CONFORMAL GEOMETRY OF THE KEPLER ORBIT SPACE

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Abstract. We present here a group theoretical analysis of the structure of the space Ω of orbits in the classical (plane) Kepler problem, and relate it to the description of the Kepler orbits as curves in configuration and in velocity spaces. A Minkowskian parametrization in Ω is introduced which allows us a clear description of many aspects of this problem. In particular, this parametrization suggests us the introduction in Ω of a Lorentzian metric, whose conformal group $SO(3,2)$ contains a seven-dimensional subgroup which is induced by point transformations in the configuration space $\mathcal{X}$. A $SO(2,1)$ subgroup of this group still acts transitively on $\mathcal{X}$, which is thus identified as a homogeneous space for $SO(2,1)$; each regular Kepler orbit is the trace of a one-dimensional subgroup whose canonical parameter automatically equals to the classical anomalies. These results are somehow a configuration space analogous of the geometrical structure of the Kepler problem in the velocity space previously known.

Key words: Two body problem, conformal geometry.

1. Introduction

It is known since Hamilton's work [1] that the hodographs of the orbits of Kepler problem, i.e., the graphs of the velocity vector $d\mathbf{x}/dt$, are either circles, for orbits of negative energy, or portion of circles, otherwise. Györgyi [2], Moser [3], Osipov [4], Belbruno [5], and Milnor [7] gave a very interesting geometric interpretation of this fact. The set $\mathcal{V}_E$ of velocity vectors $\mathbf{v}$ such that $\mathbf{v}.\mathbf{v} > 2E$, for a fixed energy E, can be endowed with a Riemannian structure given by the length element $dl^2_{v,E} = 4/[(\mathbf{v}^2 - 2E)^2]\, d\mathbf{v}^2$. This metric has constant curvature $K = -2E$, and its geodesics are the hodographs of the Kepler problem for this energy. Then, for elliptic orbits ($E < 0$) the manifold has a spherical geometry, in the parabolic case (of null energy) $\mathcal{V}_0$ is an Euclidean plane, and for hyperbolic orbits ($E > 0$) the manifold $\mathcal{V}_E$ is a hyperbolic plane. In either case, the arc-length along the geodesics equals to the classical eccentric (resp., parabolic, hyperbolic) anomalies. It is also remarkable that these anomalies are proportional to the "fictitious time"

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$s = \int dt/\ |\mathbf{x}(t)\ |$, called Levi-Civita parameter, which plays an important role in the regularization problem for singular orbits [3].

Once the energy E has been fixed, we can associate a geodesic in the manifold $\mathcal{V}_E$ with each Kepler orbit of energy E. In this way the space Ω_E of the Kepler orbits of energy E is identified with the space of the geodesics of $\mathcal{V}_E$. The group of the corresponding geometry (either $SO(3)$, $E(2)$ or $SO(2,1)$ respectively) acts in a natural way on this space. It is to be remarked that these groups act on the velocity space as point transformations and they are not dynamical symmetries in the usual sense. This association allows us to understand the connection between the Kepler motion and the geodesic flow on spaces of a constant curvature, a relation that has been very often used.

However, there is no similar picture for the Kepler orbits in the configuration space $\mathcal{X}$, and the only obvious point transformations in $\mathcal{X}$ mapping Kepler orbits into Kepler orbits are Euclidean rotations and homotheties; only the first subgroup maps orbits of energy E into orbits of energy E.

In this work we introduce two groups $\mathcal{G}_c$, $\mathcal{G}_v$, of local point transformations in the space Ω of all Kepler orbits, both isomorphic to the conformal group of the $(2+1)$ dimensional Minkowski space. Each one of these groups has subgroups acting as point transformations in either the configuration space and the velocity space. Starting from $\mathcal{G}_v$ and restricting to the submanifold of orbits with constant energy E we recover all the previously known results on the $SO(3)/E(2)/SO(2,1)$ homogeneous space structure for the sets $\mathcal{V}_E$ or Ω_E. New results are obtained from $\mathcal{G}_c$ which restore, even in this classical context, a remarkable symmetry between configuration and velocity spaces.

In Section 2 we present the Kepler problem, in order to unify notation and to introduce some concepts.

Section 3 is devoted to describe the Kepler problem in velocity space. We briefly review some details of Milnor's description, where some very nice properties of the geometry of the Kepler problem appear. As the hodographs are circles, we consider there the space of all circles and derive from them the metrics in Ω_E.

Next, Section 4 studies the space of all plane Kepler orbits. A particular parametrization, called Minkowskian, is introduced and used thoroughly to exhibit in a clear descriptive way the structure of this space. All collision orbits fit naturally into this description as "points at infinity" and the remaining singular orbits complete the scheme.

Once a metric in the space of all orbits has been defined, we study in Section 5 the groups of isometries and conformal transformations $\mathcal{C}_c$ of this metric. This conformal group includes a seven dimensional subgroup of transformations in the orbit space that are induced by point transformations in the configuration space, some of which are not evident at a first sight. This subgroup is isomorphic to the $2+1$ Poincaré group extended with dilations. By analyzing this construction, the configuration space $\mathcal{X}$ for the Kepler problem appears as a homogeneous space for $SO(2,1)$, obtained by taking the quotient by a one-dimensional parabolic subgroup.

The action of $SO(2,1)$ on $\mathcal{X}$ is transitive but not primitive, and therefore there is neither any invariant Riemannian metric in $\mathcal{X}$ nor a unique invariant connection. Instead, there is a whole one-parameter family of invariant connections $\Gamma^{(C)}$, and Kepler orbits are the auto-paralell lines of this family of connections. For each Kepler orbit, there is a one-parameter subgroup of point transformations in the configuration space $\mathcal{X}$ mapping the orbit into itself, and the canonical parameter for this one-parameter subgroup equals to the anomaly (eccentric / parabolic / hyperbolic) along the orbit. This has some interest, as it shows some group theoretical meaning of these parameters, long known as the "best" parameters in the Kepler problem and in connection with the regularization for collision orbits [3]. As far as we know, these groups of point transformations on the Kepler configuration space are new and they provide a configuration space counterpart for the groups arising in the Moser-Osipov-Belbruno-Milnor (MOBM) approach in velocity space. We complete this study by an independent computation of all possible sprays [8] (second order differential equation vector fields) in $\mathcal{X}$, invariant under $SO(2,1)$, with the result that only for the family corresponding to parabolic orbits the connection comes from a metric in configuration space.

In Section 6 we introduce another different construction, which leads to a new group $\mathcal{C}_v$. In the same way as some subgroups of $\mathcal{C}_c$ are related to transformation groups in configuration space, here some subgroups of $\mathcal{C}_v$ are related to point transformation groups in velocity space; in particular the groups in the MOBM approach are recovered when one restricts to the subgroups of $\mathcal{C}_v$ mapping the submanifolds of constant energy onto themselves.

Finally, some remarks and comments, as well as an outlook for future work is presented in Section 7.

2. The Kepler Problem in Configuration Space

The motion of a point of mass m in a conservative field with potential

$V(r) = -(k/r)$ is called the Kepler problem. Here k is a positive constant, and with no loss of generality we shall only consider the case $m = 1$. The motion takes place in a plane because the force is central. It is well known that the problem can be split into a geometrical part (determination of the Kepler orbits) and a non-geometrical one (time evolution along the orbit). The orbits of this problem are generically non-degenerate conics with a focus at the centre of attraction. The general equation of non-degenerate conics in polar coordinates (r, θ), with the origin at the centre of atraction is:

$$r = \frac{p}{1 + e\cos(\theta - \phi)} = \frac{p}{1 + e_x \cos\theta + e_y \sin\theta}\,, \tag{2.1}$$

where $p > 0$ is the semi-latus rectum, $\mathbf{e} = (e_x, e_y)$ and $e = |\mathbf{e}| \in [0, \infty)$ is the eccentricity of the conic.

As it is well known, if $0 \leq e < 1$ the conic is an ellipse reducing to a circle when $e = 0$, for $e = 1$ the conic is a parabola, and when $e > 1$ the conic is a hyperbola (see Figure 1a). The angle ϕ corresponds to the direction of the point on the conic closest to the origin (periastron). For $e \neq 1$, the semiaxis lengths are given by

$$a = \frac{|\,p\,|}{|\,1-e^2\,|}, \qquad b = \frac{|\,p\,|}{\sqrt{|\,1-e^2\,|}}\,. \tag{2.2}$$

When the conic is an ellipse or a parabola, the Kepler orbit coincides with the conic, but when it is a hyperbola, it is only one of the two branches of the hyperbola, (the branch nearest to the origin) which we shall refer to as the Kepler branch. From the mechanical point of view, each Kepler orbit has some associated constants of the motion as the energy E, the angular momentum L (here scalar, so that positive as well negative values are allowed) and the Laplace-Runge-Lenz vector $\boldsymbol{R}$ [9, 10], which is a vector pointing out from the origin of coordinates to the periastron. The (vector) angular momentum $\boldsymbol{L}$ is orthogonal to the plane of the motion. For every non degenerate conic there are two possible senses of motion along the conic, and hence two different Kepler orbits with opposite values of L, related with p by $L^2 = kp$.

A practical way to describe at the conic level the fact that there are two Kepler orbits for each non-degenerate conic is to allow for negative values of p. So the conic (2.1) (as a point set in configuration space) is either described by the triplet (p, e, ϕ) (with $p > 0$) or by $(-p, e, \phi + \pi)$. As to the associated Kepler orbit, when the angular momentum is positive we shall match it to the triplet (p, e, ϕ) with $p > 0$, and when angular momentum is negative we shall match it to the triplet $(-p, e, \phi + \pi)$. From now on we shall allow negative values for p by following the preceding convention.

The energy E is related with the geometrical parameters of the orbit, eccentricity and semi-latus rectum by

$$\begin{aligned} E &= -\frac{k}{2a} = -\frac{k(1-e^2)}{2\,|\,p\,|} && \text{for} \quad 0 \le e < 1 \\ E &= 0 && \text{for} \quad e = 1 \\ E &= \frac{k}{2a} = \frac{k(e^2-1)}{2\,|\,p\,|} && \text{for} \quad e > 1, \end{aligned} \tag{2.3}$$

where a is the focal semi-axis of the ellipse or the hyperbola respectively. Sometimes a different convention amounting to have positive a for elliptical orbits and negative a for hyperbolic orbits has been used [11]. Kepler orbits with the same energy have also the same value of the focal semi-axis. Formulas (2.3) and the expression for the angular momentum can be written in a more compact form

$$E = \frac{k(e^2-1)}{2\,|\,p\,|}, \qquad L = \operatorname{sign}(p)\sqrt{k|\,p\,|}. \tag{2.4}$$

The Laplace-Runge-Lenz vector $\boldsymbol{R}$ lies in the plane of the orbit, points to the periastron, and its length is ke, namely $\boldsymbol{R} = \text{sign}\,(p)\,(ke\cos\phi,\ ke\sin\phi)$. It does not depend on the sense of motion along the orbit. The four constants E, L, R_x, R_y are not functionally independent but they are related by

$$\boldsymbol{R}^2 = \left(1 + \frac{2EL^2}{k^2}\right) k^2 = k^2 + 2EL^2, \tag{2.5}$$

and therefore $2EL^2 = -k^2(1-e^2)$. Notice that the Kepler problem is the classical example of the so-called super-integrable systems [12].

The energy can take values in the real line $\mathbb{R}$. Once a value E of the energy is fixed, the ranks of the (scalar) angular momentum L and of the modulus R of $\boldsymbol{R}$ are given in the following table:

	$0 \le e < 1$ $(E < 0)$	$e = 1$ $(E = 0)$	$e > 1$ $(E > 0)$
$\boldsymbol{L}$	$[-\frac{k}{\sqrt{2\lvert E\rvert}}, \frac{k}{\sqrt{2\lvert E\rvert}}]$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$\boldsymbol{R}$	$[0, +k)$	k	$(k, +\infty)$

Note that $\boldsymbol{R}= 0$ if and only if the orbit is circular ($e = 0$).

Another constant of motion which will appear later is

$$C = \frac{2E}{L^2}. \tag{2.6}$$

From the former expressions the relationship of C with the geometrical parameters of the orbit is, for $E \neq 0$,

$$C = \frac{e^2-1}{p^2} = \text{sign}\,(e-1)\frac{1}{b^2}. \tag{2.7}$$

Hence the orbits of the family with constant value of C are ellipses with the same value of the non-focal semiaxis for $C < 0$, parabolas for $C = 0$, and hyperbolas with the same value of the non-focal semiaxis for $C > 0$.

From now on, Ω will denote the set of all (plane) Kepler orbits and $\Omega^{\pm}$ the subsets of all regular orbits, $\Omega^{\pm} = \Omega^{+} \cup \Omega^{-}$, the sign in the superscripts meaning the positive or negative orientation of the orbit, $L > 0$ or $L < 0$ respectively. They are three-dimensional manifolds. A subindex like E in Ω_E will always mean the submanifold of all orbits of energy E, with the obvious meaning for symbols like Ω_E^{+}.

In addition to all the regular orbits (characterized by $L \neq 0$), there are non-regular orbits, obtained as limits of regular orbits when some of their geometrical and/or physical parameters tend to particular values. From the point of view of configuration space, there are two kind of non-regular orbits: the collision orbits ($L = 0$ and finite E), and a set of more singular orbits, characterized by values of E and/or L equal to $\pm\infty$, whose consideration is required for reasons of mathematical convenience, although they are of course highly non-physical. We warn the reader that usually only regular and collision orbits are considered at all.

Let us first describe the collision orbits. In order to allow a continuous variation of Kepler orbits of fixed energy E as $L \to 0$ it is natural to adopt the so called Reflection Convention [13, 7]: *If an orbit has $L = 0$ ($\mathbf{x}$ and $d\mathbf{x}/dt$ are linearly dependent for some initial time t_0), the particle falls towards the origin and when it hits this point, $\mathbf{x}(t)$ is reflected back along the same half-line.* The reflection convention allows us to discuss the problem of identification between half-lines through the origin as (degenerate) conics and collision Kepler orbits. Fix E to some value (say $E < 0$) and let $L \to 0$ from positive values; the Kepler ellipse of fixed semiaxis a flattens out and tends towards a straight line segment with lenght $2a$ and one end at the origin (collision "elliptic" orbits). The same construction holds for $E = 0$ and $E > 0$ collision orbits, where parabolas or hyperbolas flatten out and tend towards a semi-infinite half-line with one end at origin (collision parabolic and hyperbolic orbits). Hence each half-line through the origin corresponds to an infinite family, parametrized by E, of collision orbits. For $E \neq 0$ these orbits are limits of orbits of fixed a when $b \to 0$.

The remaining singular orbits fall into the following classes:

1. Circular orbit at the origin. This is a single orbit, which arises, for instance, as the limit of circular orbits when $a = b$ tend towards 0. It has energy $E = -\infty$, and angular momentum $L = 0$; as a curve in configuration space it reduces to a single point at the origin.
2. Circular orbits at infinity. These are two single orbits, limits of circular orbits when $a = b$ tends towards ∞. For these orbits $E = 0$ and $L = \pm\infty$; they correspond to a circle of infinite radius.
3. Hyperbolic "angle" orbits. When a and b tend towards 0 with fixed b/a, the Kepler branch of hyperbolic orbits tends to two half-asymptotes, making an angle with vertex at the origin. Formally such conics have a finite $e > 1$, but $E = \infty$ and $L = 0$. There is one such orbit for each oriented angle with vertex at the origin (a half-asymptote marked as *in* and the other as *out*); note that this includes the case where the half-asymptotes are colinear.
4. Hyperbolic "straight" orbits. When $b \to \infty$, but $ae - a$ is constant, the Kepler branch of hyperbolas tends towards a straight line not passing through the origin but at a finite distance of it. Formally these orbits have $e = \infty$, $E = \infty$ and $L = \pm\infty$.

From the point of view of configuration space, these are the relevant classes of singular orbits; note that we have considered the circular orbit at the origin as a single orbit, though formally we could have considered as being different all the limits of ellipses when $a \to 0$, $b \to 0$ while b/a is fixed. In a similar way, we could consider other formal limits, in particular of parabolic and hyperbolic orbits "at

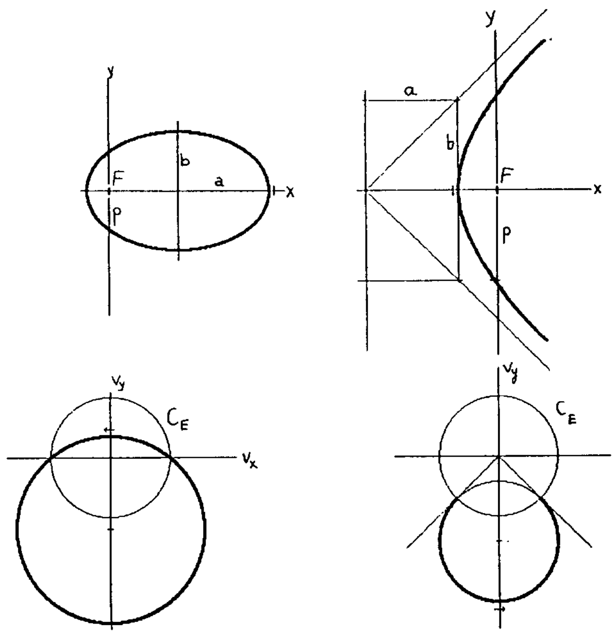

Fig. 1. a) Geometrical elements of elliptical and hyperbolical Kepler orbits in configuration space $\mathcal{X}$. b) The hodographs for the same orbits in the velocity space $\mathcal{V}$.

infinity" in configuration space in addition to the single "circular orbit at infinity". A more explicit and detailed description of the structure of all regular, collision and singular orbits is in progress and will appear in a forthcoming paper.

3. The Kepler Problem in Velocity Space

As explained in the Introduction, the hodographs of regular motions in the Kepler's problem, i.e., graphs of the velocity vector $d\mathbf{x}/dt$, are either circles (for negative energy) or arcs of circles (for zero or positive energy). These circles degenerate into straight lines through the origin in velocity space for collision orbits (see [7] for a detailed description). Before entering into the nice geometrical structure hidden behind these velocity circles we shall give an explicit description. Let us start with regular orbits. As t varies, the velocity vector $d\mathbf{x}/dt$, moves along a circle, which together with its orientation, determines a Kepler orbit in a unique way (Figure 1b).

Notice that two orbits differing only in the sense of motion have associated the same velocity circle, traversed with different orientations. Consider first regular orbits with $L > 0$, and let us introduce cartesian coordinates (v_x, v_y) in the velocity plane, so that $v_x^2 + v_y^2 + 2gv_x + 2fv_y + c = 0$ is the equation of a circle centred at the point $(-g, -f)$ and with a radius $r = \sqrt{g^2 + f^2 - c}$. Let us remark that in order to the parameters f, g and c represent a circle they must satisfy $g^2 + f^2 - c > 0$ and therefore circles are represented by points under the paraboloid $c = g^2 + f^2$ in the space of triplets (g, f, c) (see e.g. [14]). In this open set, (g, f, c) gives a coordinate system for the space of circles of a given orientation, which are in a one-to-one correspondence with the set of regular orbits of a given sign of L. Using simple trigonometry the angle α between two circles is given by

$$\cos\alpha = \frac{1}{2r_1r_2}[2g_1g_2 + 2f_1f_2 - c_1 - c_2], \tag{3.1}$$

and therefore the angle $d\alpha$ between two neighbouring circles can be written as

$$d\alpha^2 = \frac{1}{r^4}\Big[(f^2 - c)dg^2 + (g^2 - c)df^2 - \frac{1}{4}dc^2 + g\,dg\,dc + f\,df\,dc - 2gf\,df\,dg\Big]. \tag{3.2}$$

The angle between two neighbouring circles with the same value of the parameter c is

$$d\alpha^2 = \frac{1}{r^4}\left[(f^2 - c)dg^2 + (g^2 - c)df^2 - 2gf\,df\,dg\right]. \tag{3.3}$$

The ratio $\frac{\sqrt{g^2+f^2}}{r}$ will be called the eccentricity of the circle with respect to the origin; the reason for that denomination is, of course, that it equals to the value of the eccentricity of the corresponding Kepler orbit in configuration space.

The relation between (g, f, c) and the parameters of the Kepler orbit in configuration space and with the constants of motion is

$$(g, f) = \text{sign}\,(p)\sqrt{\frac{k}{|\,p\,|}}(e\sin\phi, -e\cos\phi) = \frac{\boldsymbol{R}^*}{|\,L\,|}$$

$$c = \frac{k(e^2-1)}{|\,p\,|} = 2E$$

$$r = \sqrt{k}|\,p\,| = \frac{k}{|\,L\,|}, \tag{3.4}$$

where $\boldsymbol{R}^*$ is the dual vector (in the plane) to $\boldsymbol{R}$, $\boldsymbol{R}^* = (R_y, -R_x)$.

Conversely, for the orbits associated to the velocity circle (g, f, c), the values of the energy, angular momentum, Laplace-Runge-Lenz vector and the constant C in (2.6) are:

$$E = \frac{c}{2}$$

$$L = \frac{k}{\sqrt{g^2 + f^2 - c}}$$

$$R = \frac{k}{\sqrt{g^2 + f^2 - c}}(-f, g)$$

$$C = \frac{c(g^2 + f^2 - c)}{k^2}. \tag{3.5}$$

Let us denote by C_E the circle of radius $\sqrt{2 \mid E \mid}$ and centered at the origin. It is clear from (3.4) and (3.5) that for $E < 0$ the velocity circles intersect C_E in two diametrically opposite points, they pass through the origin for $E = 0$, and they are orthogonal to the circle C_E for $E > 0$. Notice that for positive energy the centre of the velocity circle lie outside of the circle C_E, and that the possible velocity vectors satisfy $v^2 > 2E$, so the hodograph point moves along the part of the circle external to C_E.

There is a completely similar description for regular orbits with $L < 0$, so there is a one-to-one correspondence between regular Kepler orbits and oriented circles in velocity plane $\mathcal{V}$.

Hodographs of collision orbits are straight lines passing through the origin in the velocity space. They fit neatly in the former scheme: for each value of energy E there are two (oriented) straight lines through the origin that automatically satisfy the corresponding requirement (cutting the circle C_E in antipodal points for $E < 0$, passing through the origin for $E = 0$ and cutting orthogonally the circle C_E for $E > 0$). Moser [3] gave a picture of the space of all elliptic orbits of a fixed energy. The idea for such a construction is implicit in the previous description, as for a fixed negative energy (say E), a stereographic projection of a 2-sphere of radius $\sqrt{2 \mid E \mid}$ placed in 3-dimensional Euclidean space into its equatorial plane (identified to the velocity plane), carries great circles of the 2-sphere precisely into velocity circles (or straight lines through the origin) associated with Kepler orbits for energy E.

Following Osipov [4], and Belbruno [5, 6], Milnor [7] pointed out that an essentially similar construction could be performed for parabolic and hyperbolic orbits of fixed energy. For hyperbolic orbits with energy E, the inverted velocity vectors $\mathbf{w} = (\mathbf{v}/\mathbf{v}^2)$, varies over the interior of the circle $|\mathbf{w}| < \frac{1}{\sqrt{2E}}$ and in terms of $\mathbf{w}$ the hodographs of Kepler problem are arcs of circles orthogonal to the circle $|\mathbf{w}| = \frac{1}{\sqrt{2E}}$ and interior to it. The Lobachevski plane can be realized as the (space-like) 2-pseudosphere of "radius" $\frac{1}{\sqrt{2E}}$ placed in a Minkowskian three-dimensional space. A stereographic projection into the two-dimensional Euclidean plane passing through the origin (again identified to the inverted velocity plane) carries the geodesics of the 2-pseudosphere (Lobachevski plane) precisely into velocity circles (or straight lines through the origin) associated with Kepler orbits with energy E. For $E = 0$, the space of inverted velocity vectors has the structure of an Euclidean plane, because circles through the origin are carried onto straight lines (geodesics of the Euclidean plane) by the inversion $\mathbf{w} = (\mathbf{v}/\mathbf{v}^2)$.

For these three cases, the metrics have the standard Riemmanian form

$$dl^2_{v,E} = \frac{\mathrm{w}^2}{(1+(-2E)\mathrm{w}^2)^2}$$

and in terms of coordinates v,

$$dl^2_{v,E} = \frac{4\,dv^2}{(v^2-2E)^2}\,.$$

The space Ω_E of all Kepler's orbits of energy E can therefore be identified with the set of oriented geodesics of the manifold $\mathcal{V}_E$; both spaces are dual in the geometric sense [15, 16]. The groups of isometries of the metric $dl^2_{v,E}$ ($SO(3), E(2), SO(2,1)$ respectively) maps geodesics into geodesics and therefore acts also on Ω_E. If the distance between two geodesics (as points in Ω_E) is defined as the angle between the geodesics (as oriented lines in $\mathcal{V}_E$), this distance comes from a metric, and the group ($SO(3), E(2), SO(2,1)$) also acts in Ω_E as isometries. As $dl^2_{v,E}$ is conformal to the euclidean metric, the angle between geodesics computed in $\mathcal{V}_E$ coincides with the euclidean angle (3.3). In terms of a new coordinate system (e,ϕ) in Ω_E^+ (Sec. 2.1), we obtain for the "dual" metrics the following results:

i) $E<0$. The metric induced on Ω_E, i.e., dual of $dl^2_{v,E}$ is described in each "half" $\Omega_E^\pm$ by

$$ds^2|_{\Omega_E^\pm} = \frac{1}{1-e^2}de^2 + e^2 d\phi^2, \qquad 0<e<1. \tag{3.6}$$

This metric is definite positive and has positive constant curvature. We will call this metric cospheric because the manifold Ω_E is dual to $\mathcal{V}_E$, whose geometry is spherical for $E<0$. Note that these coordinates cover only Ω_E^+, but the same expresions also hold when the coordinates (e,ϕ) for the set of orbits of energy E with opposite orientation (as given in 2.1) are used.

ii) $E>0$. The induced metric is now the cohyperbolic one:

$$ds^2|_{\Omega_E^\pm} = \frac{1}{e^2-1}de^2 - e^2 d\phi^2, \qquad 1<e. \tag{3.7}$$

This metric has signature $(+,-)$ and positive constant curvature. In this case the manifolds Ω_E are dual to $\mathcal{V}_E$, whose geometry is hyperbolic.

4. The Space of All Kepler Orbits

Let us first consider the regular orbits ($L \neq 0$) as curves in configuration space. These orbits lie on non-degenerate conics, whose equation can be written in the form

$$\frac{1}{r} = Z + X\cos\theta + Y\sin\theta. \tag{4.1}$$

where $Z = 1/p$, $(X, Y) = (e_x/p, e_y/p)$. Note that $p \neq 0$, but with the conventions given in Section 2, p can have negative values. Henceforth each regular orbit can be parametrized by a point (Z, X, Y) in $\mathbb{R}^3$. The admisible range of Z is $(-\infty, 0) \cup (0, \infty)$ while the ranges of X and Y are $(-\infty, \infty)$ so the domain of regular orbits is $\mathbb{R}^3 - \{ Z = 0 \}$; each half-space $Z > 0$ or $Z < 0$ corresponds to orbits in Ω^+ or Ω^- respectively. For reasons to be discussed later we shall call Minkowskian such parametrization. It is easy to obtain the expressions for the constants of motion in terms of this parametrization

$$E = -\frac{k(Z^2 - X^2 - Y^2)}{2 \mid Z \mid}$$

$$L = \text{sign}\,(Z)\sqrt{\frac{k}{\mid Z \mid}}$$

$$\boldsymbol{R} = k\text{sign}\,(Z)\left(\frac{X}{Z}, \frac{Y}{Z}\right). \tag{4.2}$$

The couple of orbits with different orientations corresponding to the same conic are characterized by (Z, X, Y) and $(-Z, X, Y)$, and thus the change of direction of motion along the conic is represented by the reflection in the plane $Z = 0$ in the model. This Minkowskian parametrization allows us to display in a simple way many interesting geometric properties of the space of orbits. The first question is: given a point (Z, X, Y), $(Z \neq 0)$ in Ω, how can we find its Kepler orbit as a curve in configuration space?. Let us identify the configuration space $\mathcal{X}$ with cartesian coordinates (x, y) with the plane $z = 0$ of an auxiliar three dimensional space $\mathbb{R}^3(z, x, y)$. Writing the equation (4.1) in the form

$$1 = Zr + Xx + Yy \tag{4.3}$$

(with $x = r\cos\theta, y = r\sin\theta$ and therefore $x^2 + y^2 = r^2$), (4.3) appears as the solution of the system of equations

$$Zz + Xx + Yy = 1$$

$$z^2 = x^2 + y^2 \tag{4.4}$$

and hence the curve corresponding to (4.3) is the projection on the Oxy plane, along the Oz axis, of the intersection of the plane (4.4a) with the cone (4.4b). Notice that the plane (4.4a) is the polar plane (with respect to the quadric $z^2+x^2+y^2 = 1$) of the point of coordinates (Z, X, Y) in the auxiliar space. Hence we obtain the following simple geometrical construction relating the point (Z, X, Y) in $\Omega^{\pm}$ $(L \neq 0)$ with the corresponding Kepler orbit:

Let us consider in $\mathbb{R}^3(z, x, y)$ *the plane with characteristic vector* (Z, X, Y) *through the point of coordinates* $\frac{1}{Z^2+X^2+Y^2}(Z, X, Y)$. *Its intersection with the*

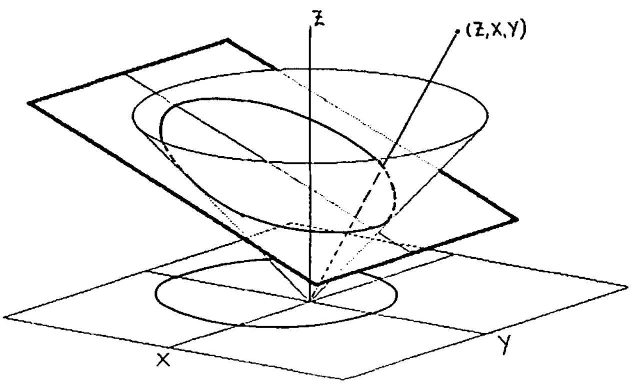

Fig. 2. Relation between the Kepler orbit as a conic in configuration space and as a point (Z, X, Y) in the space of orbits Ω. The plane is the polar of the point (Z, X, Y); its intersection with the cone, projected on the OXY plane gives the Kepler orbit.

cone $z^2 - x^2 - y^2 = 0$ *is a conic whose* (x, y) *projection is the configuration space Kepler orbit* (Z, X, Y) (Figure 2).

Indeed, it can be checked by direct calculation that the curves obtained by following this procedure are conics in the plane (x, y) with focus at the origin, and that they exhaust all regular Kepler orbits.

The converse is the following: From the Kepler orbit (the configuration space being identified with the plane (x, y) as before), lift it to the cone $z^2 - x^2 - y^2 = 0$. This gives a curve which is contained in some plane whose polar point (respect to the cuadric $z^2 + x^2 + y^2 = 1$) is on the line through the origin and normal (in euclidean sense) to the plane; if d is the distance from the plane to the origin, the polar point is at a distance $d' = 1/d$. The coordinates (z, x, y) of this polar point are the Minkowskian coordinates (Z, X, Y) of the Kepler orbit.

It follows clearly from this construction (and also from (4.2a)) that points in the interior of the cone correspond to elliptical orbits, while those on the cone describe parabolas and exterior points represent hyperbolic orbits. Notice that either the polar plane associated to the point (Z, X, Y) has no intersection with the opposite sheet of the cone (corresponding to the "opposite" sign of Z), or the intersection with the opposite sheet of the cone corresponds to the non-Keplerian branch of the hyperbola. On the other side, if a Kepler orbit has coordinates (Z, X, Y), with $Z > 0$, the point $(-Z, X, Y)$ corresponds to the Kepler orbit on the same conic but with opposite orientation, so the polar planes of points (Z, X, Y) and $(-Z, X, Y)$ are related by a reflection in the plane $Z = 0$. After that discussion, the name of Minkowskian parametrization appears more natural.

In terms of this Minkowskian parametrization the submanifolds Ω_E^ϵ of regular orbits with constant energy E are characterized according to (4.2), by the equations:

$$\left(Z + \epsilon\frac{E}{k}\right)^2 - X^2 - Y^2 = \frac{E^2}{k^2}, \qquad \text{sign}\,(Z) = \epsilon. \tag{4.5}$$

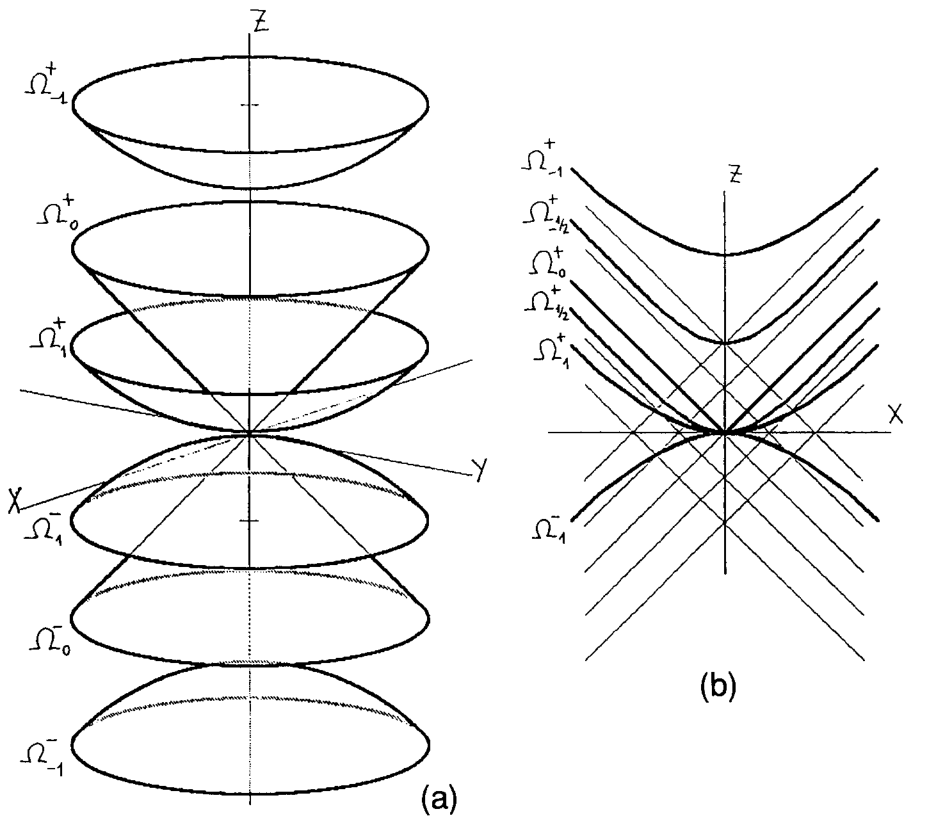

Fig. 3. (a) The submanifolds $\Omega_E^{\pm}$ of constant energy in the orbit space. (b) A section $Y = 0$ showing how the submanifolds $\Omega_E^{\pm}$ depend on E. All energies are measured in k units.

The first equation involves ϵ and E only through its product, and for $E \neq 0$, this equation is that of a two-sheet hyperboloid centred at the point with coordinates $(-\frac{\epsilon E}{k}, 0, 0)$, and vertices at $(-2\frac{\epsilon E}{k}, 0, 0)$ and $(0,0,0)$. Each sheet of this hyperboloid is completely contained in a half-space $Z \leq 0$ or $Z \geq 0$, so taking into account the second equation, the upper sheet of the hyperboloid corresponds to Ω_E^{+}, while the lower one corresponds to Ω^{-}_{E}. For $E = 0$, these hyperboloids reduce to a cone with vertex at $(0,0,0)$, whose sheets will be called Ω_0^{+} and Ω_0^{-}. This structure is depicted in Figure 3.

We can make use of this construction in order to give a geometric characterization of the submanifolds $\Omega_E^{+}, \Omega_E^{-}$. Orbits with fixed E correspond to points (Z, X, Y) satisfying (4.5). By using the relation between points (Z, X, Y) and Kepler orbits as curves in the configuration space, it is easy to translate (4.5) into a condition on the planes (polar planes of points (Z, X, Y)) whose intersection with the cone $Z^2 - X^2 - Y^2 = 0$ gives the Kepler orbit through OZ projection on the OXY plane. The result is that these polar planes are always tangent to a paraboloid inscribed in this cone. The easiest way to proof this is to note that rotational invariance allows simplification by considering only points with $Y = 0$. Let us first consider $E < 0$, so for $Z > 0$ (orbits in Ω_E^+), equation (4.5) reads:

$$\left(Z - \frac{|E|}{k}\right)^2 - X^2 - Y^2 = \left(\frac{|E|}{k}\right)^2. \tag{4.6}$$

The intersection of (4.6) with the plane $Y = 0$ is a hyperbola

$$Z = \frac{|E|}{k}(1 + \cosh\chi) \qquad X = \frac{|E|}{k}\sinh\chi, \tag{4.7}$$

and the intersection with $Y = 0$ of the corresponding polar planes are given by the equations

$$\frac{|E|}{k}\left[Z(1 + \cosh\chi) + X\sinh\chi\right] = 1. \tag{4.8}$$

The envelope of such a family is obtained by eliminating the parameter χ between the preceding equation and the one obtained by derivation of it with respect to the parameter:

$$(Z\sinh\chi + X\cosh\chi) = 0.$$

From this the parametric equation of the envelope is:

$$Z = \frac{k\cosh\chi}{|E|(1 + \cosh\chi)} \qquad X = -\frac{k\sinh\chi}{|E|(1 + \cosh\chi)}. \tag{4.9}$$

By elimination of χ and restoring Y, we obtain that all the planes polar to points in Ω_E^+ are tangent to

$$Z = \frac{|E|}{2k}(X^2 + Y^2) + \frac{k}{2|E|}. \tag{4.10}$$

This is a paraboloid inscribed in the cone $Z^2 = X^2 + Y^2$, and the point of tangency varies over the portion of the paraboloid with $\sqrt{X^2 + Y^2} < \frac{k}{|E|}$ (see Figure 4). For $E > 0$ and $L > 0$, the same procedure leads to the paraboloid

$$Z = -\tfrac{E}{2k}(X^2 + Y^2) - \tfrac{k}{2E}, \tag{4.11}$$

but now $\sqrt{X^2 + Y^2} > \frac{k}{|E|}$. Orbits with $L < 0$ give the portions of paraboloids obtained from the preceding one by a reflection in the OXY plane, and finally, when $E = 0$ the orbits correspond to intersections of the cone with planes orthogonal to its generatrices. These results appear without further comments and ascribed to A. B. Givental′ in a book by Arnold [17]. The same kind of construction also leads to descriptions for the set of all orbits with fixed angular momentum, orbits that pass through a point, etc., some of which are also reported in [17].

For instance, the set of orbits of fixed angular momentum L is a plane parallel to the OXY-plane, $Z = \text{sign}\,(L)(k/L^2)$. The submanifolds of orbits with constant

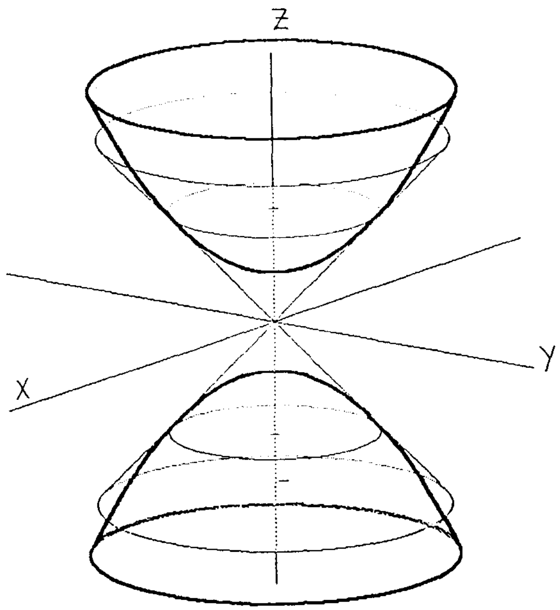

Fig. 4. Planes tangent to the paraboloids (4.10) and (4.11) give, by intersection with the cone and projection on the OXY plane all Kepler orbits of energies E and $-E$. In particular, planes tangent to the paraboloid along the circle of contact with the cone give the collision orbits.

value of $C = 2E/L^2$, to be denoted by Θ_C, are described by the equation $Z^2 - X^2 - Y^2 = -C$, and therefore they coincide for $C \neq 0$, with the family of one and two-sheet hyperboloids centred at $(0, 0, 0)$, and with the cone Ω_0 for $C = 0$.

Let us now consider the set of all Kepler orbits passing through a point P in the configuration space, say with polar coordinates (r_0, θ_0). With the parametrization (4.1), Z, X, Y must satisfy either of the two equations

$$\frac{1}{r_0} = Z + X\cos\theta_0 + Y\sin\theta_0, \tag{4.12a}$$

$$\frac{1}{-r_0} = Z - X\cos\theta_0 - Y\sin\theta_0, \tag{4.12b}$$

which are the equations of two planes in the (Z, X, Y) space, to be called respectively $\Pi_P{}^+$, and $\Pi_P{}^-$; both planes are related by a reflection in the (X, Y) plane and their intersection γ_P is a straight line in the plane $Z = 0$, which is the polar of the point P (relative to $X^2 + Y^2 = 1$), with the configuration space identified to the plane (X, Y)). It is easy to see by direct calculation that the points in $\Pi_P{}^+$ with $Z < 0$ (resp. in $\Pi_P{}^-$ with $Z > 0$) correspond to hyperbolas whose Keplerian branch does not pass through P (while the other branch pass through P), and we obtain (Figure 5a):

The Kepler orbits passing through a point P in configuration space lie on a wedge made up from two half-planes whose intersection is a straigth line γ_P.

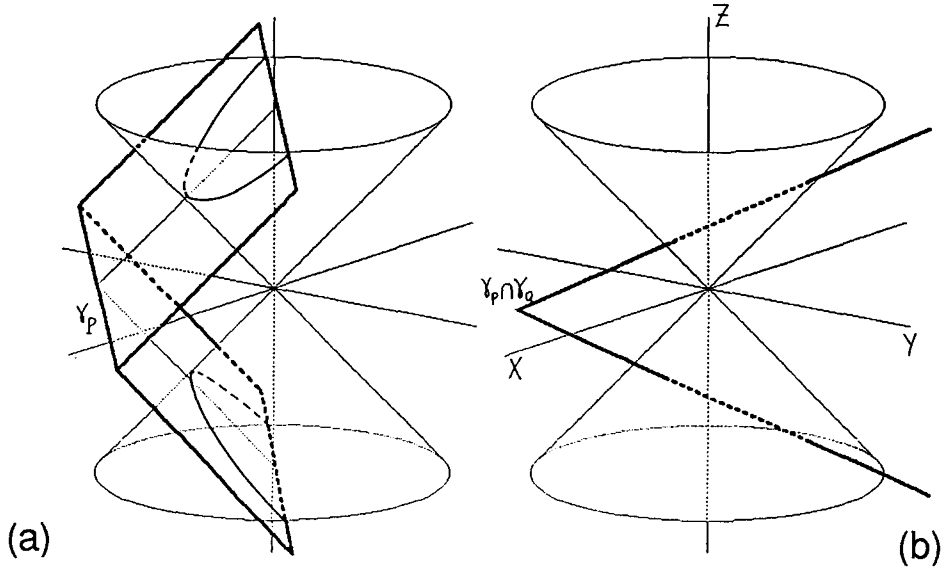

Fig. 5. (a) The "wedge" of all Kepler orbits through a fixed point P in configuration space. Each half-plane contains all orbits of a given orientation, and the intersection of each half-plane with the cone contains the parabolic orbits, which separate the elliptic orbits (in the interior of the cone) from the hyperbolic orbits (outside the cone). (b) All Kepler orbits through two fixed proper points in configuration space. Each half-line moves from the hyperbolic straight orbit $\gamma_P \cup \gamma_Q$, through hyperbolic orbits, to a parabolic orbit (in the intersection with the cone), then to elliptical orbits, and crossing again another parabolic orbit, to hyperbolic orbits towards the hyperbolical angle orbit which corresponds to the point "at infinity" along the line.

The Kepler orbits passing through two points P and Q in the configuration space lie on the intersection of the two wedges of P and of Q; the intersection consists of a "vertical" angle, made with two half lines meeting in a single point $\gamma_P \cap \gamma_Q$ *on the plane* $Z = 0$ (see Figure 5b).

It is easy to see that not every (affine) half-plane appears as associated to a point in configuration space; in order to the plane $AZ + BX + CY = D$ represents such a set of orbits its coefficients have to be related by $A^2 - B^2 - C^2 = 0$. And not every half straight line in the orbit space Ω corresponds a family of orbits through two points P, Q in configuration space. In fact, let us consider two orbits (Z_1, X_1, Y_1) and (Z_2, X_2, Y_2). The straight line determined by them is given by

$$\frac{Z - Z_1}{Z_2 - Z_1} = \frac{Y - Y_1}{Y_2 - Y_1} = \frac{X - X_1}{X_2 - X_1},$$

or in terms of the one-parameter family of planes defining the line (for $Y_2 - Y_1 \neq 0$)

$$\lambda[(Y_2 - Y_1)(X - X_1) - (X_2 - X_1)(Y - Y_1)]+$$

$$+\mu[(Y_2 - Y_1)(Z - Z_1) - (Z_2 - Z_1)(Y - Y_1)] = 0.$$

The points of intersections of orbits (Z_1, X_1, Y_1) and (Z_2, X_2, Y_2) will correspond to planes in this family satisfying the above-mentioned relation that in our case turns out to be

$$\lambda^2(Y_2 - Y_1)^2 + [\lambda(X_2 - X_1) + \mu(Z_2 - Z_1)]^2 = \mu^2(Y_2 - Y_1)^2.$$

For $\mu = 1$, the discriminant of this binomial expression in λ is

$$D = 4(Y_2 - Y_1)^2[(X_2 - X_1)^2 + (Y_2 - Y_1)^2 - (Z_2 - Z_1)^2] \tag{4.13}$$

and thus,

1. If $D > 0$ there exist two intersection points.
2. If $D = 0$ the orbits are tangent.
3. If $D < 0$ there is no intersection of the two orbits.

It turns out to be extremely useful to endow the space (Z, X, Y) with a flat Lorentzian metric, given by

$$do^2 = dZ^2 - dX^2 - dY^2 \tag{4.14}$$

When rephrased in terms of do^2, all the former results become very clear. Elliptic orbits are interior to the "light-cone" of the origin, parabolic orbits lie on the light-cone and hyperbolic orbits are in the exterior of the cone. The submanifolds of constant C are Minkowski spheres centered at the origin, the submanifolds of constant L are space-like planes orthogonal to the Z axis, and the submanifolds of constant E and $L < 0$ or $L > 0$ are the sheets of the Minkowski (space-like) spheres centered at points on the Z axis. The condition $A^2 - B^2 - C^2 = 0$ on the coefficients of the plane representing orbits through some fixed point P in configuration space means that the plane is isotropic, whereas the straight (half) lines corresponding to orbits through two points P and Q are always space-like lines that meet the ligth-cone of the origin.

The meaning of the Lorentzian character of the metric is that, according to the classification in (4.13), two near Kepler orbits can fall into three different categories:

1. they intersect at some point P,
2. they are tangent or
3. they do not intersect at all.

These three alternatives correspond respectively to $do^2 < 0$, $do^2 = 0$, $do^2 > 0$. Hence a particle (a space-craft) moving in a Kepler potential will stay in a fixed Kepler orbit until the engines are powered, and then the family of its instantaneous osculating Kepler orbits will describe a curve in the orbit space which is generically space-like, being only null for the case of an impulse along the instantaneous direction of motion; this curve cannot be time-like.

Collision Kepler orbits ($L = 0$) and singular orbits fit neatly into the scheme of this "Minkowski space of Kepler orbits" and provide a concrete realization of a compactification of this space (compare [18]). Let us first remark that collision and singular orbits are characterized either by $L = 0$ or by infinite values of the energy and/or angular momentum. Therefore, one could expect that in the (Z, X, Y) space they are either "at infinity" (as $L = 0$ implies $Z = \pm\infty$), or in the plane $Z = 0$, where the two half-spaces of regular orbits meet.

Regular orbits with energy E lies on (4.5), and if we let $L \to 0$ (say from positive values), while keeping fixed the parameter ϕ which controls the direction of the periastron of the orbit, the representative point in the space (Z, X, Y) goes to the infinity (with $Z > 0$) along one branch of a "vertical" hyperbola contained in (4.5). It is only natural to associate the limit orbit to the point "at infinity" along the asymptote, which is an isotropic line. We have a different asymptote for each value of the energy, so we obtain a whole $\mathbb{R}$ of points "at infinity" along isotropic directions with fixed ϕ, and all them correspond to collision Kepler orbits. We could also have obtained the same collision orbits as limits of orbits with fixed energy and $L < 0$, as $L \to 0$ from negative values which gives another "cylinder at infinity along isotropic lines" on the $Z < 0$ half of isotropic lines. Of course, these two cylinders must be identified via a reflection in the XY plane, because on the ground of the Reflection Convention, there is only one Kepler orbit for each E and on each half-line through the origin in $\mathcal{X}$. Collision orbits are in a one-to-one correspondence with these points "at infinity along isotropic lines" in the space (Z, X, Y) (Figure 6).

The remaining singular orbits are unphysical, but from the mathematical point of view fit very comfortably in the same scheme. For instance, the circular orbit at the origin in $\mathcal{X}$ correspond to the "point at infinity" along time-like lines; there are two such points (limits $Z \to \infty$ and $Z \to -\infty$, which must however be identified) so we get a single "point at infinity" on time-like lines through the origin. Similarly, each hyperbolic angle orbit is the single "point at infinity" along a space-like line through the origin. Notice that for all these orbits, $Z = \pm\infty$.

The remaining singular orbits are the hyperbolic straight orbits. As they have $L = \pm\infty$, we could expect that there $Z = 0$, and indeed for $Z = 0$, (4.1) is the equation of a straight line. Thus, the plane $Z = 0$ that separates the two half-spaces of regular orbits contains the hyperbolic straight orbits; equation (4.2) is even formally correct for it gives the infinite values for E and L characteristic of these orbits. There is a detail worth of notice here: each point $(0, X, Y)$ must be associated to two Kepler orbits, differing by the direction along its straight line, so really we must imagine the "plane" $Z = 0$ as two disconnected copies of $\mathbb{R}^2$, each one in the boundary of the set of orbits with a given orientation. For our purposes, the (unphysical) identification of this pair of (also highly unphysical) orbits turns out to be adequate. Finally, the two circular orbits at infinity corresponds to the two "copies" of the point $(0, 0, 0)$ in Ω.

The readers can try to see by themselves that the geometrical construction linking the point (Z, X, Y) to the Kepler orbit as a curve in configuration space still works for all the non-regular orbits, giving always the degenerate conic on which the Kepler orbit lies. Actually, all the geometrical descriptions which have

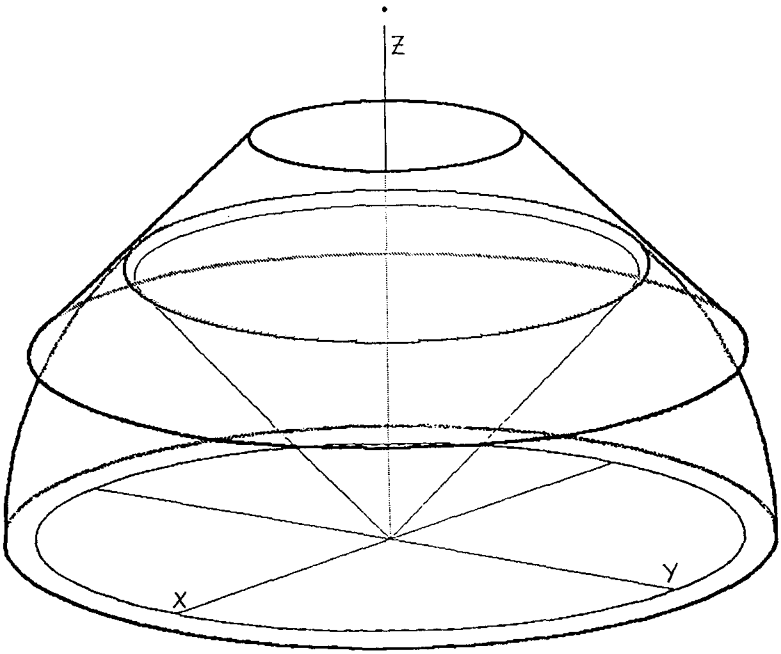

Fig. 6. A schematical diagram of the submanifolds of collision and singular orbits as points "at infinity" and on each side of the plane $Z = 0$ in $\mathbb{R}^3(Z, X, Y)$. The circular orbit at the origin appears as the point at infinity along time-like directions. Collision orbits are the points at infinity along isotropic directions, while hyperbolic angle orbits are the points at infinity along space-like directions. Orbits on $Z = 0$ are the hyperbolic straight orbits. The complete space Ω is obtained by glueing two similar halves with identification of the pair of points at infinity which correspond to each $L = 0$ non-regular orbit.

been stated only for regular orbits extend very naturally to include the singular orbits. For instance, the edge of the "wedge" of Kepler orbits passing through P corresponds to the hyperbolic straight orbits, lying on all straight lines through P in configuration space. The "vertical angle" of Kepler orbits through two points in configuration space includes, in the point $\gamma_P \cap \gamma_Q$, two hyperbolical straight orbits (on the straight line from P to Q and from Q to P in $\mathcal{X}$) and also two hyperbolic angle orbits for the points "at infinity" along the two half lines. We do not devote here more room to this description, which will be considered in more detail elsewhere.

5. The Conformal Group $\mathcal{C}_c$

There are various transformation groups of isometries or conformal transformations asssociated to the metric $d\sigma^2 = dZ^2 - dX^2 - dY^2$ in Ω, namely

1. A "Lorentz group" $\mathcal{L}_c$ of the metric (proper ortochronous linear isometries), isomorphic to the connected component of the identity of $SO(2, 1)$.

2. A “Poincaré group” $\mathcal{P}_c$ of the metric (proper ortochronous affine isometries), which includes translations and is isomorphic to the semidirect product $\mathbb{R}^3 \odot \mathcal{L}_c$.
3. A “Weyl group” $\mathcal{W}_c$, obtained by adjoining dilations to $\mathcal{P}_c$ [19]. The new transformations are not isometries, but only conformal transformations.
4. The full conformal group $\mathcal{C}_c$ of the metric, isomorphic to $SO(3,2)$.

Note that the three groups $\mathcal{L}_c$, $\mathcal{P}_c$ and $\mathcal{W}_c$ act globally in Ω, while $\mathcal{C}_c$ acts only as a local transformation group.

In this section we shall study these transformation groups, and in particular their relationship to some groups of transformations in the configuration space $\mathcal{X}$.

The standard form of the conformal group action on Minkowski space is [20]

$$J = -Y\frac{\partial}{\partial X} + X\frac{\partial}{\partial Y} \tag{5.1}$$

$$K_1 = -X\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial X}, \qquad K_2 = -Y\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial Y}$$

$$P_0 = -\frac{\partial}{\partial Z}, \qquad P_1 = -\frac{\partial}{\partial X}, \qquad P_2 = -\frac{\partial}{\partial Y}$$

$$D = Z\frac{\partial}{\partial Z} + X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}$$

$$C_0 = -(Z^2 + X^2 + Y^2)\frac{\partial}{\partial Z} - 2ZX\frac{\partial}{\partial X} - 2ZY\frac{\partial}{\partial Y}$$

$$C_1 = 2ZX\frac{\partial}{\partial Z} + (Z^2 + X^2 - Y^2)\frac{\partial}{\partial X} + 2XY\frac{\partial}{\partial Y}$$

$$C_2 = 2ZY\frac{\partial}{\partial Z} + 2XY\frac{\partial}{\partial X} + (Z^2 - X^2 + Y^2)\frac{\partial}{\partial Y}.$$

The subgroup $\mathcal{L}_c$ is generated by J, K_1, K_2 and $\mathcal{P}_c$ includes these generators and also P_0, P_1, P_2. The group $\mathcal{W}_c$, which contains $\mathcal{P}_c$ as well as the extra generator D, being an affine-linear subgroup of $\mathcal{C}_c$, maps planes into planes. In view of the previous geometrical description, this subgroup can be expected to be induced by point transformations in the configuration space $\mathcal{X}$, and as we will see next, this is indeed the case.

In terms of polar coordinates (r,θ) in $\mathcal{X}$ (identified to $\mathbb{R}^2 - \{0\}$), or (q,θ) where $q = 1/r$, the equation of the regular Kepler orbit parametrized by (Z, X, Y) is, respectively:

$$r = \frac{1}{Z + X\cos\theta + Y\sin\theta}$$

$$q = Z + X\cos\theta + Y\sin\theta. \tag{5.2}$$

Let us start by asking whether some of the transformations in $\mathcal{C}_c$ are induced by point transformations in the configuration space $\mathcal{X}$. The criterion for this is the following: Let

$$\Xi(q, \theta; Z, X, Y) \equiv Z + X\cos\theta + Y\sin\theta - q = 0 \tag{5.3}$$

be the equation of an orbit in configuration space (for fixed Z, X, Y); alternatively (5.3) could be looked at as the equation of the set of orbits through a fixed point in configuration space. A one parameter group $\exp(\boldsymbol{A}\epsilon)$ of point transformations in the configuration space, generated by $\boldsymbol{A}$, induces a one-parameter group of transformations, say $\exp(A\epsilon)$ in the space of orbits if it maps Kepler orbits into Kepler orbits, this is, if there exists a function $\lambda(q, \theta; Z, X, Y; \epsilon)$ such that

$$\Xi(\exp \boldsymbol{A}\epsilon(q,\theta); Z, X, Y) = \lambda(q, \theta; Z, X, Y; \epsilon)\Xi(q, \theta; \exp A\epsilon(Z, X, Y))$$

because the zero set of the function $\Xi(\exp(\boldsymbol{A}\epsilon)(q,\theta); Z, X, Y)$ transforms in the zero set of $\Xi(q, \theta; \exp(A\epsilon)(Z, X, Y))$. Taking the derivative at $\epsilon = 0$, we obtain

$$A\Xi - \boldsymbol{A}\Xi = \Lambda\Xi, \tag{5.4}$$

where $\Lambda = (d/d\epsilon)\lambda(q, \theta; Z, X, Y; \epsilon)|_{\epsilon=0}$. More explicitly, a vector field in Ω,

$$A = a^Z(Z,X,Y)\frac{\partial}{\partial Z} + a^X(Z,X,Y)\frac{\partial}{\partial X} + a^Y(Z,X,Y)\frac{\partial}{\partial Y}$$

can be considered as induced from an infinitesimal transformation in configuration space if there exists an infinitesimal generator

$$\boldsymbol{A} = a^\theta(q,\theta)\frac{\partial}{\partial \theta} + a^q(q,\theta)\frac{\partial}{\partial q}$$

in $\mathcal{X}$ and a function $\Lambda(q, \theta; Z, X, Y)$ such that (5.4) holds.

Furthermore, in this case the geometric interpretation already tell us that the subgroup of $\mathcal{C}_c$ for which these equations have solutions includes as generators at least J, K_1, K_2, P_1, P_2, P_0 and D. An explicit study of the equations confirms this, and also shows that (5.4) cannot be satisfied for any generator A linearly independent of this set. The corresponding generators in configuration space are given by

$$\boldsymbol{J} = \frac{\partial}{\partial\theta}, \quad \boldsymbol{P}_0 = \frac{\partial}{\partial q}, \quad \boldsymbol{P}_1 = \cos\theta\frac{\partial}{\partial q}, \quad \boldsymbol{P}_2 = \sin\theta\frac{\partial}{\partial q}, \quad \boldsymbol{D} = q\frac{\partial}{\partial q},$$

$$\boldsymbol{K}_1 = q\cos\theta\frac{\partial}{\partial q} + \sin\theta\frac{\partial}{\partial\theta}, \quad \boldsymbol{K}_2 = q\sin\theta\frac{\partial}{\partial q} - \cos\theta\frac{\partial}{\partial\theta}. \tag{5.5}$$

Hence the $(2+1)$ Weyl group $\mathcal{W}_c$ (which acts in the standard way in Ω) also acts in a non-standard way as a transitive group of point transformations in the configuration space $\mathcal{X}$ of the Kepler problem. The Lorentz subgroup $\mathcal{L}_c$ acts on Ω in a non-transitive way, but transitively on the submanifolds Θ_C, $Z^2 - X^2 - Y^2 = -C$, which corresponds to Kepler orbits with a fixed value of $C = \frac{2E}{L^2}$. By exponentiation of (5.5) we obtain this action as:

$$\exp(\alpha J)(r,\theta) = (r,\theta+\alpha)$$

$$\exp(\chi K_x)(r,\theta) = (r(\cosh\chi + \sinh\chi\cos\theta), \theta')$$

$$\exp(\chi K_y)(r,\theta) = (r(\cosh\chi + \sinh\chi\sin\theta), \theta'') \tag{5.6}$$

where θ' and θ'' are functions of χ and θ, given respectively by

$$\sin\theta' = \frac{\sin\theta}{\cosh\chi + \sinh\chi\cos\theta}, \qquad \cos\theta' = \frac{\sinh\chi + \cosh\chi\cos\theta}{\cosh\chi + \sinh\chi\cos\theta} \tag{5.7a}$$

and

$$\sin\theta'' = \frac{\sinh\chi + \cosh\chi\sin\theta}{\cosh\chi + \sinh\chi\sin\theta}, \qquad \cos\theta'' = \frac{\cos\theta}{\cosh\chi + \sinh\chi\sin\theta}, \tag{5.7b}$$

This action is not primitive [21] (as $\theta =$ const. gives an invariant foliation), and the fundamental vector fields are

$$\boldsymbol{X}_J = -\frac{\partial}{\partial\theta}$$

$$\boldsymbol{X}_{K_1} = -r\cos\theta\frac{\partial}{\partial r} + \sin\theta\frac{\partial}{\partial\theta}$$

$$\boldsymbol{X}_{K_2} = -r\sin\theta\frac{\partial}{\partial r} - \cos\theta\frac{\partial}{\partial\theta}, \tag{5.8a}$$

and in coordinates (q,θ) these expressions become

$$\boldsymbol{X}_J = -\frac{\partial}{\partial\theta}$$

$$\boldsymbol{X}_{K_x} = q\cos\theta\frac{\partial}{\partial q} + \sin\theta\frac{\partial}{\partial\theta}$$

$$\boldsymbol{X}_{K_y} = q\sin\theta\frac{\partial}{\partial q} - \cos\theta\frac{\partial}{\partial\theta}. \tag{5.8b}$$

Notice that there are three different strata for the action of $\mathcal{L}_c$ in Ω according to the sign of C:

1. The strata $C < 0$ includes all elliptical Kepler orbits. For a fixed value, say $C = -\kappa^2$, the Kepler ellipses have the same non-focal semiaxis, and all possible eccentricities $e < 1$. As a representative in Θ_C we can choose a circle $q = \kappa$.
2. If $C = 0$ the eccentricity is $e = 1$, i.e., the conics in this stratum are all parabolas and a convenient representative is $q = 1 + \cos\theta$.
3. For $C > 0$, the Kepler branches of hyperbolas appear; as a particular representative for fixed $C = \kappa^2$ we can choose the hyperbola $q = \kappa(1 + \sqrt{2}\cos\theta)$.

The point to be remarked now is that for each fixed value of C, the Kepler orbits in Θ_C determine a two-parameter family of conics in $\mathcal{X}$, and we can ask whether they can be considered as the geodesic curves of a connection in $\mathcal{X}$ and if so, whether this connection is the Levi-Civita connection defined by a Riemannian metric in $\mathcal{X}$. Therefore we now investigate whether a second order differential equation vector field Γ of the spray type [8] exists such that its integral curves are the given family of curves.

A method for the determination of such sprays for each family consists on elliminating (Z, X, Y) in terms of q, v_q, θ and v_θ by making use of the total time derivative of the equations of the generic conic in the family. We prefer a different procedure and we will study each case separately.

5.1. $C = 0$ ORBITS. (PARABOLAS)

Take now the two-dimensional family of Kepler orbits Θ_0 (i.e., $E = 0$, and hence parabolic orbits). Let $q = (1/p)(1 + e_x \cos\theta + e_y \sin\theta)$ be the general equation of such a conic depending on two parameters (here $e_x^2 + e_y^2 = 1$). Note that the basic fundamental vector fields of the action of $SO(2,1)$ are given in (5.8b), so any fundamental vector field $\boldsymbol{X}$ can be written as a linear combination of them. Then, there must exist coefficients a_Z, a_X and a_Y such that

$$\boldsymbol{X} = a_Z\Big(-\frac{\partial}{\partial\theta}\Big) + a_X\Big[q\cos\theta\frac{\partial}{\partial q} + \sin\theta\frac{\partial}{\partial\theta}\Big] + a_Y\Big[q\sin\theta\frac{\partial}{\partial q} - \cos\theta\frac{\partial}{\partial\theta}\Big], \quad (5.9)$$

and the requirement of $\boldsymbol{X}$ to be tangent to the conic $\Xi = 0$ is $\boldsymbol{X}\Xi|_{\Xi=0} = 0$, i.e., explicitly:

$$a_Z(-e_x \sin\theta + e_y\cos\theta) + a_X[e_x + \cos\theta] + a_Y[e_y + \sin\theta] = 0,$$

and as this must be true for any angle θ, we obtain

$$a_X = -a_Z e_y \qquad a_Y = a_Z e_x.$$

Any vector field $\boldsymbol{X}$ tangent to the given conic is therefore proportional to

$$\boldsymbol{X}_0 = q(-e_y\cos\theta + e_x\sin\theta)\frac{\partial}{\partial q} - (1 + e_y\sin\theta + e_x\cos\theta)\frac{\partial}{\partial\theta} \quad (5.10a)$$

or from (5.8b)

$$\boldsymbol{X}_0 = \boldsymbol{X}_J - e_y\boldsymbol{X}_{K_1} + e_x\boldsymbol{X}_{K_2}. \quad (5.10b)$$

We can now look for local coordinates adapted to the vector field $\boldsymbol{X}_0$ (see e.g., [8] p. 153). We choose the family of parabolas

$$q = \frac{1+\cos\theta}{p}, \quad (5.11)$$

where the eccentricity vector points in the positive Ox-direction, namely $e_x = 1$ and $e_y = 0$; any other parabola can be obtained from (5.11) by some rotation around the origin. The vector field tangent to these orbits is:

$$\boldsymbol{X}_0 = \boldsymbol{X}_J + \boldsymbol{X}_{K_2}. \tag{5.12}$$

We are now looking for local "adapted" coordinates s_1 and s_2 such that

$$\boldsymbol{X}_0 s_1 = 1, \qquad \boldsymbol{X}_0 s_2 = 0. \tag{5.13}$$

This leads to a system of partial differential equations

$$q \sin\theta \frac{\partial s_1}{\partial q} - (1+\cos\theta)\frac{\partial s_1}{\partial \theta} = 1$$

$$q \sin\theta \frac{\partial s_2}{\partial q} - (1+\cos\theta)\frac{\partial s_2}{\partial \theta} = 0, \tag{5.14}$$

to be solved by the method of the characteristics. In particular, a solution of the system is given by

$$s_1 = -\int \frac{d\theta}{1+\cos\theta} = -\tan\frac{\theta}{2}$$

$$s_2 = \frac{q}{1+\cos\theta}. \tag{5.15}$$

As the result is independent of p, the same parameter s_1 is appropriate for all conics in this family. A parametric description of the orbits based on the natural parameter $h = s_1$ is given by

$$q(h) = \frac{2}{p(1+h^2)}$$

$$\theta(h) = -2\arctan h. \tag{5.16}$$

Now, taking time derivatives twice in each expression of (5.16) and eliminating p and h it is easy to check that

$$\ddot{q} = \frac{3\dot{q}^2 - q^2\dot{\theta}}{2q}$$

$$\ddot{\theta} = \frac{\dot{q}\dot{\theta}^2}{q}, \tag{5.17}$$

from which we obtain the expression for the spray giving rise to the parabolic orbits

$$\Gamma^{(C=0)} = v_q \frac{\partial}{\partial q} + v_\theta \frac{\partial}{\partial \theta} + \frac{3v_q^2 - q^2 v_\theta^2}{2q}\frac{\partial}{\partial v_q} + \frac{v_q v_\theta}{q}\frac{\partial}{\partial v_\theta}. \tag{5.18}$$

It is notewhorthy that the natural parameter $h = s_1$ (5.15a) is but the so-called parabolic anomaly which is the analog for $E = 0$ of the better known eccentric anomaly.

5.2. $C < 0$ ORBITS. (ELLIPSES)

Let $C = -\kappa^2 < 0$. Here as $E < 0$ we have $e < 1$. The equation of a Kepler orbit in the family of fixed C depends on two independent parameters e_x, e_y, (with ${e_x}^2 + {e_y}^2 < 1$), and is

$$q = \frac{\kappa}{\sqrt{(1-e^2)}}(1 + e_x \cos\theta + e_y \sin\theta). \tag{5.19}$$

A vector field $\boldsymbol{X}$ as in (5.9) is tangent to such an elliptical orbit if

$$a_Z(-e_x \sin\theta + e_y \cos\theta) + a_X(e_x + \cos\theta) + a_Y(e_y + \sin\theta) = 0$$

and this leads to

$$a_X = -a_Z e_y, \qquad a_Y = a_Z e_x,$$

so the general solution for $\boldsymbol{X}$ is proportional to

$$\boldsymbol{X}_- = \frac{1}{\sqrt{(1-e^2)}}[\boldsymbol{X}_J - e_y \boldsymbol{X}_{K_1} + e_x \boldsymbol{X}_{K_2}]. \tag{5.20}$$

As $\boldsymbol{X}_-$ generates a compact subgroup, the factor $\frac{1}{\sqrt{(1-e^2)}}$ has been introduced in (5.20) in order to get a range from zero to 2π for the natural parameter in a complete turn. Coordinates adapted to this vector field are found in a similar way to the former case. Specializing the calculations to the family of orbits with $e_y = 0$, they have to be determined from the system of partial differential equations

$$\frac{1}{\sqrt{(1-e^2)}}\left[q\, e \sin\theta \,\frac{\partial s_1}{\partial q} - (1 + e\cos\theta)\,\frac{\partial s_1}{\partial \theta}\right] = 1$$

$$q\, e \sin\theta \,\frac{\partial s_2}{\partial q} - (1 + e\cos\theta)\,\frac{\partial s_2}{\partial \theta} = 0. \tag{5.21}$$

The equation determining s_1 leads to

$$-\frac{d\theta}{1 + e\cos\theta} = \frac{ds_1}{\sqrt{(1-e^2)}}, \tag{5.22}$$

and therefore s_1 is given (up to an additive constant) by

$$\tan\frac{s_1}{2} = -\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}. \tag{5.23}$$

Notice that (5.23) is the very well known expression for the eccentric anomaly, which appears here as a group theoretical parameter which describes the unfolding of a Kepler conic as the orbit (in the group theoretical sense) of a point in configuration space under the one-parameter subgroup generated by $\boldsymbol{X}_-$. The same happens in the parabolic and hyperbolic orbits. We stress the fact that these anomalies, whose role in this problem is well known, appear here with a group theoretical meaning.

The computation of the spray giving rise to these elliptic orbits is made up similarly to the preceding case. In terms of the natural parameter $\xi = s_1$ the orbits are written

$$\theta(\xi) = -2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\frac{\xi}{2}\right]$$

$$q(\xi) = \kappa\sqrt{\frac{1-e}{1+e}}\,\frac{1+\tan^2\frac{\xi}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\xi}{2}}, \tag{5.24}$$

and by elimination of the parameters e and ξ, by taking time derivatives, and after a tedious computation we will arrive to

$$\ddot{\theta} = \frac{\dot{q}\dot{\theta}}{q}, \qquad \ddot{q} = \frac{1}{2}\left(\frac{3\dot{q}^2}{q} - q\dot{\theta}^2\right) + \frac{c^2}{2}\frac{\dot{q}^2}{q}, \tag{5.25}$$

or in other words, the spray is given by

$$\Gamma^{(C=-\kappa^2)} = v_q\frac{\partial}{\partial q} + v_\theta\frac{\partial}{\partial\theta} + \frac{v_q v_\theta}{q}\frac{\partial}{\partial v_\theta} +$$
$$\left[\frac{1}{2}\left(\frac{3v_q^2}{q} - qv_\theta^2\right) + \frac{\kappa^2}{2}\frac{v_\theta^2}{q}\right]\frac{\partial}{\partial v_q}. \tag{5.26}$$

5.3. $C > 0$ Orbits. (Hyperbolas)

The complete discussion is very similar to the preceding one. In this case the adapted coordinate s_1 is given by:

$$\tan\frac{s_1}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}, \tag{5.27}$$

and also equals to the hyperbolic anomaly. The spray, not very surprisingly, turns out to be

$$\Gamma^{(C=\kappa^2)} = v_q\frac{\partial}{\partial q} + v_\theta\frac{\partial}{\partial\theta} + \frac{v_q v_\theta}{q}\frac{\partial}{\partial v_\theta} + \left[\frac{1}{2}\left(\frac{3v_q^2}{q} - qv_\theta^2\right) - \frac{\kappa^2}{2}\frac{v_\theta^2}{q}\right]\frac{\partial}{\partial v_q}. \tag{5.28}$$

The one-parameter subgroups mapping a given Kepler orbit (Z, X, Y) into itself can also be found in a more systematic way. The vector field $\boldsymbol{X}_{(Z,X,Y)}$ must be the generator of the isotopy group of the point (Z, X, Y) in Ω, so this point must be a zero of the vector field $\boldsymbol{X}_{(Z,X,Y)} = \nu \boldsymbol{X}_J + \lambda \boldsymbol{X}_{K_1} + \mu \boldsymbol{X}_{K_2}$. Taking into account the explicit expressions for the fundamental vector fields in Ω,

$$\boldsymbol{X}_J = -Y\frac{\partial}{\partial X} + X\frac{\partial}{\partial Y}$$

$$\boldsymbol{X}_{K_x} = X\frac{\partial}{\partial Z} + Z\frac{\partial}{\partial X}$$

$$\boldsymbol{X}_{K_y} = Y\frac{\partial}{\partial Z} + Z\frac{\partial}{\partial Y} \tag{5.29}$$

we obtain

$$\lambda X + \mu Y = \lambda Z - \nu Y = \mu Z + \nu X = 0, \tag{5.30}$$

and the general solution for this system gives a vector field proportional to

$$\boldsymbol{X}_{(Z,X,Y)} = Z\boldsymbol{X}_J - Y\boldsymbol{X}_{K_1} + X\boldsymbol{X}_{K_2} \tag{5.31}$$

which has (5.10) and (5.20) as particular cases.

The family of sprays $\Gamma^{(C)}$ whose geodesics are the Kepler orbits can also be found by a direct computation of the more general geodesic spray in $\mathcal{X}$ invariant under the action (5.6) of $\mathcal{L}_c$. To find this more general invariant spray, let us lift the action (5.6) to a new action on the tangent bundle $T(\mathbb{R}^2 - \{0\})$; the complete lifts of the fundamental vector fields (5.6) are given by

$$\boldsymbol{X}_J^c = -\frac{\partial}{\partial \theta}$$

$$\boldsymbol{X}_{K_1}^c = -r\cos\theta\frac{\partial}{\partial r} + \sin\theta\frac{\partial}{\partial \theta} - (v_r\cos\theta - r\sin\theta v_\theta)\frac{\partial}{\partial v_r} + \cos\theta v_\theta\frac{\partial}{\partial v_\theta}$$

$$\boldsymbol{X}_{K_2}^c = -r\sin\theta\frac{\partial}{\partial r} - \cos\theta\frac{\partial}{\partial \theta} - (v_r\sin\theta - r\cos\theta v_\theta)\frac{\partial}{\partial v_r} + \sin\theta v_\theta\frac{\partial}{\partial v_\theta}. \tag{5.32}$$

The invariance condition for the spray Γ is expressed by

$$[\boldsymbol{X}_J^c, \Gamma] = \lambda\Gamma$$

$$[\boldsymbol{X}_{K_1}^c, \Gamma] = \mu\Gamma, \tag{5.33}$$

with λ and μ arbitrary functions, because $[\boldsymbol{X}_{K_2}^c, \Gamma] = \nu\Gamma$ is a consequence of the previous ones. Let us assume that the field Γ is locally expressed as

$$\Gamma = v_r\frac{\partial}{\partial r} + v_\theta\frac{\partial}{\partial\theta} + f\frac{\partial}{\partial v_r} + g\frac{\partial}{\partial v_\theta},$$

where f and g are functions of r, θ, v_r, v_θ, quadratic in the velocities. Then the first invariance condition (5.33) means that f and g do not depend on θ, and λ vanishes because of

$$[\boldsymbol{X}_J^c, \Gamma] = \frac{\partial f}{\partial\theta}\frac{\partial}{\partial v_r} + \frac{\partial g}{\partial\theta}\frac{\partial}{\partial v_\theta}.$$

In a similar way the invariance under $\boldsymbol{X}_{K_1}^c$ leads to $\mu = 0$, and to

$$r\frac{\partial g}{\partial r} + v_r\frac{\partial g}{\partial v_r} - v_\theta\frac{\partial g}{\partial v_\theta} + g = 0$$

$$-v_\theta\frac{\partial g}{\partial v_r} - {v_\theta}^2 = 0$$

$$r\frac{\partial f}{\partial r} + v_r\frac{\partial f}{\partial v_r} - v_\theta\frac{\partial f}{\partial v_\theta} - f + r{v_\theta}^2 = 0$$

$$-rv_\theta\frac{\partial f}{\partial v_r} + 2v_\theta v_r + rg = 0. \tag{5.34}$$

The general solution for the first pair of equations (5.34), taking into account that f and g must be quadratic in the velocities is

$$g = -\frac{v_\theta v_r}{r} + v_\theta l(rv_\theta),$$

where l is an homogeneous function of degree one, and as a conseqence of the regularity conditions we get:

$$g = -\frac{v_\theta v_r}{r} + arv_\theta{}^2, \tag{5.35}$$

with a some real number. Analogously, the quadratic solution of the second subsystem is

$$f = \frac{v_r}{2r} + arv_rv_\theta + \left(\frac{1}{2}r + br^3\right){v_\theta}^2, \tag{5.36}$$

where b is a real number.

Therefore the expression of the spray field Γ depends on two parameters, a and b, and is given by

$$\Gamma^{a,b} = v_r\frac{\partial}{\partial r} + v_\theta\frac{\partial}{\partial\theta} + \left[\frac{{v_r}^2}{2r} + arv_rv_\theta + \left(\frac{1}{2}r + br^3\right){v_\theta}^2\right] \times \frac{\partial}{\partial v_r}$$
$$-\left[\frac{v_\theta v_r}{r} - arv_\theta{}^2\right]\frac{\partial}{\partial v_\theta}, \tag{5.37a}$$

and in the coordinates induced by q and θ,

$$\Gamma^{a,b} = v_q\frac{\partial}{\partial q} + v_\theta\frac{\partial}{\partial\theta} + \left[\frac{3}{2q}v_q^2 + \frac{a}{q}v_q v_\theta - \left(\frac{q}{2} + \frac{b}{q}\right){v_\theta}^2\right] \times \frac{\partial}{\partial v_q}$$

$$+\left[\frac{v_q v_\theta}{q} + \frac{a}{q}{v_\theta}^2\right]\frac{\partial}{\partial v_\theta}. \tag{5.37b}$$

This is the most general spray invariant under the action (5.6), and the family $\Gamma^{(C)}$ is included as the particular case $a = 0, b = C$.

Moreover since the coefficients of the connection $\Gamma^{a,b}$ in the local basis $\{\frac{\partial}{\partial q}, \frac{\partial}{\partial\theta}\}$ are given in the general case by (5.37b), we see that the non-zero components of the conection are

$$\Gamma^1_{11} = -\frac{3}{2q}, \quad \Gamma^1_{21} = \Gamma^1_{12} = -\frac{a}{2q}, \quad \Gamma^1_{22} = \frac{q}{2} + \frac{b}{2q} \tag{5.38a}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = -\frac{1}{2q}, \quad \Gamma^2_{22} = -\frac{a}{q}. \tag{5.38b}$$

From (5.38), the curvature tensor

$$R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} - \Gamma^e_{bc}\Gamma^a_{ed} + \Gamma^e_{bd}\Gamma^a_{ec}, \tag{5.39}$$

can be computed; for instance

$$R^1_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} - \Gamma^e_{21}\Gamma^1_{e2} + \Gamma^e_{22}\Gamma^1_{e1} = \frac{b}{q^2} + \frac{a^2}{4q^2}. \tag{5.40}$$

Another interesting point is whether or not the sprays $\Gamma^{(C)}$ (i.e., (5.37) for $a = 0$) are Riemannian or Lagrangian. To start with, the spray $\Gamma^{0,0}$ is easily seen to be geodesic, and can be considered as a Riemannian connection for the metric

$$ds^2 = \frac{1}{q^3}\,dq^2 + \frac{1}{q}\,d\theta^2, \tag{5.41}$$

or in other words, the Euler-Lagrange equations for the Lagrangian

$$L_0 = \frac{1}{q^3}\dot{q}^2 + \frac{1}{q}\dot{\theta}^2 \tag{5.42}$$

are the equations of the geodesics for the spray $\Gamma^{0,0}$. But all the other sprays in the family are neither Riemannian nor Lagrangian, i.e., associated to a regular Lagrangian L ([8], Chapter 13). To see it, take into account the preceding result for the case $a = b = 0$; in the general case the Lagrangian would be $L = L_0 + bL_1$ and then the Euler-Lagrangian equations for L_1 are

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial\dot{q}}\right) - \frac{\partial L_1}{\partial q} = -\frac{b\dot{\theta}^2}{q}, \qquad \frac{d}{dt}\left(\frac{\partial L_1}{\partial\dot{\theta}}\right) - \frac{\partial L_1}{\partial\theta} = 0. \tag{5.43}$$

The second equation tell us that there exists a function $f(\theta, q, \dot{q})$ such that $L_1 = \frac{\partial f}{\partial \theta}\dot{\theta}$, and then the first Euler-Lagrange equation becomes

$$\frac{d}{dt}\left(\frac{\partial^2 f}{\partial\theta\partial\dot{q}}\dot{\theta}\right) - \frac{\partial^2 f}{\partial\theta\partial q}\dot{\theta} \equiv \frac{b\dot{\theta}^2}{q} \tag{5.44}$$

and this equation has no solution for f when $b \neq 0$. The proof of the non existence of a Lagrangian L for the spray (5.37b) with $a = 0$ but $b \neq 0$ can also be carried out in a much more geometric way, using for instance the theory developed by Klein [22].

Returning to (5.5), we have there a local group in the configuration space, mapping Kepler orbits into Kepler orbits. Some of these transformations correspond to non-complete vector fields, and a proper discussion requires the explicit introduction of a compactification of $\mathcal{X}$. Our main purpose here has been only to discuss the group $\mathcal{L}_c$ which is the configuration space analogue of the groups involved in the MOBM construction. We do not enter in a deeper study of the action of the Weyl group in the configuration space given in (5.5).

6. The Conformal Group $\mathcal{C}_v$

In the former section we introduced a $SO(3,2)$ group acting on the space of orbits Ω, which has a subgroup of transformations $\mathcal{W}_c$ induced by point transformations in configuration space. In particular, on each submanifold Θ_C of all orbits with a fixed value of C, the subgroup of $SO(3,2)$ mapping Θ_C into itself is a $SO(2,1)$ subgroup (the "Lorentz subgroup" $\mathcal{L}_c$), and Θ_C appears as a homogeneous space for $SO(2,1)$. Furthermore, as this subgroup $\mathcal{L}_c$ is contained in $\mathcal{W}_c$, its action is induced by a group of point transformations in the configuration space $\mathcal{X}$. As a homogeneous space for $SO(2,1)$, $\mathcal{X}$ is identified to the quotient space of $SO(2,1)$ by a parabolic subgroup; this action is not primitive, but has a one-parameter family of invariant connections, $\Gamma^{(C)}$, and all Kepler orbits of fixed C, as curves in configuration space, are but the autoparalell lines of the connection $\Gamma^{(C)}$.

All this bears some analogies with the results referred in Section 1, where for each value of the energy E there is a group of point transformations in the velocity space $\mathcal{V}_E$ which also induces an action on the set Ω_E of all Kepler orbits with energy E. So both spaces $\mathcal{V}_E$ and Ω_E are homogeneous spaces for the groups $SO(3)/E(2)/SO(2,1)$, according to the sign of E.

In view of the results in Section 5, it is natural to ask whether these groups of transformations appear as subgroups of a larger group in the orbit space. The first obvious idea is to look for these groups as subgroups of $\mathcal{C}_c$ induced by point transformations in the velocity space $\mathcal{V}$. But a detailed study shows that the subgroups of $\mathcal{C}_c$ mapping each submanifold Ω_E into itself do not provide the MOBM groups when acting on Ω_E and, even more generally, that they are not induced by point transformations in $\mathcal{V}_E$.

Symmetry between configuration and velocity spaces in our problem could be restored by considering another group of transformations, to be called $\mathcal{C}_v$, in the space of all orbits. As an abstract group $\mathcal{C}_v$ is also a $SO(3,2)$ group, though the action on Ω is different. We only give a statement of the results; in some aspects the ideas are very similar to those in Sect. 5, but there are also some important differences, in particular as the orbits with $E = 0$ and the non-regular orbits fit into this new scheme in a different way.

Let us first introduce a new parameter ξ defined by

$$\xi = \frac{Z^2 - X^2 - Y^2}{2Z} = \frac{1 - e^2}{2p}. \tag{6.1}$$

In terms of ξ the energy can be rewritten as

$$E = -\text{sign}\left(\frac{\xi}{1 - e^2}\right) k\xi = \begin{cases} -k \mid \xi \mid, & \text{if} \quad 0 < e < 1 \\ k \mid \xi \mid, & \text{if} \quad 1 < e. \end{cases} \tag{6.2}$$

The focal semiaxis is $a = \frac{1}{2|\xi|}$. The triplet (ξ, e_x, e_y) is a good coordinate system in the set of all regular orbits with $E \neq 0$, and it can be used to introduce there another coordinate system (ζ, χ, v) defined by

$$\zeta = \frac{\xi}{\sqrt{1 - e^2}}, \chi = \frac{\xi e_x}{\sqrt{1 - e^2}}, v = \frac{\xi e_y}{\sqrt{1 - e^2}}, \quad \text{for} \quad E < 0, \quad (0 < e < 1),$$

$$\zeta = \frac{\xi}{\sqrt{e^2 - 1}}, \chi = \frac{\xi e_x}{\sqrt{e^2 - 1}}, v = \frac{\xi e_y}{\sqrt{e^2 - 1}}, \quad \text{for} \quad E > 0, \quad (1 < e). \tag{6.3}$$

The new coordinates are related inside the cone $Z^2 - X^2 - Y^2 = 0$ to the old ones by:

$$(\zeta, \chi, v) = \frac{\sqrt{Z^2 - X^2 - Y^2}}{2 \mid Z \mid} (Z, X, Y). \tag{6.4}$$

It is easy to derive from this that $Z^2 - X^2 - Y^2 = 4\zeta^2(\zeta^2 - \chi^2 - v^2)$. This, and similar relations which holds outside the cone, show that all the regular Kepler orbits with $E < 0$ (resp. > 0) have $\zeta^2 - \chi^2 - v^2 > 0$ (resp. < 0). The equations of the submanifolds Ω_E are:

$$E < 0\,, \quad E = -\text{sign}\,(\xi) k\xi\,, \quad \zeta^2 - \chi^2 - v^2 = \frac{E^2}{k^2}$$

$$E > 0\,, \quad E = \text{sign}\,(\xi) k\xi\,, \quad \zeta^2 - \chi^2 - v^2 = -\frac{E^2}{k^2}. \tag{6.5}$$

We now introduce an auxiliar Minkowski space, $\mathbb{R}^3(\zeta, \chi, v)$ with the flat metric $ds^2 = d\zeta^2 - d\chi^2 - dv^2$, and we identify the open subsets $\Omega_{E<0}$, and $\Omega_{E>0}$ in Ω with respectively the interior and the exterior of the cone by means of (6.3). In particular, the submanifolds Ω_E ($E \neq 0$), appear as Minkowski's spheres (space-like for $E < 0$ and time-like for $E > 0$) of the auxiliar Minkowski space. We remark that when considered as a metric in Ω, ds^2 is only defined in the open subset $\Omega_{E\neq 0}$, and when trying to extend it to $\Omega_{E=0}$, the metric becomes necessarily singular. We shall see next how the conformal group of the metric ds^2 acting in the standard way in the auxiliar Minkowski space (ζ, χ, v), contains, as subgroups mapping the submanifolds Ω_E ($E \neq 0$) into themselves, the groups arising in the MOBM construction.

The infinitesimal generators, i.e., the fundamental vector fields, are represented by the same expressions (5.1) with the replacements of (Z, X, Y) by (ζ, χ, v). Since dilations apply spheres into spheres of the same type (i.e., they conserve the sign of the energy, $E \to e^{\lambda}E$), we can restrict our study to the representative spheres Ω_k, and Ω_{-k} (i.e., $(E/k) = \pm 1$). The subgroups leaving stable a generic manifold Ω_E are conjugate to the corresponding subgroups of Ω_k, or Ω_{-k} by a dilation of rate $\mid E \mid /k$.

We shall consider first the case of Ω_{-k}. Let $\mathcal{C}_{v,-k}$ denote the subgroup of $\mathcal{C}_v$ transforming Ω_{-k} into itself. A complete set of vector fields $\boldsymbol{X}_i$ such that $\boldsymbol{X}_i f_{-k}|_{f_{-k}} = 0$, where f_E denotes the function $f_E = \zeta^2 - \chi^2 - v^2 -$ sign $(E)(E^2/k^2)$, is spanned by

$$J,\ K_1,\ K_2,\ A_i = \frac{1}{2}(C_i - P_i) \quad \text{for} \quad i = 0, 1, 2. \tag{6.6}$$

It is easy to check using the commutation rules that these vector fields generate a $SO(3,1)$ subgroup, so we have $\mathcal{C}_{v,E=-k} \approx SO(3,1)$. Subgroups of $SO(3,1)$ isomorphic to $SO(2,1)$ and $E(2)$ are generated by J, K_1, K_2 and J, $A_2 - K_2$, $-A_1 + K_1$ respectively, while J, A_1 and A_2 generate a compact maximal subgroup isomorphic to $SO(3)$.

When a similar study for $E = k$ and Ω_k is carried out, the subgroup preserving Ω_k is generated by

$$J,\ K_1,\ K_2,\ B_i = \frac{1}{2}(C_i + P_i) \quad \text{for} \quad i = 0, 1, 2, \tag{6.7}$$

and the group $\mathcal{C}_{v,E=k}$ is now a group isomorphic to $SO(2,2)$. Another interesting $SO(2,1)$ subgroup (see later on) is the one generated by J, B_1 and B_2. Other $SO(1,2)$ and $E(2)$ subgroups are also contained in $\mathcal{C}_{v,E=k} \approx SO(2,2)$.

The relation with the groups of isometries of the metrics (3.6) and (3.7) in Section 3 becomes much more transparent in terms of parameters (ξ, e, ϕ):

$$ds^2 = \begin{cases} d\xi^2 - \dfrac{\xi^2}{1-e^2}\Big(\dfrac{1}{1-e^2}\,de^2 + e^2 d\phi^2\Big), & \text{if} \quad 0 < e < 1 \\ -d\xi^2 - \dfrac{\xi^2}{e^2-1}\Big(\dfrac{1}{1-e^2}\,de^2 + e^2 d\phi^2\Big), & \text{if} \quad 1 < e \end{cases} \tag{6.8}$$

while the expresions for the basic fundamental vector fields of C_v in these coordinates are for $0 < e < 1$

$$J = \frac{\partial}{\partial\phi} \tag{6.9a}$$

$$K_1 = -\cos\phi(1-e^2)\frac{\partial}{\partial e} + \frac{\sin\phi}{e}\frac{\partial}{\partial\phi}$$

$$K_2 = -\sin\phi(1-e^2)\frac{\partial}{\partial e} - \frac{\cos\phi}{e}\frac{\partial}{\partial\phi}$$

$$P_0 = -\frac{r^2}{\sqrt{1-e^2}}\frac{\partial}{\partial r} - e\sqrt{1-e^2}\,r\frac{\partial}{\partial e}$$

$$P_1 = \cos\phi\frac{e}{\sqrt{1-e^2}}\frac{\partial}{\partial r} - \cos\phi\frac{\sqrt{1-e^2}}{r}\frac{\partial}{\partial e} + \sin\phi\frac{\sqrt{1-e^2}}{er}\frac{\partial}{\partial\phi}$$

$$P_2 = \sin\phi\frac{e}{\sqrt{1-e^2}}\frac{\partial}{\partial r} - \sin\phi\frac{\sqrt{1-e^2}}{r}\frac{\partial}{\partial e} - \cos\phi\frac{\sqrt{1-e^2}}{er}\frac{\partial}{\partial\phi}$$

$$D = r\frac{\partial}{\partial r}$$

$$C_0 = -\frac{1}{\sqrt{1-e^2}}\frac{\partial}{\partial r} + e\frac{\sqrt{1-e^2}}{r}\frac{\partial}{\partial e}$$

$$C_1 = \cos\phi\frac{e}{\sqrt{1-e^2}}r^2\frac{\partial}{\partial r} - \cos\phi\sqrt{1-e^2}\,r\frac{\partial}{\partial e} - r\sin\phi\frac{\sqrt{1-e^2}}{e}\frac{\partial}{\partial\phi}$$

$$C_2 = \sin\phi\frac{e}{\sqrt{1-e^2}}r^2\frac{\partial}{\partial r} - \sin\phi\sqrt{1-e^2}\,r\frac{\partial}{\partial e} + r\cos\phi\frac{\sqrt{1-e^2}}{e}\frac{\partial}{\partial\phi},$$

and for $1 < e$

$$J = \frac{\partial}{\partial\phi} \tag{6.9b}$$

$$K_1 = -\cos\phi\,(e^2-1)\frac{\partial}{\partial e} + \frac{\sin\phi}{e}\frac{\partial}{\partial\phi}$$

$$K_2 = \sin\phi\,(e^2-1)\frac{\partial}{\partial e} - \frac{\cos\phi}{e}\frac{\partial}{\partial\phi}$$

$$P_0 = -\frac{1}{\sqrt{e^2-1}}\frac{\partial}{\partial r} - e\sqrt{e^2-1}\,r\frac{\partial}{\partial e}$$

$$P_1 = -\cos\phi\,\frac{e}{\sqrt{e^2-1}}\frac{\partial}{\partial r} - \cos\phi\,\frac{\sqrt{e^2-1}}{r}\frac{\partial}{\partial e} + \sin\phi\frac{\sqrt{e^2-1}}{er}\frac{\partial}{\partial\phi}$$

$$P_2 = -\sin\phi\frac{e}{\sqrt{e^2-1}}\frac{\partial}{\partial r} - \sin\phi\frac{\sqrt{e^2-1}}{r}\frac{\partial}{\partial e} - \cos\phi\frac{\sqrt{e^2-1}}{er}\frac{\partial}{\partial\phi}$$

$$D = r\frac{\partial}{\partial r}$$

$$C_0 = -\frac{r^2}{\sqrt{e^2-1}}\frac{\partial}{\partial r} - e\frac{\sqrt{e^2-1}}{r}\frac{\partial}{\partial e}$$

$$C_1 = \cos\phi\frac{e}{\sqrt{e^2-1}}r^2\frac{\partial}{\partial r} - \cos\phi\sqrt{e^2-1}\,r\frac{\partial}{\partial e} - r\sin\phi\,\frac{\sqrt{e^2-1}}{e}\frac{\partial}{\partial\phi}$$

$$C_2 = \sin\phi\frac{e}{\sqrt{e^2-1}}r^2\frac{\partial}{\partial r} - \sin\phi\sqrt{e^2-1}\,r\frac{\partial}{\partial e} - r\cos\phi\,\frac{\sqrt{e^2-1}}{e}\frac{\partial}{\partial\phi}.$$

The restriction of these vector fields and of the metric to the submanifolds Ω_E is very simple because a constant E implies either $0 \le e < 1$ or $1 < e$, and reduces there to a constant ξ. Thus, up to a global sign ($-$ for $0 \le e < 1$, and $+$ for $1 < e$),

$$ds^2|_{\Omega_E} = \pm\frac{\frac{E^2}{k^2}}{1-e^2}\left(\frac{1}{1-e^2}\,de^2 + e^2 d\phi^2\right). \tag{6.10}$$

For every fixed value of E the subgroup $C_{v,E}$ leaves conformally invariant the metric (6.10). Coming back to the particular case of $\Omega_{\pm k}$ we have the following results:

i) Case Ω_{-k}:

All the elements of $C_{v,E=-k}$ act by conformal transformations of the metric (6.10). In particular, the elements of the subgroup $SO(3)$ generated by J, A_1, A_2, while being conformal transformations of (6.10), are also isometries of the following metric (of course conformal to (6.10))

$$ds^2 = \frac{1}{1-e^2}\,de^2 + e^2 d\phi^2, \qquad 0 < e < 1. \tag{6.11}$$

This can be easily checked by calculating the Killing vectors of the metric (6.11) and comparing with the restrictions to $\xi = \text{const.}$ of (6.9a/b). This metric coincides with (3.6), and therefore the action of $SO(3)$ on the submanifold $\Omega_{E=-k}$ coincides with the action in the MOBM construction. The relation with the standard spherical metric which has the expresion $ds^2_{\text{sph}} = d\theta^2 + \sin^2\theta\, d\phi^2$ in geographical coordinates is made through the coordinate change $e = \sin\theta$. Points onto the equatorial circle correspond to collision orbits; the model automaticaly regularizes collision orbits, as it was the case for the model based on the group $\mathcal{C}_c$ and its subgroups.

The other subgroups $SO(2,1)$, and $E(2)$ act by conformal transformations of the metric ds^2. Eventually, we can remark that the usual dynamical group for the negative energy orbits of the two-dimensional Kepler problem is $SO(2,1)$.

ii) Case Ω_k:

The discussion is quite similar to the above case and here the subgroup $SO(2,1)$ generated by J, $\frac{1}{2}(C_i + P_i)$, $i = 1,2$, acts by isometries of the metric conformal to (6.10),

$$ds^2 = \frac{1}{e^2-1}\, de^2 - e^2 d\phi^2, \qquad 1 < e. \tag{6.12}$$

The metric (6.12) coincides with the cohyperbolic metric (3.7), and therefore the action of this subgroup $SO(2,1)$ is the same as the action in the MOBM construction. Other subgroups $SO(1,2)$ and $E(2)$ act by conformal transformations of the metric.

7. Summary and Outlook

The dynamical symmetry of the Kepler problem has been known since a long time. The groups $SO(2,1)$, $E(2)$ and $SO(3)$ are the conventional dynamical symmetry groups for the plane case. In this paper, however, we adopt a different point of view, and completely disregard the problem of time evolution along the orbit. By considering only the description of the space of orbits, we get some sort of new "symmetry groups" for this classical problem.

In particular, a group-theoretical interpretation of the structure of the space Ω of all orbits is carried out. We introduce a Lorentzian metric do^2 in Ω, and then we consider its conformal group $\mathcal{C}_c$. A seven-dimensional subgroup of $\mathcal{C}_c$ is induced by a group of local point transformations in the configuration space $\mathcal{X}$. In particular, a $SO(2,1)$ subgroup acts transitively on $\mathcal{X}$; the traces of its one-dimensional subgroups are the Kepler orbits and the classical anomalies appear as the canonical parameters of the group action for regular orbits.

All these results closely paralell the Moser-Osipov-Belbruno-Milnor construction in velocity space $\mathcal{V}_E$. For a fixed value $E > 0$ (resp. $E = 0$, $E < 0$), $\mathcal{V}_E$ is also a homogeneous space of the group $SO(2,1)$ (resp. $E(2)$, $SO(3)$). Starting with the group of conformal transformations $\mathcal{C}_v$ of the metric ds^2 in Ω, we recover completely the MOBM construction.

For constant energy the invariant metric in $\mathcal{V}_E$ induces a metric in Ω_E. The distance between near orbits is equal to the angle between hodographs. In our construction the metric do^2 has not a direct meaning, but its use gives a clear image of global structure of the set of all orbits and of many other aspects of the problem. Any motion in a Kepler potential follows a curve in Ω which is always space-like or isotropic for the metric do^2. The minimal energy orbital transfer between two circular orbits (Hohmann transfer) corresponds to the extreme case of a isotropic curve in Ω connecting the two orbits. However, there is no any simple general relation of the metric do^2 with the minimal energy expense for an orbital maneouver.

Consider a space-craft in orbit; in order to modify its trajectory it shoots a jet of gas out. Let assume that the velocity of emission of the gas from the space-craft relative to this one has a constant value c. In the reference frame of the space-craft the total linear momentum after the gas ejection is $\mathbf{p}_f \approx m\Delta\ \mathbf{v}+c\,\Delta m$, where Δ $\mathbf{v}$ is the variation of the space-craft velocity.

Taking into account that the initial momentum is zero, we see that $c\Delta m \approx m \mid \Delta\ \mathbf{v}|$. Then, the energy lost in the process is $E \approx \frac{1}{2}\Delta m\, c^2 \approx \frac{1}{2}mc \mid \Delta v \mid$ and therefore proportional to $\mid \Delta\ \mathbf{v}|$. Therefore, a reasonable "metric" for orbits can be defined as the modulus of the difference of velocities at the intersection point. When integrated along a path, this should give a measure of the energy employed in the orbital change. It is not difficult to obtain the expression for this "metric" in terms of the parameters (E, e, ϕ)

$$ds^2_{vE} = dv^2 + v^2 d\beta^2 = \frac{1}{v^2}\, dE^2 + v_i^2 d\beta^2 =$$

$$= \frac{1}{v_i^2}\Big[\Big(1 + \frac{e^2-1}{E^2}\Big(\frac{v_i^2}{2} - E\Big)^2\Big)dE^2 +$$

$$\Big(\frac{v_i^2}{2} - E\Big)^2\Big(\frac{e}{E}\, dE de + de^2 + e^2 d\phi^2\Big)\Big]$$

where v_i is the velocity on the orbit (E, e, ϕ) at the point of intersection with the orbit $(E + dE, e + de, \phi + d\phi)$ and β is the angle of intersection. However, this "metric" in the orbit space is not Riemannian, but of Finsler type, and therefore does not coincide with do^2.

The interpretation of the metric do^2 is also related with the study of the action of the $(2+1)$ "Poincaré" group on the configuration space. In terms of the space of orbits, this group acts as isometries of do^2. Another point worth mentioning is the concrete model of a compactification of the Minkowski space obtained for the set of all Kepler orbits. The natural parameters for the Kepler evolution are the anomalies, which regularize the collision orbits. As the anomalies appear here with a group theoretical meaning, the description in this paper includes also very smoothly the non-regular orbits.

A last point worthing notice is that the structure of homogeneous space of the configuration space $\mathcal{X}$ provides a physical realization of a $SO(2,1)$ geometry which is not of Cayley-Klein type. The same $SO(2,1)$ geometry (the Lorentz group action on the light cone) describes light propagation in a plane (i.e., transformation properties of both frecuency and direction), and is a kind of contraction of the Lobachevski geometry "around the infinity" (See also [23]). We intend to study this geometry and its different physical realizations in a subsequent paper.

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