RELATIVISTIC EFFECTS IN THE CRITICAL INCLINATION PROBLEM IN ARTIFICIAL SATELLITE THEORY

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(Received 22 February, 1991; accepted 10 July, 1991)

Abstract. It is well known that in artifical satellite theory special techniques must be employed to construct a formal solution whenever the orbital inclination is sufficiently close to the critical value $\cos^{-1}(1/\sqrt{5})$. In this article the authors investigate the consequences of introducing certain relativistic effects into the motion of a satellite about an oblate primary. Particular attention is paid to the critical inclination(s), and for such critical motions an appropriate method of solution is formulated.

Key words: Artificial satellite, relativity, critical inclination.

1. Introduction

In an untypical article Brumberg and Kovalevsky (1986) set out to define and classify the unsolved problems in Celestical Mechanics. One of the *mathematical* problems they identified concerns the relativistic effects on the motion of artificial or natural satellites about an oblate primary, not necessarily Earth. In particular, they drew attention to the possible development of power series solutions for cases involving "relativistic resonance". This problem they classified as Type BV; that is an important mathematical problem of (now) age less than 20 years.

In two recent papers, Soffel *et al.* (1988) and Heimberger *et al.* (1990), the relativistic effects on the motion of a satellite about an oblate body were investigated. Using Gauss's form of the planetary equations Soffel et al. determined the secular, the short-periodic and the long-periodic relativistic perturbations in the Keplerian elements to the first order in J_2, the usual oblateness parameter. In the later paper the authors treated essentially the same problem using canonical Lie series methods. In addition, an important second-order "mixed" perturbation, due to the Newtonian quadrupole field and the Schwarzschild acceleration, was included in the analysis. Neither of the aforementioned papers, however, addressed the phenomenon of resonance associated witht he *Critical Inclination Problem* in artificial satellite theory. The purpose of the current contribution is to investigate how the relativistic effects can be suitably incorporated into the resonance theory when the orbital inclination of the satellite is at or close to a critical value.

The Critical Inclination Problem is a well-known and extensively researched problem in the theory of artificial satellite motion. The review article (Jupp, 1988) documents the discovery, analysis and controversial development of this subject. In its simplest form the Critical Inclination Problem is an example of the more general *Ideal Resonance Problem,* first identified by Garfinkel (1966). Detailed investigations of the Ideal Resonance Problem, principally by Garfinkel and Jupp, have led to formal series solutions. A feature of these formal solutions is that the series are developed in terms of the square root of a small parameter, a device used first by Bohlin, and subsequently described by Poincaré (1893). The essential features of the Ideal Resonance Problem and its solutions may be gleaned from Jupp and Abdulla (1984, 1985) and Jupp (1987), and the other relevant papers referenced therein.

In the current article the aforementioned knowledge of the Critical Inclination Problem and Ideal Resonance Problem is applied to the relativistic motion of a satellite about a general oblate body. It is assumed here that the oblateness effect is at least as significant as the relativistic effect. Consequently solutions are constructed in powers of $J_2^{1/2}$. If the relativistic terms were to dominate, expansions in powers of c^{-1} might be more appropriate.

2. The Construction of the Long-Period Hamiltonian

Since the phenomenon of resonance is here associated with the long-period behaviour of the satellite's motion, the construction of the required long-period Hamiltonian will be described only in outline form; more details are given in Heimberger et al. Adopting their notation, the Lagrangian of a satellite moving about a central body, in the Einstein form of the Post Newtonian gravitational field is,

$$L = \frac{1}{2}\, v^2 + U + \frac{1}{c^2}\left[\frac{1}{8}\, v^4 - \frac{1}{2}\, U^2 + \frac{3}{2}\, U v^2\right], \tag{1}$$

where v is the satellite's velocity and U its gravitational potential energy. Retaining only the principal (Keplerian) and the second zonal harmonic (J_2) terms in U the corresponding Hamiltonian may be written.

$$\begin{aligned} H =& -\frac{\mu^2}{2L^2} + J_2\,\frac{\mu^4 R^2}{L^6}\left(\frac{a}{r}\right)^3\left[\frac{1}{4} - \frac{3}{4}\,\frac{H^2}{G^2} - \frac{3}{4}\left(1 - \frac{H^2}{G^2}\right)\cos 2(f+g)\right] + \\ & - \frac{\mu^4}{c^2 L^4}\left[3\left(\frac{a}{r}\right)^2 - 2\,\frac{a}{r} + \frac{1}{8}\right] + \\ & + \frac{J_2 \mu^6 R^2}{4c^2 L^8}\left[1 - 3\,\frac{H^2}{G^2} - 3\left(1 - \frac{H^2}{G^2}\right)\cos 2(f+g)\right] \times \\ & \times \left[2\left(\frac{a}{r}\right)^4 - \frac{3}{2}\left(\frac{a}{r}\right)^3\right], \end{aligned} \tag{2}$$

in which $\big(L, G, H, (l), g, (h)\big)$ are the usual Delaunay variables, μ is the product of the universal constant of graviation and the central body's mass and R is the mean equatorial radius of the central body. Further, r is the radial distance and f the true anomaly of the satellite, a is the semi-major axis of the instantaneous elliptic orbit and c is, as usual, the velocity of light. Of course, both r and f are dependent upon L, G and l through the standard relations of Keplerian motion.

If the terms factored by c^{-2} in (2) are omitted there remains what is usually called the *main problem* in artificial satellite theory. This is the problem which was so elegantly solved by Brouwer (1959). Borrowing from earlier ideas of von Zeipel he tackled this problem in three stages. First, the short-period terms (i.e. those dependent upon l) were "removed" using a canonical transformation of variables. The remaining long-period problem is then governed by a Hamiltonian depending upon L', H', G' and g' only, where the primes indicate the transformed variables. This single-degree-of-freedom system was then further simplified using a second canonical transformation of variables to "remove" the long-period terms (i.e. those dependent upon g'). Finally, the remaining secular problem, being independent of all angle variables, was very simply solved.

Heimberger *et al.* used essentially the same approach in their analysis of the relativistic problem associated with (2). However, instead of employing what is generally known as the von Zeipel method they chose instead to use a method based on Lie series, as expounded by Hori (1966) and Deprit (1969). The removal of the short-period terms, as outlined above, leads to the long-period Hamiltonian.

$$H' = H'_0 + H'_1 + H'_2 + \dots , \tag{3}$$

with

$$H'_0 = -\frac{\mu^2}{2L^2} , \tag{4}$$

$$H'_1 = \frac{\mu^4 J_2 R^2}{4L^3G^3}\left(1 - 3\frac{H^2}{G^2}\right) - \frac{3\mu^4}{c^2L^4}\left(\frac{L}{G} - \frac{5}{8}\right) , \tag{5}$$

$$\begin{aligned} H'_2 = & -3\frac{\mu^6 J_2^2 R^4}{32L^{10}}\left[\frac{5}{4}\frac{L^5}{G^5}\left(1 - \frac{18}{5}\frac{H^2}{G^2} + \frac{H^4}{G^4}\right) + \frac{L^6}{G^6}\left(1 - 3\frac{H^2}{G^2}\right)^2 + \right. \\ & - \frac{5}{4}\frac{L}{G}\left(1 - 2\frac{H^2}{G^2} - 7\frac{H^4}{G^4}\right) - \frac{1}{2}\frac{L^7}{G^7}\left(\frac{G^2}{L^2} - 1\right) \times \\ & \left. \times \left(1 - 16\frac{H^2}{G^2} + 15\frac{H^4}{G^4}\right)\cos 2g\right] + \\ & + \frac{\mu^6 J_2 R^2}{8c^2L^3G^5}\left[\left(1 - 3\frac{H^2}{G^2}\right)\left(36 + 18\frac{G}{L} - 29\frac{G^2}{L^2}\right) + \right. \\ & \left. - 9\left(1 - \frac{H^2}{G^2}\right)\left(1 - \frac{G^2}{L^2}\right)\cos 2g\right] . \end{aligned} \tag{6}$$

For convenience and simplification of presentation the primes on the Delaunay variables in (4)–(6) have been dropped. The expressions for H'_0 and H'_1 have been taken directly from Heimberger et al. The second-order term has been assembled as follows:

$$H_2' = -F_2^* + \frac{1}{2}\left(\langle H_2\rangle + \langle \Gamma_2\rangle\right). \tag{7}$$

Here F_2^* is the non-relativistic term, taken from Brouwer (1959), with $k_2 = J_2R^2/2$, while $\frac{1}{2}\left(\langle H_2\rangle + \langle \Gamma_2\rangle\right)$ is the relativistic term assembled from the formulas given by Heimberger et al. The Delaunay variables L' and H' (written here as L and H), incorporating the short-period perturbations, are clearly constants. The reduced problem has just one degree of freedom and, as such, may readily be written in the form of the Ideal Resonance Problem.

3. A Résumé of the Ideal Resonance Problem

In its standard form the Ideal Resonance Problem (I.R.P.) is characterised by the Hamiltonian F, satisfying the equations

$$\begin{gathered} -F(x,y) = B(y) + 2\varepsilon^2 A(y)\sin^2 x\,, \\ \frac{dx}{dt} = \frac{\partial F}{\partial y}\,, \quad \frac{dy}{dt} = -\frac{\partial F}{\partial x}\,. \end{gathered} \tag{8}$$

The resonance manifests itself through the function $B(y)$, which is such that its first derivative $\partial B/\partial y$ vanishes for some value of y, say y_0. Further, it is assumed that ε is small and that in the neighbourhood of y_0 the product $A\partial^2 B/\partial y^2 = AB''$ is of the same order of magnitude as B. Lastly it is necessary that AB'' is positive within the domain of resonance. If, however, this condition is not fulfilled a simple change of variables is sufficient to recover the standard form of the I.R.P.

This is not the place to describe the various methods which have been devised to construct formal series solutions of this system. The interested reader should initially consult Garfinkel (1966), Jupp (1969) and Jupp and Abdulla (1984). It is sufficient here to enumerate some of the chief features of the solutions.

(i) The series solutions are expansions in powers of ε.

(ii) The solutions involve Jacobi elliptic functions and integrals.

(iii) The phase plane associated with system (8) comprises regions of libration and circulation, partitioned by separatrices – analogous to the simple pendulum phase plane. Indeed, the I.R.P. describes a perturbed simple pendulum.

Of the two solutions, i.e. those of Garfinkel and of Jupp, it has been established in the last cited paper and its sequel that the solution due to Jupp is appreciably the more accurate in deep resonance. The essential differences between the two solutions are described in the 1984 article. Further, it should be noted that the singularity at the separatrix, alluded to at the bottom of page 412 in that article has subsequently successfully been removed (Jupp, 1987). In virtue of the preceding statements the solution due to Jupp will be utilised in the current work. Accordingly, the first-order *libration* solution, as given by Equations (13)–(15) in the 1984 paper, is

$$x = \sin^{-1}(ksnw) + \frac{1}{3}\,\varepsilon p_0 \tilde{b}_3\, kcnw\, \sin^{-1}(ksnw)\,, \tag{9}$$

$$y = y_0 + \varepsilon\, p_0\, kcnw + - \frac{1}{6}\,\varepsilon^2 p_0^2 k \left[k(\tilde{b}_3 cn^2 w + \tilde{a}_1 sn^2 w) + 2\tilde{b}_3 snw\, dnw\, \sin^{-1}(ksnw)\right], \tag{10}$$

$$w = w_{00} - \tilde{\omega}_0 t\,. \tag{11}$$

To avoid confusion with earlier notation the small parameter ε here replaces μ of the previously published papers. The definitions of the constant coefficients p_0, $\tilde{b}_3$, $\tilde{a}_1$ and $\tilde{\omega}_0$ are to be found in the last cited paper. In these expressions the Jacobi elliptic functions have modulus k and the constants k and ω_{00} are determined from the initial conditions.

The non-singular *circulation* solution is displayed in Equations (13)–(15) in the 1987 paper; for reasons of economy of space this solution is not reproduced here. Then, by letting $k \to 1$ in the libration solution or $\kappa \to 1$ in the circulation solution, the first-order *separatrix* solution is obtained; the result is given by Equations (17)–(19) in the 1987 paper.

It can be seen that in each of these first-order solutions the expression for y, the momentum variable, includes terms in ε^2. In fact this bonus is an important and useful feature of Jupp's method; indeed *all* the second-order terms in ε^2 in the expressions for y are given by each of the three first-order solutions. Similarly, the second-order solutions contain all the terms in ε^3 in the formulas for y.

It was stated earlier that it is necessary that the product AB'' is of the same order of magnitude as B. Should this not be the case then the problem is classified as "abnormal", and alternative methods of solutions must be used. An example of such a problem arises in the *main problem* of artificial satellite theory when the orbital eccentricity is small and, at the same time, the orbital inclination is close to the critical value $\cos^{-1}(1/\sqrt{5})$.

4. Application to the Relativistic Satellite Problem

A comparison of the long-period Hamiltonian, given by Equations (3)–(6), with the Ideal Resonance Problem, defined by Equations (8), suggests the following identifications:

$$x = g\,, \quad y = G\,, \tag{12}$$

$$B(G) = -\frac{\mu^4 R^2}{4L^3G^3}\left(1 - 3\,\frac{H^2}{G^2}\right) + \frac{3\mu^4}{c^2L^4J_2}\left(\frac{L}{G} - \frac{5}{8}\right) + \text{smaller terms}\,, \tag{13}$$

$$\varepsilon^2 A(G) = -\frac{3\mu^6 R^2}{64L^3G^7}\left(1-\frac{G^2}{L^2}\right)\left(1-\frac{H^2}{G^2}\right)\left[J_2R^2\left(1-15\frac{H^2}{G^2}\right)+24\frac{G^2}{c^2}\right]. \tag{14}$$

Without any loss of generality the leading term in H' has been omitted in the expression for B since L is a constant of the motion. [It is perhaps timely to recall that while L and H are constants in (13) and (14), they do incorporate the short-period perturbations.] As a further convenience the Hamiltonian H' has been factored by J_2^{-1} before the identifications were made – this is essentially equivalent to a linear scaling of time.

Resonance is associated with the vanishing of the first partial derivative of B with respect to the momentum variable G. Accordingly, the differentiation of Equation (13) leads to

$$B' = \frac{\partial B}{\partial G} = 3\frac{\mu^4 R^2}{4L^3G^4}\left(1-5\frac{H^2}{G^2}-4\beta\frac{G^2}{H^2}\right)+O(J_2)\,, \tag{15}$$

in which β is the positive non-dimensional relativistic parameter defined by

$$\beta = H^2/R^2c^2J_2\,. \tag{16}$$

Neglecting, for the time being, the terms of $O(J_2)$, it is easily seen that in the non-relativistic case, with $\beta = 0$, B' vanishes when $H^2/G^2 = 1/5$; this corresponds to the orbital inclination $i = \cos^{-1}(1/\sqrt{5})$, called the *critical inclination.* For satellites whose orbital inclinations are at or close to this critical value, the "classical" theory of Brouwer is inapplicable and other theories must be adopted; for example, the theory of the I.R.P.

In virtue of Equation (15), in the relativistic case, the zeroes of the leading term in B' corresponds to

$$H^2/G^2 = \cos^2 i_0 = [1 \pm \sqrt{(1-80\beta)}]/10\,. \tag{17}$$

Clearly the roots are real provided $\beta \leq 1/80$. For $0 \leq \beta < 1/80$ two critical values of the inclination exist for each value of β, while for $\beta = 1/80$ the single critical value is $i_0 = \cos^{-1}(1\sqrt{10}) \simeq 71.6°$. Since B' is singular for $i = \pi/2$ and $\beta > 0$, polar orbits need to be treated as a special case of the relativistic problem; this will not be attempted here. The effect of the omitted terms of order J_2 is to shift by a small measure the values of the exact resonance from those given by (17). The small shifts from these critical values depend upon the size of the orbit, as governed by the major axis, the orbital eccentricity and also the relativistic parameter β.

It was stated earlier that the Ideal Resonance theory is not applicable when AB'' is other than the same order of magnitude as B. On differentiation with respect to G, Equation (15) yields

$$B'' = \frac{3\mu^2R^2}{2L^3G^5}\left(-2+15\cos^2 i+4\beta\sec^2 i\right)+\ldots \tag{18}$$

and, setting $\varepsilon = J_2^{1/2}$, (14) may be re-written

$$A = -3\,\frac{\mu^6 R^2 e^2 \sin^2 i}{64 L^3 G^7}\,(1 - 15\cos^2 i + 24\beta \sec^2 i)\,. \tag{19}$$

The libration centre corresponds to $G = G_0$, $g = 0$ where $B'(G_0) = 0$. There $i = i_0 + O(J_2)$, as given by (17), and $e = e_0 = \sqrt{(1 - G_0^2/L^2)}$. Accordingly, evaluating B'' and A at $G = G_0$ there result

$$B_0'' = \frac{\pm 3\mu^4 R^2}{2L^3 G_0^5}\sqrt{(1-80\beta)} + \ldots\,, \tag{20}$$

$$A_0 = -\,\frac{3\mu^6 R^4 e_0^2 \sin^2 i_0}{128 L^3 G_0^7}\left(5 \mp 9\sqrt{(1-80\beta)}\right), \tag{21}$$

where the upper and lower signs correspond with the upper and lower signs respectively in (17).

It follows that the relativistic problem is an "abnormal" case of the I.R.P. if either $\beta \simeq 1/80$ or $\beta \simeq 7/810$ or $e_0 \simeq 0$. On the contrary, if none of the above conditions applies then the relativistic Critical Inclination Problem may be classified as an I.R.P. Moreover, the product $A_0 B_0''$ is positive, as required, provided $0 < \beta < 7/810$. If, however, $7/180 > \beta > 1/80$ then $A_0 B_0''$ is negative. In this case replacing g by $\hat{g} + \pi/2$ readily converts the problem into I.R.P. standard form.

As has already been stated, the I.R.P. is a perturbed simple pendulum, whose phase plane incorporates regions of libration and circulation partitioned by separatrices. Garfinkel (1966) introduced the practical distinction between *deep* and *shallow* resonance. In cases of deep resonance the standard "classical" theories, e.g. Brouwer (1959), are inappropriate and special theories, e.g. the I.R.P. theory, must be employed. Deep resonance in the present application includes all the librationary motions, the separatrices and the circulatory motions close to the separatrices. The remainder of the circulatory motions make up the shallow resonance region, in which the classical theories are adequate to describe the motion with sufficient accuracy.

It has been shown in the relativistic problem that if $0 < \beta < 1/80$ then *two* critical values of the inclination exist for each value of β. Theoretical difficulties could be encountered if the two regions of deep resonance, associated with each of the critical values, are such that "overlapping" occurs. In the absence of overlapping, however, each critical inclination may be treated as a separate and distinct I.R.P. problem. Garfinkel (1966) showed that in deep resonance the inclination is always within 1° of the critical value, so the scope for overlapping is quite small.

The period of libration, T, is determined from the formulas (Jupp, 1969)

$$T = 4K/\varepsilon p_0 B_0'' \tag{22}$$

$$p_0^2 = 4A_0/B_0''\,. \tag{23}$$

In virtue of (20) and (21) it is easily seen that

$$p_0 = \frac{\mu \mathrm{Re}_0 \sin i_0}{4G_0} |9 \mp 5(1 - 80\beta)^{-1/2}|^{1/2} . \qquad (24)$$

The period of libration is minimised at the libration centre, where $K = \pi/2$. Accordingly, substitution yields

$$T = \frac{16\pi\sqrt{5}}{3J_2^{3/2} n} \left(\frac{a}{R}\right)^3 \frac{(1 - e_0^2)^3}{e_0} \frac{1}{D(\beta)} , \qquad (25)$$

with

$$D(\beta) = |43(1 - 80\beta) \mp 9(3 - 40\beta)\sqrt{(1 - 80\beta)}|^{1/2} , \qquad (26)$$

and in which the mean motion n is defined by $n^2 a^3 = \mu$.

In the non-relativistic case $\beta = 0$, $D(0) = 4$ and Equation (25) agrees with Garfinkel's calculations.

The period is minimised for $a(1 - e_0) = R$ and $e_0 = 0.5$, so that

$$T > 36 \frac{\pi\sqrt{5}}{J_2^{3/2} n D(\beta)} . \qquad (27)$$

5. Conclusions

The principal results established in the earlier sections are here summarised. The nature of the problem, and its subsequent solution, depends upon the value of the relativistic parameter β defined by (16) as follows:

(i) $\beta = 0$: The non-relativistic critical inclination case, which may be solved using a direct application of the I.R.P.

(ii) $\beta \simeq 0$: In this case it is likely that the omitted higher-value zonal harmonic terms, e.g. those in $J_2^2, J_3, J_4, \ldots$, are as significant as the relativistic term.

(iii) $0 < \beta < 7/810, 7/810 < \beta < 1/80$: Two critical values of the inclination occur for each value of β. The theory of the I.R.P. may be used, unless there is an overlapping of the resonances.

(iv) $\beta \simeq 7/810$, $\beta \simeq 1/80$: Two *abnormal* cases, for which the standard I.R.P. theory does not apply.

(v) $\beta > 1/80$: In this case no critical inclination exists. However, for values of β close to $1/80$, B' may be sufficiently small to require an I.R.P. treatment. On the contrary, when β is such that B' is not small (i.e. $O(J_2)^{1/2}$) the problem is no longer one of deep resonance; in these circumstances a classical theory (e.g. Brouwer, 1959) suffices.

(vi) $\beta \gg 1$: The relativistic terms dominate the oblateness terms so that expansions in powers of c^{-1}, rather than the more usual c^{-2}, would be appropriate.

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