FREQUENCY MODIFIED FOURIER TRANSFORM AND ITS APPLICATION TO ASTEROIDS

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Abstract. Recently a method has been suggested to analyze the chaotic behaviour of a conservative dynamical system by numerical analysis of the fundamental frequencies. Frequencies and amplitudes are determined step by step. As the frequencies are not generally orthogonal, a Gramm-Schmidt orthogonalization is made and for each new frequency the old amplitudes of previously determined frequencies are corrected. For a chaotic trajectory variations of the frequencies and amplitudes determined over different time periods are expected. The change of frequencies in such a calculation is a measure of the chaoticity of the trajectory. While amplitudes are corrected, the frequencies (once determined) are constant. We suggest here simple linear corrections of frequencies for the effect of other close frequencies. The improvement of frequency determination is demonstrated on a model case. This method is applied to the first fifty numbered asteroids.

Key words: Asteroids – Chaos – Fourier Transform

1. Algorithm for Modified Fourier Transform

The Modified Fourier Transform (MFT) was introduced by Laskar (1988) for analysis of the numerical solution to his secular system for planetary motion. The method was described in more detail in Laskar (1990) and applied to the standard mapping (Laskar *et al.*, 1992) and to multi-dimensional systems (Laskar, 1993). For regular motion the method yields an analytical representation of the solutions. For chaotic system the frequency changes calculated for two time intervals give a measure of chaos.

In this section we give a recursive algorithm for MFT, while in Section 2 the modification of the algorithm to correct the frequencies and amplitudes will be presented. Let us assume a complex function $f(t) = k(t) + ih(t)$ ($k(t)$ and $h(t)$ are real) defined on the interval $[0, 2\tau]$. We shall denote $\nu_0 = \pi/\tau$. Usually $f(t)$ is sampled at evenly spaced intervals in time. We introduce the scalar products of two such functions

$$\langle f, g \rangle = \frac{1}{2\tau} \int_0^{2\tau} f(t)\bar{g}(t) \left(1 - \cos \frac{\pi t}{\tau}\right) dt \tag{1}$$

corresponding to introducing the Hanning window filter on the interval $[0, 2\tau]$. Such integrals can be calculated by the Simpson's rule or other more sophisticated methods. Then

$$\langle e^{i\nu t}, e^{i\omega t}\rangle = e^{i(\nu-\omega)\tau} Q(\nu,\omega), \tag{2}$$

where $Q(\nu,\omega)$ is real valued function

$$\begin{aligned} Q(\nu,\omega) &= \frac{\sin(\nu-\omega)\tau}{(\nu-\omega)\tau}\frac{\pi^2}{\pi^2-(\nu-\omega)^2\tau^2} && \text{for } \nu \neq \omega, \\ Q(\nu,\omega) &= 1 && \text{for } \nu = \omega. \end{aligned} \tag{3}$$

If

$$f(t) = \sum_{j=1}^{\infty} A'_j \exp i\nu'_j t, \tag{4}$$

with

$$|A'_1| \geq |A'_2| \geq \cdots,$$

then the MFT tries to approximate $f(t)$ with the first N terms

$$f(t) = \sum_{j=1}^{N} A_j \exp i\nu_j t, \tag{5}$$

where A_j and ν_j are very close to the corresponding A'_j, ν'_j. In the first step of the MFT we determine ν_1 as the value σ, for which the absolute value of the function

$$\varphi_1(\sigma) = \langle f(t), \exp i\sigma t\rangle \tag{6}$$

takes on a maximum value. Then

$$A_1 = \varphi_1(\nu_1). \tag{7}$$

For the next step we use as input function

$$f_1(t) = f(t) - A_1 \exp i\nu_1 t. \tag{8}$$

To give the necessary formulas for the mth step in which the frequency ν_m is determined, let us make some preparatory definitions. We will define unit vectors

$$e_j = \exp i\nu_j(t-\tau). \tag{9}$$

As $\langle e_j, e_k\rangle = Q(\nu_j,\nu_k)$, the vectors e_j are not orthonormal. We will form an orthogonal basis

$$b_j = \sum_{k=1}^{j} \alpha_{jk} e_k. \tag{10}$$

In mth step e_m is determined first and then m coefficients α_{mk} for $k = 1, \ldots, m$ follow from the orthonormality conditions for vectors $b_1, \ldots, b_m$. We will choose $\alpha_{11} = 1$. Then we can choose all α_{ij} real. In mth step we calculate coefficient S_m in the development

$$f = \sum_{j=1}^{m} S_j b_j + f_m. \tag{11}$$

Here

$$S_j = \langle f, b_j \rangle, \tag{12}$$

so that

$$\langle f_m, b_j \rangle = 0 \quad \text{for } j \le m. \tag{13}$$

For mth step we need to know the function $f_{m-1}(t)$, coefficients α_{nk} (for $n < m$ and $k \le n$), and the coefficients S_k (for $k < m$). This input is certainly known for $m = 2$, since we know f_1, $\alpha_{11} = 1$, and $S_1 = A_1$. The algorithm for the mth step consists of following substeps:

1. Calculation of ν_m as the value σ for which the function
   $$\varphi_m = \langle f_{m-1}, \exp i\sigma t \rangle \tag{14}$$
   takes on a maximum value. This is achieved by applying the FFT to f_{m-1} for rough estimate (accuracy $\sim \nu_0$) followed by Brent's method for a much more accurate determination of ν_m. As our program is written in C++, we used a slightly modified routine 'brent' from Numerical Recipes in C (Press *et al.*, 1992). As a result we have not only ν_m but also value of φ at its maximum
   $$\varphi_m(\nu_m) = F_m \exp i\delta_m \tag{15}$$
   e.g. its amplitude F_m and phase δ_m.
2. Calculation of b_m, or equivalently the coefficients α_{mk} for $k \le m$. The vectors b_j for $j < m$ are known from the previous steps. Writing
   $$b_m = \alpha_{mm} \left(\sum_{j=1}^{m-1} B_j^{(m)} b_j + e_m \right), \tag{16}$$
   we obtain the coefficients $B_j^{(m)}$ from the orthogonality of b_m to all b_j for $j < m$.
   $$B_j^{(m)} = -\sum_{s=1}^{j} \alpha_{js} Q(\nu_m, \nu_s) \quad \text{for } j = 1, \ldots, m-1 \tag{17}$$
   and coefficient α_{mm} from condition $\langle b_m, b_m \rangle = 1$ which yields
   $$\alpha_{mm} = \left(1 - \sum_{j=1}^{m-1} |B_j^{(m)}|^2 \right)^{-\frac{1}{2}} \tag{18}$$

and finally

$$\alpha_{mj} = \alpha_{mm} \sum_{s=j}^{m-1} B_s^{(m)} \alpha_{sj} \quad \text{for } j < m. \tag{19}$$

3. Calculation of f_m. From Eq. (11)

$$f_m = f_{m-1} - \langle f_{m-1}, b_m \rangle b_m. \tag{20}$$

Employing Eqs. (16) and (13) we have

$$\begin{aligned} f_m &= f_{m-1} - \alpha_{mm} \langle f_{m-1}, e_m \rangle \sum_{j=1}^{m} \alpha_{mj} e_j \\ &= f_{m-1} - \alpha_{mm} F_m \sum_{j=1}^{m} \alpha_{mj} \exp i \left[\nu_j t + (\nu_m - \nu_j)\tau + \delta_m\right]. \end{aligned} \tag{21}$$

For the real and imaginary part of f_m we have the following recurrence relations

$$\begin{aligned} k_m &= k_{m-1} - \alpha_{mm} F_m \sum_{j=1}^{m} \alpha_{mj} \cos i \left[\nu_j t + (\nu_m - \nu_j)\tau + \delta_m\right], \\ h_m &= h_{m-1} - \alpha_{mm} F_m \sum_{j=1}^{m} \alpha_{mj} \sin i \left[\nu_j t + (\nu_m - \nu_j)\tau + \delta_m\right]. \end{aligned} \tag{22}$$

4. Calculation of S_m from Eq. (12).

$$\begin{aligned} S_m &= \langle f, b_m \rangle = \langle f_{m-1}, b_m \rangle = \\ &= \sum_{j=1}^{m} \alpha_{mj} \langle f_{m-1}, e_j \rangle. \end{aligned} \tag{23}$$

But f_{m-1} is orthogonal to $b_1, \ldots, b_{m-1}$ and, therefore, to $e_1, \ldots, e_{m-1}$.

$$S_m = \alpha_{mm} F_m \exp i(\nu_m \tau + \delta_m) \tag{24}$$

After N steps we have

$$f = \sum_{j=1}^{N} S_j b_j + f_N = \sum_{s=1}^{N} e_s \sum_{j=s}^{N} S_j \alpha_{js} + f_N. \tag{25}$$

Neglecting f_N and comparing Eq. (25) with (5) we find

$$A_s = \sum_{j=s}^{N} \alpha_{jj} \alpha_{js} F_j \exp i \left[(\nu_j - \nu_s)\tau + \delta_j\right]. \tag{26}$$

The program for calculation of ν_j and A_j can be written in the form of a loop with N steps. The function f_j is always calculated and written to a file with the same sampling period as f. Our program ort.cc is still more general in incorporating the frequency corrections described in following section.

2. The Frequency Modified Fourier Transform (FMFT)

If we increase the number N of calculated frequencies and amplitudes to $N+1$ it follows from Eq. (26) that amplitudes previously determined are corrected with each additional step. On the other hand frequencies, once determined, are not changed. It is, however, clear that the maximum amplitude of $\varphi_k(\sigma)$ is shifted from ν_k mostly because of the existence of nearby frequencies (several ν_0 apart from ν_k). We shall assume that no two frequencies with significant amplitudes are closer than ν_0. This could lead to calculation of a false frequency somewhere in between. Our assumption is that the error of the MFT calculation of most important frequencies is small, and that a linear correction of the calculated frequencies can be given.

Let us first assume we have only two frequencies

$$f = C_1 \exp i(\omega_1 t + \beta_1) + C_2 \exp i(\omega_2 t + \beta_2), \tag{27}$$

with real positive $C_1 > C_2$; then

$$\varphi(\sigma) = \langle f, \exp i\sigma t \rangle = \sum_{j=1}^{2} C_j Q(\omega_j, \sigma) \exp i\left[(\omega_j - \sigma)\tau + \beta_j\right]. \tag{28}$$

Calculating ν_1 as σ for which the amplitude of $\varphi(\sigma)$ takes on a maximum would give $\nu_1 = \omega_1$ only if $Q(\omega_2, \sigma)$ is negligible for $\sigma \sim \nu_1$. We determine ν_1 from the condition

$$\frac{d}{d\sigma}|\varphi(\sigma)|_{\sigma=\nu_1} = 0, \tag{29}$$

Inserting Eq. (28) in (29) we obtain an equation satisfied by ν_1:

$$\begin{aligned} &C_1^2 \;\; Q(\omega_1, \nu_1)Q'(\omega_1, \nu_1) + C_2^2 Q(\omega_2, \nu_1)Q'(\omega_2, \nu_1) + \\ &C_1 \;\; C_2 \left[Q(\omega_1, \nu_1)Q'(\omega_2, \nu_1) + Q(\omega_2, \nu_1)Q'(\omega_1, \nu_1)\right] \\ &\cos \; \left[(\omega_1 - \omega_2)t + \beta_1 - \beta_2\right] = 0, \end{aligned} \tag{30}$$

where $Q'(\omega, \nu)$ is the derivative of $Q(\omega, \nu)$ with respect to the second argument. Introducing the error ϵ in approximating ω_1 by ν_1

$$\nu_1 = \omega_1 + \epsilon, \tag{31}$$

we can make a linearization of Eq. (30) in ϵ and obtain

$$\epsilon = -\frac{C_2 Q'(\omega_2, \omega_1) \cos\left[(\omega_1 - \omega_2)\tau + \beta_1 - \beta_2\right]}{C_1 Q''(\omega_1, \omega_1)}, \tag{32}$$

where $Q''(\omega_1, \omega_1)$ is the second derivative of $Q(\omega_1, \omega_1)$ again with respect to the second argument. We introduce

$$Q(y) = \frac{\sin y}{y} \frac{\pi^2}{\pi^2 - y^2}. \tag{33}$$

Then

$$Q'(y) = \frac{1}{y}\frac{\pi^2}{\pi^2 - y^2}\left[\cos y + \frac{\sin y}{y}\frac{3y^2 - \pi^2}{\pi^2 - y^2}\right] \tag{34}$$

and

$$Q''(0) = \left(\frac{2}{\pi^2} - \frac{1}{3}\right). \tag{35}$$

The final algorithm for the calculation of the corrections ϵ_j to ν_j (for N frequencies) to produce new frequencies $\nu_j - \epsilon_j$ is to use

$$\epsilon_j = \sum_{s>j}^{N} \frac{C_s Q'(y_{sj})}{C_j Q''(0)\tau} \cos(y_{sj} + \beta_s - \beta_j), \tag{36}$$

where

$$y_{sj} = (\nu_s - \nu_j)\tau. \tag{37}$$

Real values C_s, β_s are determined from relation

$$A_s = C_s \exp i\beta_s \tag{38}$$

and $Q'(y)$, $Q''(0)$ are given by Eqs. (34), (35) and (37).

The FMFT method consists of the MFT as described in Section 1 followed by a correction of frequencies via Eq. (36) a and new determination of the amplitudes by again employing the algorithm of Section 1, with only the frequencies given a priori.

An alternative way to correct the frequencies determined by the MFT is provided by the following simple method. One first applies the MFT to the original function $f(t) = \sum C_j \exp i(\nu_j t + \beta_j)$ and gets its development $f'(t) = \sum C'_j \exp i(\nu'_j t + \beta'_j)$. The error in the frequency determination is the small quantity

$$\epsilon_k = \epsilon_k(\nu_j, C_j, \beta_j) = \nu_k - \nu'_k. \tag{39}$$

Similarly, the errors in the amplitudes and phases are also small quantities. A second MFT applied to $f'(t)$ leads again to a slightly different development $f''(t) = \sum C''_j \exp i(\nu''_j t + \beta''_j)$ with small errors in the frequencies

$$\epsilon'_k = \epsilon_k(\nu'_j, C'_j, \beta'_j) = \nu'_k - \nu''_k. \tag{40}$$

Substituting $\nu'_j = \nu_j - \epsilon_j$ into Eq. (40), and developing function $\epsilon_k()$ into a Taylor series (on the assumption that its first derivatives are small quantities), one can neglect the linear and higher order terms and write $\epsilon_k = \epsilon'_k$. Thus, in the above approximation, $\nu_k = \nu'_k + \epsilon'_k$, whose right-hand side can be simply evaluated since ν'_k and ϵ'_k are known. A similar approach provides the amplitudes $C_j = C'_j + (C'_j - C''_j)$ and phases $\beta_j = \beta'_j + (\beta'_j - \beta''_j)$.

In order to distinguish between the two FMFT methods described in this section, we call the first FMFT_1 and the second FMFT_2.

TABLE I
The reconstruction of frequencies ("/year) by MFT and FMFT

j	Original	MFT	$FMFT_1$	$FMFT_2$
1	4.2488163	4.2488183	4.2488163	4.2488163
2	28.2206942	28.2206916	28.2206942	28.2206942
3	3.0895148	3.0895150	3.0895149	3.0895148
4	52.1925732	52.1925732	52.1925732	52.1925732
5	27.0613982	27.0611601	27.0613983	27.0613957
6	29.3799573	29.3802248	29.3799577	29.3799611
7	28.8679427	28.8679114	28.8679409	28.8679426
8	27.5734578	27.5734593	27.5734593	27.5734579
9	5.4070444	5.4070414	5.4070413	5.4070444
10	0.6671228	0.6671228	0.6671228	0.6671228

3. Simple Test of the Method

Both $FMFT_1$ and $FMFT_2$ were programmed and first tested on the function

$$f(t) = k(t) + ih(t) = \sum_{j=1}^{10} C_j \exp i(\nu_j t + \beta_j), \tag{41}$$

where the coefficients ν_j, C_j, β_j were taken from a development of $f = e \exp i\tilde{\omega}$ for Jupiter ($\tilde{\omega}$ is its longitude of perihelion and e its eccentricity) as given by Laskar (1990).

The sampling period was 120 000 days and the number of sampling points was 32 768 so that the time interval 2τ for reconstruction of the frequencies was about 10.8 Myr. It is possible to distinguish frequencies $\nu_0 = \pi/\tau = 0.25$"/year apart, which is well sufficient as the closest frequencies, ν_8 and ν_5, are separated by 0.51"/year.

Table 1 shows the original and reconstructed frequencies using MFT, $FMFT_1$ and $FMFT_2$. It is clear that both $FMFT_1$ and $FMFT_2$ improve the result. The first frequency is determined exactly up to seven decimal digits and close frequencies 5 – 8 are computed by several orders more precisely than by MFT. This improvement is significant in amplitudes and phases as well, and leads to better decomposition of $f(t)$. $FMFT_1$ and $FMFT_2$ are roughly comparable in the quality of reconstruction. Frequencies 5 and 6 are better determined by $FMFT_1$, while frequencies 7, 8 and 9 are better from $FMFT_2$. There are slight differences in the computed amplitudes and phases as well, but both methods can be considered to work with similar precision.

TABLE II
The g and s for first 50 asteroids, for which a good decomposition was obtained

No.	g "/year	s "/year	No.	g "/year	s "/year
1	54.07464	-59.10736	24	131.99292	-103.31862
3	43.63334	-61.22194	26	55.26366	-55.66960
4	36.48091	-39.20237	27	38.09601	-43.25415
7	37.94909	-46.06354	28	57.32945	-66.64166
9	38.36402	-41.63721	29	49.43937	-46.98591
11	40.41837	-42.82523	30	38.18240	-40.13398
12	33.71960	-40.34126	34	55.14321	-58.69910
13	35.72530	-45.27553	37	54.53217	-57.80620
14	48.58297	-56.04458	39	53.70462	-56.40034
15	42.49709	-51.80465	40	34.27385	-34.75085
16	76.75683	-73.06070	42	39.67433	-46.66578
17	43.41116	-45.97697	44	40.89591	-45.98494
19	41.50964	-44.81832	45	56.46531	-58.19380
20	40.47798	-44.74191	46	58.62063	-48.61045
21	41.12942	-44.24252	47	70.20357	-69.61741
23	48.42881	-64.34235	48	105.30808	-81.97421

4. An Application to Asteroids

The time evolution of $e \exp \tilde{\omega}$ and $\sin I/2 \exp \Omega$ (I is the inclination and Ω is the longitude of node) for the first 50 asteroids under the force of four outer planets was obtained by numerical integration using the MSI integrator (Šidlichovský and Nesvorný, 1994). Initial conditions were taken from Milani's `oscelnum.92` at `gauss.dm.unipi.it` and corrected to the barycenter of the inner solar system. The integration step was 10 days, the output was filtered by sequential application of two low-pass filters (Nesvorný and Ferraz-Mello, 1996) and sampled at 120 000 days. The number of points (32 768) implies a time interval over which the decompositions were computed of about 10.8 Myr. During the integration, the distance to Jupiter was checked every 10 days. All the studied asteroids stayed farther than 1 AU from Jupiter, ensuring the accuracy of the integrator. The coefficients of decomposition were determined by FMFT_2 and in several cases checked by FMFT_1.

In Tab. 2, we give the main secular frequencies g in $e \exp \tilde{\omega}$ and s in $\sin I/2 \exp \Omega$ for those asteroids among the first 50 for which a good decomposition was obtained. The others, for which the frequencies were not shown, are likely to be in some secular resonance or at least in its close vicinity, and their decomposition can not be obtained in 10.8 Myr due to presence of very low frequencies. Similarly, chaoticity on a shorter time scale than 10.8 Myr would also not allow for a good orbital decomposition.

TABLE III
The coefficients for Ceres

	$e \exp \tilde{\omega}$			$\sin(I/2) \exp \Omega$		
j	ν_j "/year	C_j	β_j deg	ν_j "/year	C_j	β_j deg
1	54.07464	0.115573	152.273	-59.10736	0.080378	99.223
2	4.24465	0.030770	29.497	-60.94878	0.014701	338.786
3	28.23886	0.019704	300.863	-57.26561	0.014650	39.140
4	52.23318	0.008700	32.124	-0.00001	0.013683	107.648
5	-172.28929	0.003359	226.036	-58.42389	0.008938	129.248
6	53.39115	0.001435	302.504	-59.79006	0.008786	248.077
7	-174.13056	0.001307	105.332	-26.33878	0.005271	313.763
8	3.08658	0.001257	119.333	-57.95748	0.001899	201.575
9	55.91697	0.001077	272.615	-61.62944	0.001780	124.544
10	-170.44725	0.001017	165.590	-56.57933	0.001459	64.909

Nobili et al. (1989) discusses the effect of the inner planets on the fundamental frequencies of the outer solar system. If the quadrupole force term of the inner planets is not included in the model, the fundamental frequency g_5 of Jupiter differs by about 0.012 "/yr from the real value. Asteroid g will differ even more and the values in Tab. 2 must be regarded, although very precise in the frame of studied model, as preliminary and underestimating the real frequencies. Indeed, our recent integration of the first 20 asteroids together with seven planets (Venus to Neptune with the mass of Mercury added to Sun) led to $g = 54.25081$ "/yr and $s = -59.23250$ "/yr for Ceres, which in comparison with Tab. 2 indicates 0.18 "/yr and 0.13 "/yr differences. Similarly, the frequencies of other asteroids differ at order 0.1 "/yr. Including of the inner planets is thus necessary for good estimation of the real frequencies and thus computations which include the full effect of Venus, Earth, and Mars are in progress.

4.1. Asteroids with good decomposition

The coefficients for Ceres' $e \exp i\tilde{\omega}$ and $\sin(I/2) \exp i\Omega$ are shown in Tab. 3. We checked the quality of both decompositions by comparison with the filtered output of the integration over 10.8 Myr. In the case of $f(t) = k(t) + ih(t) = e \exp i\tilde{\omega}$, the maximum error in k was 0.009 and in h 0.008. We show the first 1 Myr in Fig. 1. We decided to consider the Ceres as a marginal case of an asteroid with good decomposition and listed in Tab. 2 only those asteroids, which have the maximum errors over 10.8 Myr in both k and h less than 0.01.

Additional prolonged integration showed that the maximum error of Ceres' decomposition over 43.2 Myr is about 0.03 in both h and k, which is

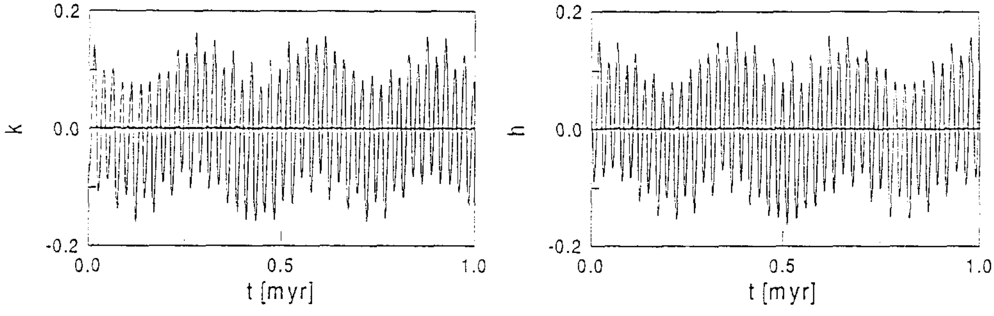

Fig. 1. Comparison between original and reconstructed k and h evolution for Ceres. Here the filtered output of the integrated k and h is plotted with its difference in the time evolution obtained from Tab. 3. The difference is the rough line at almost zero. The maximum errors on this interval in k and h are both about 0.006.

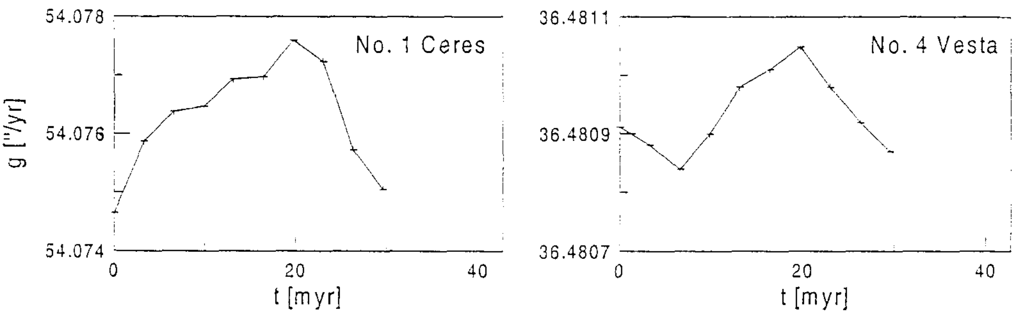

Fig. 2. Time dependence of g for Ceres (left) and No. 4 Vesta (right)

a rather large disagreement. Moreover, the decomposition obtained from the whole 43.2 Myr, which would better distinguish the low frequencies, does not represent the real filtered evolution any better. We infer that the motion of Ceres can't be properly represented by a decomposition with fixed frequencies and claim that the frequencies are changing due to orbital chaoticity. The trajectory of No. 4 Vesta (listed in Tab. 2) is much better represented by the decomposition obtained from the 10.8 Myr interval. Indeed, the error over 43.2 Myr is only 0.004.

In Fig. 2 we show the time dependence of g for Ceres and Vesta. We computed several FMFT_2 while shifting the interval for the frequency computation by 10 000 points (roughly by 3.28 Myr). Notice that g for Ceres changes by one order of magnitude more than Vesta's g. This is the reason why the decomposition for Vesta with fixed $g = 36.48091$ "/year represents its trajectory more precisely than fixed $g = 54.07464$ "/year of Ceres.

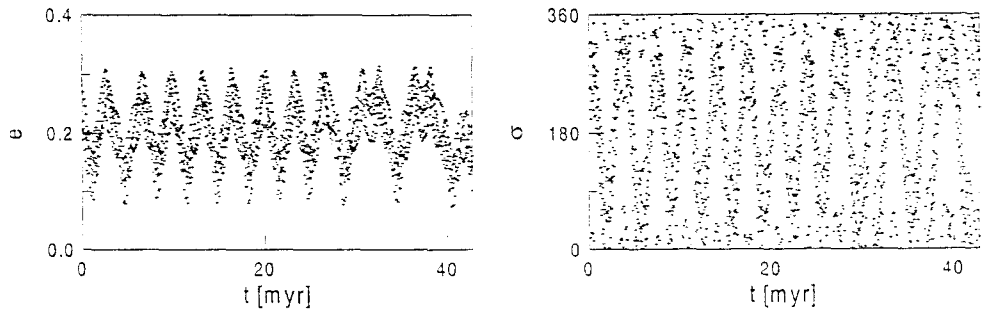

Fig. 3. Eccentricity and resonant angle $\sigma = \tilde{\omega} + \tilde{\omega}_J - 2\tilde{\omega}_S$ for No. 5 Astraea

4.2. Asteroids with bad decomposition

Asteroid No. 5 Astraea is, at least in our model neglecting the quadrupole moment of the orbits of the inner planets, is in the secular resonance $g + g_5 - 2g_6 \sim 0$, where g, g_5 and g_6 are the main secular frequencies of the asteroid's, Jupiter's and Saturn's $e \exp i\tilde{\omega}$. Fig. 3 (right) shows the resonant angle $\tilde{\omega} + \tilde{\omega}_J - 2\tilde{\omega}_S$ versus time. The period of libration is about 3.5 Myr. It is impossible to obtain, in this case, a good decomposition from only 10.6 Myr, since the low frequency forms important harmonics close to g, as discussed in Nesvorný and Ferraz-Mello (1996). Moreover, at about 30 Myr, the resonant angle begins to circulate indicating a strong orbital chaoticity.

The asteroids not shown in Tab. 2. likely have characteristic similar to Astraea; they are in or very near the secular resonances and low frequencies do not allow us to obtain a good decomposition of their trajectories.This was tested and verified for several of them. For instance, asteroids No. 6 Hebe and No. 8 Flora fulfil $g - g_6 - g_7 \sim 0$, where g_7 is the secular frequency of longitude of perihelion of Uranus.

5. Summary

Our preliminary calculations show what behaviour one may expect when investigating asteroids with the MFT. If the inner planets are taken into account the fundamental frequencies of the system change slightly and conclusions about secular resonances from a model with outer planets only might not be correct. The more exact calculations taking into account the inner planets are in progress. Such calculations are time consuming as the step of numerical integration must be much shorter.

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References

Laskar, J.: 1988, 'Secular evolution of the solar system over 10 million years', *Astron. Astrophys.* **198**, pp. 341–362

Laskar, J.: 1990, 'The Chaotic Motion of the Solar System: A Numerical Estimate of the Size of the Chaotic Zones', *Icarus* **88**, pp. 266–291

Laskar, J., Froeschlé , Cl. and Celleti, A.: 1992 ,' The Measure of Chaos by the Numerical Analysis of the Fundamental Frequencies. Application to the Standard Mapping.', *Physica D* **56**, pp. 253–269

Laskar, J.: 1993, 'Frequency analysis for multi-dimensional systems. Global dynamics and diffusion', *Physica D* **67**, pp. 257–281

Nesvorný, D. and Ferraz-Mello, S.: 1996, 'Chaotic diffusion in the 2/1 asteroidal resonance', submitted to *Astron. Astrophys.*

Nobili, A.M., Milani, A. and Carpino, M.: 1989, 'Fundamental frequencies and small divisors in the orbits of the outer planets', *Astron. Astrophys.* **210**, pp. 313–336

Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P.: 1992, Numerical Recipes in C, Cambridge University Press, Cambridge, UK

Šidlichovský, M. and Nesvorný, D.: 1994, 'Temporary capture of grains in exterior resonances with the Earth: Planar circular restricted three-body problem with Poynting-Robertson drag', *Astron. Astrophys.* **289**, pp. 972–982