first author thanks to the reserve shown by Kinoshita and Kubo with respect to the old values.

The text is completed by some comments concerning the secular motion and the Chandler period and appendices detailing mathematical formulations used in the work.

2. Approaching the Problem

As pointed out in the introduction, we are concerned with the study of the rotational motion of the Earth for a model with a deformable elastic mantle. In this section we will describe, in a schematic way, the different terms that will make up the corresponding Hamiltonian, which will be developed later.

Since the distinction between *perturbing bodies* and *perturbed bodies* is essential when performing the analytical integration of the system, we examine in detail at the beginning that distinction.

2.1. A NOTE ON PERTURBING AND PERTURBED BODY

Let us consider that an elastic body is deformed by the action of two perturbations: *tidal deformation*, due to the gravitational attraction of external bodies, these bodies being considered as point masses with known orbits, and *centrifugal* or *rotational deformation*, due to the rotation of the elastic body itself. Thus, the bodies which cause tidal deformation will be *perturbing bodies* (in the case of the elastic Earth, the perturbing bodies considered will be the Moon and the Sun).

On the other hand, the external bodies can also be considered as *perturbed bodies*, since the potential created by the Earth acts upon them, the Earth being considered as a non-spherical deformable solid, subject to tidal and centrifugal deformation.

We thus see that the external bodies to the one which suffers the deformation behave as perturbing bodies, that is to say, they cause the deformation of the relevant elastic body; on the other hand, they behave as perturbed bodies, upon which the potential created by that elastic body acts. Nevertheless, since the derivatives of the canonical equations must be taken precisely with respect to the coordinates of the perturbed bodies (Peale, 1973, Kaula, 1964) it is necessary to make the distinction from the beginning. Thus we shall represent the coordinates of the *perturbing* body with the symbol $\sim$. Moreover, we shall conserve the notation m^* and r^* for the mass and the distance from the center to the external body (perturbing and perturbed), in the case when there is no risk of confusion.

The same reasoning must be applied to the case of the rotational deformation. So, we shall also use the symbol $\sim$ upon the corresponding coordinates when acting as *perturbing* ones.

2.2. Tensor of Inertia

Due to the deformation of the elastic mantle, the tensor of inertia will suffer an increase. We can then break down this tensor into two parts:

$$\Pi = \Pi_0 + \Pi_d ,$$

where Π_0 is the tensor in absence of deformation. When it is referred to the principal axes of inertia, it has the well known expression:

$$\Pi_0 = \begin{pmatrix} A_0 & 0 & 0 \\ 0 & B_0 & 0 \\ 0 & 0 & C_0 \end{pmatrix} , \tag{2.1}$$

where A_0, B_0 and C_0 are the principal moments of inertia of the Earth without deformation.

Now, the increase Π_d, due to the deformation, depends on the perturbing potential, so it will be a function of the coordinates of the *perturbing bodies*. In order to emphasize this fact, and according to what was explained in the previous paragraph, it will be written as $\tilde{\Pi}_d$, so that we can write:

$$\Pi = \Pi_0 + \tilde{\Pi}_d . \tag{2.2}$$

2.3. Kinetic Energy and Canonical Moments in Euler Variables

Taking the Euler angles according to the notation of Goldstein (1972) (see Figure 1), and using the matrices:

$$q = \begin{pmatrix} \theta \\ \phi \\ \psi \end{pmatrix} , \; \dot{q} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix} , \; W = \begin{pmatrix} \cos\psi & \sin\theta\sin\psi & 0 \\ -\sin\psi & \sin\theta\cos\psi & 0 \\ 0 & \cos\theta & 1 \end{pmatrix} , \tag{2.3}$$

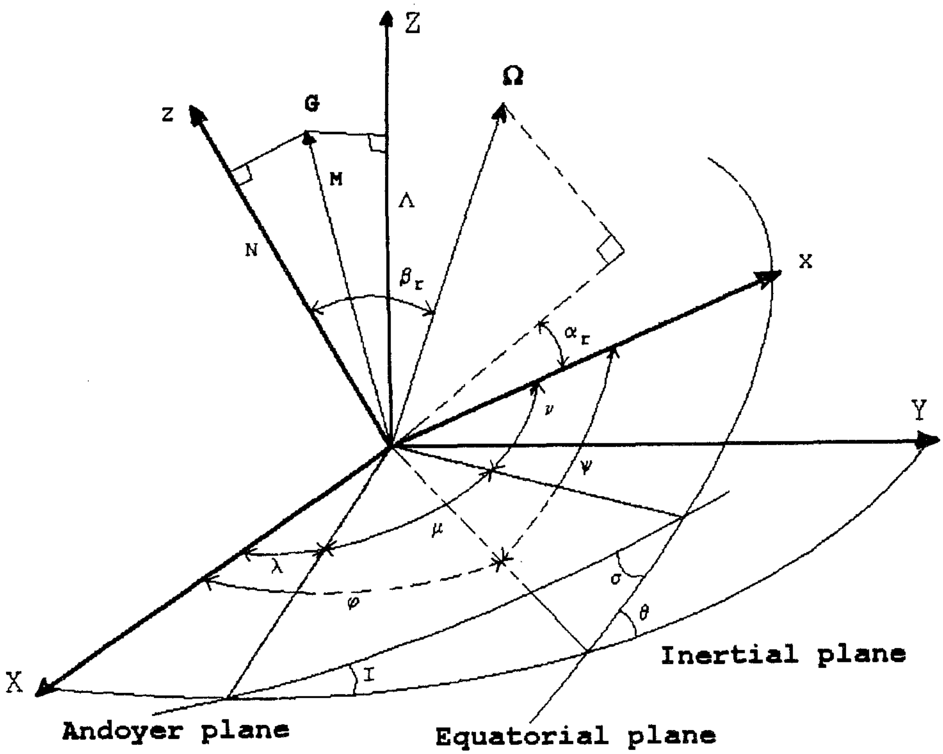

Fig. 1. Euler and Andoyer angles

the kinetic energy will be

$$T = \frac{1}{2}\omega^t \Pi \omega = \frac{1}{2}\dot{q}^t W^t \Pi W \dot{q} \,. \tag{2.4}$$

In order to define strictly the canonical moments, we must take into account the dependence of the centrifugal potential on the velocity, as studied in Getino and Ferrándiz (1992). However, in this work we will take the usual simplification of considering that the centrifugal deformation is produced at a constant velocity, so that the canonical moments will be:

$$P = \begin{pmatrix} P_\theta \\ P_\phi \\ P_\psi \end{pmatrix} = \frac{\partial T}{\partial \dot{q}} = W^t \Pi W \dot{q} = W^t(\Pi_0 + \widetilde{\Pi}_d) W \dot{q} \,. \tag{2.5}$$

It is interesting to point out that, from (2.5) it can clearly be seen that the moments thus defined depend on the deformation, so we shall call them Euler's elastic moments (Getino and Ferrándiz, 1990). Then, if we ignore the effect of the non-sphericity of the Earth when we consider the perturbation $\widetilde{\Pi}_d$, the tensor Π will be symmetrical, so that the canonical expression of the kinetic energy will be:

$$T = \frac{1}{2}P^t W^{-1} \Pi^{-1} W^{-t} P = \frac{1}{2}P^t W^{-1}(\Pi_0 + \widetilde{\Pi}_d)^{-1} W^{-t} P. \tag{2.6}$$

2.4. Andoyer's Canonical Moments

As in Getino and Ferrándiz (1990), we perform a canonical transformation to convert the former set into a new one similar to the Andoyer's canonical set, the new variables being denoted by $(\lambda,\ \mu,\ \nu,\ \Lambda,\ M,\ N)$ (see Figure 1). Two auxiliary angles σ, I are used, related to the canonical momenta through the identities $\cos\sigma = N/M,\ \cos I = \Lambda/M$. It is also useful to consider the expression relating the angular momentum, this set of variables, and the set in 2.3:

$$G = \begin{pmatrix} M \ \sin\sigma \ \sin\nu \\ M \ \sin\sigma \ \cos\nu \\ M \ \cos\sigma \ = \ N \end{pmatrix} = W^{-t}P = \Pi W\dot{q} = (\Pi_0 + \tilde{\Pi}_d)W\dot{q}\,. \tag{2.7}$$

The meaning of the new Andoyer-like variables is quite similar to the meaning of the classic Andoyer system for a rigid body. But, in our case, the angular momentum contains also the effect of the elastic deformation of the mantle, and the new canonical moments too. Due to this, we called them Andoyer's elastic moments, for the sake of the shortness. The plane perpendicular to G, in which the variable μ is measured, is simply referred to as Andoyer's plane. Variable σ is the angle between the angular momentum axis and the figure axis, and I is the angle of the first vector with the polar axis of the inertial plane (the ecliptic reference system at a given epoch). So, the differences with the usual Andoyer system for a rigid body lies in the fact that in the last system the angular momentum does not contain any deformation contributions and is given simply by $\Pi_0\, W\, \dot{q}$ in the previous notation.

There are, of course, also a function of the deformation, so that we have used the name of Andoyer's "elastic" moments (Getino and Ferrándiz, 1990).

Thus, the kinetic energy in these variables is:

$$T = \frac{1}{2}G^t\Pi^{-1}G = \frac{1}{2}G^t(\Pi_0 + \tilde{\Pi}_d)^{-1}G\,, \tag{2.8}$$

with G given in the first equality (2.7).

2.5. Potential Energy

As for the inertia tensor, the potential can be divided into two terms:

$$V = V_0 + V_d, \tag{2.9}$$

where V_0 is the potential due to the Earth in the absence of deformation, and V_d is the additional potential due to the redistribution of mass by the deformation. This potential acts upon the external bodies, the Moon and Sun, now considered as *perturbed bodies*, so V_d will not carry the symbol $\sim$.

2.6. Elastic Energy

When a planet is deformed, energy is stored inside in the form of elastic strain energy. The expressions of this energy can be found in Getino and Ferrándiz (1991a) and Getino (1992). Now, given the fact that the order of magnitude of the elastic energy due to the tidal deformation is very small with respect to the kinetic and potential energies, and that the elastic energy caused by the rotation is constant under the hypotheses with which we are working, in that follows it is not necessary to consider these terms.

2.7. Reference to a Moving Plane

We can express the kinetic energy with Andoyer's *elastic* variables, $(\Lambda, M, N, \lambda, \mu, \nu)$, which relate the fixed system (that is, the ecliptic of the epoch), to the moving system of Earth's principal axes, *Oxyz*. Nevertheless the theories which deal with the motion of the Earth's center of mass are referred to the ecliptic of date. Then we must unify the terms to one reference system only.

Given the great complexity of these theories, it is preferable to express the kinetic energy with parameters related to the ecliptic of date, introducing a new system of variables, $(\Lambda', M', N', \lambda', \mu', \nu')$ which can be interpreted as Andoyer *elastic* variables relating the reference system of the ecliptic of date to that of the principal axes. This change of variables leads to the addition of a complementary component, R_E, to the Hamiltonian referred to the previous inertial frame, as described in Kinoshita (1977).

In what follows, we shall suppress the primes of the new canonical system. That will not cause any confusion.

3. Expression of the Hamiltonian

According to what was pointed out in the previous section, the Hamiltonian is:

$$\mathcal{H} = T + V + R_E. \tag{3.1}$$

In this section we shall develop each one of these terms and we shall make the necessary prior transformations to obtain an adequate expression of the Hamiltonian in order to proceed with the first order analytical integration.

3.1. Development of the Kinetic Energy

First of all, starting from (2.8) we develop the kinetic energy. To do this we need to express the inverse matrix $(\Pi_0 + \widetilde{\Pi}_d)^{-1}$, which can be expanded as follows:

$$\Pi^{-1} = (\Pi_0 + \widetilde{\Pi}_d)^{-1} = \Pi_0^{-1}(I + \widetilde{\Pi}_d \Pi_0^{-1})^{-1}, \tag{3.2}$$

I being the unit matrix. Taking into account that the order of magnitude of the perturbation $\widetilde{\Pi}_d$ is small with respect to the unperturbed part, Π_0 , we can perform an expansion in series as follows

$$(I + \widetilde{\Pi}_d \Pi_0^{-1})^{-1} = I - \widetilde{\Pi}_d \Pi_0^{-1} + [\widetilde{\Pi}_d \Pi_0^{-1}]^2 + O[\widetilde{\Pi}_d \Pi_0^{-1}]^3. \tag{3.3}$$

Up to the second order in $\widetilde{\Pi}_d \Pi_0^{-1}$, sufficient for our study, we have:

$$\Pi^{-1} \simeq \Pi_0^{-1} - \Pi_0^{-1} \widetilde{\Pi}_d \Pi_0^{-1} + \Pi_0^{-1} \widetilde{\Pi}_d \Pi_0^{-1} \widetilde{\Pi}_d \Pi_0^{-1}. \tag{3.4}$$

We can then break down T as follows:

$$T = T_0 + T_1 + T_2 \rightarrow \begin{cases} T_0 = \frac{1}{2} G^t \Pi_0^{-1} G\,, \\ T_1 = -\frac{1}{2} G^t (\Pi_0^{-1} \widetilde{\Pi}_d \Pi_0^{-1}) G\,, \\ T_2 = \frac{1}{2} G^t (\Pi_0^{-1} \widetilde{\Pi}_d \Pi_0^{-1} \widetilde{\Pi}_d \Pi_0^{-1}) G\,, \end{cases} \tag{3.5}$$

where T_0 is the energy corresponding to the rigid body, and T_1 and T_2 the perturbations of first and second order by the deformation.

3.2. Tensor of Deformation

As already pointed out, we consider two causes in the deformation: the *tidal deformation*, which comes from the lunisolar attraction, and the *rotational deformation*, due to the Earth's centrifugal potential. Thus, the matrix of the deformation will be:

$$\widetilde{\Pi}_d = \widetilde{\Pi}_t + \widetilde{\Pi}_r\,. \tag{3.6}$$

As explained in Getino and Ferrándiz (1990), the perturbing potential causing the tidal deformation, at the second order and by unit mass is:

$$W_t = \frac{Gm^*}{r^{*3}} r^2 P_2(\cos S)\,, \quad \text{with} \quad \cos S = \frac{\mathbf{r}\mathbf{r}^*}{rr^*}\,, \tag{3.7}$$

where G is the gravitational constant, $\mathbf{r}$ the vector from the origin to the point within the Earth where the potential is evaluated, r its modulus, and $m^*, \mathbf{r}^*, r^*$ are the mass, the vector from the center of the Earth to the external body (Moon, Sun), and its modulus.

In the aforementioned work, it was shown that, considering a symmetrically spherical Earth, under the influence of this perturbing potential, $\widetilde{\Pi}_t$ is given by:

$$\widetilde{\Pi}_t = D_t \left(\frac{a^*}{r^*}\right)^3 \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}\,, \tag{3.8}$$

with

$$\begin{aligned} t_{11} &= 2P_2(\sin\tilde{\delta}) - P_2^2(\sin\tilde{\delta})\cos 2\tilde{\alpha}\,, & t_{12} &= -P_2^2(\sin\tilde{\delta})\sin 2\tilde{\alpha}\,, \\ t_{22} &= 2P_2(\sin\tilde{\delta}) + P_2^2(\sin\tilde{\delta})\cos 2\tilde{\alpha}\,, & t_{13} &= -2P_2^1(\sin\tilde{\delta})\cos\tilde{\alpha}\,, \\ t_{33} &= -4P_2(\sin\tilde{\delta})\,, & t_{23} &= -2P_2^1(\sin\tilde{\delta})\sin\tilde{\alpha}\,, \end{aligned} \tag{3.9}$$

$\tilde{\alpha}, \tilde{\delta}$ being the longitude and latitude of the perturbing body referred to the principal axes of the Earth, a^* the semi-major axis of its orbit, and the coefficient D_t is given by:

$$\begin{aligned} D_t = \frac{Gm^*}{a^{*3}}\frac{2\pi}{15}\int_r \Big[2\rho_0 r^4\left(5F_2(r) + r^2G_2(r)\right) - \\ -r^5\frac{d\rho_0}{dr}\left(2F_2(r) + r^2G_2(r)\right)\Big]\, dr\,, \end{aligned} \tag{3.10}$$

the integral being spread over the mantle. The functions ρ_0, F_2 and G_2 depend on the Earth Model used. In Getino and Ferrándiz (1991a) these functions were computed by Takeuchi's Model 2, providing the values:

$$D_t = \begin{cases} 6.953379 \times 10^{36}\text{c.g.s. for the Moon}\,, \\ 3.185508 \times 10^{36}\text{c.g.s. for the Sun}\,. \end{cases} \tag{3.11}$$

As for rotation, the disturbing potential by unit mass is

$$W_r = \frac{1}{3}\Omega^2 r^2 - \frac{1}{3}\Omega^2 r^2 P_2(\cos S')\,, \quad \text{with} \;\; \cos S' = \frac{\mathbf{r}\boldsymbol{\Omega}}{r\Omega}\,, \tag{3.12}$$

where Ω is the spin angular velocity and r is the position of the field point relative to the center of mass. The term $\Omega^2 r^2/3$ can be absorbed by the general central field of the body and will not be considered further (Peale, 1973).

Following a study similar to that of the tidal deformation we find that the additional contribution in the inertial tensor due to the centrifugal deformation is (Getino and Ferrándiz, 1991b):

$$\tilde{\Pi}_r = D_r \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{pmatrix}, \tag{3.13}$$

with

$$\begin{aligned} r_{11} &= 2P_2(\cos\tilde{\beta}_r) - P_2^2(\cos\tilde{\beta}_r)\cos 2\tilde{\alpha}_r\,, & r_{12} &= -P_2^2(\cos\tilde{\beta}_r)\sin 2\tilde{\alpha}_r\,, \\ r_{22} &= 2P_2(\cos\tilde{\beta}_r) + P_2^2(\cos\tilde{\beta}_r)\cos 2\tilde{\alpha}_r\,, & r_{13} &= -2P_2^1(\cos\tilde{\beta}_r)\cos\tilde{\alpha}_r\,, \\ r_{33} &= -4P_2(\cos\tilde{\beta}_r)\,, & r_{23} &= -2P_2^1(\cos\tilde{\beta}_r)\sin\tilde{\alpha}_r\,, \end{aligned} \tag{3.14}$$

$\tilde{\alpha}_r, \tilde{\beta}_r$ being the longitude and colatitude of $\boldsymbol{\Omega}$ referred to the principal axes of the Earth. These angles are related to the Andoyer variables in the form

$$\tilde{\beta}_r = \tilde{\sigma}_r\,, \quad \tilde{\alpha}_r = \pi/2 - \tilde{\nu}_r\,. \tag{3.15}$$

Furthermore, taking into account that $\sigma \simeq 10^{-6}$, and neglecting the terms of the second order in σ, we can perform the simplifications (Kinoshita, 1977, Kubo, 1991):

$$\widetilde{\sigma}_r \simeq \sigma\,, \quad \widetilde{\nu}_r \simeq \nu\,, \tag{3.16}$$

but, as we have indicated in subsection 2.1, these simplifications must be done only after taking the necessary derivatives in the equations of motion.

On the other hand, the coefficient D_r is:

$$\begin{aligned} D_r \;=\; & -\frac{\Omega^2}{3}\frac{2\pi}{15}\int_r \Big[2\rho_0 r^4\left(5F_2(r)+r^2G_2(r)\right) - \\ & \qquad -r^5\frac{d\rho_0}{dr}\left(2F_2(r)+r^2G_2(r)\right)\Big]\,dr\,, \end{aligned} \tag{3.17}$$

which, for the Earth Model used and taking $\Omega \simeq \omega_3$, has a value of

$$D_r = -1.422689 \times 10^{41} \text{ c.g.s.}. \tag{3.18}$$

As seen above (3.5), T_1 can be broken down as follows:

$$T_1 = T_{1r} + T_{1t} \rightarrow \begin{cases} T_{1r} = -\dfrac{1}{2}\, G^t\, \Pi_0^{-1}\, \widetilde{\Pi}_r\, \Pi_0^{-1}\, G\,, \\ T_{1t} = -\dfrac{1}{2}\, G^t\, \Pi_0^{-1}\, \widetilde{\Pi}_t\, \Pi_0^{-1}\, G\,. \end{cases} \tag{3.19}$$

As for T_2 , taking into account the expression (3.5), we will have in a similar way

$$T_2 = T_{2r} + T_{2t} + T_{2rt}\,.$$

However, since the orders of magnitude of the coefficients D_r and D_t (of dimensions ML^2) are

$$\frac{D_r}{C_0} \simeq 1.7 \times 10^{-4}\,, \quad \frac{D_t}{C_0} \simeq 8 \times 10^{-9}\,,$$

we can deduce:

$$\begin{aligned} \frac{T_{2r}}{T_0} &\simeq \left(\frac{D_r}{C_0}\right)^2 && \simeq 3 \times 10^{-8}\,, \\ \frac{T_{2t}}{T_0} &\simeq \left(\frac{D_t}{C_0}\right)^2 && \simeq 6 \times 10^{-17}\,, \\ \frac{T_{2rt}}{T_0} &\simeq \left(\frac{D_r}{C_0}\right)\left(\frac{D_t}{C_0}\right) && \simeq 1 \times 10^{-12}\,, \end{aligned}$$

and, for our purposes, it is sufficient to only consider in this term the effect of the centrifugal deformation, that is:

$$T_2 \simeq T_{2r} = \frac{1}{2}\, G^t\, (\Pi_0^{-1}\, \widetilde{\Pi}_r\, \Pi_0^{-1}\, \widetilde{\Pi}_r\, \Pi_0^{-1})\, G\,. \tag{3.20}$$

Finally, the kinetic energy can be expressed as:

$$T = T_0 + T_{1r} + T_{1t} + T_{2r}\,. \tag{3.21}$$

3.3. Potential Energy

The general expression of the potential of a solid Earth acting upon an external point (Moon, Sun) has the classical form:

$$V = \frac{GMm^*}{r^*}\left\{-1 + \sum_{n=2}^{\infty}\left(\frac{R_\oplus}{r^*}\right)^n \left[J_n P_n(\sin\delta) + \sum_{m=1}^{n} P_n^m(\sin\delta)\left(C_n^m \cos m\alpha + S_n^m \sin m\alpha\right)\right]\right\}, \tag{3.22}$$

J_n, C_{nm} and S_{nm} being the usual coefficients which represent the Earth's mass distribution with respect to the reference system being used.

In our case, the main disturbing term corresponds to $n = 2$, for which we have the relationships

$$J_2 = \frac{2C - A - B}{2MR_\oplus^2}\,, C_2^1 = \frac{-F}{MR_\oplus^2}\,, S_2^1 = \frac{-E}{MR_\oplus^2}\,, C_2^2 = \frac{A - B}{4MR_\oplus^2}\,, S_2^2 = \frac{-D}{2MR_\oplus^2}\,,$$

where A, B and C are the principal inertia moments, and $-D, -E$ and $-F$ the inertia products, we can express it in the most convenient form

$$V = \frac{Gm^*}{r^{*3}}\left\{\frac{2C - A - B}{2}P_2(\sin\delta) + \left[-F\cos\alpha - E\sin\alpha\right]P_2^1(\sin\delta) + \left[\frac{A - B}{4}\cos 2\alpha + \frac{-D}{2}\sin 2\alpha\right]P_2^2(\sin\delta)\right\}. \tag{3.23}$$

According to (2.2) and (3.6), the inertia tensor of our problem can be broken down in the form

$$\Pi = \Pi_0 + \widetilde{\Pi}_r + \widetilde{\Pi}_t\,.$$

The potential itself can be expanded as follows:

$$V = V_0 + V_r + V_t\,, \tag{3.24}$$

and taking into account (2.1), (3.8) and (3.13) we get:

$$V_0 = \frac{Gm^*}{r^{*3}}\left[\frac{2C_0 - A_0 - B_0}{2}P_2(\sin\delta) + \frac{A_0 - B_0}{4}P_2^2(\sin\delta)\cos 2\alpha\right], \tag{3.25}$$

$$V_r = \frac{Gm^*}{r^{*3}}D_r\left[\frac{2r_{33} - r_{11} - r_{22}}{2}P_2(\sin\delta) + \right.$$

$$+ (r_{13}\cos\alpha + r_{23}\sin\alpha)\, P_2^1(\sin\delta) + \\ + \left(\frac{r_{11}-r_{22}}{4}\cos 2\alpha + \frac{1}{2}\, r_{12}\sin 2\alpha\right) P_2^2(\sin\delta)\Bigg] \,, \tag{3.26}$$

$$V_t = \frac{Gm^*}{r^{*3}} D_t \left[\frac{2t_{33}-t_{11}-t_{22}}{2} P_2(\sin\delta) + \right. \\ + (t_{13}\cos\alpha + t_{23}\sin\alpha)\, P_2^1(\sin\delta) + \\ \left. + \left(\frac{t_{11}-t_{22}}{4}\cos 2\alpha + \frac{1}{2}\, t_{12}\sin 2\alpha\right) P_2^2(\sin\delta)\right] . \tag{3.27}$$

Using (3.11) and (3.18) it is easy to find the order of magnitude of each component with respect to main term T_0:

$$\frac{V_0}{T_0} \simeq 6\times 10^{-8}\,, \qquad \frac{V_r}{T_0} \simeq 3\times 10^{-9}\,, \qquad \frac{V_t}{T_0} \simeq 2\times 10^{-13}\,.$$

The term V_t is much smaller than the others and we can neglect it, so that the potential energy is reduced to

$$V \simeq V_0 + V_r\,. \tag{3.28}$$

By substituting the definitions of r_{ij} given in (3.14), we obtain the required expression for V_r:

$$V_r = K_r' \left[(1 - \frac{3}{2}\sin^2\widetilde{\sigma}_r) P_2(\sin\delta) + \frac{1}{2}\sin 2\widetilde{\sigma}_r P_2^1(\sin\delta)\sin(\alpha+\widetilde{\nu}_r) - \right. \\ \left. - \frac{1}{4}\sin^2\widetilde{\sigma}_r P_2^2(\sin\delta)\cos(2\alpha + 2\widetilde{\nu}_r)\right] , \tag{3.29}$$

where

$$K_r' = -6D_r\frac{Gm^*}{r^{*3}}\,. \tag{3.30}$$

It is worth noting the fact that, in this derivation, both the rotational potential, V_r, and the tidal potential, V_t, are given respectively a function of the coefficients D_r and D_t defined from an integral depending on the radial distance and the rheological parameters of the chosen Earth model. Thus, these coefficients are calculated principally as a function of geophysical data. In the traditional approach (Kaula, 1964, Peale, 1973), the corresponding terms of the potential are written in a straightforward manner as:

$$V_t = \frac{Gm^* m' R_\oplus^5}{r^{*3} r'^3} k_2 P_2(\cos S')\,, \quad \text{with} \quad \cos S' = \frac{\mathbf{r}^*\cdot\mathbf{r}'}{r^* r'}\,, \tag{3.31}$$

where k_2 is the Love number, and

$$V_r = -\frac{1}{3} k_2 \Omega^2 \frac{m^* R_\oplus^5}{r^{*3}} P_2(\cos S'')\,, \quad \text{with} \quad \cos S'' = \frac{\boldsymbol{\Omega}\cdot\mathbf{r}^*}{\Omega r^*}\,. \tag{3.32}$$

Equations (3.26) and (3.3) can be given by formulae similar to the last two ones, in terms of a coefficient that would take the role of a global Love number, whereas k_2 is the usual Love number at the Earth surface. There is no contradiction between these different formulations, as the previous (3.31) and (3.32) are strictly true only when some hypotheses are verified (Love, 1927), which is not the case for an accurate model of Earth. Thus these formulations should be understood as approximations. As for our derivation, it is valid in more general hypotheses (Getino and Ferrándiz, 1991a) which can be verified in more modern Earth models (Takeuchi, 1951, Gilbert and Dziewonski, 1975), and so can be considered more rigorous.

In this sense, our equations should represent better approximations, as the procedure followed to obtain them is closer to real phenomenon of the deformations that actually occur.

3.4. Reference to a Moving Ecliptic

The complementary term R_E is written (Kinoshita, 1977):

$$R_E = M \sin I \, R_{E1} + \Lambda R_{E2} \, , \tag{3.33}$$

where:

$$R_{E1} = \sin\pi \cos(\lambda - \Pi)\frac{d\Pi}{dt} - \sin(\lambda - \pi)\frac{d\pi}{dt}\,; \quad R_{E2} = (1 - \cos\pi)\frac{d\Pi}{dt} \,, \tag{3.34}$$

π and Π being the angles defining the moving reference plane.

4. First Order Integration for a Rigid Earth

As explained in the previous paragraph, the complete Hamiltonian of the problem is, in our approximation,

$$\mathcal{H} = T_0 + T_{1r} + T_{1t} + T_{2r} + V_0 + V_r + R_E \, . \tag{4.1}$$

We now proceed to the first order analytical integration using Hori's perturbation method (1966), which eliminates short period terms by the use of a Lie transformation and an averaging method. However, since the expressions that appear are very complicated, and since that perturbation method is linear at the first order of integration, we believe it is preferable to study separately the effects corresponding to a rigid Earth, to the centrifugal perturbation and to the tidal perturbation. This will allow us to work with simpler expressions, which will make the reading less difficult, and on the other hand, to understand more clearly the effects of each of the above mentioned perturbations.

We begin this section by performing an integration corresponding to a rigid Earth, that is, prior to the deformation. The Hamiltonian is reduced to

$$\mathcal{H}_{rigid} = T_0 + V_0 + R_E \, . \tag{4.2}$$

The procedure used can be considered as a simplification of the more general approach of Kinoshita (1977), since instead of having recourse to the use of action–angles variables, a reordering of the unperturbed Hamiltonian is performed (passing the terms in the variable ν to the perturbation), which allows us to shorten the calculations. This procedure has already been used in other problems (as in Henrard and Moons, 1978, for the case of the Moon) although as far as we know it has never been published in the case of the Earth.

4.1. Preliminary Arrangements

To make the integration easier, we will first consider each one of the terms in (4.2). Thus, the kinetic energy T_0 (see (2.1) and (3.5)) can be written as:

$$T_0 = \frac{M^2 - N^2}{2}\left(\frac{\sin^2\nu}{A_0} + \frac{\cos^2\nu}{B_0}\right) + \frac{N^2}{2C_0}, \tag{4.3}$$

which can be broken down in the following way:

$$T_0 = T_0^a + T_0^b \rightarrow \begin{cases} T_0^a = \frac{1}{4}M^2\left(\frac{1}{A_0} + \frac{1}{B_0}\right) + \frac{1}{4}N^2\left(\frac{2}{C_0} - \frac{1}{A_0} - \frac{1}{B_0}\right), \\ T_0^b = \frac{1}{4}(M^2 - N^2)\left(\frac{1}{B_0} - \frac{1}{A_0}\right)\cos 2\nu . \end{cases} \tag{4.4}$$

In the same manner, the term V_0 can be split into:

$$V_0 = V_0^a + V_0^b \rightarrow \begin{cases} V_0^a = \frac{1}{3}K_0'\left(\frac{a^*}{r^*}\right)^3 P_2(\sin\delta), \\ V_0^b = \frac{Gm^*}{a^{*3}}\frac{A_0 - B_0}{4}\left(\frac{a^*}{r^*}\right)^3 P_2^2(\sin\delta)\cos 2\alpha . \end{cases} \tag{4.5}$$

where we have introduced the coefficient:

$$K_0' = 3\frac{Gm^*}{a^{*3}}\frac{2C_0 - A_0 - B_0}{2}. \tag{4.6}$$

According to the orders of magnitude of these terms (Kinoshita, 1977)

$$\frac{V_0^a}{T_0} \cong 10^{-8}, \quad \frac{V_0^b}{T_0} \cong 10^{-13}, \quad \frac{R_E}{T_0} \cong 10^{-7},$$

the Hamiltonian can be broken down into

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 \rightarrow \begin{cases} \mathcal{H}_0 = T_0^a, \\ \mathcal{H}_1 = T_0^b + R_E + V_0^a, \\ \mathcal{H}_2 = V_0^b . \end{cases} \tag{4.7}$$

4.2. Elimination of Periodic Terms

To perform the integration at the first order we divide the Hamiltonian into an unperturbed part, $\mathcal{H}_0$, and a perturbed part $\mathcal{H}_1$, that is: $\mathcal{H}_{rigid} = \mathcal{H}_0 + \mathcal{H}_1$. By using Hori's method, the initial Hamiltonian is transformed into a new one, easier to integrate, marked with the symbol * :

$$\mathcal{H}_{rigid} = \mathcal{H}_0 + \mathcal{H}_1 \rightarrow \mathcal{H}^*_{rigid} = \mathcal{H}^*_0 + \mathcal{H}^*_1 . \tag{4.8}$$

The new disturbing term $\mathcal{H}^*_1$ will be chosen following a criterion of averages, that is, $\mathcal{H}^*_1$ is defined as the time average of $\mathcal{H}_1$ with respect to the solution to the unperturbed problem, with Hamiltonian $\mathcal{H}^*_0 = \mathcal{H}_0$. In the absence of resonances this procedure leads to an asymptotical solution that differs from the real one to the order of ϵ in time intervals of longitude

$1/\epsilon$, ϵ being the relative magnitude of the perturbation (Sanders and Verhulst, 1985). Thus,the new Hamiltonian (for the first order) will be:

$$\mathcal{H}^*_0 = \mathcal{H}_0 \; ; \qquad \mathcal{H}^*_1 = \mathcal{H}_{1\,\mathrm{sec}} . \tag{4.9}$$

As the secular part of $\mathcal{H}_1$ we take the terms which do not contain the angular variables μ, ν, nor those giving the position of the Moon and Sun. Then, we have:

$$\mathcal{H}^*_0 = T^a_0 = \frac{M^{*^2}}{4}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)+\frac{N^{*^2}}{4}\left(\frac{2}{C_0}-\frac{1}{A_0}-\frac{1}{B_0}\right), \tag{4.10}$$

$$\mathcal{H}^*_1 = R_E + V^a_{0\,\mathrm{sec}} = M^* \sin I^* R_{E1} + \Lambda^* R_{E2} + K'_0(1 - \frac{3}{2}\sin^2\sigma^*)B^*_{\hat{0}}, \tag{4.11}$$

where the constant coefficient $B_{\hat{0}} = B_{(00000)}$ corresponds to the only secular contribution of the expansion of $P_2(\sin\delta)$ (see Appendix I). Note that we have used asterisks to indicate the new variables that result from the canonical transformation.

4.3. Generating Function and First-Order Perturbations

In this section, the asterisks used with the canonically transformed variables are omitted for the sake of simplicity. The generating function of the transformation is, at the first order:

$$W = \int(\mathcal{H}_1 - \mathcal{H}^*_1)dt = \int \mathcal{H}_{1per}dt, \tag{4.12}$$

where this integral must be performed along the solution to the unperturbed system (Hori, 1966). The solutions which are not constant will concern only the variables μ, ν, whose mean motions are:

$$n_\mu = \frac{d\mu}{dt} = \frac{1}{2}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)M_0 ,$$

$$n_\nu = \frac{d\nu}{dt} = \frac{1}{2}\left(\frac{2}{C_0}-\frac{1}{A_0}-\frac{1}{B_0}\right)N_0 , \tag{4.13}$$

M_0 and N_0 being constants of the motion. Carrying out the integration along the solution of the unperturbed Hamiltonian we obtain:

$$\begin{aligned} W &= \left(\frac{1}{B_0} - \frac{1}{A_0}\right) \frac{M^2 - N^2}{4} \frac{\sin 2\nu}{2n_\nu} + \\ &+ K_0' \left[\frac{1}{2}(3\cos^2\sigma - 1)W_a - \frac{1}{2}\sin 2\sigma W_b + \frac{1}{4}\sin^2\sigma W_c\right], \end{aligned} \tag{4.14}$$

where the terms which belong to the secular part, that is to say, those corresponding to $B_{\hat{0}} = B_{(00000)}$, are assumed to be excluded. The functions W_i are (see Appendix I):

$$\begin{aligned} W_a &= \sum_i \frac{B_i}{n_i} \sin\Theta_i, \\ W_b &= \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i} \sin(\mu - \tau\Theta_i), \\ W_c &= \sum_\tau \sum_i \frac{D_i(\tau)}{2n_\mu - \tau n_i} \sin(2\mu - \tau\Theta_i), \end{aligned} \tag{4.15}$$

with $n_i = d\Theta_i/dt$. Obviously, both the secular Hamiltonian and the generating function are basically the same as those of Kinoshita (1977), as we are referring, in this paragraph, to a rigid Earth (without deformation), although the expressions are not identical since he used action-angle variables. The perturbations, both periodic and secular, will then also be equivalent. However, for reasons of clarity, we prefer to calculate them, even in a succinct manner, since they will serve as a basis for the study of the deformations, which will be done later.

The associated canonical transformation can be obtained at the first order by the equations of perturbation (Hori, 1966), which can be written in a symbolical form as:

$$\Delta(\Lambda, M, N) = -\frac{\partial W}{\partial(\lambda, \mu, \nu)}; \qquad \Delta(\lambda, \mu, \nu) = \frac{\partial W}{\partial(\Lambda, M, N)}.$$

As the generating function depends on the variables σ, I, related to the moments by the equations:

$$\cos\sigma = \frac{N}{M}; \qquad \cos I = \frac{\Lambda}{M},$$

it is convenient to carry out the derivations with the help of these variables. The derivation operators are:

$$\begin{aligned} \frac{\partial}{\partial M} &= \left(\frac{\partial}{\partial M}\right) + \frac{1}{M}\cot\sigma\frac{\partial}{\partial\sigma} + \frac{1}{M}\cot I\frac{\partial}{\partial I}, \\ \frac{\partial}{\partial N} &= \left(\frac{\partial}{\partial N}\right) - \frac{1}{M\sin\sigma}\frac{\partial}{\partial\sigma}, \end{aligned}$$

$$\frac{\partial}{\partial \Lambda} = \left(\frac{\partial}{\partial \Lambda}\right) - \frac{1}{M \sin I}\frac{\partial}{\partial I} . \tag{4.16}$$

The symbolical derivatives inside the parentheses refer to the partial derivatives, in an explicit way, with respect to the variables Λ, M, N.

It is also interesting to get the expressions for the variation of the angles σ, I, which are necessary to fix the position of fundamental planes of the Earth's rotation. We can deduce:

$$\Delta\sigma = \frac{1}{M \sin\sigma}\left(\frac{\partial W}{\partial \nu} - \cos\sigma \frac{\partial W}{\partial \mu}\right) ,$$

$$\Delta I = \frac{1}{M \sin I}\left(\frac{\partial W}{\partial \lambda} - \cos I \frac{\partial W}{\partial \mu}\right) . \tag{4.17}$$

Given that $\sigma \simeq 10^{-6}$ rad., once the corresponding derivatives have been carried out, we take the simplification

$$\sin\sigma \simeq 0 , \qquad \cos\sigma \simeq 1 ,$$

obtaining, in the first place, the following results (neglecting terms in σ^2)

$$\Delta_0\mu \simeq K_0 \left[-3W_a - \frac{1}{\sin\sigma}W_b + \frac{1}{2}W_c + \frac{\cos I}{\sin I}\frac{\partial W_a}{\partial I}\right] , \tag{4.18}$$

$$\Delta_0\nu \simeq K_0 \left[3W_a + \frac{1}{\sin\sigma}W_b - \frac{1}{2}W_c\right] , \tag{4.19}$$

$$\Delta_0(\mu + \nu) \simeq K_0 \frac{\cos I}{\sin I}\frac{\partial W_a}{\partial I} , \tag{4.20}$$

$$\Delta_0\sigma \simeq K_0 \frac{\partial W_b}{\partial \mu} , \tag{4.21}$$

where we have introduced the new coefficient:

$$K_0 = \frac{K_0'}{M} = 3\frac{Gm^*}{a^{*3}}\frac{2C_0 - A_0 - B_0}{2M} , \tag{4.22}$$

while we use the notation Δ_0 to indicate the perturbations of a rigid Earth. The most important results for our purposes are those that refer to the motion of the three fundamental planes, which we shall now study.

4.4. Motion of the Andoyer's plane

The Andoyer plane, perpendicular to the angular-momentum axis, is determined by the angles λ, longitude of the node, and I, inclination. Their nutations are, with the said simplifications in the angle σ :

$$\Delta_0\lambda \simeq -K_0\frac{1}{\sin I}\frac{\partial W_a}{\partial I} = -K_0\sum_i \frac{1}{\sin I}\frac{\partial}{\partial I}\left(\frac{B_i}{n_i}\right)\sin\Theta_i \, , \tag{4.23}$$

$$\Delta_0 I \simeq K_0\frac{1}{\sin I}\frac{\partial W_a}{\partial \lambda} = \frac{K_0}{\sin I}\sum_i(-m_5)\frac{B_i}{n_i}\cos\Theta_i \, . \tag{4.24}$$

4.5. Motion of the Equatorial Plane

This plane is determined by the Euler angles ϕ and θ. To use a notation similar to that from Kinoshita, we shall call them λ_f and I_f respectively. To obtain their perturbations we will refer them to the Andoyer variables. According to Kinoshita (1977), neglecting the terms in σ^2 we have:

$$\lambda_f \simeq \lambda + \sigma\frac{\sin\mu}{\sin I} \, ; \qquad I_f \simeq I + \sigma\cos\mu \, . \tag{4.25}$$

Once the corresponding derivatives have been calculated, we obtain

$$\begin{aligned}\Delta_0(\delta\lambda_f) &= \Delta_0(\lambda_f - \lambda) \simeq \frac{K_0}{\sin I}\left[\frac{\partial W_b}{\partial\mu}\sin\mu - W_b\cos\mu\right] = \\ &= \frac{K_0}{\sin I}\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i}\sin\Theta_i \, ,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\Delta_0(\delta I_f) &= \Delta_0(I_f - I) \simeq K_0\left[\frac{\partial W_b}{\partial\mu}\cos\mu + W_b\sin\mu\right] = \\ &= K_0\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos\Theta_i \, .\end{aligned} \tag{4.27}$$

The second members of (4.26) and (4.27) are known as *Oppolzer terms for the plane of figure.*

4.6. Motion of the Plane Perpendicular to the Rotation Axis

The longitude of the node and the inclination of this plane are designated by λ_r and I_r . Their expressions as functions of Andoyer variables are (Kinoshita, 1977):

$$\begin{aligned}\lambda_r &\simeq \lambda + \left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\sigma\frac{\sin\mu}{\sin I} \, , \\ I_r &\simeq I + \left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\sigma\cos\mu \, .\end{aligned} \tag{4.28}$$

Their perturbations will be:

$$\Delta_0(\delta\lambda_r) = \Delta_0(\lambda_r - \lambda) \simeq \left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\Delta_0(\delta\lambda_f)\,, \tag{4.29}$$

$$\Delta_0(\delta I_r) = \Delta_0(I_r - I) \simeq \left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\Delta_0(\delta I_f)\,, \tag{4.30}$$

the second members of (4.29) and (4.30) also being known as *Oppolzer terms* (for the plane perpendicular to the rotation axis).

4.7. Perturbations of the Polar Motion

Polar motion is defined as the motion of the rotation axis relative to the figure axis (Kinoshita, 1977), specified usually by the pair of coordinates

$$x_p \simeq \frac{C_0}{A_0}(1 - e/2)\sigma \sin\nu\,, \qquad y_p \simeq -\frac{C_0}{B_0}(1 + e/2)\sigma\cos\nu\,, \tag{4.31}$$

where e is a measure of the triaxiality of the Earth. Calculating their perturbations we obtain:

$$\Delta_0 x_p \simeq \frac{C_0}{A_0}(1 - e/2)K_0 \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\sin(\mu + \nu - \tau\Theta_i)\,, \tag{4.32}$$

$$\Delta_0 y_p \simeq -\frac{C_0}{B_0}(1 + e/2)K_0 \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos(\mu + \nu - \tau\Theta_i)\,. \tag{4.33}$$

4.8. Secular Perturbations

Given that the Hamiltonian $\mathcal{H}^*$ contains the angular variable λ (see (4.11) and (3.34)) and its conjugate momentum Λ (through the angle I), we cannot obtain the perturbations corresponding to λ and I analytically, so it is convenient to obtain a solution in power series of time (Kinoshita, 1977). As for the variables μ, ν, we have:

$$n_\mu^* = \frac{d\mu^*}{dt} = \frac{\partial\mathcal{H}^*}{\partial M^*} = \frac{\partial(\mathcal{H}_0^* + \mathcal{H}_1^*)}{\partial M^*}\,,$$

$$n_\nu^* = \frac{d\nu^*}{dt} = \frac{\partial\mathcal{H}^*}{\partial N^*} = \frac{\partial(\mathcal{H}_0^* + \mathcal{H}_1^*)}{\partial N^*}\,, \tag{4.34}$$

Once these derivatives have been determined, and with the simplifications: $\sin\sigma \simeq 0$, $\cos\sigma \simeq 1$, we finally arrive at the expression:

$$n_\mu^* = n_\mu + \frac{R_{E1}^*}{\sin I^*} + K_0\left[\frac{\cos I^*}{\sin I^*}\frac{\partial B_{\hat{0}}^*}{\partial I^*} - 3B_{\hat{0}}^*\right]\,, \tag{4.35}$$

$$n_\nu^* = n_\nu + K_0[3B_{\hat{0}}^*]\,, \tag{4.36}$$

with which:

$$n_\mu^* + n_\nu^* = n_\mu + n_\nu + \frac{R_{E1}^*}{\sin I^*} + K_0 \frac{\cos I^*}{\sin I^*} \frac{\partial B_{\widehat{0}}^*}{\partial I^*} . \tag{4.37}$$

4.9. Remark

It is clear that on studying the rigid Earth in this section, the results obtained for both periodic and secular perturbations are the same as those given by Kinoshita's theory, if we only substitute the principal moments of inertia A, B and C for the corresponding ones A_0, B_0 and C_0 in the absence of deformation. For the same reason, Kinoshita's K coefficient, function of A, B and C, is substituted by K_0, as a function of A_0, B_0 and C_0.

5. Effect of the Rotational Deformation

In this section we shall study, following the same steps as before, the perturbations due to the centrifugal deformation. According to (4.1), the contribution to be added to the Hamiltonian, due to the effect of the rotation is:

$$\mathcal{H}_{rot} = T_{1r} + T_{2r} + V_r . \tag{5.1}$$

Before proceeding to its integration, we shall study each one of its terms.

5.1. Preliminary Arrangements

By means of (3.13), (3.14), (3.15) and (3.19), the expression of T_{1r}, after some calculations, will be:

$$\begin{aligned} T_{1r} = D_r \Bigg\{ & \left[\frac{2N^2}{C_0^2} - (M^2 - N^2) \left(\frac{\sin^2 \nu}{A_0^2} + \frac{\cos^2 \nu}{B_0^2} \right) \right] (1 - \frac{3}{2} \sin^2 \widetilde{\sigma}_r) - \\ & - \frac{3}{2} (M^2 - N^2) \left(\frac{\sin^2 \nu}{A_0^2} - \frac{\cos^2 \nu}{B_0^2} \right) \sin^2 \widetilde{\sigma}_r \cos 2\widetilde{\nu}_r + \\ & + \frac{3}{2} \frac{M^2 - N^2}{A_0 B_0} \sin 2\nu \sin^2 \widetilde{\sigma}_r \sin 2\widetilde{\nu}_r + \\ & + \frac{3}{2} \frac{M^2}{C_0} \sin 2\sigma \sin 2\widetilde{\sigma}_r \left(\frac{1}{A_0} \sin \nu \sin \widetilde{\nu}_r + \frac{1}{B_0} \cos \nu \cos \widetilde{\nu}_r \right) \Bigg\} . \end{aligned} \tag{5.2}$$

As in the previous section, it is convenient to transform this expression in order to separate first order terms from those with the coefficient $B_0^{-1} - A_0^{-1}$. Taking into account the trigonometric relationships:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) , \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x) ,$$

after some calculations we get:

$$T_{1r} = T_{1r}^a + T_{1r}^b \,, \tag{5.3}$$

where

$$\begin{aligned} T_{1r}^a &= D_r \left\{ \left[\frac{2N^2}{C_0^2} - \frac{1}{2}(M^2 - N^2)\left(\frac{1}{A_0^2} + \frac{1}{B_0^2}\right)\right] (1 - \frac{3}{2}\sin^2\widetilde{\sigma}_r) + \right. \\ &\quad + \frac{3}{8}(M^2 - N^2)\left(\frac{1}{A_0} + \frac{1}{B_0}\right)^2 \sin^2\widetilde{\sigma}_r \cos 2(\nu - \widetilde{\nu}_r) + \\ &\quad \left. + \frac{3}{4}\frac{M^2}{C_0}\left(\frac{1}{A_0} + \frac{1}{B_0}\right) \sin 2\sigma \sin 2\widetilde{\sigma}_r \cos(\nu - \widetilde{\nu}_r) \right\} , \end{aligned} \tag{5.4}$$

$$\begin{aligned} T_{1r}^b &= D_r \left(\frac{1}{B_0} - \frac{1}{A_0}\right) \left\{ -\frac{M^2 - N^2}{2}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)(1 - \frac{3}{2}\sin^2\widetilde{\sigma}_r)\cos 2\nu + \right. \\ &\quad + \frac{3}{4}(M^2 - N^2)\sin^2\widetilde{\sigma}_r \left[\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\cos 2\widetilde{\nu}_r + \right. \\ &\quad \left. + \frac{1}{2}\left(\frac{1}{B_0} - \frac{1}{A_0}\right)\cos 2(\nu + \widetilde{\nu}_r)\right] + \\ &\quad \left. + \frac{3}{4}\frac{M^2}{C_0} \sin 2\sigma \sin 2\widetilde{\sigma}_r \cos(\nu + \widetilde{\nu}_r) \right\} . \end{aligned} \tag{5.5}$$

From (3.20) we can get the expression of T_{2r}. In this formal expression, this term is very complicated, but we have shown that $\widetilde{\sigma}_r \simeq \sigma \simeq 10^{-6}$ rad., and, in addition,

$\widetilde{\sigma}_r$ behaves as constant when calculating the perturbation equations. Thus, we shall neglect the terms in $\widetilde{\sigma}_r^2$, so that:

$$T_{2r} = T_{2r}^a + T_{2r}^b \,, \tag{5.6}$$

where:

$$\begin{aligned} T_{2r}^a &= D_r^2 \left\{ \left[\frac{8N^2}{C_0^3} + (M^2 - N^2)\left(\frac{1}{A_0^3} + \frac{1}{B_0^3}\right)\right] + \frac{3}{2}\frac{M^2}{C_0} \times \right. \\ &\quad \left. \times \left[\frac{2}{C_0}\left(\frac{1}{A_0} + \frac{1}{B_0}\right) - \left(\frac{1}{A_0^2} + \frac{1}{B_0^2}\right)\right] \sin 2\sigma \sin 2\widetilde{\sigma}_r \cos(\nu - \widetilde{\nu}_r) \right\} , \end{aligned} \tag{5.7}$$

and:

$$\begin{aligned} T_{2r}^b &= D_r^2 \left\{ (M^2 - N^2)\left(\frac{1}{B_0^3} - \frac{1}{A_0^3}\right)\cos 2\nu + \frac{3}{2}\frac{M^2}{C_0} \times \right. \\ &\quad \left. \times \left[\frac{2}{C_0}\left(\frac{1}{B_0} - \frac{1}{A_0}\right) - \left(\frac{1}{B_0^2} - \frac{1}{A_0^2}\right)\right] \sin 2\sigma \sin 2\widetilde{\sigma}_r \cos(\nu + \widetilde{\nu}_r) \right\} , \end{aligned} \tag{5.8}$$

Finally, from (3.29) and (3.28) we have:

$$V_r = \frac{1}{3}K_r'\left[(1-\frac{3}{2}\sin^2\widetilde{\sigma}_r)P_0 + \frac{1}{2}\sin 2\widetilde{\sigma}_r P_{1r} - \frac{1}{4}\sin^2\widetilde{\sigma}_r P_{2r}\right], \quad (5.9)$$

where, for the sake of brevity, we have used the notation,

$$\begin{aligned} P_0 &= (a^*/r^*)^3 P_2(\sin\delta), \\ P_{1r} &= (a^*/r^*)^3 P_2^1(\sin\delta)\sin(\alpha+\widetilde{\nu}_r), \\ P_{2r} &= (a^*/r^*)^3 P_2^2(\sin\delta)\cos 2(\alpha+\widetilde{\nu}_r). \end{aligned}$$

The expression of P_0, obtained directly from Appendix I, is:

$$\begin{aligned} \left(\frac{a}{r}\right)^3 P_2(\sin\delta) &= \frac{3}{2}(3\cos^2\sigma - 1)\sum_i B_i\cos\Theta_i - \\ &\quad -\frac{3}{2}\sin 2\sigma\sum_\tau\sum_i C_i(\tau)\cos(\mu-\tau\Theta_i) + \\ &\quad +\frac{3}{4}\sin^2\sigma\sum_\tau\sum_i D_i(\tau)\cos(2\mu-\tau\Theta_i), \end{aligned} \quad (5.10)$$

while those of remaining P_{1r} and P_{2r} are:

$$\begin{aligned} P_{1r} &= 3\left[\frac{3}{4}\sin 2\sigma\sum_\tau\sum_i B_i\cos(\nu-\widetilde{\nu}_r-\tau\Theta_i) + \right. \\ &\quad +\frac{1}{2}(1+\cos\sigma)(-1+2\cos\sigma)\sum_\tau\sum_i C_i(\tau)\cos(\mu+\nu-\widetilde{\nu}_r-\tau\Theta_i) - \\ &\quad \left. -\frac{1}{4}\sin\sigma(1+\cos\sigma)\sum_\tau\sum_i D_i(\tau)\cos(2\mu+\nu-\widetilde{\nu}_r-\tau\Theta_i)\right], \end{aligned} \quad (5.11)$$

$$\begin{aligned} P_{2r} &= -3\left[\frac{3}{2}\sin^2\sigma\sum_\tau\sum_i B_i\cos(2\nu-2\widetilde{\nu}_r-\tau\Theta_i) + \right. \\ &\quad +\sin\sigma(1+\cos\sigma)\sum_\tau\sum_i C_i(\tau)\cos(\mu+2\nu-2\widetilde{\nu}_r-\tau\Theta_i) + \\ &\quad \left. +\frac{1}{4}(1+\cos\sigma)^2\sum_\tau\sum_i D_i(\tau)\cos(2\mu+2\nu-2\widetilde{\nu}_r-\tau\Theta_i)\right]. \end{aligned} \quad (5.12)$$

5.2. Elimination of Periodic Terms

In a similar way as in the previous section we break down the Hamiltonian part of the rotational perturbation as follows:

$$\mathcal{H}_{rot} = \mathcal{H}_{1r} + \mathcal{H}_{2r} \rightarrow \begin{cases} \mathcal{H}_{1r} = T_{1r}^a + T_{2r}^a + V_r, \\ \mathcal{H}_{2r} = T_{1r}^b + T_{2r}^b. \end{cases} \quad (5.13)$$

For the first order integration we consider the transformation:

$$\mathcal{H}_{1r} \to \mathcal{H}^*_{1r} = \mathcal{H}_{1sec} = T^a_{1r} + T^a_{2r} + V_{rsec}\,, \tag{5.14}$$

where T^a_{1r} and T^a_{2r} are given by (5.4) and (5.7) respectively, while, taking into account (5.11), (5.12) and Appendix I:

$$V_{rsec} = \frac{1}{3}K'_r\left[(1-\frac{3}{2}\sin^2\widetilde{\sigma}_r)P_0 + \frac{1}{2}\sin 2\widetilde{\sigma}_r P_{1r} - \frac{1}{4}\sin^2\widetilde{\sigma}_r P_{2r}\right]_{sec}\,, \tag{5.15}$$

with

$$\begin{aligned} P_{0sec} &= 3\,(1-\frac{3}{2}\sin^2\sigma)\,B_{\widehat{0}}\,,\\ P_{1rsec} &= \frac{9}{2}\sin 2\sigma\,B_{\widehat{0}}\cos(\nu-\widetilde{\nu}_r)\,,\\ P_{2rsec} &= -9\sin^2\sigma\,B_{\widehat{0}}\cos(2\nu-2\widetilde{\nu}_r)\,. \end{aligned} \tag{5.16}$$

We have maintained the terms in $\nu - \widetilde{\nu}_r$ in $\mathcal{H}^*_{r1}$ since, once the corresponding derivatives have been carried out, after identifying $\widetilde{\sigma}_r \simeq \sigma$ and $\widetilde{\nu}_r \simeq \nu$, these terms will give secular contributions.

5.3. Generating Function and First-Order Perturbations

The corresponding generating function at the first order, and omitting the asterisk for the sake of simplicity, will be:

$$W_r = \int(\mathcal{H}_{1r} - \mathcal{H}^*_{1r})\,dt = \int \mathcal{H}_{1rper}\,dt = \int V_{rper}\,dt\,,$$

resulting in

$$W_r = \frac{1}{3}K'_r\left[(1-\frac{3}{2}\sin^2\widetilde{\sigma}_r)W^r_0 + \frac{1}{2}\sin 2\widetilde{\sigma}_r W^r_1 - \frac{1}{4}\sin^2\widetilde{\sigma}_r W^r_2\right]\,, \tag{5.17}$$

with

$$\begin{aligned} W^r_0 &= 3\left[(1-\frac{3}{2}\sin^2\sigma)W_a - \frac{1}{2}\sin 2\sigma W_b + \frac{1}{4}\sin^2\sigma W_c\right]\,,\\ W^r_1 &= 3\left[\frac{3}{4}\sin 2\sigma W_{1a} + \frac{1}{2}(1+\cos\sigma)(-1+2\cos\sigma)W_{1b} - \right.\\ &\qquad \left. -\frac{1}{4}\sin\sigma(1+\cos\sigma)W_{1c}\right]\,,\\ W^r_2 &= -3\left[\frac{3}{2}\sin^2\sigma W_{2a} + \sin\sigma(1+\cos\sigma)W_{2b} + \frac{1}{4}(1+\cos\sigma)^2 W_{2c}\right]\,, \end{aligned} \tag{5.18}$$

where W_a, W_b and W_c are those shown in (4.15), the rest being:

$$\begin{aligned}
W_{1a} &= \sum_i \frac{B_i}{-\tau n_i} \sin(\nu - \widetilde{\nu}_r - \tau\Theta_i)\,, \\
W_{1b} &= \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i} \sin(\mu + \nu - \widetilde{\nu}_r - \tau\Theta_i)\,, \\
W_{1c} &= \sum_\tau \sum_i \frac{D_i(\tau)}{2n_\mu - \tau n_i} \sin(2\mu + \nu - \widetilde{\nu}_r - \tau\Theta_i)\,, \\
W_{2a} &= \sum_i \frac{B_i}{-\tau n_i} \sin(2\nu - 2\widetilde{\nu}_r - \tau\Theta_i)\,, \\
W_{2b} &= \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i} \sin(\mu + 2\nu - 2\widetilde{\nu}_r - \tau\Theta_i)\,, \\
W_{2c} &= \sum_\tau \sum_i \frac{D_i(\tau)}{2n_\mu - \tau n_i} \sin(2\mu + 2\nu - 2\widetilde{\nu}_r - \tau\Theta_i)\,.
\end{aligned} \tag{5.19}$$

Note that to obtain the above expressions we have taken into account that $n_\nu \simeq n_{\widetilde{\nu}_r}$.

After performing the derivatives with respect to the variables which do not have the symbol $\sim$, we can express $\widetilde{\sigma}_r$ and $\widetilde{\nu}_r$ as functions of the Andoyer variables. Following Kinoshita (1977) we get

$$\widetilde{\sigma}_r = \frac{C}{A}\sigma \; + \; O(\sigma^3)\,,$$

but, the effects we are dealing with being very small, we can make the approximation (Kubo, 1991):

$$\widetilde{\sigma}_r \simeq \sigma, \qquad \widetilde{\nu}_r \simeq v.$$

Now, with the simplifications $\sin\sigma = 0, \cos\sigma = 1$, except for the terms where $\sin\sigma$ appears as a divisor, by means of (4.16) and following a similar procedure to that described in subsection 4.3, the canonical equations of the perturbation are:

$$\begin{aligned}
\Delta_r\mu \simeq K_r \Bigg[& -3W_a - \frac{1}{\sin\sigma} W_b + \frac{1}{2} W_c + \frac{\cos I}{\sin I}\frac{\partial W_a}{\partial I} + \\
& + \frac{3}{2} W_{1a} - \frac{1}{2} W_{1c} \Bigg]\,,
\end{aligned} \tag{5.20}$$

$$\Delta_r\nu \simeq K_r \left[3W_a + \frac{1}{\sin\sigma} W_b - \frac{1}{2} W_c - \frac{3}{2} W_{1a} + \frac{1}{2} W_{1c} \right]\,, \tag{5.21}$$

$$\Delta_r(\mu + \nu) \simeq K_r \frac{\cos I}{\sin I}\frac{\partial W_a}{\partial I}\,, \tag{5.22}$$

$$\Delta_r \sigma \simeq K_r \frac{\partial W_b}{\partial \mu}, \tag{5.23}$$

where we have introduced the new coefficient,

$$K_r = \frac{3K_r'}{M} = -18\, D_r \frac{Gm^*}{a^{*3} M}, \tag{5.24}$$

and we use the notation Δ_r to indicate the perturbations due to the centrifugal deformation.

5.4. Perturbations of the Fundamental Planes

Andoyer's Plane:

With the previous results, and following a similar approach to the case of the rigid Earth, we obtain:

$$\Delta_r \lambda \simeq -K_r \frac{1}{\sin I} \frac{\partial W_a}{\partial I} = -K_r \sum_i \frac{1}{\sin I} \frac{\partial}{\partial I} (\frac{B_i}{n_i}) \sin \Theta_i, \tag{5.25}$$

$$\Delta_r I \simeq K_r \frac{1}{\sin I} \frac{\partial W_a}{\partial \lambda} = \frac{K_r}{\sin I} \sum_i (-m_5) \frac{B_i}{n_i} \cos \Theta_i. \tag{5.26}$$

Equatorial Plane:

In the same way, we have:

$$\Delta_r(\delta\lambda_f) \simeq \frac{K_r}{\sin I} \left[\frac{\partial W_b}{\partial \mu} \sin \mu - W_b \cos \mu \right] = \frac{K_r}{\sin I} \sum_\tau \sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i} \sin \Theta_i, \tag{5.27}$$

$$\Delta_r(\delta I_f) \simeq K_r \left[\frac{\partial W_b}{\partial \mu} \cos \mu + W_b \sin \mu \right] = K_r \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i} \cos \Theta_i. \tag{5.28}$$

Plane perpendicular to the rotation axis:

It is not so easy, as in the previous case, to get the perturbations corresponding to this plane. This is because the angles λ_r and I_r which determine it are obtained through the inertia tensor (see Kinoshita, 1977), so that, when dealing with a deformable Earth, their expression as a function of Andoyer variables is not that given by (4.28), corresponding to the rigid Earth. To make the present part easier to read, we have included the method followed to obtain these new expressions in Appendix II.

Then, the expressions of the angles related to the plane perpendicular to the rotation axis when considering the centrifugal deformation, and with the simplifications detailed in the Appendix II, are:

$$\lambda_r \simeq \lambda + \frac{1}{\sin I}\left[\left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\sigma \sin\mu + \right.$$
$$\left. +3D_r\left(\frac{1}{A_0} + \frac{1}{B_0}\right)[\sigma\sin\mu - \tilde{\sigma}_r \sin(\mu + \nu - \tilde{\nu}_r)]\right], \tag{5.29}$$

$$I_r \simeq I + \left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\sigma\cos\mu +$$
$$+3D_r\left(\frac{1}{A_0} + \frac{1}{B_0}\right)[\sigma\cos\mu - \tilde{\sigma}_r\cos(\mu + \nu - \tilde{\nu}_r)]. \tag{5.30}$$

Taking into account the results above, the perturbations due to the centrifugal deformation, taking $\nu = \tilde{\nu}_r$ after carrying out the corresponding derivations, are:

$$\Delta_r(\delta\lambda_r) \simeq \left[K_r\left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right) + 3\,D_r(K_0 + K_r)\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\right] \times$$
$$\times \frac{1}{\sin I}\sum_\tau \sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i}\sin\Theta_i\,, \tag{5.31}$$

$$\Delta_r(\delta I_r) \simeq \left[K_r\left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right) + 3\,D_r(K_0 + K_r)\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\right] \times$$
$$\times \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos\Theta_i\,. \tag{5.32}$$

The above equations have been written in that way in order to obtain a more compact expressions of the (6.36) and (6.37), which include all the effects. Let us note however that the term in $D_r\,K_r$ is of a greater order than the rest, since it is of the second order in D_r, and the others of the order of $K_0\,D_r$. Nevertheless, it cannot be suppressed since its value is only slightly less (see (3.18)).

5.5. Perturbations of the Polar Motion

The polar motion is also obtained from the inertia tensor, so its expression must be modified. After the calculation which is detailed in Appendix III, we have:

$$x_p \simeq \frac{C_0}{A_0}\left[\left(1 - \frac{6\,D_r}{C_0}\right)\sigma\sin\nu + \frac{6\,D_r}{C_0}\,\tilde{\sigma}_r\sin\tilde{\nu}_r)\right], \tag{5.33}$$

$$y_p \simeq -\frac{C_0}{B_0}\left[\left(1 - \frac{6\,D_r}{C_0}\right)\sigma\cos\nu + \frac{6\,D_r}{C_0}\,\tilde{\sigma}_r\cos\tilde{\nu}_r)\right]. \tag{5.34}$$

By means of the equations (5.20)–(5.23), the corresponding perturbations due to the centrifugal deformation are:

$$\Delta_r x_p \simeq \frac{C_0}{A_0}\left[K_0 - \frac{6\,D_r}{C_0}(K_0+K_r)\right]\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\sin(\mu+\nu-\tau\Theta_i) \quad (5.35)$$

$$\Delta_r y_p \simeq -\frac{C_0}{B_0}\left[K_0 - \frac{6\,D_r}{C_0}(K_0+K_r)\right]\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos(\mu+\nu-\tau\Theta_i) \quad (5.36)$$

5.6. Secular Perturbations

As we have seen in (5.14), the contribution to the perturbed secular Hamiltonian is:

$$\mathcal{H}^*_{1r} = T^a_{1r} + T^a_{2r} + V_{rsec}\,,$$

whose terms are given by (5.4), (5.7) and (5.2). To obtain the corresponding perturbations, we carry out the adequate derivations with respect to the variables without the symbol $\sim$. Once taken, we can identify $\sigma = \widetilde{\sigma}_r$ and $\nu = \widetilde{\nu}_r$, obtaining the following results, after neglecting the terms in σ^2:

$$\delta_r n^*_\mu \simeq K_r \frac{\cos I^*}{\sin I^*}\frac{\partial B^*_{\widehat{0}}}{\partial I^*} + D_r M^* Q_a + 2D_r^2 M^* Q_b\,, \quad (5.37)$$

$$\delta_r n^*_\nu \simeq D_r N^* Q_c + 2D_r^2 N^* Q_d\,, \quad (5.38)$$

where

$$Q_a = \frac{3}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right) - \frac{1}{A_0^2} - \frac{1}{B_0^2}\,,$$

$$Q_b = \frac{6}{C_0^2}\left(\frac{1}{A_0}+\frac{1}{B_0}\right) - \frac{3}{C_0}\left(\frac{1}{A_0^2}+\frac{1}{B_0^2}\right) + \frac{1}{A_0^3} + \frac{1}{B_0^3}\,,$$

$$Q_c = \frac{4}{C_0^2} + \frac{1}{A_0^2} + \frac{1}{B_0^2} - \frac{3}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right),$$

$$Q_d = \frac{8}{C_0^3} - \frac{1}{A_0^3} - \frac{1}{B_0^3} - \frac{6}{C_0^2}\left(\frac{1}{A_0}+\frac{1}{B_0}\right) + \frac{3}{C_0}\left(\frac{1}{A_0^2}+\frac{1}{B_0^2}\right). \quad (5.39)$$

We have used the symbol δ_r to indicate the perturbation due to the rotation. Finally, from (5.37) and (5.38) it is clear that

$$\delta_r(n^*_\mu + n^*_\nu) \simeq K_r\left[\frac{\cos I^*}{\sin I^*}\frac{\partial B^*_{\widehat{0}}}{\partial I^*}\right] + \frac{4N^*}{C_0^2}\left[D_r + \frac{4D_r^2}{C_0}\right]. \quad (5.40)$$

Note that in these last expressions we have taken into account the simplification

$$M - N = M(1-\cos\sigma) \simeq 0\,.$$

5.7. Discussion of the Results

Periodic Perturbations:

Let us study the periodic perturbations including the effect of the rigid Earth and of the centrifugal deformation. From expressions (4.20), (4.21), (5.22) and (5.23) we have, using the notation Δ_{0r} to indicate the perturbations due to these two effects together:

$$\Delta_{0r}(\mu+\nu) \simeq (K_0+K_r)\frac{\cos I}{\sin I}\frac{\partial W_a}{\partial I} = K_{0r}\frac{\cos I}{\sin I}\frac{\partial W_a}{\partial I}\,, \tag{5.41}$$

$$\Delta_{0r}\sigma \simeq (K_0+K_r)\frac{\partial W_b}{\partial \mu} = K_{0r}\frac{\partial W_b}{\partial \mu}\,, \tag{5.42}$$

where we have introduced the coefficient:

$$K_{0r} = K_0 + K_r = \frac{3Gm^*}{a^{*3}M}\left[\frac{2C_0-A_0-B_0}{2} - 6\,D_r\right]. \tag{5.43}$$

Comparing (5.41) and (5.42) with the corresponding equations of Kinoshita (1977) for a rigid Earth, we see that they are the same if we replace Kinoshita's coefficient K by our K_{0r}, so that the perturbations have the same expression with only a change of coefficients. This change is quite logical: Kinoshita's coefficient K comes from the secular part of the potential energy, which in this case is solely the rigid Earth potential U_0 only, while, when the centrifugal deformation is considered, we have two potentials, U_0 and U_r,from which we get the new coefficient $K_{0r} = K_0 + K_r$.

Now we study the variation of the fundamental planes. From (4.23), (4.24), (5.25) and (5.26) we have, for the Andoyer's plane:

$$\Delta_{0r}\lambda \simeq -K_{0r}\sum_i \frac{1}{\sin I}\frac{\partial}{\partial I}\left(\frac{B_i}{n_i}\right)\sin\Theta_i\,, \tag{5.44}$$

$$\Delta_{0r}I \simeq \frac{K_{0r}}{\sin I}\sum_i(-m_5)\frac{B_i}{n_i}\cos\Theta_i\,. \tag{5.45}$$

As for the Equatorial plane, from (4.26), (4.27), (5.27) and (5.28), the Oppolzer terms are

$$\Delta_{0r}(\delta\lambda_f) \simeq \frac{K_{0r}}{\sin I}\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i}\sin\Theta_i\,, \tag{5.46}$$

$$\Delta_{0r}(\delta I_f) \simeq K_{0r}\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos\Theta_i\,. \tag{5.47}$$

Comparing these expressions with those of Kinoshita, the correspondence between the coefficients K and K_{0r} is also clear. The effect of the centrifugal deformation is reduced to an expression similar to that corresponding to the rigid Earth.

In the same way, for the Oppolzer terms of the plane perpendicular to the rotation axis, from (4.29), (4.30), (5.31) and (5.32) we have:

$$\Delta_{0r}(\delta\lambda_r) \simeq \frac{K_{0r}}{\sin I}\left[\left(1-\frac{C_0}{2A_0}-\frac{C_0}{2B_0}\right)+3\,D_r\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\right]\times \times\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu-\tau n_i}\sin\Theta_i\,, \quad (5.48)$$

$$\Delta_{0r}(\delta I_r) \simeq K_{0r}\left[\left(1-\frac{C_0}{2A_0}-\frac{C_0}{2B_0}\right)+3\,D_r\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\right]\times \times\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu-\tau n_i}\cos\Theta_i\,, \quad (5.49)$$

while for the polar motion, neglecting the triaxiality of the Earth, from (4.32), (4.33), (5.35) and (5.36), we have:

$$\Delta_r x_p \simeq \frac{C_0}{A_0}K_{0r}\left(1-\frac{6\,D_r}{C_0}\right)\sum_\tau\sum_i\frac{C_i(\tau)}{n_\mu-\tau n_i}\sin(\mu+\nu-\tau\Theta_i)\,, \quad (5.50)$$

$$\Delta_r y_p \simeq -\frac{C_0}{B_0}K_{0r}\left(1-\frac{6\,D_r}{C_0}\right)\sum_\tau\sum_i\frac{C_i(\tau)}{n_\mu-\tau n_i}\cos(\mu+\nu-\tau\Theta_i)\,. \quad (5.51)$$

Secular Perturbations:

Using the notation δ_{0r} to indicate the secular perturbations due to the rigid Earth and the centrifugal deformation, from (4.35), (4.36), (4.37), (5.37), (5.38) and (5.40), we have:

$$\delta_{0r}n_\mu^* \simeq \frac{R_{E1}^*}{\sin I^*}+K_{0r}\frac{\cos I^*}{\sin I^*}\frac{\partial B_{\widehat{0}}^*}{\partial I^*}-3K_0B_{\widehat{0}}^*+D_rM^*Q_a + 2D_r^2M^*Q_b\,, \quad (5.52)$$

$$\delta_{0r}n_\nu^* \simeq 3\,K_0B_{\widehat{0}}^*+D_rN^*Q_c+2\,D_r^2N^*Q_d\,, \quad (5.53)$$

And finally:

$$\delta_{0r}(n_\mu^*+n_\nu^*) \simeq \frac{R_{E1}^*}{\sin I^*}+K_{0r}\frac{\cos I^*}{\sin I^*}\frac{\partial B_{\widehat{0}}^*}{\partial I^*}+\frac{4N^*}{C_0^2}\left[D_r+\frac{4D_r^2}{C_0}\right]\,. \quad (5.54)$$

Here we can remark the appearance of new terms. To evaluate their order of magnitude we need to know the value of the coefficient K_{0r}, which will be studied in section 7.

6. Effect of the Tidal Deformation

We study now the effect of the tidal deformation. From the complete expression of the Hamiltonian (4.1), we can observe that the tidal perturbation is reduced to the term:

$$\mathcal{H}_t = T_{1t}\,. \tag{6.1}$$

Let us now look at the expression of this perturbation.

6.1. Preliminary Arrangements

Starting from (3.8), (3.9) and (3.19), the term T_{1t} will be:

$$\begin{aligned} T_{1t} \;=\; D_t \Bigg\{ & \left[\frac{2N^2}{C_0^2} - (M^2 - N^2)\left(\frac{\sin^2\nu}{A_0^2} + \frac{\cos^2\nu}{B_0^2}\right)\right]\widetilde{P}_0 + \\ & + \frac{1}{2}\frac{M^2 - N^2}{A_0 B_0}\sin 2\nu \widetilde{P}_2 \sin 2\widetilde{\alpha} + \\ & + \frac{1}{2}(M^2 - N^2)\left(\frac{\sin^2\nu}{A_0^2} - \frac{\cos^2\nu}{B_0^2}\right)\widetilde{P}_2 \cos 2\widetilde{\alpha} + \\ & + \frac{M}{C_0^2}\sin 2\sigma \widetilde{P}_1\left(\frac{1}{A_0}\sin\nu\cos\widetilde{\alpha} + \frac{1}{B_0}\cos\nu\sin\widetilde{\alpha}\right)\Bigg\}\,, \end{aligned} \tag{6.2}$$

where we have used the notation

$$\widetilde{P}_m = (a^*/r^*)^3 P_2^m(\sin\widetilde{\delta})\,, \qquad m = 0, 1, 2\,.$$

Using a procedure analogous to that of section 5 for the case of the centrifugal deformation, this expression can be broken down into two parts:

$$T_{1t} = T_{1t}^a + T_{1t}^b\,, \tag{6.3}$$

where

$$\begin{aligned} T_{1t}^a \;=\; D_t \Bigg\{ & \left[\frac{2N^2}{C_0^2} - \frac{1}{2}(M^2 - N^2)\left(\frac{1}{A_0^2} + \frac{1}{B_0^2}\right)\right]\widetilde{P}_0 - \\ & - \frac{1}{8}(M^2 - N^2)\left(\frac{1}{A_0} + \frac{1}{B_0}\right)^2 \widetilde{P}_2 \cos 2(\widetilde{\alpha} + \nu) + \\ & + \frac{1}{2}\frac{M^2}{C_0}\sin 2\sigma\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\widetilde{P}_1 \sin(\widetilde{\alpha} + \nu)\Bigg\}\,, \end{aligned} \tag{6.4}$$

$$T_{1t}^b \;=\; D_t \left\{ \frac{M^2 - N^2}{2}\left[\left(\frac{1}{A_0^2} - \frac{1}{B_0^2}\right)\left(\widetilde{P}_0 \cos 2\nu + \widetilde{P}_2 \cos 2\widetilde{\alpha}\right) - \right.\right.$$

$$
-\frac{1}{4}\left(\frac{1}{A_0}-\frac{1}{B_0}\right)^2 \widetilde{P}_2 \cos 2(\widetilde{\alpha}-\nu)\Bigg] -
$$

$$
-\frac{1}{2}\frac{M^2}{C_0}\left(\frac{1}{A_0}-\frac{1}{B_0}\right)\sin 2\sigma\, \widetilde{P}_1 \sin(\widetilde{\alpha}-\nu)\Bigg\} . \tag{6.5}
$$

As concerning the term T^a_{1t}, the expression of P_0 can be found directly in Appendix I, and for the remaining two spherical functions it can be easily deduced that:

$$
\begin{aligned}
\widetilde{P}_1 \sin(\widetilde{\alpha}+\nu) = 3 \Bigg[& \frac{3}{4}\sin 2\widetilde{\sigma} \sum_\tau \sum_i \widetilde{B}_i \cos(\widetilde{\nu}-\nu-\tau\widetilde{\Theta}_i) + \\
& + \frac{1}{2}(1+\cos\widetilde{\sigma})(-1+2\cos\widetilde{\sigma}) \sum_\tau \sum_i \widetilde{C}_i(\tau)\cos(\widetilde{\mu}+\widetilde{\nu}-\nu-\tau\widetilde{\Theta}_i) - \\
& - \frac{1}{4}\sin\widetilde{\sigma}(1+\cos\widetilde{\sigma}) \sum_\tau \sum_i \widetilde{D}_i(\tau)\cos(2\widetilde{\mu}+\widetilde{\nu}-\nu-\tau\widetilde{\Theta}_i)\Bigg] ,
\end{aligned} \tag{6.6}
$$

$$
\begin{aligned}
\widetilde{P}_2 \cos 2(\widetilde{\alpha}+\nu) = -3 \Bigg[& \frac{3}{2}\sin^2\widetilde{\sigma} \sum_\tau \sum_i \widetilde{B}_i \cos(2\widetilde{\nu}-2\nu-\tau\widetilde{\Theta}_i) + \\
& + \sin\widetilde{\sigma}(1+\cos\widetilde{\sigma}) \sum_\tau \sum_i \widetilde{C}_i(\tau)\cos(\widetilde{\mu}+2\widetilde{\nu}-2\nu-\tau\widetilde{\Theta}_i) + \\
& + \frac{1}{4}(1+\cos\widetilde{\sigma})^2 \sum_\tau \sum_i \widetilde{D}_i(\tau)\cos(2\widetilde{\mu}+2\widetilde{\nu}-2\nu-\tau\widetilde{\Theta}_i)\Bigg] .
\end{aligned} \tag{6.7}
$$

6.2. ELIMINATION OF PERIODIC TERMS

Using a similar procedure to that followed in previous sections, $\mathcal{H}_t$ is broken down into:

$$
\mathcal{H}_t = \mathcal{H}_{1t} + \mathcal{H}_{2t} \rightarrow \begin{cases} \mathcal{H}_{1t} = T^a_{1t} , \\ \mathcal{H}_{2t} = T^b_{1t} . \end{cases} \tag{6.8}
$$

For the first order integration we carry out the transformation:

$$
\begin{aligned}
& \mathcal{H}_{1t} \rightarrow \mathcal{H}^*_{1t} = \mathcal{H}_{1tsec} = \\
& = D_t \widetilde{B}_0 \Bigg\{ \left[\frac{2N^2}{C_0^2} - \frac{1}{2}(M^2-N^2)\left(\frac{1}{A_0^2}+\frac{1}{B_0^2}\right)\right] 3(1-\frac{3}{2}\sin^2\widetilde{\sigma}) + \\
& \qquad + \frac{9}{8}(M^2-N^2)\left(\frac{1}{A_0}+\frac{1}{B_0}\right)^2 \sin^2\widetilde{\sigma}\, \cos 2(\widetilde{\nu}-\nu) + \\
& \qquad + \frac{9}{4}\frac{M^2}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\sin 2\sigma\, \sin 2\widetilde{\sigma}\, \cos(\widetilde{\nu}-\nu)\Bigg\} ,
\end{aligned} \tag{6.9}
$$

where, for the sake of simplicity, we have omitted the asterisks. As in previous sections, for the secular part we have retained the terms which do not contain the variables μ, ν, or those of the position of Moon and Sun, and also preserving the terms in $\widetilde{\nu} - \nu$, which will give secular contribution as explained in subsection 5.2

6.3. Generating Function and First-Order Perturbations

The generating function corresponding to the first order transformation (6.9) is (omitting the asterisks again):

$$W_t = \int (\mathcal{H}_{1t} - \mathcal{H}_{1t}^*)\, dt = \int \mathcal{H}_{1tper}\, dt\,,$$

from which we get

$$\begin{aligned} W_t \;=\; & D_t \left\{ \left[\frac{2N^2}{C_0^2} - \frac{1}{2}(M^2 - N^2)\left(\frac{1}{A_0^2} + \frac{1}{B_0^2}\right)\right] \widetilde{W}_0^t + \right. \\ & \left. + \frac{M^2 - N^2}{8}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)^2 \widetilde{W}_2^t + \frac{1}{2}\frac{M^2}{C_0}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\sin 2\sigma\, \widetilde{W}_1^t \right\} \end{aligned} \quad (6.10)$$

with

$$\begin{aligned} \widetilde{W}_0^t &= 3\left[(1 - \frac{3}{2}\sin^2\widetilde{\sigma})\widetilde{W}_a - \frac{1}{2}\sin 2\widetilde{\sigma}\,\widetilde{W}_b + \frac{1}{4}\sin^2\widetilde{\sigma}\,\widetilde{W}_c\right], \\ \widetilde{W}_1^t &= 3\left[\frac{3}{4}\sin 2\widetilde{\sigma}\,\widetilde{W}_{1a} + \frac{1}{2}(1 + \cos\widetilde{\sigma})(-1 + 2\cos\widetilde{\sigma})\widetilde{W}_{1b} - \right. \\ & \qquad \left. -\frac{1}{4}\sin\widetilde{\sigma}(1 + \cos\widetilde{\sigma})\widetilde{W}_{1c}\right], \\ \widetilde{W}_2^t &= 3\left[\frac{3}{2}\sin^2\widetilde{\sigma}\,\widetilde{W}_{2a} + \sin\widetilde{\sigma}(1 + \cos\widetilde{\sigma})\widetilde{W}_{2b} + \frac{1}{4}(1 + \cos\widetilde{\sigma})^2\widetilde{W}_{2c}\right], \end{aligned} \quad (6.11)$$

where each one of these functions is

$$\begin{aligned} \widetilde{W}_a &= \sum_i \frac{\widetilde{B}_i}{n_i}\sin\widetilde{\Theta}_i\,, \\ \widetilde{W}_b &= \sum_\tau \sum_i \frac{\widetilde{C}_i(\tau)}{n_\mu - \tau n_i}\sin(\widetilde{\mu} - \tau\widetilde{\Theta}_i)\,, \\ \widetilde{W}_c &= \sum_\tau \sum_i \frac{\widetilde{D}_i(\tau)}{2n_\mu - \tau n_i}\sin(2\widetilde{\mu} - \tau\widetilde{\Theta}_i)\,, \\ \widetilde{W}_{1a} &= \sum_i \frac{\widetilde{B}_i}{-\tau n_i}\sin(\widetilde{\nu} - \nu - \tau\widetilde{\Theta}_i)\,, \end{aligned}$$

$$
\begin{aligned}
\widetilde{W}_{1b} &= \sum_{\tau}\sum_{i} \frac{\widetilde{C}_i(\tau)}{n_\mu - \tau n_i} \sin(\widetilde{\mu} + \widetilde{\nu} - \nu - \tau\widetilde{\Theta}_i)\,, \\
\widetilde{W}_{1c} &= \sum_{\tau}\sum_{i} \frac{\widetilde{D}_i(\tau)}{2n_\mu - \tau n_i} \sin(2\widetilde{\mu} + \widetilde{\nu} - \nu - \tau\widetilde{\Theta}_i)\,, \\
\widetilde{W}_{2a} &= \sum_{i} \frac{\widetilde{B}_i}{-\tau n_i} \sin(2\widetilde{\nu} - 2\nu - \tau\widetilde{\Theta}_i)\,, \\
\widetilde{W}_{2b} &= \sum_{\tau}\sum_{i} \frac{\widetilde{C}_i(\tau)}{n_\mu - \tau n_i} \sin(\widetilde{\mu} + 2\widetilde{\nu} - 2\nu - \tau\widetilde{\Theta}_i)\,, \\
\widetilde{W}_{2c} &= \sum_{\tau}\sum_{i} \frac{\widetilde{D}_i(\tau)}{2n_\mu - \tau n_i} \sin(2\widetilde{\mu} + 2\widetilde{\nu} - 2\nu - \tau\widetilde{\Theta}_i)\,.
\end{aligned}
\tag{6.12}
$$

Through the perturbation equations and taking into account the fact that the variables with the symbol $\sim$ behave as constants, we obtain, with the usual simplifications in σ:

$$
\begin{aligned}
\Delta_t\mu \simeq\; & 3MD_t\Bigg[-\left(\frac{1}{A_0^2}+\frac{1}{B_0^2}\right)\widetilde{W}_a + \frac{1}{4}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)^2\widetilde{W}_{2c} + \\
& + \frac{1}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\left(\frac{3}{2}\widetilde{W}_{1a} + \frac{1}{\sin\sigma}\widetilde{W}_{1b} - \frac{1}{2}\widetilde{W}_{1c}\right)\Bigg]\,,
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
\Delta_t\nu \simeq\; & 3D_t\Bigg\{N\left[\left(\frac{4}{C_0^2}+\frac{1}{A_0^2}+\frac{1}{B_0^2}\right)\widetilde{W}_a - \frac{1}{4}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)^2\widetilde{W}_{2c}\right] - \\
& - \frac{M}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\left(\frac{3}{2}\widetilde{W}_{1a} + \frac{1}{\sin\sigma}\widetilde{W}_{1b} - \frac{1}{2}\widetilde{W}_{1c}\right)\Bigg\}\,,
\end{aligned}
\tag{6.14}
$$

$$
\Delta_t(\mu+\nu) \simeq 12D_t\frac{N}{C_0^2}\widetilde{W}_a\,,
\tag{6.15}
$$

$$
\Delta_t\sigma \simeq 3D_t\frac{N}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\frac{\partial\widetilde{W}_{1b}}{\partial\nu}\,,
\tag{6.16}
$$

using once again the simplification $M \simeq N$, and the notation Δ_t to indicate that these perturbations are produced by the tidal deformation. Once these derivatives have been carried out we can identify $\sigma = \widetilde{\sigma}$, $\nu = \widetilde{\nu}$.

6.4. Perturbations of Fundamental Planes

Andoyer's Plane:

The nutations of this plane are obtained directly from the perturbations equations. With the usual simplifications, we have:

$$
\Delta_t\lambda \simeq 0\,, \qquad \Delta_t I \simeq 0\,.
\tag{6.17}
$$

Equatorial Plane:

Through (4.25) we obtain:

$$\Delta_t(\delta\lambda_f) \simeq \frac{D_t}{\sin I}\frac{3M}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\left[\sin\mu\,\frac{\partial\widetilde{W}_{1b}}{\partial\nu}+\cos\mu\,\widetilde{W}_{1b}\right], \tag{6.18}$$

$$\Delta_t(\delta I_f) \simeq D_t\frac{3M}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right)\left[\cos\mu\,\frac{\partial\widetilde{W}_{1b}}{\partial\nu}-\sin\mu\,\widetilde{W}_{1b}\right]. \tag{6.19}$$

Taking into account the expression of $\widetilde{W}_{1b}$ in (6.12), and identifying the variables after taking the derivatives, we finally obtain:

$$\Delta_t(\delta\lambda_f) \simeq -K_t\frac{1}{\sin I}\sum_\tau\sum_i\frac{\tau C_i(\tau)}{n_\mu-\tau n_i}\sin\Theta_i\,, \tag{6.20}$$

$$\Delta_t(\delta I_f) \simeq -K_t\sum_\tau\sum_i\frac{C_i(\tau)}{n_\mu-\tau n_i}\cos\Theta_i\,, \tag{6.21}$$

where we have introduced the new coefficient

$$K_t = 3D_t\frac{M}{C_0}\left(\frac{1}{A_0}+\frac{1}{B_0}\right), \tag{6.22}$$

which gives the order of magnitude of the tidal perturbation.

Plane perpendicular to the rotation axis:

As in the case of the centrifugal deformation, the tidal deformation would give rise to a variation in the equations (4.28) of the rigid Earth. This dependence will be given quantitatively by the coefficient D_t. However, as $D_t/C_0 \simeq 8\times10^{-8}$ (see section 3), we can neglect this effect, and we shall calculate the tidal perturbation of these angles starting from the equations (4.28).

Thus, with equation (4.28), and by means of the previous results (6.13) to (6.17), we obtain:

$$\Delta_t(\delta\lambda_r) \simeq -\frac{K_t}{\sin I}\left(1-\frac{C_0}{2A_0}-\frac{C_0}{2B_0}\right)\sum_\tau\sum_i\frac{\tau C_i(\tau)}{n_\mu-\tau n_i}\sin\Theta_i\,, \tag{6.23}$$

$$\Delta_t(\delta I_r) \simeq -K_t\left(1-\frac{C_0}{2A_0}-\frac{C_0}{2B_0}\right)\sum_\tau\sum_i\frac{C_i(\tau)}{n_\mu-\tau n_i}\cos\Theta_i\,. \tag{6.24}$$

6.5. Perturbations of the Polar Motion

For the effect of the tidal perturbation on the polar motion the procedure used for the plane perpendicular to the rotation axis described above is valid, so that, after neglecting the effect of the triaxiality of the Earth, we get:

$$\Delta_t x_p \simeq -K_t \frac{C_0}{A_0} \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i} \sin(\mu + \nu - \tau\Theta_i) , \tag{6.25}$$

$$\Delta_t y_p \simeq K_t \frac{C_0}{B_0} \sum_\tau \sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i} \cos(\mu + \nu - \tau\Theta_i) . \tag{6.26}$$

6.6. Secular Perturbations

Starting from (6.9), with the help of (4.16) and the results (6.13) to (6.17), the canonical equations for the secular perturbations are:

$$\delta_t n_\mu^* \simeq 3D_t B_0^* M^* \left[\frac{3}{C_0} \left(\frac{1}{A_0} + \frac{1}{B_0} \right) - \frac{1}{A_0^2} - \frac{1}{B_0^2} \right] , \tag{6.27}$$

$$\delta_t n_\nu^* \simeq 3D_t B_0^* \left[N^* \left(\frac{4}{C_0^2} + \frac{1}{A_0^2} + \frac{1}{B_0^2} \right) - M^* \frac{3}{C_0} \left(\frac{1}{A_0} + \frac{1}{B_0} \right) \right] , \tag{6.28}$$

where the symbol δ_t is used to indicate the tidal perturbation. From these two expressions above and taking into account $M \simeq N$ once more, we can write

$$\delta_t (n_\mu^* + n_\nu^*) \simeq D_t B_0^* \frac{12N^*}{C_0^2} . \tag{6.29}$$

6.7. Discussion of the Results

Periodic Perturbations:

We now express the total periodic perturbations, by adding the effects corresponding to a rigid Earth, to the rotational deformation and to the tidal deformation. From the expressions (5.41), (5.42), (6.15) and (6.16) we have, using the notation Δ to indicate the total periodic perturbations:

$$\Delta(\mu + \nu) \simeq K_{0r} \frac{\cos I}{\sin I} \frac{\partial W_a}{\partial I} + 12\, D_t \frac{M}{C_0^2} \widetilde{W}_a ,$$

$$\Delta\sigma \simeq K_{0r} \frac{\partial W_b}{\partial \mu} + K_t \frac{\partial \widetilde{W}_{1b}}{\partial \nu} ,$$

which becomes, when identifying variables after taking derivatives:

$$\Delta(\mu + \nu) \simeq K_{0r} \frac{\cos I}{\sin I} \sum_i \frac{\partial}{\partial I} \left(\frac{B_i}{n_i} \right) \sin \Theta_i + D_t \frac{12M}{C_0^2} \sum_i \frac{B_i}{n_i} \sin \Theta_i , \tag{6.30}$$

$$\Delta\sigma \simeq (K_{0r} - K_t)\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos(\mu - \tau\Theta_i)\,. \tag{6.31}$$

Let us now study the variation of the fundamental planes. From (5.44), (5.45) and (6.17) we have, for *Andoyer's plane*, or plane perpendicular to the angular momentum:

$$\Delta\lambda \simeq -K_{0r}\sum_i \frac{1}{\sin I}\frac{\partial}{\partial I}\left(\frac{B_i}{n_i}\right)\sin\Theta_i\,, \tag{6.32}$$

$$\Delta I \simeq \frac{K_{0r}}{\sin I}\sum_i(-m_5)\frac{B_i}{n_i}\cos\Theta_i\,. \tag{6.33}$$

As for the *plane of the equator of figure*, from (5.46), (5.47), (6.20) and (6.21), the Oppolzer terms are:

$$\Delta(\delta\lambda_f) \simeq (K_{0r} - K_t)\frac{1}{\sin I}\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i}\sin\Theta_i\,, \tag{6.34}$$

$$\Delta(\delta I_f) \simeq (K_{0r} - K_t)\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos\Theta_i\,. \tag{6.35}$$

For the plane *perpendicular to the rotation axis*, from (5.48), (5.49), (6.23) and (6.24), we have:

$$\Delta(\delta\lambda_r) \simeq \left[(K_{0r} - K_t)\left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right) + 3\,D_r\,K_{0r}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\right]\times$$
$$\times\frac{1}{\sin I}\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i}\sin\Theta_i\,, \tag{6.36}$$

$$\Delta(\delta I_r) \simeq \left[(K_{0r} - K_t)\left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right) + 3\,D_r\,K_{0r}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\right]\times$$
$$\times\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i}\cos\Theta_i\,. \tag{6.37}$$

Finally, for the polar motion, from (5.50), (5.51), (6.25) and (6.26):

$$\Delta_r x_p \simeq \frac{C_0}{A_0}\left[(K_{0r} - K_t) - K_{0r}\,\frac{6\,D_r}{C_0}\right]\times$$
$$\times\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\sin(\mu + \nu - \tau\Theta_i)\,, \tag{6.38}$$

$$\Delta_r y_p \simeq -\frac{C_0}{B_0}\left[(K_{0r} - K_t) - K_{0r}\,\frac{6\,D_r}{C_0}\right]\times$$
$$\times\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos(\mu + \nu - \tau\Theta_i)\,. \tag{6.39}$$

Once the expressions (6.32)-(6.39) have been obtained, we can interpret the role the tidal deformation plays, the following results standing out:

i) Concerning the Andoyer's plane, normal to the angular-momentum vector, the tidal perturbation gives a negligible contribution, at the order of accuracy with which we are working.

ii) However, this perturbation acts upon the Oppolzer terms of the planes perpendicular to the axis of figure and the axis of rotation, as well as the motion of the pole. Its contribution consists in diminishing the effect of the perturbations for rigid Earth and centrifugal deformation, as the coefficient K_t appears with a negative sign. This coincides with the results exposed by Kubo (1991).

Finally, we should point out that just as we have included the coefficients K_0 and K_r respectively for the effects of the rigid Earth and of the centrifugal deformation in a single coefficient: $K_{0r} = K_0 + K_r$, we could add to K_{0r} the coefficient of the tidal deformation K_t, as in the previous expressions it appears as $K_{0r} - K_t$. Nevertheless for reasons which will be exposed in the following section we prefer to group K_0 and K_r in one single coefficient.

Secular Perturbations:

Using the notation δ to indicate the total secular perturbations, from the equations (5.52), (5.53), (5.54), (6.27), (6.28) and (6.29) we have:

$$\delta n_\mu^* \simeq \frac{R_{E1}^*}{\sin I^*} + K_{0r}\frac{\cos I^*}{\sin I^*}\frac{\partial B_{\widehat{0}}^*}{\partial I^*} - 3(K_0 - K_t)B_{\widehat{0}}^* + D_r\, M^*\, Q_a + \\ +2\, D_r^2\, M^*\, Q_b - 3\, D_t\, M^*\, B_{\widehat{0}}^* \left(\frac{1}{A_0^2} + \frac{1}{B_0^2}\right), \qquad (6.40)$$

$$\delta n_\nu^* \simeq 3(K_0 - K_t)B_{\widehat{0}}^* + D_r\, N^*\, Q_c + 2\, D_r^2\, N^*\, Q_d + \\ +3\, D_t\, N^*\, B_{\widehat{0}}^* \left(\frac{4}{C_0^2} + \frac{1}{A_0^2} + \frac{1}{B_0^2}\right), \qquad (6.41)$$

$$\delta(n_\mu^* + n_\nu^*) \simeq \frac{R_{E1}^*}{\sin I^*} + K_{0r}\frac{\cos I^*}{\sin I^*}\frac{\partial B_{\widehat{0}}^*}{\partial I^*} + \frac{4N^*}{C_0^2}\left[D_r + 3D_t B_{\widehat{0}}^* + \frac{4D_r^2}{C_0}\right]. \qquad (6.42)$$

which clearly shows the secular contribution of the tidal perturbation.

7. Numerical Representation of the Earth's Rotation

In this section we give the numerical values of the perturbations we have studied in the previous ones and which are represented by the equations (6.32)-(6.42). For this, we need to evaluate numerically the coefficients K_{0r} and K_t.

7.1. CALCULATION OF COEFFICIENTS

The value of K_t can be obtained directly. Starting from (6.22), and making the approximations $M = C_0\omega_3$ (see (2.7)) and $A_0 \simeq B_0$, we have

$$K_t = D_t \frac{6M}{A_0 C_0} \simeq 6D_t \frac{\omega_3}{A_0} \,.$$

A_0, the moment in absence of deformation, can be written: $A_0 = A - \Delta A$, where A is the total moment and ΔA the increment due to the deformation. As explained in subsection 3.2, this increment is: $\Delta A = \Delta_r A + \Delta_t A$, corresponding respectively to the rotational and to the tidal deformations. The orders of magnitude of these increments with respect to the unperturbed moment of inertia are given by the coefficients D_r and D_t, being 1.7×10^{-4} and 8×10^{-9} respectively. So that, we can write:

$$K_t \simeq 6\ D_t\ \omega_3\ \frac{1}{A - \Delta A} \simeq 6\ D_t\ \omega_3\ \frac{1}{A} \,,$$

neglecting the products of the deformations. Then, using the values $\omega_3 = 4.746599 \times 10^{10}$ sec.arc/Julian century (Seidelmann, 1982), and $A = 8.094 \times 10^{44} gr.cm^2$. (Danby, 1962), and taking into account the value of D_t (3.11), we obtain

$$K_{t☾} = 2446''.6204/cy.\,, \qquad K_{t\odot} = 1120''.8549/cy.\,, \tag{7.1}$$

However, the value of K_{0r} cannot be obtained directly. Starting from its definition (5.43), and taking once again the approximation $M = C_0\,\omega_3$, it can be expressed as

$$K_{0r} = 3\frac{Gm^*}{a^{*3}\omega_3}\left[\frac{2C_0 - A_0 - B_0}{2C_0} - 6\frac{D_r}{C_0}\right] \,, \tag{7.2}$$

that depends on the value of $(2C_0 - A_0 - B_0)/(2C_0)$, which, of course, is different from the value of $(2C - A - B)/(2C)$ corresponding to the usual dynamical ellipticity $\mathbf{H}$ of Kinoshita (1977) for a rigid Earth.

The value of K_{0r} can be found by a procedure similar to that developed by Kinoshita (1977) to obtain $\mathbf{H}$, and improved by Kinoshita and Souchay (1990). We think there is no need to reproduce the method in this paper, but only to summarize some facts. First, the resulting Hamiltonian after the eliminations of periodic terms is

$$\begin{aligned}\mathcal{H}^* \simeq\ & \frac{M^{*2}}{4}\left(\frac{1}{A_0} + \frac{1}{B_0}\right) + \frac{N^{*2}}{4}\left(\frac{2}{C_0} - \frac{1}{A_0} - \frac{1}{B_0}\right) + M^* \sin I^* R^*_{E1} + \\ & + \Lambda^* R^*_{E2} + (K'_0 + 3K'_r)B^*_{\widehat{0}} + \frac{2N^{*2}}{C_0^2}\left(D_r + D_r^2\frac{4}{C_0} + 3D_t\widetilde{B}^*_{\widehat{0}}\right)\,, \end{aligned} \tag{7.3}$$

where all the terms in $\sin\sigma$ have been neglected since no derivatives have to be taken with respect either to M or σ. Notice that $K_0' + 3\,K_r' \;=\; M\,K_{0r}$, according to (4.22), (5.24) and (5.43), and that the term $D_t\,\widetilde{B}_{\widehat{0}}$ does not modify the calculation, since it is very small and is affected by the symbol "$\sim$" (no derivatives of it are to be taken). In order to compute $d\lambda^*/dt$ and dI^*/dt the function (7.3) is the same as that found by Kinoshita. Moreover, the accuracy of the solution can be improved in a standard way by adding to this Hamiltonian the second order secular terms. Because of their smallness, we have only to take into account the second order terms corresponding to a rigid Earth, that can be taken from Kinoshita and Souchay.

Then, we can apply the procedure used by those authors to the Hamiltonian and get completely analogous results, after replacing the dynamical ellipticity $\mathbf{H}$ by $\mathbf{H}_0 + \mathbf{H}_r$ (proportional to K_{0r}), where

$$\mathbf{H}_0 = \frac{2C_0 - A_0 - B_0}{2C_0}\,, \qquad \mathbf{H}_r = -6\,\frac{D_r}{C_0}\,. \tag{7.4}$$

Taking the most up to date values of Kinoshita and Souchay (1990) for $K_{0r(}$, $K_{0r\odot}$ and $\mathbf{H}$, we have:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_r = 0.0032739567\,, \tag{7.5}$$

with which we get:

$$\begin{aligned} K_{0r(} &= 3\frac{Gm_(}{a_(^3\omega_3}\,[\mathbf{H}_0 + \mathbf{H}_r] = \\ &= 3\,[\mathbf{H}_0 + \mathbf{H}_r]\,\frac{m_(}{m_(+ m_\oplus}\,\frac{1}{F_2^3}\,\frac{n_(^2}{\omega_3} = 7567''.768157/\mathrm{J}.cy\,, \end{aligned} \tag{7.6}$$

$$\begin{aligned} K_{0r\odot} &= 3\frac{Gm_\odot}{a_\odot^3\omega_3}\,[\mathbf{H}_0 + \mathbf{H}_r] = \\ &= 3\,[\mathbf{H}_0 + \mathbf{H}_r]\,\frac{m_\odot}{m_(+ m_\odot + m_\oplus}\,\frac{n_\odot^2}{\omega_3} = 3475''.413512/\mathrm{J}.cy\,, \end{aligned} \tag{7.7}$$

in arc sec. per Julian century. From these values we can then calculate $\mathbf{H}_0$. Taking $C_0 \simeq C = 8.11 \times 10^{44}$ (Danby, 1962), and $D_r = -1.422689 \times 10^{41}$ (both in c.g.s. units) (3.18), we get:

$$\mathbf{H}_r = 0.00105251 \quad\Longrightarrow\quad \mathbf{H}_0 = \mathbf{H} - \mathbf{H}_r = 0.00222145\,. \tag{7.8}$$

The coefficients $\mathbf{H}_0$ and $\mathbf{H}_r$ have a simple dynamical interpretation. $\mathbf{H}$ represents the dynamical ellipticity of the actual Earth, and $\mathbf{H}_r$ represents the part of the ellipticity induced by the rotation. A similar concept already appears in Newcomb (1892), who carried out a rough estimation of the ratio between the said ellipticity, ϵ' in his notation, and the actual ellipticity, ϵ, obtaining a value $\epsilon'/\epsilon = 292/849 \simeq 0.34$, very close to that determined following our much more elaborated procedure: $\mathbf{H}_r/\mathbf{H} \simeq 0.32$. On the other hand, $\mathbf{H}_0$ would represents the hypothetical ellipticity corresponding to the Earth without the deformation due to the rotation.

With the help of these results we can now numerically evaluate the different perturbations.

7.2. Periodic Perturbations

Motion of Andoyer's Plane:

Let us first calculate the variations of the Andoyer's plane, which, as we have shown, is not affected by the tidal perturbation. Developing the expressions (6.32), (6.33), and according to the Appendix I, we have, for the nutation in obliquity:

$$\begin{aligned}\Delta\epsilon &= -\Delta I = \\ &= K_{0r(}\sum_i \frac{-m_5}{n_i}\left(A_i^{1)}\cos\epsilon_0 - A_i^{0)}\frac{3\cos^2\epsilon_0 - 1}{6\sin\epsilon_0} - \frac{1}{4}A_i^{2)}\sin\epsilon_0\right)_(\cos\Theta_i + \\ &\quad + K_{0r\odot}\sum_i -\frac{m_5}{n_i}\left(-A_i^{0)}\frac{3\cos^2\epsilon_0 - 1}{6\sin\epsilon_0} - \frac{1}{4}A_i^{2)}\sin\epsilon_0\right)_\odot \cos\Theta_i\,, \end{aligned} \qquad (7.9)$$

while for the nutation in longitude, taking into account the fact that (Kinoshita, 1977):

$$\frac{\partial}{\partial I}\left(\frac{B_i}{n_i}\right) = \frac{1}{n_i}\frac{\partial B_i}{\partial I} - m_5\frac{B_i}{n_i^2}\frac{\partial n_\Omega}{\partial I}\,,$$

we get:

$$\begin{aligned}\Delta\psi &= -\Delta\lambda = \\ &= K_{0r(}\left\{\sum_i \frac{1}{n_i}\left[\left(A_i^{0)} - \frac{1}{2}A_i^{2)}\right)\cos\epsilon_0 + \frac{\cos 2\epsilon_0}{\sin\epsilon_0}A_i^{1)}\right] - \right. \\ &\qquad \left. - m_5\frac{B_i}{n_i^2\sin\epsilon_0}\frac{\partial n_\Omega}{\partial\epsilon_0}\right\}_(\sin\Theta_i + \\ &\quad + K_{0r\odot}\sum_i \frac{1}{n_i}\left(A_i^{0)} - \frac{1}{2}A_i^{2)}\right)_\odot \cos\epsilon_0 \sin\Theta_i\,. \end{aligned} \qquad (7.10)$$

The expression of $\partial n_\Omega/\partial\epsilon_0$ can be found in Kinoshita and Souchay (1990), eq. (2.14.2). For the coefficients $A_i^{j)}$ we have used the values of Kinoshita (1977) with the corrections given by Kinoshita and Souchay (1990).

The corresponding perturbation series can be found in Table I and II.

TABLE I. Coefficients of cosine for nutation in OBLIQUITY. Epoch J2000.0. Unit = 0.001 arc sec.

	Period	Argument (cosine)					RIGID and ROTATIONAL EFFECTS				TIDAL EFFECT		RIGID, ROTATIONAL and TIDAL EFFECTS				
							Oppolzer				Oppolzer			Oppolzer			
No.	(days)	l	l'	F	D	Ω	Fig.	Rot.	$-\Delta_{or} I_f$	$-\Delta_{or} I_r$	Fig.	Rot.	$-\Delta I$	Fig.	Rot.	$-\Delta I_f$	$-\Delta I_r$
1	9.2	3	0	0	0	0	-0.007	0.000	-0.007	0.000	0.002	-0.000	0.000	-0.004	0.000	-0.004	0.000
2	131.7	2	1	0	-2	0	-0.000	0.000	-0.000	0.000	0.000	-0.000	0.000	-0.000	0.000	-0.000	0.000
3	-1095.2	2	0	-2	0	0	0.000	-0.000	0.000	-0.000	-0.000	0.000	0.000	0.000	-0.000	0.000	-0.000
4	205.9	2	0	0	-2	0	-0.009	0.000	-0.009	0.000	0.003	-0.000	0.000	-0.006	0.000	-0.006	0.000
5	-15.9	2	0	0	-4	0	-0.003	0.000	-0.003	0.000	0.001	-0.000	0.000	-0.002	0.000	-0.002	0.000
6	7.1	2	0	0	2	0	-0.003	0.000	-0.003	0.000	0.001	-0.000	0.000	-0.002	0.000	-0.002	0.000
7	13.8	2	0	0	0	0	-0.081	0.000	-0.081	0.000	0.026	-0.000	0.000	-0.055	0.000	-0.055	0.000
8	-3232.9	1	-1	0	-1	0	-0.000	0.000	-0.000	0.000	0.000	-0.000	0.000	-0.000	0.000	-0.000	0.000
9	-29.3	1	-1	0	-2	0	0.001	-0.000	0.001	-0.000	-0.000	0.000	0.000	0.001	-0.000	0.001	-0.000
10	29.8	1	-1	0	0	0	-0.006	0.000	-0.006	0.000	0.002	-0.000	0.000	-0.004	0.000	-0.004	0.000
11	-34.9	1	1	0	-2	0	-0.008	0.000	-0.008	0.000	0.003	-0.000	0.000	-0.005	0.000	-0.005	0.000
12	25.6	1	1	0	0	0	0.005	-0.000	0.005	-0.000	-0.002	0.000	0.000	0.003	-0.000	0.003	-0.000
13	-9.5	1	0	-2	-2	0	-0.003	0.000	-0.003	0.000	0.001	-0.000	0.000	-0.002	0.000	-0.002	0.000
14	32.8	1	0	-2	2	0	0.001	-0.000	0.001	-0.000	-0.000	0.000	0.000	0.000	-0.000	0.000	-0.000
15	-26.9	1	0	-2	0	0	0.006	-0.000	0.006	-0.000	-0.002	0.000	0.000	0.004	-0.000	0.004	-0.000
16	23.8	1	0	2	-2	0	0.001	-0.000	0.001	-0.000	-0.000	0.000	0.000	0.001	-0.000	0.001	-0.000
17	9.1	1	0	2	0	0	-0.014	0.000	-0.014	0.000	0.005	-0.000	0.000	-0.009	0.000	-0.009	0.000
18	411.8	1	0	0	-1	0	0.000	-0.000	0.000	-0.000	-0.000	0.000	0.000	0.000	-0.000	0.000	-0.000
19	-31.8	1	0	0	-2	0	-0.188	0.001	-0.188	0.001	0.061	-0.000	0.000	-0.128	0.000	-0.128	0.000
20	-10.1	1	0	0	-4	0	-0.005	0.000	-0.005	0.000	0.002	-0.000	0.000	-0.003	0.000	-0.003	0.000
21	9.6	1	0	0	2	0	-0.026	0.000	-0.026	0.000	0.008	-0.000	0.000	-0.018	0.000	-0.018	0.000
22	27.6	1	0	0	0	0	-0.986	0.003	-0.986	0.003	0.319	-0.001	0.000	-0.667	0.003	-0.667	0.003
23	-329.8	0	1	-2	2	0	-0.000	0.000	-0.000	0.000	0.000	-0.000	0.000	-0.000	0.000	-0.000	0.000
24	117.5	0	1	2	-2	0	0.000	-0.000	0.000	-0.000	-0.000	0.000	0.000	0.000	-0.000	0.000	-0.000
25	-15.4	0	1	0	-2	0	-0.011	0.000	-0.011	0.000	0.003	-0.000	0.000	-0.007	0.000	-0.007	0.000
26	14.2	0	1	0	2	0	0.002	-0.000	0.002	-0.000	-0.001	0.000	0.000	0.001	-0.000	0.001	-0.000

TABLE I. Continued.

No.	Period (days)	Argument (cosine) l	l'	F	D	Ω	RIGID and ROTATIONAL EFFECTS Oppolzer Fig.	Rot.	$-\Delta_{or} I_f$	$-\Delta_{or} I_r$	TIDAL EFFECT Oppolzer Fig.	Rot.	RIGID, ROTATIONAL and TIDAL EFFECTS $-\Delta I$	Oppolzer Fig.	Rot.	$-\Delta I_f$	$-\Delta I_r$
27	27.3	0	1	0	1	0	-0.001	0.000	-0.001	0.000	0.000	-0.000	0.000	-0.001	0.000	-0.001	0.000
28	173:3	0	0	2	-2	0	0.005	-0.000	0.005	-0.000	-0.002	0.000	0.000	0.003	-0.000	0.003	-0.000
29	13.6	0	0	2	0	0	-0.073	0.000	-0.073	0.000	0.023	-0.000	0.000	-0.049	0.000	-0.049	0.000
30	14.8	0	0	0	2	0	-0.164	0.001	-0.164	0.001	0.053	-0.000	0.000	-0.111	0.000	-0.111	0.000
31	29.5	0	0	0	1	0	0.005	-0.000	0.005	-0.000	-0.002	0.000	0.000	0.004	-0.000	0.004	-0.000
32	35.0	-1	-1	0	2	1	-0.001	0.000	-0.038	-0.037	0.000	-0.000	-0.037	-0.001	0.000	-0.038	-0.037
33	-32.6	-1	0	2	-2	1	-0.002	0.000	0.101	0.104	0.001	-0.000	0.104	-0.002	0.000	0.102	0.104
34	9.5	-1	0	2	2	1	0.042	-0.000	0.508	0.466	-0.014	0.000	0.466	0.029	-0.000	0.494	0.466
35	27.0	-1	0	2	0	1	-0.029	0.000	-1.035	-1.005	0.010	-0.000	-1.005	-0.020	0.000	-1.025	-1.005
36	-9.6	-1	0	0	-2	1	-0.002	0.000	0.026	0.028	0.001	-0.000	0.028	-0.001	0.000	0.026	0.028
37	32.0	-1	0	0	2	1	-0.018	0.000	-0.769	-0.751	0.006	-0.000	-0.751	-0.012	0.000	-0.763	-0.751
38	-388.3	-1	0	0	1	1	0.000	-0.000	-0.034	-0.034	-0.000	0.000	-0.034	0.000	-0.000	-0.034	-0.034
39	-27.4	-1	0	0	0	1	-0.079	0.000	2.966	3.045	0.026	-0.000	3.045	-0.054	0.000	2.991	3.045
40	1305.5	-2	0	2	0	1	-0.001	0.000	-2.407	-2.405	0.000	-0.000	-2.405	-0.001	0.000	-2.406	-2.405
41	-199.8	-2	0	0	2	1	-0.001	0.000	0.279	0.280	0.000	-0.000	0.280	-0.001	0.000	0.279	0.280
42	-13.8	-2	0	0	0	1	-0.006	0.000	0.119	0.125	0.002	-0.000	0.125	-0.004	0.000	0.121	0.125
43	-943.2	2	0	-2	0	1	0.000	-0.000	-0.037	-0.037	-0.000	0.000	-0.037	0.000	-0.000	-0.037	-0.037
44	12.8	2	0	2	-2	1	-0.003	0.000	-0.051	-0.048	0.001	-0.000	-0.048	-0.002	0.000	-0.050	-0.048
45	6.9	2	0	2	0	1	0.031	-0.000	0.264	0.233	-0.010	0.000	0.233	0.021	-0.000	0.254	0.233
46	212.3	2	0	0	-2	1	-0.001	0.000	-0.211	-0.211	0.000	-0.000	-0.211	-0.001	0.000	-0.211	-0.211
47	13.8	2	0	0	0	1	-0.006	0.000	-0.109	-0.103	0.002	-0.000	-0.103	-0.004	0.000	-0.107	-0.103
48	-34.7	1	1	0	-2	1	-0.001	0.000	0.030	0.031	0.000	-0.000	0.031	-0.000	0.000	0.030	0.031
49	23.9	1	0	2	-2	1	-0.009	0.000	-0.293	-0.284	0.003	-0.000	-0.284	-0.006	0.000	-0.290	-0.284
50	5.6	1	0	2	2	1	0.010	-0.000	0.065	0.056	-0.003	0.000	0.056	0.006	-0.000	0.062	0.056
51	9.1	1	0	2	0	1	0.224	-0.001	2.568	2.343	-0.072	0.000	2.344	0.152	-0.001	2.496	2.344
52	-31.7	1	0	0	-2	1	-0.015	0.000	0.657	0.672	0.005	-0.000	0.672	-0.010	0.000	0.662	0.672

TABLE I. Continued.

	Period	Argument (cosine)					RIGID and ROTATIONAL EFFECTS				TIDAL EFFECT		RIGID, ROTATIONAL and TIDAL EFFECTS				
							Oppolzer				Oppolzer			Oppolzer			
No.	(days)	l	l'	F	D	Ω	Fig.	Rot.	$-\Delta_{or}I_f$	$-\Delta_{or}I_r$	Fig.	Rot.	$-\Delta I$	Fig.	Rot.	$-\Delta I_f$	$-\Delta I_r$
53	9.6	1	0	0	2	1	0.004	-0.000	0.048	0.044	-0.001	0.000	0.044	0.003	-0.000	0.047	0.044
54	27.7	1	0	0	0	1	-0.089	0.000	-3.196	-3.107	0.029	-0.000	-3.108	-0.060	0.000	-3.167	-3.107
55	346.6	0	-1	2	-2	1	0.001	-0.000	0.260	0.259	-0.000	0.000	0.259	0.000	-0.000	0.259	0.259
56	14.2	0	-1	2	0	1	0.002	-0.000	0.034	0.032	-0.001	0.000	0.032	0.001	-0.000	0.033	0.032
57	-346.6	0	-1	0	0	1	-0.001	0.000	0.526	0.528	0.000	-0.000	0.528	-0.001	0.000	0.527	0.528
58	119.6	0	1	2	-2	1	-0.001	0.000	-0.188	-0.186	0.000	-0.000	-0.186	-0.001	0.000	-0.187	-0.186
59	13.1	0	1	2	0	1	-0.002	0.000	-0.041	-0.038	0.001	-0.000	-0.038	-0.002	0.000	-0.040	-0.038
60	386.0	0	1	0	0	1	0.002	-0.000	0.790	0.788	-0.000	0.000	0.788	0.001	-0.000	0.789	0.788
61	-169.0	0	0	-2	2	1	0.000	-0.000	-0.042	-0.042	-0.000	0.000	-0.042	0.000	-0.000	-0.042	-0.042
62	-13.6	0	0	-2	0	1	-0.002	0.000	0.034	0.036	0.001	-0.000	0.036	-0.001	0.000	0.035	0.036
63	177.8	0	0	2	-2	1	-0.028	0.000	-6.633	-6.605	0.009	-0.000	-6.605	-0.019	0.000	-6.624	-6.605
64	7.1	0	0	2	2	1	0.037	-0.000	0.328	0.291	-0.012	0.000	0.291	0.025	-0.000	0.316	0.291
65	13.6	0	0	2	0	1	1.114	-0.004	19.428	18.311	-0.360	0.001	18.314	0.754	-0.003	19.068	18.312
66	-14.7	0	0	0	-2	1	-0.012	0.000	0.255	0.268	0.004	-0.000	0.268	-0.008	0.000	0.259	0.268
67	14.8	0	0	0	2	1	0.017	-0.000	0.317	0.300	-0.005	0.000	0.300	0.011	-0.000	0.311	0.300
68	-6798.4	0	0	0	0	1	-1.019	0.003	9228.556	9229.578	0.329	-0.001	9229.575	-0.689	0.003	9228.885	9229.577
69	9.8	-1	-1	2	2	2	0.011	-0.000	0.118	0.107	-0.004	0.000	0.107	0.008	-0.000	0.115	0.107
70	9.1	-1	0	4	0	2	-0.005	0.000	-0.048	-0.043	0.002	-0.000	-0.043	-0.003	0.000	-0.046	-0.043
71	5.8	-1	0	2	4	2	0.010	-0.000	0.064	0.054	-0.003	0.000	0.054	0.007	-0.000	0.061	0.054
72	9.6	-1	0	2	2	2	0.246	-0.001	2.498	2.252	-0.079	0.000	2.253	0.166	-0.001	2.419	2.252
73	27.1	-1	0	2	0	2	-0.176	0.001	-5.139	-4.962	0.057	-0.000	-4.962	-0.119	0.000	-5.082	-4.962
74	-27.3	-1	0	0	0	2	0.002	-0.000	-0.057	-0.059	-0.001	0.000	-0.059	0.001	-0.000	-0.058	-0.059
75	7.4	-2	0	2	4	2	0.007	-0.000	0.051	0.044	-0.002	0.000	0.044	0.004	-0.000	0.049	0.044
76	14.6	-2	0	2	2	2	-0.004	0.000	-0.058	-0.054	0.001	-0.000	-0.054	-0.002	0.000	-0.057	-0.054
77	1615.7	-2	0	2	0	2	0.000	-0.000	0.135	0.135	-0.000	0.000	0.135	0.000	-0.000	0.135	0.135
78	8.8	3	0	2	-2	2	-0.004	0.000	-0.039	-0.035	0.001	-0.000	-0.035	-0.003	0.000	-0.038	-0.035
79	5.5	3	0	2	0	2	0.021	-0.000	0.123	0.101	-0.007	0.000	0.101	0.014	-0.000	0.116	0.101

TABLE I. Continued.

No.	Period (days)	Argument (cosine) l	l'	F	D	Ω	RIGID and ROTATIONAL EFFECTS Oppolzer Fig.	Rot.	$-\Delta_{or} I_f$	$-\Delta_{or} I_r$	TIDAL EFFECT Oppolzer Fig.	Rot.	RIGID, ROTATIONAL and TIDAL EFFECTS $-\Delta I$	Oppolzer Fig.	Rot.	$-\Delta I_f$	$-\Delta I_r$
80	12.8	2	0	2	-2	2	-0.020	0.000	-0.269	-0.249	0.006	-0.000	-0.249	-0.013	0.000	-0.263	-0.249
81	4.7	2	0	2	2	2	0.009	-0.000	0.047	0.037	-0.003	0.000	0.037	0.006	-0.000	0.044	0.037
82	6.9	2	0	2	0	2	0.180	-0.001	1.306	1.126	-0.058	0.000	1.126	0.122	-0.000	1.248	1.126
83	9.4	1	-1	2	0	2	0.012	-0.000	0.121	0.109	-0.004	0.000	0.109	0.008	-0.000	0.117	0.109
84	22.5	1	1	2	-2	2	-0.002	0.000	-0.054	-0.052	0.001	-0.000	-0.052	-0.002	0.000	-0.053	-0.052
85	8.9	1	1	2	0	2	-0.011	0.000	-0.104	-0.093	0.004	-0.000	-0.093	-0.007	0.000	-0.101	-0.093
86	23.9	1	0	2	-2	2	-0.046	0.000	-1.190	-1.144	0.015	-0.000	-1.144	-0.031	0.000	-1.175	-1.144
87	5.6	1	0	2	2	2	0.055	-0.000	0.325	0.270	-0.018	0.000	0.270	0.037	-0.000	0.307	0.270
88	9.1	1	0	2	0	2	1.300	-0.004	12.627	11.323	-0.420	0.001	11.327	0.880	-0.003	12.207	11.324
89	27.8	1	0	0	0	2	0.003	-0.000	0.082	0.079	-0.001	0.000	0.079	0.002	-0.000	0.081	0.079
90	7.2	0	-1	2	2	2	0.015	-0.000	0.112	0.097	-0.005	0.000	0.097	0.010	-0.000	0.107	0.097
91	14.2	0	-1	2	0	2	0.020	-0.000	0.299	0.279	-0.006	0.000	0.279	0.013	-0.000	0.292	0.279
92	13.2	0	1	2	0	2	-0.023	0.000	-0.317	-0.294	0.007	-0.000	-0.294	-0.015	0.000	-0.309	-0.294
93	409.2	0	1	0	0	2	-0.000	0.000	-0.032	-0.032	0.000	-0.000	-0.032	-0.000	0.000	-0.032	-0.032
94	12.7	0	0	4	-2	2	-0.003	0.000	-0.039	-0.036	0.001	-0.000	-0.036	-0.002	0.000	-0.038	-0.036
95	25.4	0	0	2	-1	2	0.001	-0.000	0.031	0.029	-0.000	0.000	0.029	0.001	-0.000	0.030	0.029
96	4.8	0	0	2	4	2	0.006	-0.000	0.029	0.024	-0.002	0.000	0.024	0.004	-0.000	0.027	0.024
97	7.1	0	0	2	2	2	0.216	-0.001	1.622	1.406	-0.070	0.000	1.407	0.146	-0.001	1.553	1.406
98	9.3	0	0	2	1	2	-0.007	0.000	-0.070	-0.063	0.002	-0.000	-0.063	-0.005	0.000	-0.067	-0.063
99	13.7	0	0	2	0	2	6.505	-0.021	95.021	88.495	-2.103	0.005	88.516	4.402	-0.017	92.918	88.500
100	-3399.2	0	0	0	0	2	0.024	-0.000	-90.343	-90.367	-0.008	0.000	-90.367	0.017	-0.000	-90.350	-90.367
101	182.6	0	2	0	0	0	-0.003	0.000	-0.003	0.000	0.001	-0.000	0.000	-0.002	0.000	-0.002	0.000
102	365.3	0	1	0	0	0	-0.138	0.000	-0.138	0.000	0.045	-0.000	0.000	-0.094	0.000	-0.094	0.000
103	365.2	0	-1	2	-2	2	-0.023	0.000	-9.260	-9.237	0.008	-0.000	-9.237	-0.016	0.000	-9.253	-9.237
104	91.3	0	2	2	-2	2	0.007	-0.000	0.662	0.656	-0.002	0.000	0.656	0.005	-0.000	0.660	0.656
105	121.8	0	1	2	-2	2	0.165	-0.001	21.678	21.513	-0.053	0.000	21.513	0.112	-0.000	21.625	21.513
106	182.6	0	0	2	-2	2	2.803	-0.009	553.401	550.589	-0.904	0.002	550.598	1.899	-0.007	552.497	550.591

TABLE II. Coefficients of sine for nutation in LONGITUDE. Epoch J2000.0. Unit = 0.001 arc sec.

No.	Period (days)	Argument (sine) l	l'	F	D	Ω	RIGID and ROTATIONAL EFFECTS Oppolzer Fig.	Rot.	$-\Delta_{or}\lambda_f$	$-\Delta_{or}\lambda_r$	TIDAL EFFECT Oppolzer Fig.	Rot.	RIGID, ROTATIONAL and TIDAL EFFECTS $-\Delta\lambda$	Oppolzer Fig.	Rot.	$-\Delta\lambda_f$	$-\Delta\lambda_r$
1	9.2	3	0	0	0	0	0.002	-0.000	0.150	0.149	-0.001	0.000	0.149	0.001	-0.000	0.150	0.149
2	131.7	2	1	0	-2	0	0.000	-0.000	0.096	0.096	-0.000	0.000	0.096	0.000	-0.000	0.096	0.096
3	-1095.2	2	0	-2	0	0	0.000	-0.000	1.113	1.113	-0.000	0.000	1.113	0.000	-0.000	1.113	1.113
4	205.9	2	0	0	-2	0	0.000	-0.000	4.486	4.486	-0.000	0.000	4.486	0.000	-0.000	4.486	4.486
5	-15.9	2	0	0	-4	0	-0.000	0.000	-0.122	-0.122	0.000	-0.000	-0.122	-0.000	0.000	-0.122	-0.122
6	7.1	2	0	0	2	0	0.001	-0.000	0.056	0.055	-0.000	0.000	0.055	0.001	-0.000	0.056	0.055
7	13.8	2	0	0	0	0	0.015	-0.000	2.789	2.774	-0.005	0.000	2.774	0.010	-0.000	2.785	2.774
8	-3232.9	1	-1	0	-1	0	-0.000	0.000	-0.333	-0.333	0.000	-0.000	-0.333	-0.000	0.000	-0.333	-0.333
9	-29.3	1	-1	0	-2	0	0.000	-0.000	0.091	0.091	-0.000	0.000	0.091	0.000	-0.000	0.091	0.091
10	29.8	1	-1	0	0	0	0.001	-0.000	0.451	0.451	-0.000	0.000	0.451	0.000	-0.000	0.451	0.451
11	-34.9	1	1	0	-2	0	-0.001	0.000	-0.702	-0.702	0.000	-0.000	-0.702	-0.000	0.000	-0.702	-0.702
12	25.6	1	1	0	0	0	-0.001	0.000	-0.324	-0.323	0.000	-0.000	-0.323	-0.000	0.000	-0.324	-0.323
13	-9.5	1	0	-2	-2	0	-0.001	0.000	-0.061	-0.060	0.000	-0.000	-0.060	-0.000	0.000	-0.061	-0.060
14	32.8	1	0	-2	2	0	-0.000	0.000	-0.055	-0.055	0.000	-0.000	-0.055	-0.000	0.000	-0.055	-0.055
15	-26.9	1	0	-2	0	0	0.001	-0.000	0.387	0.386	-0.000	0.000	0.386	0.000	-0.000	0.386	0.386
16	23.8	1	0	2	-2	0	-0.000	0.000	-0.073	-0.073	0.000	-0.000	-0.073	-0.000	0.000	-0.073	-0.073
17	9.1	1	0	2	0	0	0.004	-0.000	0.319	0.315	-0.001	0.000	0.315	0.003	-0.000	0.318	0.315
18	411.8	1	0	0	-1	0	-0.000	0.000	-0.284	-0.284	0.000	-0.000	-0.284	-0.000	0.000	-0.284	-0.284
19	-31.8	1	0	0	-2	0	-0.015	0.000	-14.959	-14.944	0.005	-0.000	-14.944	-0.010	0.000	-14.955	-14.944
20	-10.1	1	0	0	-4	0	-0.001	0.000	-0.128	-0.126	0.000	-0.000	-0.126	-0.001	0.000	-0.127	-0.126
21	9.6	1	0	0	2	0	0.007	-0.000	0.628	0.621	-0.002	0.000	0.621	0.005	-0.000	0.625	0.621
22	27.6	1	0	0	0	0	0.091	-0.000	67.779	67.689	-0.029	0.000	67.689	0.061	-0.000	67.750	67.689
23	-329.8	0	1	-2	2	0	-0.000	0.000	-0.079	-0.079	0.000	-0.000	-0.079	-0.000	0.000	-0.079	-0.079
24	117.5	0	1	2	-2	0	-0.000	0.000	-0.063	-0.063	0.000	-0.000	-0.063	-0.000	0.000	-0.063	-0.063
25	-15.4	0	1	0	-2	0	-0.002	0.000	-0.416	-0.414	0.001	-0.000	-0.414	-0.001	0.000	-0.415	-0.414
26	14.2	0	1	0	2	0	-0.000	0.000	-0.064	-0.064	0.000	-0.000	-0.064	-0.000	0.000	-0.064	-0.064

TABLE II. Continued.

	Period	Argument (sine)					RIGID and ROTATIONAL EFFECTS				TIDAL EFFECT		RIGID, ROTATIONAL and TIDAL EFFECTS				
							Oppolzer				Oppolzer			Oppolzer			
No.	(days)	l	l'	F	D	Ω	Fig.	Rot.	$-\Delta_{or}\lambda_f$	$-\Delta_{or}\lambda_r$	Fig.	Rot.	$-\Delta\lambda$	Fig.	Rot.	$-\Delta\lambda_f$	$-\Delta\lambda_r$
27	27.3	0	1	0	1	0	0.000	-0.000	0.054	0.054	-0.000	0.000	0.054	0.000	-0.000	0.054	0.054
28	173.3	0	0	2	-2	0	-0.000	0.000	-2.059	-2.059	0.000	-0.000	-2.059	-0.000	0.000	-2.059	-2.059
29	13.6	0	0	2	0	0	0.014	-0.000	2.467	2.453	-0.004	0.000	2.453	0.009	-0.000	2.462	2.453
30	14.8	0	0	0	2	0	0.028	-0.000	6.048	6.019	-0.009	0.000	6.019	0.019	-0.000	6.038	6.019
31	29.5	0	0	0	1	0	-0.000	0.000	-0.384	-0.384	0.000	-0.000	-0.384	-0.000	0.000	-0.384	-0.384
32	35.0	-1	-1	0	2	1	0.003	-0.000	0.072	0.069	-0.001	0.000	0.069	0.002	-0.000	0.071	0.069
33	-32.6	-1	0	2	-2	1	0.008	-0.000	-0.187	-0.194	-0.003	0.000	-0.194	0.005	-0.000	-0.189	-0.194
34	9.5	-1	0	2	2	1	-0.135	0.000	-1.007	-0.872	0.044	-0.000	-0.872	-0.091	0.000	-0.964	-0.872
35	27.0	-1	0	2	0	1	0.097	-0.000	1.980	1.882	-0.031	0.000	1.883	0.066	-0.000	1.948	1.882
36	-9.6	-1	0	0	-2	1	0.007	-0.000	-0.045	-0.052	-0.002	0.000	-0.052	0.005	-0.000	-0.047	-0.052
37	32.0	-1	0	0	2	1	0.061	-0.000	1.467	1.406	-0.020	0.000	1.406	0.041	-0.000	1.447	1.406
38	-388.3	-1	0	0	1	1	-0.000	0.000	0.064	0.064	0.000	-0.000	0.064	-0.000	0.000	0.064	0.064
39	-27.4	-1	0	0	0	1	0.274	-0.001	-5.430	-5.704	-0.088	0.000	-5.703	0.185	-0.001	-5.518	-5.704
40	1305.5	-2	0	2	0	1	0.005	-0.000	4.510	4.505	-0.002	0.000	4.505	0.003	-0.000	4.508	4.505
41	-199.8	-2	0	0	2	1	0.004	-0.000	-0.521	-0.524	-0.001	0.000	-0.524	0.002	-0.000	-0.522	-0.524
42	-13.8	-2	0	0	0	1	0.022	-0.000	-0.212	-0.234	-0.007	0.000	-0.234	0.015	-0.000	-0.219	-0.234
43	-943.2	2	0	-2	0	1	-0.000	0.000	0.069	0.069	0.000	-0.000	0.069	-0.000	0.000	0.069	0.069
44	12.8	2	0	2	-2	1	0.010	-0.000	0.100	0.090	-0.003	0.000	0.090	0.007	-0.000	0.097	0.090
45	6.9	2	0	2	0	1	-0.098	0.000	-0.534	-0.436	0.032	-0.000	-0.436	-0.066	0.000	-0.503	-0.436
46	212.3	2	0	0	-2	1	0.003	-0.000	0.397	0.395	-0.001	0.000	0.395	0.002	-0.000	0.396	0.395
47	13.8	2	0	0	0	1	0.020	-0.000	0.213	0.193	-0.006	0.000	0.193	0.014	-0.000	0.207	0.193
48	-34.7	1	1	0	-2	1	0.002	-0.000	-0.055	-0.057	-0.001	0.000	-0.057	0.001	-0.000	-0.056	-0.057
49	23.9	1	0	2	-2	1	0.031	-0.000	0.563	0.532	-0.010	0.000	0.532	0.021	-0.000	0.553	0.532
50	5.6	1	0	2	2	1	-0.029	0.000	-0.134	-0.105	0.009	-0.000	-0.105	-0.020	0.000	-0.125	-0.105
51	9.1	1	0	2	0	1	-0.713	0.002	-5.104	-4.388	0.231	-0.001	-4.391	-0.483	0.002	-4.873	-4.389
52	-31.7	1	0	0	-2	1	0.053	-0.000	-1.206	-1.259	-0.017	0.000	-1.259	0.036	-0.000	-1.223	-1.259

TABLE II. Continued.

No.	Period (days)	Argument (sine) l	l'	F	D	Ω	RIGID and ROTATIONAL EFFECTS Oppolzer Fig.	Rot.	$-\Delta_{or}\lambda_f$	$-\Delta_{or}\lambda_r$	TIDAL EFFECT Oppolzer Fig.	Rot.	RIGID, ROTATIONAL and TIDAL EFFECTS $-\Delta\lambda$	Oppolzer Fig.	Rot.	$-\Delta\lambda_f$	$-\Delta\lambda_r$
53	9.6	1	0	0	2	1	-0.013	0.000	-0.096	-0.083	0.004	-0.000	-0.083	-0.009	0.000	-0.092	-0.083
54	27.7	1	0	0	0	1	0.292	-0.001	6.113	5.819	-0.095	0.000	5.820	0.198	-0.001	6.018	5.820
55	346.6	0	-1	2	-2	1	-0.002	0.000	-0.487	-0.485	0.001	-0.000	-0.485	-0.001	0.000	-0.486	-0.485
56	14.2	0	-1	2	0	1	-0.006	0.000	-0.066	-0.060	0.002	-0.000	-0.060	-0.004	0.000	-0.064	-0.060
57	-346.6	0	-1	0	0	1	0.004	-0.000	-0.984	-0.988	-0.001	0.000	-0.988	0.003	-0.000	-0.985	-0.988
58	119.6	0	1	2	-2	1	0.004	-0.000	0.353	0.349	-0.001	0.000	0.349	0.003	-0.000	0.352	0.349
59	13.1	0	1	2	0	1	0.008	-0.000	0.080	0.072	-0.003	0.000	0.072	0.005	-0.000	0.077	0.072
60	386.0	0	1	0	0	1	-0.005	0.000	-1.482	-1.476	0.002	-0.000	-1.476	-0.004	0.000	-1.480	-1.476
61	-169.0	0	0	-2	2	1	-0.001	0.000	0.079	0.079	0.000	-0.000	0.079	-0.000	0.000	0.079	0.079
62	-13.6	0	0	-2	0	1	0.006	-0.000	-0.062	-0.068	-0.002	0.000	-0.068	0.004	-0.000	-0.064	-0.068
63	177.8	0	0	2	-2	1	0.096	-0.000	12.522	12.426	-0.031	0.000	12.426	0.065	-0.000	12.491	12.426
64	7.1	0	0	2	2	1	-0.117	0.000	-0.662	-0.545	0.038	-0.000	-0.545	-0.079	0.000	-0.624	-0.545
65	13.6	0	0	2	0	1	-3.607	0.012	-37.903	-34.284	1.166	-0.003	-34.296	-2.441	0.009	-36.737	-34.287
66	-14.7	0	0	0	-2	1	0.044	-0.000	-0.458	-0.502	-0.014	0.000	-0.501	0.030	-0.000	-0.472	-0.502
67	14.8	0	0	0	2	1	-0.054	0.000	-0.616	-0.561	0.018	-0.000	-0.562	-0.037	0.000	-0.598	-0.562
68	-6798.4	0	0	0	0	1	3.437	-0.011	-17281.759	-17285.208	-1.111	0.002	-17285.197	2.326	-0.009	-17282.871	-17285.205
69	9.8	-1	-1	2	2	2	-0.031	0.000	-0.277	-0.247	0.010	-0.000	-0.247	-0.021	0.000	-0.267	-0.247
70	9.1	-1	0	4	0	2	0.013	-0.000	0.113	0.099	-0.004	0.000	0.099	0.009	-0.000	0.108	0.099
71	5.8	-1	0	2	4	2	-0.028	0.000	-0.152	-0.124	0.009	-0.000	-0.124	-0.019	0.000	-0.143	-0.124
72	9.6	-1	0	2	2	2	-0.662	0.002	-5.858	-5.194	0.214	-0.000	-5.196	-0.448	0.002	-5.644	-5.194
73	27.1	-1	0	2	0	2	0.480	-0.002	11.926	11.444	-0.155	0.000	11.446	0.325	-0.001	11.771	11.445
74	-27.3	-1	0	0	0	2	-0.005	0.000	0.131	0.137	0.002	-0.000	0.137	-0.004	0.000	0.133	0.137
75	7.4	-2	0	2	4	2	-0.018	0.000	-0.120	-0.102	0.006	-0.000	-0.103	-0.012	0.000	-0.114	-0.103
76	14.6	-2	0	2	2	2	0.010	-0.000	0.135	0.125	-0.003	0.000	0.125	0.007	-0.000	0.131	0.125
77	1615.7	-2	0	2	0	2	-0.000	0.000	-0.311	-0.310	0.000	-0.000	-0.310	-0.000	0.000	-0.311	-0.310
78	8.8	3	0	2	-2	2	0.011	-0.000	0.092	0.081	-0.004	0.000	0.081	0.008	-0.000	0.088	0.081
79	5.5	3	0	2	0	2	-0.057	0.000	-0.290	-0.234	0.018	-0.000	-0.234	-0.038	0.000	-0.272	-0.234

TABLE II. Continued.

	Period	Argument (sine)					RIGID and ROTATIONAL EFFECTS				TIDAL EFFECT		RIGID, ROTATIONAL and TIDAL EFFECTS				
							Oppolzer				Oppolzer			Oppolzer			
No.	(days)	l	l'	F	D	Ω	Fig.	Rot.	$-\Delta_{or}\lambda_f$	$-\Delta_{or}\lambda_r$	Fig.	Rot.	$-\Delta\lambda$	Fig.	Rot.	$-\Delta\lambda_f$	$-\Delta\lambda_r$
80	12.8	2	0	2	-2	2	0.053	-0.000	0.628	0.575	-0.017	0.000	0.575	0.036	-0.000	0.611	0.575
81	4.7	2	0	2	2	2	-0.025	0.000	-0.111	-0.085	0.008	-0.000	-0.086	-0.017	0.000	-0.103	-0.086
82	6.9	2	0	2	0	2	-0.482	0.002	-3.080	-2.596	0.156	-0.000	-2.598	-0.326	0.001	-2.924	-2.596
83	9.4	1	-1	2	0	2	-0.033	0.000	-0.283	-0.251	0.011	-0.000	-0.251	-0.022	0.000	-0.273	-0.251
84	22.5	1	1	2	-2	2	0.006	-0.000	0.125	0.119	-0.002	0.000	0.119	0.004	-0.000	0.123	0.119
85	8.9	1	1	2	0	2	0.030	-0.000	0.245	0.215	-0.010	0.000	0.215	0.020	-0.000	0.235	0.215
86	23.9	1	0	2	-2	2	0.126	-0.000	2.765	2.638	-0.041	0.000	2.639	0.085	-0.000	2.724	2.638
87	5.6	1	0	2	2	2	-0.146	0.000	-0.768	-0.622	0.047	-0.000	-0.623	-0.099	0.000	-0.721	-0.622
88	9.1	1	0	2	0	2	-3.502	0.011	-29.629	-26.115	1.132	-0.003	-26.126	-2.370	0.009	-28.496	-26.117
89	27.8	1	0	0	0	2	-0.007	0.000	-0.191	-0.183	0.002	-0.000	-0.183	-0.005	0.000	-0.188	-0.183
90	7.2	0	-1	2	2	2	-0.039	0.000	-0.263	-0.224	0.013	-0.000	-0.224	-0.026	0.000	-0.250	-0.224
91	14.2	0	-1	2	0	2	-0.053	0.000	-0.697	-0.643	0.017	-0.000	-0.644	-0.036	0.000	-0.680	-0.643
92	13.2	0	1	2	0	2	0.061	-0.000	0.740	0.678	-0.020	0.000	0.679	0.041	-0.000	0.720	0.679
93	409.2	0	1	0	0	2	0.000	-0.000	0.074	0.074	-0.000	0.000	0.074	0.000	-0.000	0.074	0.074
94	12.7	0	0	4	-2	2	0.008	-0.000	0.090	0.082	-0.002	0.000	0.082	0.005	-0.000	0.087	0.082
95	25.4	0	0	2	-1	2	-0.003	0.000	-0.071	-0.068	0.001	-0.000	-0.068	-0.002	0.000	-0.070	-0.068
96	4.8	0	0	2	4	2	-0.016	0.000	-0.070	-0.054	0.005	-0.000	-0.054	-0.011	0.000	-0.065	-0.054
97	7.1	0	0	2	2	2	-0.579	0.002	-3.823	-3.243	0.187	-0.000	-3.245	-0.392	0.001	-3.636	-3.243
98	9.3	0	0	2	1	2	0.019	-0.000	0.163	0.144	-0.006	0.000	0.144	0.013	-0.000	0.157	0.144
99	13.7	0	0	2	0	2	-17.615	0.058	-221.779	-204.107	5.695	-0.013	-204.165	-11.920	0.045	-216.085	-204.120
100	-3399.2	0	0	0	0	2	-0.067	0.000	209.025	209.092	0.022	-0.000	209.092	-0.045	0.000	209.047	209.092
101	182.6	0	2	0	0	0	0.000	-0.000	1.584	1.584	-0.000	0.000	1.584	0.000	-0.000	1.584	1.584
102	365.3	0	1	0	0	0	0.001	-0.000	125.836	125.835	-0.000	0.000	125.835	0.001	-0.000	125.835	125.835
103	365.2	0	-1	2	-2	2	0.064	-0.000	21.369	21.305	-0.021	0.000	21.305	0.043	-0.000	21.349	21.305
104	91.3	0	2	2	-2	2	-0.018	0.000	-1.531	-1.512	0.006	-0.000	-1.512	-0.012	0.000	-1.525	-1.512
105	121.8	0	1	2	-2	2	-0.451	0.001	-50.071	-49.619	0.145	-0.000	-49.620	-0.305	0.001	-49.926	-49.619
106	182.6	0	0	2	-2	2	-7.674	0.025	-1277.640	-1269.942	2.475	-0.005	-1269.967	-5.199	0.020	-1275.166	-1269.947

Motion of the Equatorial plane of Figure:

Starting from expressions (6.34) and (6.35), the Oppolzer terms corresponding to this plane will be:

$$\begin{aligned}\Delta(\delta\lambda_f) &= (K_{0r\leftmoon} - K_{t\leftmoon})\times\\ &\times\sum_i\left[\frac{1}{n_\mu - n_i}\left(-\frac{1}{2}A_i^{0)}\cos\epsilon_0 + \frac{1}{4}(1+\cos\epsilon_0)A_i^{2)} - \right.\right.\\ &\left.-\frac{1}{2\sin\epsilon_0}(1+\cos\epsilon_0)(-1+2\cos\epsilon_0)A_i^{1)}\right) + \\ &+\frac{1}{n_\mu + n_i}\left(\frac{1}{2}A_i^{0)}\cos\epsilon_0 + \frac{1}{4}(1-\cos\epsilon_0)A_i^{2)} + \right.\\ &\left.\left.+\frac{1}{2\sin\epsilon_0}(1-\cos\epsilon_0)(-1-2\cos\epsilon_0)A_i^{1)}\right)\right]_{\leftmoon}\sin\Theta_i + \\ &+(K_{0r\odot} - K_{t\odot})\times\\ &\times\sum_i\left[\frac{1}{n_\mu - n_i}\left(-\frac{1}{2}A_i^{0)}\cos\epsilon_0 + \frac{1}{4}(1+\cos\epsilon_0)A_i^{2)}\right) + \right.\\ &\left.+\frac{1}{n_\mu + n_i}\left(\frac{1}{2}A_i^{0)}\cos\epsilon_0 + \frac{1}{4}(1-\cos\epsilon_0)A_i^{2)}\right)\right]_{\odot}\sin\Theta_i\,,\end{aligned} \tag{7.11}$$

$$\begin{aligned}\Delta(\delta I_f) &= (K_{0r\leftmoon} - K_{t\leftmoon})\times\\ &\times\sum_i\left[\frac{1}{n_\mu - n_i}\left(\frac{1}{4}A_i^{0)}\sin 2\epsilon_0 - \frac{1}{4}\sin\epsilon_0(1+\cos\epsilon_0)A_i^{2)} + \right.\right.\\ &\left.+\frac{1}{2}(1+\cos\epsilon_0)(-1+2\cos\epsilon_0)A_i^{1)}\right) + \\ &+\frac{1}{n_\mu + n_i}\left(\frac{1}{4}A_i^{0)}\sin 2\epsilon_0 + \frac{1}{4}\sin\epsilon_0(1-\cos\epsilon_0)A_i^{2)} + \right.\\ &\left.\left.+\frac{1}{2}(1-\cos\epsilon_0)(-1-2\cos\epsilon_0)A_i^{1)}\right)\right]_{\leftmoon}\cos\Theta_i + \\ &+(K_{0r\odot} - K_{t\odot})\times\\ &\times\sum_i\left[\frac{1}{n_\mu - n_i}\left(\frac{1}{4}A_i^{0)}\sin 2\epsilon_0 - \frac{1}{4}\sin\epsilon_0(1+\cos\epsilon_0)A_i^{2)}\right) + \right.\\ &\left.+\frac{1}{n_\mu + n_i}\left(\frac{1}{4}A_i^{0)}\sin 2\epsilon_0 + \frac{1}{4}\sin\epsilon_0(1-\cos\epsilon_0)A_i^{2)}\right)\right]_{\odot}\cos\Theta_i\,.\end{aligned} \tag{7.12}$$

As for the values of the coefficients, from (7.1), (7.6) and (7.7):

$$K_{0r\leftmoon} - K_{t\leftmoon} = 5121''.147757/\mathrm{J.}cy\,, \quad K_{0r\odot} - K_{t\odot} = 2354''.558612/\mathrm{J.}cy\,, \tag{7.13}$$

Thus we remark that the tidal perturbation diminishes considerably the amplitude of the Oppolzer terms.

The previous series (7.11) and (7.12) gather the first order contributions to the nutation of a rigid Earth and the corrections due to the elasticity. Nevertheless, in order to make the comparison with other studies easier, and to increase the accuracy, we have found convenient to add to those series a few second order corrections of lunar origin. As pointed out firstly by Kubo (1982), some significant second order contributions to the nutations in obliquity and longitude appear when considering the whole Hamiltonian for the rotation of the Earth and the orbital motion of the Moon. A more complete study of this effect has been carried out by Kinoshita and Souchay (1990).

More precisely, we have added to the series the corrections

$$\begin{aligned} -\Delta\lambda &= -0.0433\sin\Omega + 1.191\sin 2\Omega - \\ &\quad - 0.083\sin(2F-2D+\Omega) - 0.016\sin(2F+\Omega), \\ -\Delta I &= 0.102\cos\Omega - 0.227\cos 2\Omega + \\ &\quad + 0.074\cos(2F-2D+\Omega) + 0.012\cos(2F+\Omega). \end{aligned} \tag{7.14}$$

The three remaining terms in expressions (7.38) and (7.39) of Kinoshita and Souchay have not been included since they do not share the frequency with any first order term.

The resulting nutation series are given in Tables I and II. Let us remark that the corrections (7.14) have been used in all the concerned terms throughout the rest of the paper. These tables are referred to the Epoch J2000.0, and have been arranged in the same way. In the first column a label number is given to identify each term, and in the second one the period, with a sign corresponding to the sign of the argument of the trigonometric function. In the third column we have the argument, of the cosine for the obliquity and of the sine for the longitude. The next columns give the coefficients corresponding to the different contributions to the nutations, using 1 mas as the unit. Columns 4 – 7 show the effects of the rigid Earth with rotational deformation only, giving first of all the Oppolzer terms for the axes of figure and of rotation, and then the nutations of both axes. These nutations were obtained by adding corresponding Oppolzer terms to that of Andoyer's plane (Column 10, headed by $-\Delta I$ for the obliquity and $-\Delta\lambda$ for the longitude). Columns 8 and 9 contain the contribution to the Oppolzer terms of tidal origin. The last five columns show the final coefficients for our model of a rigid Earth with elastic mantle. Column 10 gives the nutation of the angular momentum, which is not altered by tidal deformation. Columns 11 and 12 show the Oppolzer terms (obtaining by adding up columns 4 and 8, and 5 and 9 respectively) whose amplitudes decrease with respect to a rigid Earth or a rigid Earth with rotational deformation. The last columns give the total nutations of the axes of figure $(-\Delta I_f, -\Delta\lambda_f)$ and of rotation $(-\Delta I_r, -\Delta\lambda_r)$.

Motion of the plane perpendicular to the rotation axis:

Putting together the expressions (6.36) and (6.37) which give the Oppolzer terms corresponding to this plane, we can write:

$$\Delta(\delta\lambda_r) = \frac{K_a}{\sin I}\sum_\tau\sum_i \frac{\tau C_i(\tau)}{n_\mu - \tau n_i}\sin\Theta_i\,, \tag{7.15}$$

$$\Delta(\delta I_r) = K_a\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos\Theta_i\,, \tag{7.16}$$

where

$$K_a = (K_{0r} - K_t)\left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right) + 3\,K_{0r}\,D_r\left(\frac{1}{A_0} + \frac{1}{B_0}\right), \tag{7.17}$$

and we treat (7.15) and (7.16) in a way similar to (7.11) and (7.12). With the value of D_r given by (3.18), the simplifications used in Subsection 7.1, and taking into account that $(1 - C_0/2A_0 - C_0/2B_0) \simeq -H_0$, we have, in arc sec. per Julian century:

$$K_{a(} = -19''\!.321938/\mathrm{J.}cy\,, \qquad K_{a\odot} = -8''\!.879428/\mathrm{J.}cy\,. \tag{7.18}$$

The results are gathered in Tables I and II.

Polar motion:

Starting from (6.38) and (6.39) we get:

$$\Delta x_p = \frac{C_0}{A_0}K_b\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\sin(\mu + \nu - \tau\Theta_i)\,, \tag{7.19}$$

$$-\Delta y_p = \frac{C_0}{B_0}K_b\sum_\tau\sum_i \frac{C_i(\tau)}{n_\mu - \tau n_i}\cos(\mu + \nu - \tau\Theta_i)\,, \tag{7.20}$$

where

$$K_b = (K_{0r} - K_t) - 6\,K_{0r}\,\frac{D_r}{C_0}\,. \tag{7.21}$$

To evaluate this coefficient we take $C_0 = 8.11 \times 10^{44}$c.g.s., (Danby, 1962), which gives

$$K_{b(} = 5129''\!.113168/\mathrm{J.}cy\,, \qquad K_{b\odot} = 2358''\!.216639/\mathrm{J.}cy\,. \tag{7.22}$$

The most significant terms are listed in Table III.

7.3. Secular Perturbations and Chandler Period

The secular perturbations can be evaluated using equations (6.40), (6.41) and (6.42). Let us look specifically at the most significant fact, referring to the Chandler period.

TABLE III

Coefficients of POLAR MOTION. Epoch J2000.0. Units: amplitude = 0.001 arc sec.; period = days. The table shows the coefficients of $(C/A)\sin()$ and $-(C/B)\cos()$ for x_p and y_p respectively

	Argument (Θ)					COEFFICIENTS			
						$\mu+\nu-\Theta$		$\mu+\nu+\Theta$	
No.	l	l'	F	D	Ω	Period	Amplit.	Period	Amplit.
19	1	0	0	-2	0	0.9742	0.062	1.0378	0.066
22	1	0	0	0	0	1.0430	0.346	0.9696	0.322
30	0	0	0	2	0	1.0783	0.059	0.9409	0.052
39	-1	0	0	0	1	0.9695	0.064	1.0432	-0.010
51	1	0	2	0	1	1.1294	-0.172	0.9052	0.020
54	1	0	0	0	1	1.0428	0.069	0.9697	-0.009
65	0	0	2	0	1	1.0850	-0.864	0.9360	0.109
68	0	0	0	0	1	1.0048	0.809	1.0051	-0.118
72	-1	0	2	2	2	1.1230	-0.172	0.9094	0.006
73	-1	0	2	0	2	1.0437	0.125	0.9690	-0.005
82	2	0	2	0	2	1.1775	-0.126	0.8766	0.004
88	1	0	2	0	2	1.1293	-0.913	0.9053	0.031
97	0	0	2	2	2	1.1707	-0.151	0.8804	0.005
99	0	0	2	0	2	1.0848	-4.579	0.9361	0.170
105	0	1	2	-2	2	1.0133	-0.117	0.9967	0.115
106	0	0	2	-2	2	1.0105	-1.987	0.9995	1.962

From (4.13) and (6.41) we can write:

$$
\begin{aligned}
n_\nu^* = n_\nu + \delta n_\nu \; &= \; \frac{1}{2}\left(\frac{2}{C_0} - \frac{1}{A_0} - \frac{1}{B_0}\right) N_0 + 3\,K_0\,B_{\hat{0}}^* + \\
&+ N^*\left(D_r + 3\,D_t B_{\hat{0}}^*\right) Q_c + 2\,D_r^2\,N^*\,Q_d\,, \qquad (7.23)
\end{aligned}
$$

where N_0 is the constant value of the variable N^*. To evaluate this expression we can arrange the different terms as follows:

$$
\frac{1}{2}\left(\frac{2}{C_0} - \frac{1}{A_0} - \frac{1}{B_0}\right) = -\frac{2C_0 - A_0 - B_0}{2C_0^2} - \frac{1}{2C_0^2}\left[\frac{(C_0 - A_0)^2}{A_0} + \frac{(C_0 - B_0)^2}{B_0}\right],
$$

$$
Q_c = -\frac{2}{C_0^2}\frac{2C_0 - A_0 - B_0}{2C_0} + \frac{1}{2C_0^3}\left[\frac{(C_0 - A_0)^3}{A_0^2} + \frac{(C_0 - B_0)^3}{B_0^2}\right],
$$

$$
Q_d \; = \; -\frac{6}{C_0^3}\frac{2C_0 - A_0 - B_0}{2C_0} - \frac{A_0 - C_0}{C_0^4 A_0^3}\left[2A_0(A_0 - C_0)^2 + (A_0^3 - C_0^3)\right] -
$$

$$-\frac{B_0 - C_0}{C_0^4 B_0^3}\left[2B_0(B_0 - C_0)^2 + (B_0^3 - C_0^3)\right].$$

Then, neglecting the products of the differences $A_0 - C_0$, $B_0 - C_0$, taking into account the expression of K_0 in (4.22), and performing the simplifications $N_0 \simeq N^* \simeq C_0\,\omega_3$, we obtain finally:

$$n_\nu^* \simeq -\frac{2C_0 - A_0 - B_0}{2C_0}\,\omega_3(1 + K), \qquad (7.24)$$

with

$$K = 2\frac{D_r}{C_0} + 12\frac{D_r^2}{C_0^2} + \frac{6}{C_0}(D_{t(}\,B_(+ D_{t\odot}\,B_\odot) - \frac{3Gm_(}{a_(^3\omega_3^2}B_(- \frac{3Gm_\odot}{a_\odot^3\omega_3^2}B_\odot\,. \qquad (7.25)$$

Giving numerical values, it results that $K = -3.11 \times 10^{-4}$. Then, we can conclude that the main part of n_ν^* is:

$$n_\nu^* \simeq -\frac{2C_0 - A_0 - B_0}{2C_0}\,\omega_3 = -\mathbf{H}_0\,\omega_3\,, \qquad (7.26)$$

where $\mathbf{H}_0$ is the component of the actual ellipticity $\mathbf{H}$ introduced in (7.4), whose value is given by (7.8). Thus, the period P of the variable ν, in absolute value, becomes:

$$P \simeq \frac{1}{\mathbf{H}_0} \simeq 450\text{days}\,. \qquad (7.27)$$

This period corresponds to the free Eulerian oscillation of the Earth (in this case for the deformable model considered), and according to the usual terminology (Jeffreys and Vicente, 1957), P can be referred to as the Chandler period.

It is well known that for a rigid Earth, the use of the ellipticity $\mathbf{H} = (2C - A - B)/(2C) \simeq 0.00327$ leads to a value of 305 days for the Eulerian period of ν, which differs notably from the observational results. However, in our derivation we get a period of around 450 days, which fits reasonably well with the observed values of the Chandler period (Lambert, 1980 and 1988, Melchior, 1983).

Thus the previous mathematical developments prove that for an elastic Earth, the Chandler period does not depend on the actual (or usual) ellipticity of the planet, but on a hypothetical ellipticity $\mathbf{H}_0$ obtained from the former by removing the contribution to it of the centrifugal deformation, that is, the ellipticity induced by the rotation itself. The authors cannot resist the temptation to consider that ellipticity $\mathbf{H}_0$ as being in some sense "free", since it is not affected or "forced" by the rotational deformation and besides, it is in harmony with Chandler's period which plays the role of "free" Eulerian period.

In any case, to give an intuitive physical explanation of this phenomenon that is both brief and at the same time clear, is not an easy task, and the literature is of no great help. Thus, a textbook of a general nature with the prestige of that of

Goldstein says of Chandler's period: *"In effect some part of the earth follows along with the shift in the rotation axis, which has the effect of reducing the difference in the principal moments of inertia and therefore increasing the period"* (p. 212, second edition). This description does not seem to be very exact, and perhaps it would be better to speak of an increase in the difference of moments of inertia due to the rotation since, according to our mathematical derivations, it does not affect the period of precession.

For the reader who are interested in the physical explanation of our results we would refer them to the above–mentioned article by Newcomb (1892) who, on using a very simplified mathematical model, insists on physical considerations that are possibly not very easy to understand in the light of the few occasions he is quoted.

Finally, we must point out that although the obtained value of 450 days is slightly superior to those given by the latest experimental results (436-440 days), the fact that we have not considered the deformation produced in the liquid core must be taken into account as it shortens the value of this period (Vicente, 1961, Herring *et al.*, 1991).

8. Discussion of Results

In order to evaluate the theory, some tables are given in which our numerical results are compared to those obtained by other authors. In Table IV and V, the values obtained for the nutations of the three fundamental planes are compared to those given by the theories of a rigid Earth, of first order (Kinoshita, 1977, reduced to the Epoch J2000.0) and of second order (Kinoshita and Souchay, 1990). For the sake of brevity, only the terms of greatest amplitude have been considered. The number of the first column is the same as the label term in Tables I and II. For the Andoyer plane the agreement with the Kinoshita and Souchay values is very good as expected, since that plane is not affected by the tidal deformation. The slights differences should be originated by the second order rigid corrections.

Concerning the nutations of the axes of figure and of rotation, discrepancies do appear (less than 1 mas) due to the decrease in amplitude of the Oppolzer terms caused by the elasticity.

In Table VI we show the comparison between Oppolzer terms calculated by us and those given by other authors (Sasao et al., 1980, and Kubo, 1991) with similar hypotheses about the elastic properties of the Earth. The values shown are the differences with those corresponding to Kinoshita (1977). It can be seen that the greatest differences amount to a few 0.01 mas, with the exception of the term 99, semimonthly, for which the differences are of the order of 0.1 mas, with the value of Kubo in obliquity and that of Sasao et al. in longitude.

In the last table, the principal nutations of the figure axis are compared to those of other theories for a non-rigid Earth. We have chosen those of Capitaine (1978)

TABLE IV. Comparison with RIGID theories for nutation in OBLIQUITY. Epoch J2000.0. Unit = 0.001 arc sec. K = Kinoshita (1977). KS = Kinoshita and Souchay (1990).

No.	Period (days)	Argument (cosine) l	l'	F	D	Ω	$-\Delta I$ Rigid Earth K	$-\Delta I$ Rigid Earth KS	$-\Delta I$ Non-Rigid Earth Authors	$-\Delta I_f$ Rigid Earth K	$-\Delta I_f$ Rigid Earth KS	$-\Delta I_f$ Non-Rigid Earth Authors	$-\Delta I_r$ Rigid Earth K	$-\Delta I_r$ Rigid Earth KS	$-\Delta I_r$ Non-Rigid Earth Authors
19	-31.8	1	0	0	-2	0	0.0	0.000	0.000	-0.2	-0.187	-0.128	0.0	0.001	0.000
22	27.5	1	0	0	0	0	0.0	0.000	0.000	-1.0	-0.976	-0.667	0.0	0.003	0.003
30	14.8	0	0	0	2	0	0.0	0.000	0.000	-0.2	-0.162	-0.111	0.0	0.001	0.000
34	9.5	-1	0	2	2	1	0.5	0.466	0.466	0.5	0.508	0.494	0.5	0.466	0.466
39	-27.4	-1	0	0	0	1	3.0	3.045	3.045	3.0	2.966	2.991	3.0	3.045	3.045
51	9.1	1	0	2	0	1	2.3	2.344	2.344	2.6	2.566	2.496	2.3	2.343	2.344
54	27.7	1	0	0	0	1	-3.1	-3.107	-3.108	-3.2	-3.194	-3.167	-3.1	-3.107	-3.107
64	7.1	0	0	2	2	1	0.3	0.291	0.291	0.3	0.328	0.316	0.3	0.291	0.291
65	13.6	0	0	2	0	1	18.3	18.319	18.314	19.4	19.421	19.068	18.3	18.315	18.312
68	-6798.4	0	0	0	0	1	9228.6	9229.578	9229.575	9227.7	9228.570	9228.885	9228.6	9229.581	9229.577
72	9.6	-1	0	2	2	2	2.3	0.225	2.253	2.5	2.495	2.419	2.3	2.251	2.252
73	27.1	-1	0	2	0	2	-5.0	-4.962	-4.962	-5.1	-5.137	-5.082	-5.0	-4.961	-4.962
82	6.9	2	0	2	0	2	1.1	1.126	1.126	1.3	1.304	1.248	1.1	1.125	1.126
86	23.9	1	0	2	-2	2	-1.1	-1.144	-1.144	-1.2	-1.190	-1.175	-1.1	-1.144	-1.144
87	5.6	1	0	2	2	2	0.3	0.270	0.270	0.3	0.323	0.307	0.3	0.270	0.270
88	9.1	1	0	2	0	2	11.3	11.330	11.327	12.6	12.615	12.207	11.3	11.326	11.324
97	7.1	0	0	2	2	2	1.4	1.405	1.407	1.6	1.618	1.553	1.4	1.404	1.406
99	13.7	0	0	2	0	2	88.5	88.521	88.516	95.0	94.956	92.918	88.5	88.500	88.500
100	-3399.2	0	0	0	0	2	-90.2	-90.368	-90.367	-90.2	-90.344	-90.350	-90.2	-90.368	-90.367
102	365.3	0	1	0	0	0	0.0	0.000	0.000	-0.1	-0.014	-0.094	0.0	0.000	0.000
103	365.2	0	-1	2	-2	2	-9.2	-9.190	-9.237	-9.3	-9.213	-9.253	-9.2	-9.190	-9.237
105	121.8	0	1	2	-2	2	21.4	21.432	21.513	21.6	21.594	21.625	21.4	21.431	21.513
106	182.6	0	0	2	-2	2	550.6	549.660	550.598	553.4	552.430	552.497	550.6	549.651	550.591

TABLE V. Comparison with RIGID theories for nutation in LONGITUDE. Epoch J2000.0. Unit = 0.001 arc sec. K = Kinoshita (1977). KS = Kinoshita and Souchay (1990).

	Period	Argument (sine)					$-\Delta\lambda$			$-\Delta\lambda_f$			$-\Delta\lambda_r$		
							Rigid Earth		Non-Rigid Earth	Rigid Earth		Non-Rigid Earth	Rigid Earth		Non-Rigid Earth
No.	(days)	l	l'	F	D	Ω	K	KS	Authors	K	KS	Authors	K	KS	Authors
19	-31.8	1	0	0	-2	0	-14.9	-14.945	-14.944	-15.0	-14.960	-14.955	-14.9	-14.945	-14.944
22	27.5	1	0	0	0	0	67.7	67.688	67.689	67.8	67.776	67.750	67.7	67.688	67.689
30	14.8	0	0	0	2	0	6.0	6.017	6.019	6.0	6.044	6.038	6.0	6.017	6.019
34	9.5	-1	0	2	2	1	-0.9	-0.873	-0.872	-1.0	-1.007	-0.964	-0.9	-0.873	-0.872
39	-27.4	-1	0	0	0	1	-5.7	-5.704	-5.703	-5.4	-5.434	-5.518	-5.7	-5.705	-5.704
51	9.1	1	0	2	0	1	-4.4	-4.391	-4.391	-5.1	-5.096	-4.873	-4.4	-4.389	-4.389
54	27.7	1	0	0	0	1	5.8	5.819	5.820	6.1	6.109	6.018	5.8	5.818	5.820
64	7.1	0	0	2	2	1	-0.5	-0.545	-0.545	-0.7	-0.661	-0.624	-0.5	-0.545	-0.545
65	13.6	0	0	2	0	1	-34.3	-34.304	-34.296	-37.8	-37.873	-36.737	-34.2	-34.292	-34.287
68	-6798.4	0	0	0	0	1	-17284.9	-17285.202	-17285.197	-17281.5	-17281.798	-17282.870	-17284.9	-17 285.213	-17285.205
72	9.6	-1	0	2	2	2	-5.2	-5.194	-5.196	-5.8	-5.848	-5.644	-5.2	-5.192	-5.194
73	27.1	-1	0	2	0	2	11.4	11.446	11.446	11.9	11.921	11.771	11.4	11.444	11.445
82	6.9	2	0	2	0	2	-2.6	-2.597	-2.598	-3.1	-3.073	-2.924	-2.6	-2.595	-2.596
86	23.9	1	0	2	-2	2	2.6	2.640	2.639	2.8	2.765	2.724	2.6	2.640	2.638
87	5.6	1	0	2	2	2	-0.6	-0.623	-0.623	-0.8	-0.767	-0.721	-0.6	-0.623	-0.622
88	9.1	1	0	2	0	2	-26.1	-26.134	-26.126	-29.6	-29.597	-28.496	-26.1	-26.123	-26.117
97	7.1	0	0	2	2	2	-3.2	-3.242	-3.245	-3.8	-3.814	-3.636	-3.2	-3.240	-3.243
99	13.7	0	0	2	0	2	-204.1	-204.175	-204.165	-221.5	-221.602	-216.085	-204.1	-204.118	-204.120
100	-3399.2	0	0	0	0	2	207.9	209.095	209.092	207.9	209.029	209.047	207.9	209.095	209.092
102	365.3	0	1	0	0	0	125.5	127.234	125.835	125.5	127.235	125.835	125.5	127.234	125.835
103	365.2	0	-1	2	-2	2	21.2	21.197	21.305	21.3	21.260	21.349	21.2	21.197	21.305
105	121.8	0	1	2	-2	2	-49.5	-49.433	-49.620	-50.0	-48.988	-49.926	-49.5	-49.433	-49.619
106	182.6	0	0	2	-2	2	-1269.9	-1267.799	-1269.967	-1277.5	-1275.382	-1275.166	-1269.9	1267.774	-1269.947

TABLE VI

TIDAL EFFECTS of Oppolzer terms for the FIGURE PLANE. The results show the differences with Kinoshita (1977). Unit 0.0001 arc sec. S.O.S. = Sasao, Okubo and Saito (1980). Kubo = Kubo (1991).

No.	Period (days)	Argument l	l'	F	D	Ω	OBLIQUITY S.O.S.	Kubo	Authors	LONGITUDE S.O.S.	Kubo	Authors
22	27.5	1	0	0	0	0	+3.2	+3.0	+3.03	-0.3	-0.3	-0.29
68	-6798.4	0	0	0	0	1	+3.3	+3.1	+3.11	-11.0	-10.5	-10.64
99	13.7	0	0	2	0	2	-20.7	-19.9	-20.08	+56.1	+53.8	+54.50
100	-3399.2	0	0	0	0	2	-0.1	-0.1	-0.03	+0.2	+0.2	+0.25
102	365.3	0	1	0	0	0	+0.4	+0.4	+0.46	+0.0	+0.0	+0.01
103	365.2	0	-1	2	-2	2	+0.1	+0.1	+0.04	-0.2	-0.2	-0.17
105	121.7	0	1	2	-2	2	-0.5	-0.5	-0.48	+1.4	+1.4	+1.35
106	182.6	0	0	2	-2	2	-9.0	-8.6	-8.71	+24.5	+23.5	+23.71

whose hypotheses are similar to those of this paper, and Wahr (1981). The values of the first have been taken from Capitaine (1978), referring them to the Epoch J2000.0 with the help of the secular variations given by Kinoshita. For the obliquity, the agreement with Capitaine's values is very good. The differences reach 0.1 mas only for the terms in Ω and in $l' + 2F - 2D + 2\Omega$, so that all the significant figures given by the said author are identical to ours, with the above mentioned exception. In longitude the discrepancies with Capitaine reach 1 mas for the term in Ω, and are of the order of 0.1 mas for the other ones. As expected, the differences with the Wahr's calculations are greater, since this theory includes a liquid core.

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Appendix 1. Development of Spherical Functions

In order to transform the spherical functions which appear in the perturbed terms of the kinetic energy and of the potential, a particular case of Wigner's theorem

TABLE VII

Comparison with NON-RIGID theories for nutation in OBLIQUITY and LONGITUDE of the FIGURE PLANE. Epoch J2000.0. Unit = 0.001 arc sec.

No.	Period (days)	Argument l	l'	F	D	Ω	$-\Delta I_f$ Capit.	Wahr	Authors	$-\Delta\lambda_f$ Capit.	Wahr	Authors
19	-31.8	1	0	0	-2	0		-0.1	-0.128		-15.8	-14.955
22	27.5	1	0	0	0	0	-0.7	-0.7	-0.667	67.7	71.2	67.750
30	14.8	0	0	0	2	0		-0.2	-0.111		6.3	6.038
34	9.5	-1	0	2	2	1		0.5	0.494		-1.0	-0.964
39	-27.4	-1	0	0	0	1		3.2	2.991		-5.8	-5.518
51	9.1	1	0	2	0	1		2.7	2.496		-5.1	-4.873
54	27.7	1	0	0	0	1		-3.3	-3.167		6.3	6.018
64	7.1	0	0	2	2	1		0.3	0.316		-0.7	-0.624
65	13.6	0	0	2	0	1	19.1	20.0	19.068	-36.9	-38.6	-36.737
68	-6798.4	0	0	0	0	1	9227.9	9202.5	9228.885	17282.7	-17199.6	-17282.870
72	9.6	-1	0	2	2	2		2.6	2.419		-5.9	-5.644
73	27.1	-1	0	2	0	2		-5.3	-5.082		12.3	11.771
82	6.9	2	0	2	0	2		1.3	1.248		-3.1	-2.924
86	23.9	1	0	2	-2	2		-1.2	-1.175		2.9	2.724
87	5.6	1	0	2	2	2		0.3	0.307		-0.8	-0.721
88	9.1	1	0	2	0	2	12.2	12.9	12.207	-28.6	-30.1	-28.496
97	7.1	0	0	2	2	2		1.6	1.553		-3.8	-3.636
99	13.7	0	0	2	0	2	92.9	97.7	92.918	-216.4	-227.4	-216.085
100	-3399.2	0	0	0	0	2	-90.1	-89.5	-90.350	207.9	206.2	209.047
102	365.3	0	1	0	0	0	-0.1	5.4	-0.094	125.5	142.6	125.835
103	365.2	0	-1	2	-2	2		-9.5	-9.253		21.7	21.349
105	121.8	0	1	2	-2	2	21.5	22.4	21.625	-49.7	-51.7	-49.926
106	182.6	0	0	2	-2	2	552.5	573.6	552.497	-1275.2	-1318.7	-1275.166

(1959), given by Kinoshita, will be used. The application of this theorem is detailed in Kinoshita (1977), and Getino (1989), obtaining the following results:

$$\left(\frac{a}{r}\right)^3 P_2(\sin\delta) = \frac{3}{2}(3\cos^2\sigma - 1)\sum_i B_i\cos\Theta_i -$$
$$-\frac{3}{2}\sin 2\sigma \sum_\tau\sum_i C_i(\tau)\cos(\mu - \tau\Theta_i) +$$
$$+\frac{3}{4}\sin^2\sigma \sum_\tau\sum_i D_i(\tau)\cos(2\mu - \tau\Theta_i)\,, \tag{A1.1}$$

$$\left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\cos\alpha = \frac{9}{4}\sin 2\sigma \sum_\tau \sum_i B_i \sin(\nu - \tau\Theta_i) +$$
$$+\frac{3}{2}\sum_\rho \rho(1+\rho\cos\sigma)(-1+2\rho\cos\sigma)\sum_\tau\sum_i C_i(\tau)\sin(\mu+\rho\nu-\tau\Theta_i) -$$
$$-\frac{3}{4}\sum_\rho \sin\sigma(1+\rho\cos\sigma)\sum_\tau\sum_i D_i(\tau)\sin(2\mu+\rho\nu-\tau\Theta_i)\,, \qquad \text{(A1.2)}$$

$$\left(\frac{a}{r}\right)^3 P_2^1(\sin\delta)\sin\alpha = \frac{9}{4}\sin 2\sigma \sum_\tau \sum_i B_i \cos(\nu - \tau\Theta_i) +$$
$$+\frac{3}{2}\sum_\rho (1+\rho\cos\sigma)(-1+2\rho\cos\sigma)\sum_\tau\sum_i C_i(\tau)\cos(\mu+\rho\nu-\tau\Theta_i) -$$
$$-\frac{3}{4}\sum_\rho \rho\sin\sigma(1+\rho\cos\sigma)\sum_\tau\sum_i D_i(\tau)\cos(2\mu+\rho\nu-\tau\Theta_i)\,, \qquad \text{(A1.3)}$$

$$\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\cos 2\alpha = -\frac{9}{2}\sin^2\sigma \sum_\tau \sum_i B_i \cos(2\nu - \tau\Theta_i) -$$
$$-3\sum_\rho \rho\sin\sigma(1+\rho\cos\sigma)\sum_\tau\sum_i C_i(\tau)\cos(\mu+2\rho\nu-\tau\Theta_i) -$$
$$-\frac{3}{4}\sum_\rho (1+\rho\cos\sigma)^2\sum_\tau\sum_i D_i(\tau)\cos(2\mu+2\rho\nu-\tau\Theta_i)\,, \qquad \text{(A1.4)}$$

$$\left(\frac{a}{r}\right)^3 P_2^2(\sin\delta)\sin 2\alpha = \frac{9}{2}\sin^2\sigma \sum_\tau \sum_i B_i \sin(2\nu - \tau\Theta_i) +$$
$$+3\sum_\rho \sin\sigma(1+\rho\cos\sigma)\sum_\tau\sum_i C_i(\tau)\sin(\mu+2\rho\nu-\tau\Theta_i) +$$
$$+\frac{3}{4}\sum_\rho \rho(1+\rho\cos\sigma)^2\sum_\tau\sum_i D_i(\tau)\sin(2\mu+2\rho\nu-\tau\Theta_i)\,, \qquad \text{(A1.5)}$$

where $\rho = \pm 1, \tau = \pm 1$, and the functions $B_i, C_i(\tau)$ and $D_i(\tau)$ are:

$$B_i = -\frac{1}{6}(3\cos^2 I - 1)A_i^{0)} - \frac{1}{2}\sin 2I\, A_i^{1)} - \frac{1}{4}\sin^2 I\, A_i^{2)}\,,$$

$$C_i(\tau) = -\frac{1}{4}\sin 2I\, A_i^{0)} + \frac{1}{2}(1+\tau\cos I)(-1+2\tau\cos I)A_i^{1)} + $$
$$+\frac{\tau}{4}\sin I\,(1+\tau\cos I)A_i^{2)}\,,$$

$$D_i(\tau) = -\frac{1}{2}\sin^2 I\, A_i^{0)} + \tau\sin I\,(1+\tau\cos I)A_i^{1)} - \frac{1}{4}(1+\tau\cos I)^2 A_i^{2)}\,. \qquad \text{(A1.6)}$$

The numerical values of the coefficients $A_i^{j)}$ were given in the said paper by Kinoshita. In our computation we have used the updated values by Kinoshita and Souchay (1990). As for the argument Θ_i, we have:

$$\begin{aligned}
\Theta_i &= m_1 l_{(} + m_2 l_{\odot} + m_3 F + m_4 D + m_5 \Omega\,, \\
&\quad \text{with } i = (m_1, m_2, m_3, m_4, m_5)\,, \\
F &= l_{(} + g_{(}\,, \\
D &= l_{(} + g_{(} + h_{(} - l_{\odot} - g_{\odot} - h_{\odot}\,, \\
\Omega &= h_{(} - \lambda\,,
\end{aligned}$$

where $l_{(}$, $g_{(}$, $h_{(}$ are the Delaunay variables for the Moon, and $l_{\odot}$, $g_{\odot}$, $h_{\odot}$ are the Delaunay variables for the Sun.

Appendix 2. Plane Perpendicular to the Rotation Axis

In this appendix we briefly describe the procedure followed to obtain the expressions of the longitude of the node, λ_r, and inclination I_r, of the plane perpendicular to the rotation axis of a deformable Earth. This method is an adaptation of that used by Kinoshita (1977) for a rigid Earth. Thus, using the same notation for the Euler angles, h_f, I_f and ϕ, we first of all introduce the matrices:

$$\dot{q} = \begin{pmatrix} \dot{h}_f \\ \dot{I}_f \\ \dot{\phi} \end{pmatrix}, \quad W = \begin{pmatrix} \sin I_f \sin\phi & \cos\phi & 0 \\ \sin I_f \cos\phi & -\sin\phi & 0 \\ \cos\phi & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix},$$

where G gives the angular moment. Then, through (2.7) we have:

$$G = \begin{pmatrix} M \sin\sigma \sin\nu \\ M \sin\sigma \cos\nu \\ M \cos\sigma \end{pmatrix} = \Pi W \dot{q} = (\Pi_0 + \widetilde{\Pi}_d) W \dot{q}\,,$$

where Π is the inertia tensor, which, as we have seen in section 2 is broken down into an unperturbed and a perturbed part, which we show as:

$$\Pi_0 = \begin{pmatrix} A_0 & 0 & 0 \\ 0 & B_0 & 0 \\ 0 & 0 & C_0 \end{pmatrix}, \qquad \widetilde{\Pi}_d = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{pmatrix},$$

d_{ij} being the components of the deformation. So, we can deduce:

$$\dot{q} = W^{-1}(\Pi_0 + \widetilde{\Pi}_d)^{-1} G\,.$$

Developing the inverse of the inertia tensor in power series (see (3.4)), we can break down the matrix $\dot{q}$ as follows:

$$\dot{q} = \dot{q}_0 + \delta\dot{q} \rightarrow \begin{cases} \dot{q}_0 = W^{-1}\Pi_0^{-1} G\,, \\ \delta\dot{q} = -W^{-1}\Pi_0^{-1}\widetilde{\Pi}_d\Pi_0^{-1} G\,, \end{cases}$$

$\dot{q}_0$ corresponding to the rigid Earth, and $\delta\dot{q}$ to the deformation. The first step in this process consists in determining $\dot{q}$ as a function of the Andoyer variables.

Using the above expression and performing the corresponding operations we obtain $\dot{q}$ as function of the Euler angles, q, and the components of the angular moment G, as follows:

$$\begin{aligned}
\dot{h}_{f0} &= \frac{1}{\sin I_f}\left(\frac{\sin\phi}{A_0}G_x + \frac{\cos\phi}{B_0}G_y\right),\\
\dot{I}_{f0} &= \frac{\cos\phi}{A_0}G_x - \frac{\sin\phi}{B_0}G_y,\\
\dot{\phi}_0 &= \frac{G_z}{C_0} - \cot I_f\left(\frac{\sin\phi}{A_0}G_x + \frac{\cos\phi}{B_0}G_y\right),
\end{aligned} \tag{A2.1}$$

$$\begin{aligned}
\delta\dot{h}_f &= -\frac{1}{\sin I_f}\left[\cos\phi\left(\frac{d_{12}}{A_0B_0}G_x + \frac{d_{22}}{B_0^2}G_y + \frac{d_{23}}{B_0C_0}G_z\right) + \right.\\
&\quad \left. + \sin\phi\left(\frac{d_{11}}{A_0^2}G_x + \frac{d_{12}}{A_0B_0}G_y + \frac{d_{13}}{A_0C_0}G_z\right)\right],\\
\delta\dot{I}_f &= -\cos\phi\left(\frac{d_{11}}{A_0^2}G_x + \frac{d_{12}}{A_0B_0}G_y + \frac{d_{13}}{A_0C_0}G_z\right) + \\
&\quad + \sin\phi\left(\frac{d_{12}}{A_0B_0}G_x + \frac{d_{22}}{B_0^2}G_y + \frac{d_{23}}{B_0C_0}G_z\right),\\
\delta\dot{\phi} &= -\left(\frac{d_{13}}{A_0C_0}G_x + \frac{d_{23}}{B_0C_0}G_y + \frac{d_{33}}{C_0^2}G_z\right) - \cos I_f\delta\dot{h}_f\,.
\end{aligned} \tag{A2.2}$$

Now, taking into account the relationships of the Euler angles with those of Andoyer (Kinoshita, 1977) up to the first order in σ:

$$I_f \simeq I + \sigma\cos\mu\,, \quad h_f \simeq \lambda + \sigma\frac{\sin\mu}{\sin I}\,, \quad \phi \simeq \mu + \nu - \sigma\cot I\sin\mu\,,$$

we will obtain first of all, neglecting the terms in $\sigma(1/A_0 - 1/B_0)$:

$$\dot{h}_{f0} \simeq \frac{1}{\sin I}MP_0\,, \quad \dot{I}_{f0} \simeq MQ_0\,, \quad \dot{\phi}_0 \simeq \frac{M}{C_0} - MP_0\cot I\,, \tag{A2.3}$$

where we have used the notation (Kinoshita, 1992):

$$P_0 = \frac{1}{2}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\sigma\cos\mu\,, \quad Q_0 = -\frac{1}{2}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)\sigma\sin\mu\,. \tag{A2.4}$$

Let us now calculate the increase of these angles. As we have said in sections 5 and 6 the main perturbation is due to the rotational deformation, so the tidal deformation will not be taken into account, since its order of magnitude is greater than σ. Then, starting from (A2.2), we can make $d_{ij} = r_{ij}$, and taken into account the relationships (3.13)–(3.16), we obtain

$$\delta \dot{h}_f \simeq \frac{1}{\sin I} MRP_1 \,, \; \delta \dot{I}_f \simeq MRQ_1 \,, \; \delta\dot{\phi} \simeq 2R\frac{M}{C_0} - MRP_1 \cot I \,, \qquad \text{(A2.5)}$$

with the notation:

$$\begin{aligned} R &= 2\frac{D_r}{C_0} \,, \\ P_1 &= -\frac{1}{2}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)[\sigma\cos\mu - 3\widetilde{\sigma}_r \cos(\mu + \nu - \widetilde{\nu}_r)] \,, \\ Q_1 &= \frac{1}{2}\left(\frac{1}{A_0} + \frac{1}{B_0}\right)[\sigma\sin\mu - 3\widetilde{\sigma}_r \sin(\mu + \nu - \widetilde{\nu}_r)] \,. \end{aligned} \qquad \text{(A2.6)}$$

To get these expressions above, as well as neglecting the terms in $A_0 - B_0$ and those in σ^2, we have chosen to take the approximations:

$$\sigma\frac{D_r}{A_0^2} \simeq \sigma\frac{D_r}{A_0 C_0} \,, \qquad \sigma\frac{D_r}{B_0^2} \simeq \sigma\frac{D_r}{B_0 C_0} \,,$$

since the resulting error is to the order of 10^{-13}, while, on the other hand, it notably simplifies the results.

Then, from (A2.3) and (A2.5) we can deduce:

$$\dot{h}_f \simeq \frac{1}{\sin I} MP \,, \; \dot{I}_f \simeq MQ \,, \; \dot{\phi} \simeq (1 + 2R)\frac{M}{C_0} - MP\cot I \,, \qquad \text{(A2.7)}$$

where now:

$$P = P_0 + RP_1 \,, \qquad Q = Q_0 + RQ_1 \,, \qquad \text{(A2.8)}$$

and in this way, we have $\dot{q}$ as a function of the Andoyer variables. Through these relations we shall find the desired expressions of λ_r and I_r. To do this let us start with the relationships (Kinoshita, 1977):

$$\begin{aligned} \dot{I}_f \cos h_f + \dot{\phi}\sin I_f \sin h_f &= \omega \sin I_r \sin h_r \,, \\ \dot{I}_f \sin h_f - \dot{\phi}\sin I_f \cos h_f &= -\omega \sin I_r \cos h_r \,, \\ \dot{h}_f + \dot{\phi}\cos I_f &= \omega \cos I_r \,. \end{aligned}$$

Introducing the expressions we have found for $\dot{q}$ and q into these relationships, and taking into account the fact that when ignoring the terms in σ^2:

$$\omega \simeq \frac{M}{C_0}(1 + 2R) \,,$$

after same calculations we obtain

$$\delta I_r = I_r - I \simeq \left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\sigma\cos\mu + +3D_r\left(\frac{1}{A_0} + \frac{1}{B_0}\right)[\sigma\cos\mu - \widetilde{\sigma}_r\cos(\mu + \nu - \widetilde{\nu}_r)]\,, \qquad \text{(A2.9)}$$

$$\delta\lambda_r = \lambda_r - \lambda \simeq \frac{1}{\sin I}\left[\left(1 - \frac{C_0}{2A_0} - \frac{C_0}{2B_0}\right)\sigma\sin\mu + +3D_r\left(\frac{1}{A_0} + \frac{1}{B_0}\right)[\sigma\sin\mu - \widetilde{\sigma}_r\sin(\mu + \nu - \widetilde{\nu}_r)]\right]\,, \qquad \text{(A2.10)}$$

which are the desired expressions.

Appendix 3. Polar Motion

Using an analogous procedure to that developed in Appendix 2, here we shall obtain the expressions of the polar motion, which is defined as the motion of the rotation axis relative to the figure axis (Kinoshita, 1977):

$$x_p = \frac{\omega_x}{\omega}\,, \qquad y_p = -\frac{\omega_y}{\omega}\,.$$

From $G = \Pi\omega$, and developing Π^{-1} as previously described, we have:

$$\omega = \omega_0 + \delta\omega \rightarrow \begin{cases} \omega_0 &= \Pi_0^{-1}G\,, \\ \delta\omega &= -\Pi_0^{-1}\widetilde{\Pi}_d\Pi_0^{-1}G\,, \end{cases}$$

which results in, neglecting terms in σ^2:

$$\omega_{x0} \simeq \frac{M}{A_0}\sigma\sin\nu\,, \quad \omega_{y0} \simeq \frac{M}{B_0}\sigma\cos\nu\,, \quad \omega_{z0} \simeq \frac{M}{C_0}\,, \qquad \text{(A3.1)}$$

while their increases, with the same simplifications as in the previous appendix, will be:

$$\begin{aligned} \delta\omega_x &\simeq \frac{M}{A_0}R(-\sigma\sin\nu + 3\widetilde{\sigma}_r\sin\widetilde{\nu}_r)\,, \\ \delta\omega_y &\simeq \frac{M}{B_0}R(-\sigma\cos\nu + 3\widetilde{\sigma}_r\cos\widetilde{\nu}_r)\,, \\ \delta\omega_z &\simeq \frac{M}{C_0}2R\,, \end{aligned} \qquad \text{(A3.2)}$$

with $R = 2D_r/C_0$. Given that, with the level of accuracy within which we are working, $\omega \simeq M(1 + 2R)/C_0$, from (A3.1) and (A3.2) we finally obtain:

$$x_p \simeq \frac{C_0}{A_0}\left[\left(1 - \frac{6\,D_r}{C_0}\right)\sigma\sin\nu + \frac{6\,D_r}{C_0}\widetilde{\sigma}_r\sin\widetilde{\nu}_r)\right]\,, \qquad \text{(A3.3)}$$

$$y_p \simeq -\frac{C_0}{B_0}\left[\left(1 - \frac{6\,D_r}{C_0}\right)\sigma\cos\nu + \frac{6\,D_r}{C_0}\widetilde{\sigma}_r\cos\widetilde{\nu}_r)\right]\,, \qquad \text{(A3.4)}$$

which are the desired expressions.

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