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Lyapunov's Direct Method in the Estimation of the Hausdorff Dimension of Attractors

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Abstract. This paper surveys results of the authors and others concerning estimates for the Hausdorff dimension of strange attractors, particularly in the case of (generalized) Lorenz systems and Rössler systems. A key idea is the interpretation of Hausdorff measure as an analogue of a Lyapunov function.

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1. Introduction

A century ago, in 1892, in Russia the thesis of A. M. Lyapunov entitled "The general problem of the stability of motion" was published. The thesis contained many fruitful ideas and profound mathematical results. So it became a certain landmark for the stability theory as well as for the whole qualitative theory of ordinary differential equations. Particular attention of mathematitians was attracted to the method which was worked out in the thesis. Later on, it was called Lyapunov's second method or Lyapunov's direct method. This method turned out to be extremely effective and a rather universal tool for investigating differential equation solutions. There are many monographs in which this method is expounded and developed, both in Russia [1–6] and abroad [7–12].

Lyapunov's second method is based on the so-called Lyapunov function, which possess certain special properties.

Let us consider the simplest theorem using Lyapunov function. Let there be given the system

$$\frac{dx}{dt} = f(x), \tag{1.1}$$

where $f(x)$ is a continuous vector function satisfying the Lipschitz condition in an open domain $D \subset R^n$.

Theorem 1.1, [1]. *Let the domain D contain the origin $x = 0$ and $f(0) = 0$. Suppose that there exists in D a continuously differentiable positive (negative) definite function $v(x)$ with a negative (positive) semi–definite derivative with respect to the system (1.1). Then the equilibrium state $x = 0$ is Lyapunov stable.*

It will be recalled that the derivative with respect to (1.1) is defined by $\dot{v}(x) = (grad\, v(x))^* f(x)$. By the asterisk, we denote the Hermitian conjugation (here, of course, the asterisk means the transpose of a vector).

The requirements on Lyapunov functions, which are contained in Theorem 1.1, are typical for the classical results of Lyapunov's direct method. The latter are devoted to various problems of qualitative theory of differential equations (different types of stability, convergence, dissipativity, etc.). But one should not think that the properties of Lyapunov functions described in Theorem 1.1 are indispensable for them.

Consider again the system (1.1) on $D = R^n$. We say that the system (1.1) is *monostable* if any bounded solution of (1.1) tends to the equilibrium set $\{x \in R^n \mid f(x) = 0\}$ as $t \to \infty$.

Theorem 1.2, [13]. *Suppose a function $v(x)$ is continuous in R^n and has the following properties:*

(1) for any solution $x(t)$ of (1.1), the function $v(x(t))$ does not increase with respect to t;

(2) if for a certain bounded solution $x(t)$, the identity $v(x(t)) \equiv \text{const}$ is valid, then $x(t)$ is an equilibrium state of (1.1).

Then system (1.1) is monostable.

Note that in Theorem 1.2 we assume neither positive or negative definitess of $v(x)$ nor its differentiability.

In the sixties, for certain finite dimensional dynamical systems, the phenomenon of the so–called strange attractors was discovered. A strange attractor is a compact invariant set in the phase space, all the trajectories on which are locally unstable. The discovery aroused interest to various dimension–like characteristics of attractors. This interest was stimulated not only by intense investigation of chaotic oscillations in finite dimensional systems [14–21] but also by O.A. Ladyzheskaya's research into the attractors of Navier–Stokes equations [22–23].

Among various dimension–like characteristics, the Hausdorff dimension is the most popular. For any compact subset K of R^n, this dimension can be defined as follows [24]. Let $d \geq 0$, $\varepsilon > 0$. Let us cover K by spherical balls with radius $r_i \leq \varepsilon$ and denote

$$\mu(K, d, \varepsilon) = \inf \sum_i r_i^d,$$

the infimum being taken over all finite covering of K.

Hausdorff d–measure is defined to be

$$\mu(K,d) = \lim_{\varepsilon \to 0} \mu(K,d,\varepsilon).$$

Hausdorff showed that there existed a critical value d_1 such that $\mu(K,d) = 0$ for all $d > d_1$ and $\mu(K,d) = +\infty$ for all $d < d_1$. We denote d_1 by $\dim K$ and call it the *Hausdorff dimension* of compact K.

It is rather difficult, it turned out, to determine experimentally the Hausdorff dimension of attractors. That is why other dimensions [25–27] were introduced which could, with comparative ease, be calculated with the help of a computer. On the other hand the problem of analytical estimates of the Hausdorff dimension became rather urgent.

Essential progress in the establishment of upper bounds for the Hausdorff dimension is connected with the theorem of A. Douady and J. Oesterlé. The latter can be formulated as follows.

Let K be a compact invariant set of the system (1.1). Let us assume that the right part of (1.1) is a continuously differentiable vector function. Let us denote by $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ the eigenvalues of the symmetrized Jacobian matrix $\frac{1}{2}(\partial f^*/\partial x + \partial f/\partial x)$.

Theorem 1.3, [28, 29]. *Suppose that there exist an integer* $d_0 \in [0,n]$ *and a number* $s \in [0,1]$ *such that the inequality*

$$\lambda_1(x) + \cdots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x) < 0 \tag{1.2}$$

holds for all $x \in K$. *Then* $\dim K < d_0 + s$.

In [30], a simple estimate for the dimension of the attractors of the famous Lorenz system [14] has been obtained. Its proof is directly based on (1.2). A number of results [29, 31–33] have been obtained by joint use of the Douady–Oesterlé and Yakubovich–Kalman frequency theorem. The latter is widely used for stability investigations of automatic control systems [13]. We must also notice the paper [34–36], where the upper bounds of the dimension of the attractors of the Lorenz system are given. Their demonstration is also closely connected with the ideas and results of paper [28].

A new view on Hausdorff dimension is proposed in [37, 38], where the Hausdorff measure is considered as an analogue of a Lyapunov function. Such a conception gives the opportunity to introduce the Lyapunov function into the estimates of Hausdorff dimension [37–39] and to prove a number of theorems involving the known results. Of these theorems we formulate here the one which generalizes Theorem 1.3.

Theorem 1.4, [37]. *Suppose that there exist a continuously differentiable function $v(x)$, an integer $d_0 \in [0, n]$, and a number $s \in [0, 1]$ such that the inequality*

$$\lambda_1(x) + \cdots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x) + \dot{v}(x) < 0$$

holds for all $x \in K$. Then $dim\, K < d_0 + s$.

Note that Theorem 1.4 does not assume positive or negative definiteness either for $v(x)$ or for $\dot{v}(x)$.

Thus, one can affirm that the set of analytical methods presently developed for establishing the upper bounds of the Hausdorff dimension is one of the branches of modern stability theory.

On the other hand, the idea of considering the Hausdorff measure of compact sets, mapped by a shift operator along trajectories, as an analogue of Lyapunov function gave the opportunity to obtain certain new results [37–39] in classical stability theory, close form to the upper bounds of the Hausdorff dimension. Let us illustrate this fact by the following assertion.

Theorem 1.5, [38]. *Let the system (1.1) possess only isolated equilibrium states. Let D be a bounded simply-connected open domain in R^n. Suppose that its boundary is crossed strictly inwards by every solution of (1.1) which meet it. Suppose also that there exists a continuously differentiable function $v(x)$ such that the inequality*

$$\lambda_1(x) + \lambda_2(x) + \dot{v}(x) < 0$$

is valid for all $x \in \overline{D}$. Then each solution of (1.1) in D converges to an equilibrium state.

When D is also a globally absorbing set for the system (1.1), Theorem 1.5 ensures its complete stability.

It will be recalled that complete stability is an analogue of global asymptotic stability for systems with multiple equilibria. The system (1.1) is called *completely stable* if any of its solution $x(t)$ tends to one of equilibrium states, as t goes to infinity [13, 40]. We must note here that the notion of complete stability that belongs to stability theory proves to be useful when investigating problems connected with chaos. Further, we shall demonstrate this assertion with the help of a certain physical system.

The first impulse to such a conception of the Hausdorff dimension was given by the notion of weakly contracting systems [41–43] introduced by Yu.S. Il'yashenko. It is clear today that a theory of weakly contracting operators is possible which would involve upper bounds of dimension as well as orbital stability and global stability [37, 44].

The structure of this paper is as follows.

In Sections 2 and 3, we present two three–dimensional systems and use them to illustrate all the main theorems proved in the paper. In Section 2, we introduce a system which is a generalization of the Lorenz system. We consider its simplest properties: by means of Lyapunov functions, the theorem about global stability is proved, the dissipativity of the system is established, and certain estimates of the dissipativity region are obtained. In Section 3, we give one of the well–known Rössler systems.

In Section 4, we expound a method of approach to estimating the Hausdorff dimension which gives the opportunity of introducing Lyapunov functions into them. Then several theorems about convergence and point–wise monostability are proved. These are obtained as consequences of the estimates established. The theorems involve well–known results. We illustrate the theorems by Rössler's system and a certain concrete specimen of generalized Lorenz equations. At the end of the section, we formulate two corollaries about estimating the Hausdorff dimension and convergence condition. Their statements are suitable for frequency methods.

In Section 5, we prove several frequency theorems about the Hausdorff dimension and convergence. On the basis of the latter, corresponding theorems for generalized Lorenz equations are proved. These theorems give the opportunity for obtaining certain results about the Lorenz system which we compare with the results of other authors. Apparently, analytical estimates of the dimension of attractors of the Lorenz system and the conditions of complete stability of the latter, are the best among other estimates and conditions currently known.

Finally, in Section 6, an application of the main theorems proved for generalized Lorenz equations are illustrated by the following concrete examples of physical systems: rigid–body rotation in a resisting medium, convection of the fluid contained within an ellipsoidal rotating cavity, interaction between waves in plasma.

2. Generalized Lorenz Equation

Consider the following three–dimensional system of ordinary differential equations

$$\begin{aligned}\frac{dx}{dt} &= -dx + dy - ayz,\\ \frac{dy}{dt} &= rx - y - xz,\\ \frac{dz}{dt} &= -bz + xy,\end{aligned} \tag{2.1}$$

where d, b, r are positive numbers and a is an arbitrary real number. For $a = 0$, this system coincides with the widely known Lorenz system for which the chaotic oscillations were discovered and investigated for the first time by numerical integration [14, 45]. Numerical experiments showed that, for $a \neq 0$, in system (2.1) there also exist strange attractors [21, 46].

The Lorenz system is interesting not only because of its strange attractors. It is also important that the Lorenz system which has appeared as a model of atmosphere convection may serve as a model for other physical processes. Recently, the Lorenz model was intensively investigated both by numerical and analytical methods. So since many systems describing various natural phenomena may be reduced to the Lorenz system, it is possible to apply numerous results of its mathematical investigation to these original systems. At the end of the section, we shall illustrate this idea by a simple system arising from the dynamics of nematic liquid crystals.

It should be noticed that the generalized Lorenz equations (2.1) essentially embrace a wider set of physical systems. We shall consider some of them in the last section.

Let us now proceed to investigate the simplest properties of (2.1).

We shall determine the equilibrium states of the system and their number depending on the values of the parameters; show that in the case of the unique equilibrium state, system (2.1) is globally asymptotically stable; prove the dissipativity of the system and establish some estimates of its dissipativity region.

Suppose $a = 0$. Then if $r < 1$, system (2.1) has the unique equilibrium $(0, 0, 0)$ and if $r > 1$ it has three equilibriums: $(0, 0, 0)$ and

$$\left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$$

[14, 45].

In the case where $a \neq 0$, it is easy to verify that the following assertion is true. If

$$d + a > 0, \quad r < 1 \quad \text{or} \quad d + a < 0, \quad r < \frac{d}{a} + 2\sqrt{-\frac{d}{a}}$$

then (2.1) has the unique equilibrium $(0,0,0)$; if $r > 1$, then (2.1) has three equilibriums: $(0,0,0)$ and $(\pm x_1, \pm y_1, z_1)$; if

$$d + a < 0, \quad \frac{d}{a} + 2\sqrt{-\frac{d}{a}} < r < 1,$$

then (2.1) has five equlibriums: $(0,0,0)$, $(\pm x_1, \pm y_1, z_1)$, $(\pm x_2, \pm y_2, z_2)$. Here

$$x_k = \frac{db\sqrt{\zeta_k}}{db + a\zeta_k}, \quad y_k = \sqrt{\zeta_k}, \quad z_k = \frac{d\zeta_k}{db + a\zeta_k} \qquad (k = 1,\ 2)$$

and the real numbers ζ_1, ζ_2 are defined as

$$\zeta_{1,2} = \frac{db}{2a^2}\left[a(r-2) - d \pm \sqrt{(ar-d)^2 + 4ad}\right].$$

For $a = 0, r < 1$, the equilibrium $(0,0,0)$ is Lyapunov asymptotically stable [14, 45]. Since the linearized system corresponding to (2.1) in the origin does not depend on a, then for $a \neq 0, r < 1$ the origin is also Lyapunov asymptotically stable. The following theorem about the global asymptotic stability generalizes for the case of an arbitrary a, the result of which is well known for $a = 0$ [14, 45].

Theorem 2.1, [32]. *The system* (2.1) *is globally asymptotically stable if one of the following two hypotheses holds*

(1) $d + a > 0$, $r < 1$;

$d + a < 0$, $r < \frac{d}{a} + 2\sqrt{-\frac{d}{a}}$.

Proof. Assume that the hypothesis (1) is valid. Let

$$v_1(x,y,z) = \frac{1}{2}[x^2 + dy^2 + (d+a)z^2].$$

Then

$$\dot{v}_1 = -b(d+a)z^2 - dx^2 + d(r+1)xy - dy^2 < 0$$

for any $(x,y,z) \neq 0$ because the quadratic form $x^2 - (r+1)xy + y^2$ is positive definite for $r < 1$.

Assume that the hypothesis (2) is valid. In this case, the inequality

$$r^2 - 2\frac{d}{a}r + \frac{d^2}{a^2} + 4\frac{d}{a} < 0$$

is true. Hence, there exists a small positive number ε such that the inequality

$$r^2 + 2\frac{d}{\widetilde{a}}r + \frac{d^2}{\widetilde{a}^2} - 4\frac{d}{\widetilde{a}} < 0 \tag{2.2}$$

holds. Here $\widetilde{a} = -a + \varepsilon$. Let

$$v_2(x, y, z) = \frac{1}{2}(x^2 + \widetilde{a}y^2 + \varepsilon z^2).$$

Then

$$\dot{v}_2 = -\varepsilon bz^2 - dx^2 + (d + \widetilde{a}r)xy - \widetilde{a}y^2 < 0$$

for any $(x, y, z) \neq 0$ becausethe quadratic form $dx^2 - (d + \widetilde{a}r)xy + \widetilde{a}y^2$ is positive definite due to the inequality (2.2). □

We infer from the theorem proved that the chaotization of the trajectories of (2.1), as well as that of the Lorenz system, is possible only when there are several equilibrium states.

We shall now show that (2.1) is dissipative and get some estimates for its dissipativity region which are necessary for the establishment of the complete stability conditions and also for the localization of its attractors in the phase space.

Let

$$v(x, y, z) = \frac{1}{2}\left[x^2 + \delta y^2 + (a + \delta)\left(z - \frac{d + \delta r}{a + \delta}\right)^2\right],$$

where δ is an arbitrary positive number such that $a + \delta > 0$. We have, for an arbitrary solution $\big(x(t), y(t), z(t)\big)$ of the system (2.1)

$$\begin{aligned}\dot{v} &= -dx^2 - \delta y^2 - b(a + \delta)z^2 + b(d + \delta r)z \\ &\leq -dx^2 - \delta y^2 - \frac{1}{2}b(a + \delta)\left(z - \frac{d + \delta r}{a + \delta}\right)^2 + \frac{1}{2}b\frac{(d + \delta r)^2}{a + \delta}.\end{aligned}$$

Denoting $c = \min(d, 1, \frac{1}{2}b)$, we get

$$\dot{v} \leq -2cv + \Gamma,$$

where

$$\Gamma = \frac{1}{2}b\frac{(d + \delta r)^2}{a + \delta}.$$

Hence

$$\overline{\lim_{t\to\infty}}\, v\,(x(t), y(t), z(t)) \le \frac{1}{2c}\Gamma$$

and all trajectories of the system (2.1) ultimately enter the ellipsoid

$$x^2 + \delta y^2 + (a+\delta)\left(z - \frac{d+\delta r}{a+\delta}\right)^2 \le \frac{b(d+\delta r)^2}{2c(a+\delta)}$$

and remain in it thereafter.

It should be noticed that the dissipativity ellipsoid obtained here may be improved by introducing several varied parameters into the function v just in the same way as it was done in [47, 49]for the Lorenz system. To make the dissipativity region of (2.1) more precise, we may also use additional quadratic forms as was done when investigating the Lorenz system [50, 51]. We should like to pay attention to a series of papers [52–59] and to papers [60, 61]. In [52–59] an approximation of attractors of the Lorenz system is developed with the help of estimates of solutions of two–dimensional ordinary differential equations. In [60, 61], an algebraic method of approach to the approximation of attractors is suggested. In this paper, however, in order to effectively use the estimates of the disposition of dissipativity region and of attractors of (2.1), we shall limit ourselves to the following simple lemmas.

Denote

$$\ell = \begin{cases} 1, & \text{for } b \le 2 \\ \frac{b}{2\sqrt{b-1}}, & \text{for } b \ge 2. \end{cases} \tag{2.3}$$

Lemma 2.1. *Let* $(x(t), y(t), z(t))$ *be an arbitrary solution of system* (2.1). *Then we have the estimate*

$$\overline{\lim_{t\to\infty}}\,[y^2(t) + (z(t) - r)^2] \le \ell^2 r^2.$$

Proof. Let

$$v(y,z) = \frac{1}{2}[y^2 + (z-r)^2].$$

For any $\mu \in (0, \mu_0)$, where $\mu_0 = min\,(1, b)$, we have

$$\begin{aligned}\dot{v} + 2\mu v =&(\mu - 1)y^2 + (\mu - b)z^2 - 2r\left(\mu - \frac{b}{2}\right)z + \mu r^2 \\ \le&(\mu - b)\left[z - \frac{r(\mu - \frac{b}{2})}{\mu - b}\right]^2 - \frac{r^2(\mu - \frac{b}{2})^2}{\mu - b} + \mu r^2 \\ \le&\left[\mu - \frac{(\mu - \frac{b}{2})^2}{\mu - b}\right] r^2 = \frac{b^2 r^2}{4(b-\mu)}.\end{aligned}$$

Hence

$$\overline{\lim_{t\to\infty}}\, v(y(t), z(t)) \le \frac{b^2 r^2}{8(b-\mu)\mu}.$$

Minimizing with respect to μ the right side of the last inequality, we get the estimate we have to prove. □

It follows from Lemma 2.1 that the system (2.1) possesses of the dissipativity region D^* for which the following inclusion is true

$$D^* \subset \{x| -\infty < x < \infty\} \times \overline{D}_1, \tag{2.4}$$

where $D_1 = \{y, z \mid y^2 + (z-r)^2 < \ell^2 r^2\}$.

Lemma 2.2. *Suppose that* $2d - b \ge 0$ *and* $a(b-2) \ge 0$. *Let* $\big(x(t), y(t), z(t)\big)$ *be an arbitrary solution of* (2.1). *Then the following estimate is true*

$$\underline{\lim_{t\to\infty}}\, [2(d-ar)z(t) - x^2(t) + ay^2(t)] \ge 0. \tag{2.5}$$

Proof. Let

$$v(x,y,z) = (d-ar)z - \frac{1}{2}x^2 + \frac{a}{2}y^2.$$

We have

$$\dot{v} = -b\left[(d-ar)z - \frac{2d}{b}\frac{1}{2}x^2 + \frac{2}{b}\frac{a}{2}y^2\right] \ge -bv.$$

This implies the inequality (2.5). □

The following two remarks concern the dissipativity region of the Lorenz system. If $a = 0$, then joining up the estimates of Lemmas 2.1 and 2.2 allows us to assert that there exists for (2.1) the dissipativity region D^* for which the following inclusion is true

$$D^* \subset \{x \mid -\infty < x < \infty\} \times (\overline{D}_1 \cap \{z \mid z \ge 0\}). \tag{2.6}$$

It should be noticed that the inclusion (2.6) also follows from [48, 49, 52]. Therefore, the estimates of Lemmas 2.1 and 2.2 may be considered as an extension of the corresponding results of these papers for the case $a \ne 0$.

Further, the inclusion (2.6) will be used when studying two questions: (a) estimates of the Hausdorff dimension of attractors; (b) conditions of the complete stability. It follows from the theorem of V.I. Yudovich [47] that the Lorenz system is completely stable for $2d - b < 0$. Thus, when investigating questions (a) and (b), we may assume without loss of generality that $2d - b \geq 0$. That is why, without further special remarks, we shall employ (2.6) for the Lorenz system without restrictions on the system parameters which are contained in Lemma 2.2.

In conclusion of this section we consider a system which arises in the dynamics of nematic liquid crystals [62, 63]

$$\frac{d}{dt}\mathbf{n} = [\Omega, \mathbf{n}],$$
$$I\frac{d}{dt}[\mathbf{n}, \frac{d}{dt}\mathbf{n}] = -\gamma\Omega + \chi(\mathbf{H}, \mathbf{n})[\mathbf{n}, \mathbf{H}],$$

where $\mathbf{n}, \Omega, \mathbf{H}$ are three–dimensional vectors; I, γ, χ are positive numbers; and $(\cdot, \cdot)$, and $[\cdot, \cdot]$ are scalar and vector products, respectively. In [62], these equations were numerically investigated by means of the following scalar variables: $\sigma = (\mathbf{H}, \mathbf{n})$, $\eta = (\mathbf{H}, [\Omega, \mathbf{n}])$, $\zeta = |\Omega|^2$. In these variables, we have

$$\begin{aligned} \frac{d\sigma}{dt} &= \eta, \\ \frac{d\eta}{dt} &= -\gamma I^{-1}\eta - \zeta\sigma + \chi I^{-1}\sigma(|\mathbf{H}|^2 - \sigma^2), \\ \frac{d\zeta}{dt} &= -2\gamma I^{-1}\zeta + 2\chi I^{-1}\sigma\eta. \end{aligned} \tag{2.7}$$

Making another change of variables

$$\sigma = \frac{\gamma}{4\sqrt{I\chi}}x, \quad \eta = \frac{\gamma^2}{8I\sqrt{I\chi}}(y - x), \quad \zeta = \frac{\gamma^2}{4I^2}\left(z - \frac{x^2}{4}\right), \quad t = \frac{2I}{\gamma}\tau,$$

close to that used in [47] we get

$$\begin{aligned} \frac{dx}{d\tau} &= -x + y, \\ \frac{dy}{d\tau} &= (1 + 4I\gamma^{-2}\chi|\mathbf{H}|^2)x - y - xz, \\ \frac{dz}{d\tau} &= -4z + xy. \end{aligned}$$

Thus, system (2.7) is reduced to the Lorenz system with the following parameters: Prandtl number $d = 1$, geometrical parameter $b = 4$, Rayleigh number $r = 1 + 4I\gamma^{-2}\chi|\mathbf{H}|^2$.

Obviously, Yudovich's condition of the complete stability $2d - b < 0$ is true. Consequently, any solution of the system (2.7) tends to one of the equilibrium states

$(0,0,0)$, $(\pm|\mathbf{H}|,0,0)$ as $t \to \infty$. Thus, the statement about the nontriviality of the dynamics described by (2.7), which has been proposed on the basis of numerical simulation, is not confirmed.

3. Rössler System

Let us consider one more three–dimensional system suggested by Rössler [17], which as well as the generalized Lorenz equations, possesses the complicated behavior of solutions

$$\begin{aligned} \frac{dx}{dt} &= -y - z, \\ \frac{dy}{dt} &= x, \\ \frac{dz}{dt} &= -bz + a(y - y^2). \end{aligned} \tag{3.1}$$

Here a, b are positive parameters.

In [17], by means of numerical simulation, it has been established that for certain values of parameters (3.1) has a compact invariant set K. In next section, we shall get for it an upper bound of the Hausdorff dimension. However unlike the case of (2.1) for the proof of this estimate we shall not employ information about the localization of compact set K in the phase space.

We have no estimates for the dissipativity region and more than that, it is unknown if (3.1) is dissipative. We must emphasize that the negativeness of the divergence of a vector field does not ensure the dissipativity. As a suitable example, we indicate the system which results from (2.1) for $a = 0$ by change of a sign before the nonlinearity in the second equation (the so–called Gleick–Lorenz system, see [64]).

It is not out of place to mention the paper [65]. There, the nondissipativity of the following two systems is proved

$$\begin{aligned} \frac{dx}{dt} &= -y - z, \\ \frac{dy}{dt} &= x + ay, \\ \frac{dz}{dt} &= bx - cz + xy \end{aligned}$$

and

$$\frac{dx}{dt} = -y - z,$$
$$\frac{dy}{dt} = x + ay,$$
$$\frac{dz}{dt} = b - cz + xy.$$

Here a, b, c are positive parameters. These systems were considered and numerically investigated by Rössler [16, 17].

4. Lyapunov Functions in the Estimates of the Hausdorff Dimension

Let L be a linear operator. We denote by $\alpha_1(L) \geq \cdots \geq \alpha_n(L)$ the eigenvalues of $(L^*L)^{\frac{1}{2}}$. For an arbitrary integer $k \geq 0$, we put

$$\omega_k(L) = \begin{cases} \alpha_1(L) \dots \alpha_k(L), & k > 0, \\ 1, \quad k = 0. \end{cases} \tag{4.1}$$

If $d \geq 0$ is an arbitrary number, then we write it in the form $d = d_0 + s$, where $d_0 \geq 0$ is an integer, $s \in [0, 1]$, and put

$$\omega_d(L) = \omega_{d_0}^{1-s}(L)\omega_{d_0+1}^{s}(L). \tag{4.2}$$

Let E be an ellipsoid in R^n. Let us denote by $\alpha_k(E)$ the length of its axes and we assume that $\alpha_1(E) \geq \cdots \geq \alpha_n(E)$. For an arbitrary number $d \geq 0$, we introduce the notation $\omega_d(E)$ by means of equalities analogous to (4.1), (4.2). It is known that the image, by L of the unit ball of R^n, is ellipsoid E with the axes $\alpha_k(E)$ coinciding with the numbers $\alpha_k(L)$ [66].

Let D be an open domain in R^n, let F be a C^1–mapping: $D \to R^n$. The latter means that, for any point $x \in D$, the increment of $F(x)$ under the transition from the point x to $x + h$ allows the representation

$$F(x + h) - F(x) = L(x)h + o(h),$$

where $L(x)$ is a linear operator which is called the derivative of the mapping F in the point x. We shall suppose that the inclusion $F(D) \subset D$ is true. Then, for an arbitrary integer $m \geq 1$ on D, the C^1–mapping F^m is defined. Denote its derivative in the point x by $L^{(m)}(x)$. It is well known that the following relation holds

$$L^{(m)}(x) = L(F^{m-1}(x)) \dots L(F(x))L(x). \tag{4.3}$$

Let $B(x, r)$ denote a ball of the radius r with the centre in the point x in R^n. Consider the restriction $F(x)$ on $B(x, r)$. Due to the Taylor formula for any h such that $|h| < r$, we have

$$|F(x+h) - F(x) - L(x)h| \leq \sup_{x' \in B(x,r)} |L(x') - L(x)||h|. \tag{4.4}$$

Denote by $\tilde{K}$ a compact set in R^n. Let $K \subset \tilde{K}$ be a compact set such that $F^m(K) \subset \tilde{K}$ for any integer $m \geq 0$. Denote by $p(x)$ a continuous function on $\tilde{K}$ $p(x) > 0$ for any $x \in \tilde{K}$

Theorem 4.1. *Suppose that the inequality*

$$\sup_{x \in \tilde{K}} \left[\frac{p(F(x))}{p(x)} \omega_d(L(x)) \right] < 1$$

is valid. Then (a) *if* $\mu(K, d) < \infty$, *then*

$$\lim_{m \to \infty} \mu(F^m(K), d) = 0;$$

(b) *if* $K \subset F(K)$, *then* $dim\, K < d$ *and for any integer* $m > 0$ $\mu(F^m(K), d) = 0$.

Notice that in [39] a version of Theorem 4.1 for Hilbert space is proved which embraces the known theorem of R. Temam about the estimation of the Hausdorff dimension of a compact [34, Theorem 5.3.1].
In the proof of Theorem 4.1, the following two propositions stated by A. Douady and J. Oesterlé are used [28].

Lemma 4.1. *Let* E *be an ellipsoid in* R^n *such that* $\alpha_1(E) \leq \delta$, $\omega_d(E) \leq \nu$ *and* $\nu \leq \delta^d$. *Then, for any* $\eta > 0$, *the set* $E + B(\eta)$ *is included into the ellipsoid* E' *for which*

$$\omega_d(E') \leq (1 + \kappa\eta)^d \nu,$$

where $\kappa = (\delta^{d_0}/\nu)^{1/s}$.

Here and further, $B(r)$ denotes the ball centered at 0 of radius r.

Lemma 4.2. *For an ellipsoid $E \subset R^n$ and a number $\varepsilon \geq [\omega_d(E)]^{\frac{1}{d}}$ the inequality*

$$\mu(E, d, \lambda\varepsilon) \leq C\omega_d(E),$$

where $\lambda = \sqrt{d_0 + 1}, \quad C = 2^{d_0}(d_0 + 1)^{\frac{d}{2}}$, *is valid.*

Proof. From the condition of the theorem ,the existence of the positive number $\nu_1 < 1$ such that

$$\sup_{x \in \widetilde{K}} \left[\frac{p(F(x))}{p(x)} \omega_d(L(x)) \right] \leq \nu_1. \tag{4.5}$$

follows. For any integer $m \geq 0$, we denote

$$\nu(m) = \nu_1^m \sup_{x \in \widetilde{K}} \frac{p(x)}{p(F^m(x))}. \tag{4.6}$$

Let $\ell > 0$ be an arbitrary number. There obviously exists $m_0 > 0$ such that for any $m > m_0$, we have

$$\nu(m) < \ell. \tag{4.7}$$

Since for any two linear operators L' and L'' the relation

$$\omega_d(L'L'') \leq \omega_d(L')\omega_d(L'')$$

is true [66], then from (4.3) we infer

$$\omega_d(L^{(m)}(x)) \leq \prod_{i=1}^{m} \omega_d(L(F^{m-i}(x))).$$

From this and (4.6) we obtain

$$\omega_d(L^{(m)}(x)) \leq \prod_{i=1}^{m} \nu_1 \frac{p(F^{m-i}(x))}{p(F^{m-i+1}(x))} = \nu_1^m \frac{p(x)}{p(F^m(x))} \leq \nu(m).$$

Thus

$$\sup_{x \in \widetilde{K}} \omega_d(L^{(m)}(x)) \leq \nu(m). \tag{4.8}$$

Use Lemma 4.1 with $\nu = \nu(m)$ and a number δ such that

$$\sup_{x\in\widetilde{K}} \|L^{(m)}(x)\| \le \delta, \quad \nu(m) \le \delta^d$$

and choose $\eta > 0$ satisfying the inequality $(1+\kappa\eta)^d \nu(m) < \ell$. This is possible due to (4.7). Take $\varepsilon > 0$ such that from (4.4) in which F and L are replaced by F^m and $L^{(m)}$, it follows that

$$|F^m(y) - F^m(x) - L^{(m)}(x)(y-x)| \le \eta|y-x| \tag{4.9}$$

for all $y \in B(x,r)$ with $r \le \varepsilon$. As we have mentioned above, we have

$$L^{(m)}(x)B(r) \subset E,$$

where $E \subset R^n$ is an ellipsoid whose axes are equal to $r\alpha_i(L^{(m)}(x))$. Therefore, from (4.9), the inclusion

$$F^m(B(x,r)) \subset F^m(x) + E + B(\eta r)$$

follows. Recalling (4.8), we have

$$\omega_d(E) = r^d \omega_d\left(\frac{1}{r}E\right) = r^d \omega_d(L^{(m)}(x)) \le \nu(m) r^d.$$

Thanks to Lemma 4.1, $E + B(\eta r)$ is included in an ellipsoid E' for which we have

$$\omega_d(E') \le (1+\kappa\eta)^d \nu(m) r^d < \ell r^d.$$

Thus if $\{B(x_i, r_i)\}$ is a covering of K by balls with radii $r_i \le \varepsilon$ then one can construct a covering of $F^m(K)$ by ellipsoids E_i' with $[\omega_d(E_i')]^{\frac{1}{d}} \le \ell^{\frac{1}{d}} r_i$ and

$$\sum_i \omega_d(E_i') \le \ell \sum_i r_i^d. \tag{4.10}$$

For any compact set K' in R^n, we put

$$\widetilde{\mu}(K', d, \varepsilon) = \inf \sum_i \omega_d(E_i),$$

where the infimum is taken over all covering of K' by a finite number of ellipsoids E_i for which $[\omega_d(E_i)]^{\frac{1}{d}} \le \varepsilon$. From (4.10), it follows that

$$\widetilde{\mu}(F^m(K), d, \ell^{\frac{1}{d}}\varepsilon) \le \ell\mu(K, d, \varepsilon). \tag{4.11}$$

We show now, employing Lemma 4.2, that for an arbitrary compact set $K' \subset \widetilde{K}$, it is true that

$$\mu(K', d, \lambda\varepsilon) \leq C\widetilde{\mu}(K', d, \varepsilon). \tag{4.12}$$

Indeed, for a covering of the compact set K' by a finite number of ellipsoids $\{E_i\}$ with $[\omega_d(E_i)]^{\frac{1}{d}} \leq \varepsilon$ we have

$$\mu(K', d, \lambda\varepsilon) \leq \mu\left(\bigcup_i E_i, d, \lambda\varepsilon\right) \leq \sum_i \mu(E_i, d, \lambda\varepsilon) \leq C\sum_i \omega_d(E_i).$$

From this we get (4.12).

Employing (4.12) with $K' = F^m(K)$, from (4.11) we obtain

$$\mu(F^m(K), d, \lambda\ell^{\frac{1}{d}}\varepsilon) \leq C\ell\mu(K, d, \varepsilon). \tag{4.13}$$

Suppose that $\mu(K, d) < \infty$. Making ε tend to 0, we shall have

$$\mu(F^m(K), d) \leq C\ell\mu(K, d). \tag{4.14}$$

From (4.7), it follows that by choosing m large enough, we can make the number ℓ and, hence, the right side of (4.14) as small as we please. Thus the assertion a) is proved.

In order to prove assertion (b) we demand that, for arbitrary ℓ, conditions $\lambda\ell^{\frac{1}{d}} < 1$ and $C\ell < 1$ are satisfied. Then $\mu(K, d, \varepsilon) \leq \mu(K, d, \lambda\ell^{\frac{1}{d}}\varepsilon)$ and from (4.13) we get

$$\mu(F^m(K), d, \lambda\ell^{\frac{1}{d}}\varepsilon) \leq C\ell\mu(K, d, \lambda\ell^{\frac{1}{d}}\varepsilon).$$

Due to the inclusion $K \subset F(K)$, from the last inequality we have

$$\mu(K, d, \lambda\ell^{\frac{1}{d}}\varepsilon) \leq C\ell\mu(K, d, \lambda\ell^{\frac{1}{d}}\varepsilon).$$

Therefore, $\mu(K, d, \lambda\ell^{\frac{1}{d}}\varepsilon) = 0$ and, hence, $\mu(F^m(K), d, \lambda\ell^{\frac{1}{d}}\varepsilon) = 0$. Making ε tend to 0 in these two equalities, we obtain $\mu(K, d) = 0$ and $\mu(F^m(K), d) = 0$. The last equality is true for all m larger than a certain m_0. But for $m \leq m_0$ its validity follows from the inclusion $K \subset F(K)$. Thus, assertion (b) is proved. □

Let us consider the application of Theorem 4.1 to ordinary differential equations. Let there be given a system

$$\frac{dx}{dt} = f(t, x), \quad (t, x) \in R_+ \times R^n, \tag{4.15}$$

where $f(t,x)$ is a continuously differentiable vector function. Denote by $x(t,x_0)$ the solution of the system (4.15) with the initial condition $x(0,x_0) = x_0$. Let us introduce the variation equations on this solution

$$\frac{dy}{dt} = \frac{\partial f}{\partial x}(t, x(t,x_0))y, \tag{4.16}$$

where $\partial f/\partial x$ is the Jacobian matrix of the right–hand side of (4.15). The derivative of shift mapping $F_t : x_0 \to x(t,x_0)$ $\partial x/\partial x_0$ coincides, as is known [67], with the Cauchy matrix $L(t,x_0)$ of the system (4.16), i.e. it is the fundamental matrix of the system (4.16) such that $L(0,x_0) = I$ (I is the unit matrix).

Theorem 4.2. *Suppose that there exist a continuously differentiable function $v(x)$, an integer $d_0 \geq 0$, numbers $s \in [0,1]$ and $\tau > 0$ such that $K \subset F_\tau(K)$ and the inequality*

$$\int_0^\tau [\lambda_1(t,x(t,x_0)) + \cdots + \lambda_{d_0}(t,x(t,x_0)) + s\lambda_{d_0+1}(t,x(t,x_0)) + \dot{v}(t,x(t,x_0))]\, dt < 0 \tag{4.17}$$

is true for all $x_0 \in K$. Then $\dim K < d_0 + s$.

Here and further, $\lambda_1 \geq \cdots \geq \lambda_n$ are eigenvalues of the symmetrized Jacobian matrix

$$\frac{1}{2}(\partial f^*/\partial x + \partial f/\partial x) \quad \text{and} \quad \dot{v}(x) = (\operatorname{grad} v(x))^* f(x).$$

We should emphasize that in Theorem 4.2 neither constancy of sign of the function v, nor constancy of sign of $\dot{v}$ is proposed.

Observe the following. (1) For $v(x) \equiv const$ the inequality (4.17) coincides with the condition which is contained in Smith's theorem about the Hausdorff dimension estimate [29]. (2) Without difficulty, theorem 4.2 can be extended on the Hilbert space [39].

Obviously (4.17) is satisfied if

$$\lambda_1(t,x) + \cdots + \lambda_{d_0}(t,x) + s\lambda_{d_0+1}(t,x) + \dot{v}(t,x) < 0 \tag{4.18}$$

for all $(t,x) \in R_+ \times K$. This inequality in the case when (4.15) is autonomous coincides with the widely known condition of Douady and Oesterlé [28, 29].

Before proving Theorem 4.2, we introduce the necessary notations and recall some known facts which we shall use in the proof.

Denote by Λ^k a Euclidean space whose elements are exterior products $\xi_1 \wedge \cdots \wedge \xi_k$, $\xi_i \in R^n$ with the scalar product $(\cdot,\cdot)_{\Lambda^k}$ and the norm $|\cdot|_{\Lambda^k}$. For a linear operator $L : R^n \to R^n$, we denote by L_k the linear operator: $\Lambda^k \to \Lambda^k$ defined by the equality

$$L_k(\xi_1 \wedge \cdots \wedge \xi_k) = L\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k + \xi_1 \wedge L\xi_2 \wedge \cdots \wedge \xi_k + \xi_1 \wedge \xi_2 \wedge \cdots \wedge L\xi_k.$$

The following relations (see [34, pp. 266, 259]) take place:

$$\omega_k(L) = \sup_{x_1,\ldots,x_k \in R^n, |x_i| \le 1} |Lx_1 \wedge \cdots \wedge Lx_k|_{\Lambda^k}, \tag{4.19}$$

and if L is a self–adjoint operator, then

$$\|L_k\| \le \alpha_1(L) + \cdots + \alpha_k(L). \tag{4.20}$$

Notice that when L is a self–adjoint operator, then the numbers $\alpha_i(L)$ obviously coincide with the eigenvalues of L.

Proof of Theorem 4.2. Let $p(x) = \exp[-v(x)]$. It is evident that due to Theorem 4.1 it is sufficient to verify the validity of the inequality

$$\omega_d(L(\tau, x_0)) < \frac{p(x_0)}{p(F_\tau(x_0))} = \exp[v(x_0) - v(F_\tau(x_0))]. \tag{4.21}$$

Thanks to (4.19) for an arbitrary $k > 0$ we have

$$\omega_k(L(\tau, x_0)) = \sup_{\xi_i \in R^n, |\xi_i| = 1} |y_1(\tau) \wedge \cdots \wedge y_k(\tau)|_{\Lambda^k}, \tag{4.22}$$

where $y_i(t) = L(t, x_0)\xi_i$ is the solution of (4.16).
Let us estimate $|y_1(\tau) \wedge \cdots \wedge y_k(\tau)|_{\Lambda^k}$. We have

$$\frac{d}{dt}|y_1 \wedge \cdots \wedge y_k|^2_{\Lambda^k} = \left(\left(\frac{\partial f^*}{\partial x} + \frac{\partial f}{\partial x}\right)_k (y_1 \wedge \cdots \wedge y_k), y_1 \wedge \cdots \wedge y_k\right)_{\Lambda^k}.$$

By using (4.20) we obtain

$$\frac{d}{dt}|y_1 \wedge \cdots \wedge y_k|^2_{\Lambda^k} \le 2(\lambda_1 + \cdots + \lambda_k)|y_1 \wedge \cdots \wedge y_k|^2_{\Lambda^k}$$

or

$$\frac{d}{dt}|y_1 \wedge \cdots \wedge y_k|_{\Lambda^k} \leq (\lambda_1 + \cdots + \lambda_k)|y_1 \wedge \cdots \wedge y_k|_{\Lambda^k}.$$

Multiplying both sides of the last inequality by

$$\exp\left[-\int_0^t (\lambda_1 + \cdots + \lambda_k)\, dt\right],$$

we find

$$\frac{d}{dt}\left\{|y_1 \wedge \cdots \wedge y_k|_{\Lambda^k} \exp\left[-\int_0^t (\lambda_1 + \cdots + \lambda_k)\, dt\right]\right\} \leq 0.$$

Hence, by integrating over the interval $[0, \tau]$, we get

$$|y_1 \wedge \cdots \wedge y_k|_{\Lambda^k} \leq |\xi_1 \wedge \cdots \wedge \xi_k|_{\Lambda^k} \exp \int_0^\tau (\lambda_1 + \cdots + \lambda_k)\, dt.$$

Since $|\xi_1 \wedge \cdots \wedge \xi_k|_{\Lambda^k} \leq |\xi_1| \dots |\xi_k|$, then it follows from the last inequality and (4.22)

$$\omega_d(L(\tau, x_0)) \leq \exp \int_0^\tau (\lambda_1 + \cdots + \lambda_k)\, dt.$$

Recalling the definition of $\omega_d(L)$ for an arbitrary $d > 0$ we get

$$\omega_d(L(\tau, x_0)) \leq \exp \int_0^\tau (\lambda_1 + \cdots + \lambda_{d_0} + s\lambda_{d_0+1})\, dt.$$

This implies that when (4.17) from the condition of the theorem is true, inequality (4.21) is also true. □

Notice that another version of the proof of Theorem 4.2 without using exterior products is suggested in [68].

Theorem 4.3. *Let $K \subset D$, where K and D are compact subsets in R^n. Suppose that for certain $\tau > 0$ the relation $F_\tau^m(x_0) \stackrel{\text{def}}{=} x(m\tau, x_0) \in D$ is true for all $x_0 \in K$ and for all integer $m \geq 0$. Suppose also that there exist a continuously differentiable function $v(x)$, an integer $d_0 \geq 0$, and a number $s \in [0,1]$ such that (4.17) is satisfied for all $x_0 \in D$. Then*

$$\lim_{m \to \infty} \mu(F_\tau^m(K), d_0 + s) = 0.$$

The proof of Theorem 4.3 is based on Theorem 4.1 and is analogous to the one of the previous theorem.

Below, by means of Theorems 4.2 and 4.3, we prove a new convergence theorem and establish new conditions of point–wise monostability for the autonomous system

$$\frac{dx}{dt} = f(x), \quad x \in R^n, \tag{4.23}$$

where $f(x)$ is a continuously differentiable function. Their proofs repeat, almost word for word the proofs of the corresponding results of R. Smith [29].

We shall assume that the system (4.23) has only isolated equilibrium states.

Theorem 4.4. *Let D be a bounded simply–connected open domain in R^n. Suppose that its boundary is crossed strictly inwards by every solution of (4.23) which meets it. Suppose also that there exists a continuously differentiable function $v(x)$ for which the inequality*

$$\lambda_1(x) + \lambda_2(x) + \dot{v}(x) < 0 \tag{4.24}$$

is true for all $x \in \overline{D}$. Then each solution of (4.23) in D converges to an equlibrium state.

Proof. Let $x(t)$ be a solution of (4.23). Let us show that any of its ω–limit point q is an equilibrium state of the system (4.23). Indeed, assume that this statement is false. Then due to Pugh's lemma about closure [69], for any $\varepsilon > 0$ there exists a system

$$\frac{dx}{dt} = g(x), \tag{4.25}$$

where $g(x)$ is a continuously differentiable vector function such that $||f - g||_{C^1} < \varepsilon$ and g–flow has a closed trajectory throught q. Choose ε so small that for the eigenvalues $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n$ of the symmetrized Jacobian matrix of the right–hand side of (4.25) the inequality (4.24) remains true, i.e.

$$\tilde{\lambda}_1(x) + \tilde{\lambda}_2(x) + \dot{v}(x) < 0$$

for all $x \in \overline{D}$. Thanks to Theorem 4.2, from the last inequality it follows that any compact set K in D which is invariant for the system (4.25) has the Hausdorff dimension less than 2. But in this case, the existence of the closed trajectory of the system (4.25) in D is impossible [29, Theorem 5].

Thus, ω–limit set of the solution $x(t)$ of the system (4.23) consists of equilibrium states. Since they are supposed isolated, then from the simply–connection of the ω–limit set it follows that it consists of the unique equilibrium state. □

Note that the paper [38] offers another proof of Theorem 4.4 which does not use Smith's result [29, Theorem 5. The proof is based on ideas connected with the Plateau problem [70, 71] and is reduced to the following.

Let us return to the closed trajectory through q. We denote this trajectory by Γ. Let us pull a smooth two–dimensional surface $K \subset \overline{D}$ with a finite area on Γ. The existence of such a surface for a smooth curve has been shown, for example, in [70]. Denote by $\mu(S)$ Hausdorff 2–measure of a smooth two–dimensional surface S. It is obvious that $F_\tau^m(\Gamma) = \Gamma$ for every integer $m \geq 0$. Therefore, we have

$$\inf_{m \geq 0} \mu(F_\tau^m(K)) > 0.$$

The latter contradicts the relation

$$\lim_{m \to \infty} \mu(F_\tau^m(K)) = 0$$

which follows from Theorem 4.3. Hence, the existence of the closed trajectory of the system (4.25) in D is impossible.

Theorem 4.5. *Suppose that there exists a continuously differentiable function $v(x)$ such that*

$$\lambda_1(x) + \lambda_2(x) + \dot{v}(x) < 0 \tag{4.26}$$

for all $x \in R^n$. Then any bounded solution of the system (4.23) tends to an equlibrium state as $t \to \infty$.

Proof. Let $x(t)$ be a bounded solution of the system (4.23). Choose the number $\rho > 0$ such that the trajectory of the solution $x(t)$ is contained in the ball centred at the origin of radius ρ. There exist a number $\sigma > \rho$ and a continuously differentiable vector function $f_\rho(x)$ having the following properties [29, Lemma 4]:

$$f_\rho(x) = f(x) \quad \forall x : |x| \leq \rho, \tag{4.27}$$

$$x^* f_\rho(x) < 0 \quad \forall x : |x| \geq \sigma, \tag{4.28}$$

$$\mu_1(x) + \mu_2(x) \leq \lambda_1(x) + \lambda_2(x) \quad \forall x \in R^n, \tag{4.29}$$

where $\mu_1 \geq \cdots \geq \mu_n$ are the eigenvalues of the symmetrized Jacobian matrix of the vector function $f_\rho(x)$.

From (4.27), it follows that $x(t)$ is a solution of the system

$$\frac{dx}{dt} = f_\rho(x). \tag{4.30}$$

Denote by D the ball, centred at the origin, of radius σ. From (4.27)–(4.29) it follows that all the hypotheses of Theorem 4.4 hold due to which $x(t)$ will tend to an equilibrium state of the system (4.30) as $t \to \infty$. Thanks to (4.27) this equilibrium state is also an equilibrium state of the original system (4.23). □

Observe that by setting in Theorems 4.4 and 4.5 $v(x) \equiv \text{const}$ we get convergence theorem and conditions of point–wise monostability of R. Smith [29].

Let us demonstrate the application of the theorems proved above. We start with the Rössler system

$$\begin{aligned} \frac{dx}{dt} &= -y - z, \\ \frac{dy}{dt} &= x, \\ \frac{dz}{dt} &= -bz + a(y - y^2), \end{aligned} \tag{4.31}$$

where a, b are positive numbers.

Theorem 4.6. *Let K be a compact set invariant for the system* (4.31)*. Then we have*

$$\dim K \leq 3 - \frac{2b}{b + \sqrt{(a+2b)^2 + b^2 + 1}}. \tag{4.32}$$

Proof. It is easy to see that the eigenvalues of the symmetrized Jacobian matrix of the right–hand side of (4.31) are the following numbers:

$$0, \quad \frac{1}{2}\{-b \pm \sqrt{b^2 + 1 + a^2(1-2y)^2}\}.$$

Hence, condition (4.18) can be written in the form

$$-(1+s)b + (1-s)\sqrt{b^2 + 1 + a^2(1-2y)^2} + 2\dot{v} < 0. \tag{4.33}$$

Choose

$$v = \frac{1}{2}(1-s)\kappa(z - bx),$$

where κ is a varied parameter. Then

$$\dot{v} = \frac{1}{2}(1-s)\kappa\left[(a+b)y - ay^2\right]$$

and inequality (4.33) is equivalent to the following

$$-(1+s)b+(1-s)\varphi(y;\kappa)<0, \tag{4.34}$$

where

$$\varphi(y;\kappa)=\sqrt{b^2+1+a^2(1-2y)^2}+\kappa\big[(a+b)y-ay^2\big].$$

Let us denote

$$m=\inf_{\kappa}\max_{y}\varphi(y;\kappa).$$

From (4.34) we get due to Theorem 4.2

$$\dim K\le 2+\frac{m-b}{m+b}=3-\frac{2b}{m+b}. \tag{4.35}$$

We have

$$\varphi(y;\kappa)=-\left(\theta\sqrt{b^2+1+a^2(1-2y)^2}-\frac{1}{2\theta}\right)^2$$
$$+\theta^2\big[b^2+1+a^2(1-2y)^2\big]+\frac{1}{4\theta^2}+\kappa\big[(a+b)y-ay^2\big].$$

Here $\theta\ne 0$ is a varied parameter. Further,

$$\begin{aligned}\varphi(y;\kappa)&\le\theta^2\big[b^2+1+a^2(1-2y)^2\big]+\frac{1}{4\theta^2}+\kappa\big[(a+b)y-ay^2\big]\\&\le\theta^2(a^2+b^2+1)+\frac{1}{4\theta^2}-(\kappa a-4\theta^2a^2)y^2-\big[4\theta^2a^2-\kappa(a+b)\big]y\\&=-(\kappa a-4\theta^2a^2)\left[y+\frac{4\theta^2a^2-\kappa(a+b)}{2(\kappa a-4\theta^2a^2)}\right]^2+\frac{\big[4\theta^2a^2-\kappa(a+b)\big]^2}{4(\kappa a-4\theta^2a^2)}\\&+\theta^2(a^2+b^2+1)+\frac{1}{4\theta^2}.\end{aligned}$$

If we take the varied parameters κ and θ as follows $\kappa a-4\theta^2a^2>0$, then

$$\varphi(y;\kappa)\le\frac{\big[4\theta^2a^2-\kappa(a+b)\big]^2}{4(\kappa a-4\theta^2a^2)}+\theta^2(a^2+b^2+1)+\frac{1}{4\theta^2}.$$

Let us take

$$\kappa=4\theta^2a\frac{a+2b}{a+b},\qquad\theta^2=\frac{1}{2\sqrt{(a+2b)^2+b^2+1}}.$$

Then we have

$$\varphi(y;\kappa) \le \sqrt{(a+2b)^2 + b^2 + 1}$$

and (4.35) implies (4.32). □

For $a = 0.386$ and $b = 0.2$ [17] from the estimate (4.32), we get $\dim K \le 2.731$.

The authors of known published works [29–36] containing the bounds of the Hausdorff dimension for attractors of concrete ordinary differential equations used the estimates of the dissipativity region as a basis. These estimates allow us to localize an attractor in the phase space. The example given above shows that using function v makes it possible to obtain the dimension estimates without such localization.

At the same time, if a system is dissipative and a Lyapunov function V is used for estimating of its dissipativity region, then it is possible to utilize in Theorems 4.2 and 4.4 the same function V or, more generally, $\psi(V)$, where ψ is a continuously differentiable function, as v. The following example illustrate this idea.

Consider the three–dimensional system arising in the research of the interaction between waves in plasma [21]

$$\begin{aligned} \frac{dx}{dt} &= hy - x - yz, \\ \frac{dy}{dt} &= hx - \nu y + xz, \\ \frac{dz}{dt} &= -z + xy, \end{aligned} \tag{4.36}$$

where h, ν are positive numbers. The analytical estimates of the Hausdorff dimension of attractors and the conditions of complete stability of this system were first stated in [32]. In [32], system (4.36) was investigated by means of its reducing to the generalized Lorenz equations (see below, Section 6).

Let us change the variables: $x \mapsto x$, $y \mapsto \alpha y$, $z \mapsto z$, where $\alpha \ne 0$ is a varied parameter. In the new variables, Equations (4.36) take the form

$$\begin{aligned} \frac{dx}{dt} &= \alpha hy - x - \alpha yz, \\ \frac{dy}{dt} &= \frac{h}{\alpha}x - \nu y + \frac{1}{a}xz, \\ \frac{dz}{dt} &= -z + \alpha xy. \end{aligned} \tag{4.37}$$

The eigenvalues of the symmetrized Jacobian matrix of the right–hand side of (4.37) are -1, and

$$\frac{1}{2}\left\{-(\nu+1) \pm \sqrt{(\nu-1)^2 + (\frac{1}{\alpha}+\alpha)^2x^2 + \left[(\frac{1}{\alpha}+\alpha)h + (\frac{1}{\alpha}-\alpha)z\right]^2}\right\}.$$

It follows from the results of Section 2 that: (1) the system (4.37) is dissipative and there is an ellipsoid $E(\alpha)$ in the phase space such that any trajectory enters it and remains in it thereafter; (2) we can take a narrower set D given by

$$D = E(\alpha) \cap \{x, z \mid V(x,z) \leq h^2\},$$

where $V(x,z) = x^2 + (z-h)^2$, as a region of dissipativity.

Theorem 4.7. *Let K be an attractor of the system (4.36). Then we have*

$$\dim K \leq 3 - \frac{2(\nu+2)}{\nu+1+\sqrt{(\nu-1)^2+\frac{16}{3}h^2}}. \tag{4.38}$$

Proof. The proof is based on Theorem 4.2. The inequality (4.18) becomes

$$\begin{aligned}&-(\nu+3) - s(\nu+1)\\ &+(1-s)\sqrt{(\nu-1)^2+(\frac{1}{\alpha}+\alpha)^2x^2+\left[(\frac{1}{\alpha}+\alpha)h+(\frac{1}{\alpha}-\alpha)z\right]^2}\\ &+2\dot{v} < 0.\end{aligned} \tag{4.39}$$

Set

$$v = \frac{\kappa}{4}V(x,z),$$

where κ is a varied parameter. Then

$$\dot{v} = \frac{\kappa}{2}(-x^2 - z^2 + hz)$$

and the inequality (4.39) is equivalent to

$$\begin{aligned}&-(\nu+3) - s(\nu+1)\\ &+(1-s)\sqrt{(\nu-1)^2+\left(\frac{1}{\alpha}+\alpha\right)^2x^2+\left[\left(\frac{1}{\alpha}+\alpha\right)h+\left(\frac{1}{\alpha}-\alpha\right)z\right]^2}\\ &+\kappa(-x^2-z^2+hz) < 0.\end{aligned} \tag{4.40}$$

The inequality (4.40) will be true if

$$\begin{aligned}&-(\nu+3) - s(\nu+1) + (1-s)\theta^2\left[(\nu-1)^2+\left(\frac{1}{\alpha}+\alpha\right)^2h^2\right]\\ &+(1-s)\frac{1}{4\theta^2}+\varphi(x,z) < 0\end{aligned} \tag{4.41}$$

is true. Here $\theta \neq 0$ is varied parameter and

$$\begin{aligned}\varphi(x,z) = &\left[(1-s)\theta^2\left(\frac{1}{\alpha}+\alpha\right)^2-\kappa\right]x^2\\&+\left[(1-s)\theta^2\left(\frac{1}{\alpha}-\alpha\right)^2-\kappa\right]z^2\\&+\left[2(1-s)\theta^2\left(\frac{1}{\alpha^2}-\alpha^2\right)+\kappa\right]hz.\end{aligned}$$

If

$$(1-s)\theta^2\left(\frac{1}{\alpha}+\alpha\right)^2-\kappa \geq 0, \tag{4.42}$$

then granting that $K \subset D$ and $x^2 \leq -z^2+2hz$ on D we shall have

$$\begin{aligned}\varphi(x,z) &\leq -4(1-s)\theta^2 z^2+\left[4(1-s)\theta^2\left(\frac{1}{\alpha^2}+1\right)-\kappa\right]hz\\&\leq \frac{[4(1-s)\theta^2(\frac{1}{\alpha^2}+1)-\kappa]^2}{16(1-s)\theta^2}h^2.\end{aligned}$$

Choose

$$\kappa = 4(1-s)\theta^2\left(\frac{1}{\alpha^2}+1\right).$$

Then, using (4.42), we get $\varphi(x,z) \leq 0$ for $\alpha \geq \sqrt{3}$. Thus, for $\alpha \geq \sqrt{3}$, the inequality (4.41) will be true if

$$-(\nu+3)-s(\nu+1)+(1-s)\theta^2\left[(\nu-1)^2+\left(\frac{1}{\alpha}+\alpha\right)^2h^2\right]+(1-s)\frac{1}{4\theta^2}<0.$$

Choose

$$\theta^2 = \frac{1}{2\sqrt{(\nu-1)^2+\left(\frac{1}{\alpha}+\alpha\right)^2h^2}}.$$

Then the last inequality can be written

$$-(\nu+3)-s(\nu+1)+(1-s)\sqrt{(\nu-1)^2+\left(\frac{1}{\alpha}+\alpha\right)^2h^2}<0.$$

This implies

$$s > \frac{-(\nu+3)+\sqrt{(\nu-1)^2+(\frac{1}{\alpha}+\alpha)^2h^2}}{\nu+1+\sqrt{(\nu-1)^2+(\frac{1}{\alpha}+\alpha)^2h^2}}$$

$$= 1-\frac{2(\nu+2)}{\nu+1+\sqrt{(\nu-1)^2+(\frac{1}{\alpha}+\alpha)^2h^2}}. \tag{4.43}$$

Finally setting $\alpha = \sqrt{3}$ in (4.43) we obtain (4.38). □

Numerical analysis [21] allowed us to establish that for $\nu = 4$, $h = 4.92$ there exists a strange attractor for the system (4.36). For these values of parameters, Theorem 4.7 yields $\dim K < 2.284$.

Notice that the upper bound $\dim K < 2.339$ was stated in [32]. Let us show that applying the method of approach considered here, it became possible to improve the result by means of introducing a function $v \not\equiv \text{const}$ into the estimate of the Hausdorff dimension. Indeed, suppose that $v(x) \equiv \text{const}$ i.e. $\dot{v} \equiv 0$ and, consequently, the inequality (4.18) coincides with to the Douady–Oesterlé's condition. The identity $\dot{v} \equiv 0$ in (4.18) is equivalent to the choice of $\kappa = 0$. The inequality (4.42) in this case is always valid. Therefore

$$\varphi(x,z) \le (1-s)\theta^2\left(\frac{1}{\alpha^2}+1\right)^2 h^2$$

and (4.41) will be true if

$$\begin{aligned} &-(\nu+3)-s(\nu+1)\\ &+(1-s)\theta^2\left\{(\nu-1)^2+\left[\left(\frac{1}{\alpha}+\alpha\right)^2+\left(\frac{1}{\alpha^2}+1\right)^2\right]h^2\right\}\\ &+(1-s)\frac{1}{4\theta^2}<0. \end{aligned} \tag{4.44}$$

Let us take

$$\theta^2 = \frac{1}{2\sqrt{(\nu-1)^2+[(\frac{1}{\alpha}+\alpha)^2+(\frac{1}{\alpha^2}+1)^2]h^2}}.$$

Then it follows from (4.44) that

$$s > 1-\frac{2(\nu+2)}{\nu+1+\sqrt{(\nu-1)^2+[(\frac{1}{\alpha}+\alpha)^2+(\frac{1}{\alpha^2}+1)^2]h^2}}. \tag{4.45}$$

The right side is maximum for $\alpha = \sqrt{2}$. Substituting this value in (4.45), we infer from Theorem 4.2

$$\dim K \leq 3 - \frac{2(\nu+2)}{\nu + 1 + \sqrt{(\nu-1)^2 + \frac{27}{4}h^2}}.$$

This estimate for $\nu = 4$, $h = 4.92$ implies $\dim K \leq 2.339$, i.e. the result stated in [32] (see also below Section 6).

Theorem 4.8. *The system* (4.36) *is completely stable if*

$$h^2 < \frac{4}{27}(13\sqrt{13} - 35)(\nu + 1). \tag{4.46}$$

Proof. The proof is based on Theorem 4.4. Just in the same way as for Theorem 4.7, we take the function

$$v = \frac{\kappa}{4}[x^2 + (z-h)^2].$$

Then (4.24) can be written as

$$\begin{aligned} &-(\nu+3) \\ &+ \sqrt{(\nu-1)^2 + \left(\frac{1}{\alpha} + \alpha\right)^2 x^2 + \left[\left(\frac{1}{\alpha} + \alpha\right) h + \left(\frac{1}{\alpha} - \alpha\right) z\right]^2} \\ &+ \kappa(-x^2 - z^2 + hz) < 0. \end{aligned} \tag{4.47}$$

The inequality (4.47) will be true if

$$-(\nu+3) + \frac{1}{4\theta^2} + \theta^2 \left[(\nu-1)^2 + \left(\frac{1}{\alpha} + \alpha\right)^2 h^2\right] + \varphi(x,z) < 0, \tag{4.48}$$

where $\theta \neq 0$ is a varied parameter, $\varphi(x,z) = Ax^2 + Bz^2 - Cz$ and

$$A = \theta^2 \left(\frac{1}{\alpha} + \alpha\right)^2 - \kappa,$$

$$B = \theta^2 \left(\frac{1}{\alpha} - \alpha\right)^2 - \kappa,$$

$$C = \left[2\theta^2 \left(\alpha^2 - \frac{1}{\alpha^2}\right) - \kappa\right] h.$$

Since, for each point in D, the relation $x^2 + z^2 - 2hz \leq 0$ is true then for $A \geq 0$, i.e. if

$$\theta^2 \left(\frac{1}{\alpha} + \alpha\right)^2 - \kappa \geq 0 \tag{4.49}$$

we have $Ax^2 \leq -Az^2 + 2Ahz$. Hence

$$\varphi \leq (B-A)z^2 - (C-2Ah)z = (B-A)\left(z - \frac{C-2Ah}{2(B-A)}\right)^2 - \frac{(C-2Ah)^2}{4(B-A)}.$$

And since $B - A = -4\theta^2$, then if (4.49) is valid, it is true that

$$\varphi \leq \frac{(C-2Ah)^2}{4(A-B)}. \tag{4.50}$$

We have

$$\begin{aligned} C - 2Ah =& h\left[2\theta^2\left(\alpha^2 - \frac{1}{\alpha^2}\right) - \kappa - 2\theta^2\left(\frac{1}{\alpha} + \alpha\right)^2 + 2\kappa\right] \\ =& h\left[\kappa - 4\theta^2\left(\frac{1}{\alpha^2} + 1\right)\right]. \end{aligned} \tag{4.51}$$

Let us choose

$$\kappa = \tau 4\theta^2\left(\frac{1}{\alpha^2} + 1\right),$$

where τ is a new varied parameter. Then it follows from (4.50), (4.51)

$$\varphi \leq h^2(\tau - 1)^2\theta^2\left(\frac{1}{\alpha^2} + 1\right)^2.$$

Take

$$\theta^2 = \frac{1}{2\sqrt{(\nu-1)^2 + \left(\frac{1}{\alpha} + \alpha\right)^2 h^2 + (\tau-1)^2\left(\frac{1}{\alpha^2} + 1\right)^2 h^2}}.$$

In such case, the inequality (4.48) is true if (4.49) and

$$\sqrt{(\nu-1)^2 + \left(\frac{1}{\alpha} + \alpha\right)^2 h^2 + (\tau-1)^2\left(\frac{1}{\alpha^2} + 1\right)^2 h^2} < \nu + 3$$

are true. The last inequality is equivalent to

$$\left[\left(\frac{1}{\alpha}+\alpha\right)^2 h^2+(\tau-1)^2\left(\frac{1}{\alpha^2}+1\right)^2\right]h^2<8(\nu+1). \tag{4.52}$$

The inequality (4.49) is valid for $\alpha^2 \geq 4\tau - 1$. Choose

$$\tau=\frac{1}{4}(\sqrt{13}-1), \qquad \alpha^2=\sqrt{13}-2.$$

Then from (4.52) we get the condition (4.46). □

For $\nu = 4$, Theorem 4.8 ensures complete stability for $h < 2.96$. Notice that, in [32], complete stability for this value of ν was stated only for $h < 2.4$.

We will emphasize that we could extend a complete stability region here in comparison with that obtained in [32] just on using in (4.24) a function $v \not\equiv$ const. Indeed, if $\dot{v} \equiv 0$, i.e. $\kappa = 0$ and, consequently $\tau = 0$, then (4.52) becomes

$$\left[\left(\frac{1}{\alpha}+\alpha\right)^2+\left(\frac{1}{\alpha^2}+1\right)^2\right]h^2<8(\nu+1). \tag{4.53}$$

The expression in the square brackets is minimum for $\alpha = \sqrt{2}$, therefore from (4.53) we get

$$h^2<\frac{32}{27}(\nu+1).$$

This condition for $\nu = 4$ yields the known result $h < 2.4$ (see also below, Section 6).

In conclusion of the section, we offer two statements which are necessary for the development of the frequency method of approach to the questions considered above. Let the system

$$\frac{dx}{dt}=f(t,x) \tag{4.54}$$

be given. Here $f(t,x)$ is a continuously differentiable τ–periodic with respect to t vector function from $R \times G$ in R^n, $G \subset R^n$ being an open domain. We denote by $J(t,x)$ the Jacobian matrix of the right–hand side of (4.54). Suppose that there exists an open domain $D \subset G$ which has the property: if $x_0 \in D$, then $x(t,x_0) \in G$ for $t \in [0,\tau]$. Let K be an arbitrary compact set in D invariant for the system (4.54).

The following statement is obtained from Theorem 4.1 when $p(x) \equiv \text{const} \neq 0$.

Corollary 4.1, [29]. *Assume that there exist a constant matrix* $H = H^* < 0$ *and a continuous* τ*–periodic with respect to* t *function* $\theta(t, x)$ *such that*

$$J^*(t,x)H + HJ(t,x) + 2\theta(t,x)H \leq 0$$

for all $(t, x) \in R \times G$. *Then, if for some* $\rho \in [0, n]$

$$\int_0^\tau \left[(n-\rho)\theta(t, x(t, x_0)) + tr\ J(t, x(t, x_0))\right] dt < 0$$

for all $x_0 \in K$ *then* $\dim K < \rho$.

Suppose now that the system (4.54) is autonomous. Let there exist a bounded simply–connected open domain D with closure $\overline{D} \subset G$. Suppose also that its boundary is crossed strictly inwards by every solution of (4.54) which meets it.
It follows from Theorem 4.4 for $v(x) \equiv \text{const}$.

Corollary 4.2, [29]. *Suppose that there exist a constant matrix* $H = H^* < 0$ *and a continuous function* $\theta(x)$ *such that*

$$J^*(x)H + HJ(x) + 2\theta(x)H \leq 0,$$
$$(n-2)\theta(x) + tr\ J(x) < 0$$

for all $x \in \overline{D}$. *Then each trajectory in* D *converges to an equilibrium state.*

5. Frequency Estimates of the Hausdorff Dimension

Consider the system

$$\frac{dx'}{dt} = Ax' + B\Phi(t, C^*x'), \quad (t, x') \in R \times G. \tag{5.1}$$

Here G is an open convex domain in R^n; A, B, C are constant $n \times n$–matrices and $\Phi(t, \sigma)$ is a continuously differentiable τ–periodic with respect to t vector function.

Suppose that there exists an open subset D in G possessing the property: if $x'_0 \in D$, then $x'(t, x'_0) \in G$ for $t \in [0, \tau]$. Let K be any compact invariant subset in D. Let $\chi(p)$ denote the transfer matrix defined by the equality

$$\chi(p) = C^*(pI - A)^{-1}B,$$

where p is a complex number, I is the unit $n \times n$–matrix.

Theorem 5.1, [32]. *Let there exist positive numbers λ and γ such that the following hypotheses are satisfied:*

(1) *the inequality*

$$\frac{1}{2}\eta^* \left[\frac{\partial\Phi^*}{\partial\sigma}(t, C^*x') + \frac{\partial\Phi}{\partial\sigma}(t, C^*x')\right] \eta \leq \lambda|\eta|^2 \tag{5.2}$$

holds for all $\eta \in R^n$ and $(t, x') \in R\times G$; (2) *the pair $\{A, B\}$ is controllable, the pair $\{A, C\}$ is observable;*

(3) *all eigenvalues of the matrix $A+\gamma I$ are situated to the right of the imaginary axis;*

(4) *the frequency condition*

$$Re\chi(i\omega - \gamma) - \lambda\chi^*(i\omega - \gamma)\chi(i\omega - \gamma) \geq 0 \quad \forall\omega \in R$$

holds.

Then, if there exists a number $\rho \in [0, n]$ for which the inequality

$$\int_0^T [(n-\rho)\gamma + tr\, J(t, x'(t, x_0'))]\, dt < 0 \quad \forall x_0' \in K \tag{5.3}$$

is true we have $\dim K < \rho$.

Recall the definition of controllability and observability. The pair $\{A, B\}$ is called controllable if the rank of the matrix $\left(B, AB, \ldots, A^{n-1}B\right)$ is equal to n. The pair $\{A, C\}$ is called observable if the rank of the matrix $\left(C, A^*C, \ldots, (A^*)^{n-1}C\right)$ is equal to n. Various criteria of the controllability and the observability can be found in [13].

Suppose now that the system (5.1) is autonomous, i.e. it can be written as

$$\frac{dx'}{dt} = Ax' + B\Phi(C^*x'), \quad x' \in G. \tag{5.4}$$

Suppose that there exists an open bounded simply–connected convex domain D such that $\overline{D} \subset G$. Suppose also that its boundary is crossed strictly inwards by every solution of (5.4) which meets it.

Theorem 5.2, [32]. *Let there exist positive numbers λ and γ such that the following hypotheses are satisfied:*

(1) *the inequality*

$$\frac{1}{2}\eta^* \left[\frac{\partial\Phi^*}{\partial\sigma}(C^*x') + \frac{\partial\Phi}{\partial\sigma}(C^*x')\right] \eta \leq \lambda|\eta|^2 \tag{5.5}$$

holds for all $\eta \in R^n$ and $x' \in D$;

(2) *the pair* $\{A, B\}$ *is controllable, the pair* $\{A, C\}$ *is observable;*

(3) *all eigenvalues of the matrix* $A+\gamma I$ *are situated to the right of the imaginary axis;*

(4) *the frequency condition*

$$Re\chi(i\omega - \gamma) - \lambda\chi^*(i\omega - \gamma)\chi(i\omega - \gamma) \geq 0 \quad \forall \omega \in R$$

holds.

Then if the inequality

$$(n-2)\gamma + tr\, J(x') < 0 \quad \forall x' \in \overline{D} \tag{5.6}$$

is true, then each trajectory of (5.4) *in* D *converges to an equilibrium state.*

Notice that the hypothesis (2) in Theorems 5.1 and 5.2 may be relaxed by strengthening the hypothesis (4) (see [31]).

The following lemma is necessary for the proof of Theorems 5.1 and 5.2. The only restriction we lay on the vector function $\Phi(t, \sigma)$ in this lemma is that it is continuously differentiable on the set $R \times \{\sigma \mid \sigma = C^*x', x' \in \overline{\Omega}\}$, where Ω is an arbitrary open convex domain in R^n.

Lemma 5.1. *Let there exist positive numbers* λ *and* γ *such that the inequality*

$$\frac{1}{2}\eta^* \left[\frac{\partial \Phi^*}{\partial \sigma}(t, C^*x') + \frac{\partial \Phi}{\partial \sigma}(t, C^*x')\right] \eta \leq \lambda|\eta|^2 \tag{5.7}$$

is true for all $\eta \in R^n$ *and* $(t, x') \in R \times \Omega$ *and the hypotheses* (2)–(4) *of Theorems 5.1, 5.2 are satisfied. Then there exists a square* $n \times n$*–matrix* $H = H^* < 0$ *such that the inequality*

$$J^*(t, x')H + HJ(t, x') + 2\gamma H \leq 0 \tag{5.8}$$

is valid for all $(t, x') \in R \times \overline{\Omega}$.

Proof. By virtue of Yakubovich–Kalman's theorem [13, Theorem 1.2.6], there exists a square $n \times n$–matrix $H = H^*$ which satisfies the inequality

$$z'^*H[(A + \gamma I)z' + B\xi] - (z'^*C\xi - \lambda|C^*z'|^2) \leq 0 \tag{5.9}$$

for all $z' \in R^n$ and $\xi \in R^n$. Hence,

$$z'^* H[(A + \gamma I)z' + B\xi] \leq 0 \tag{5.10}$$

for all $z' \in R^n$ and $\xi \in R^n$, which satisfy the quadratic relation

$$z'^* C\xi - \lambda |C^* z'|^2 \leq 0. \tag{5.11}$$

For arbitrary $t \in R$ and $x'_1, x'_2 \in \Omega$, we put $z' = x'_1 - x'_2$ and $\xi = \Phi(t, C^* x'_1) - \Phi(t, C^* x'_2)$. Since

$$\Phi(t, C^* x'_1) - \Phi(t, C^* x'_2) = \int_0^1 \frac{\partial \Phi}{\partial \sigma} \left[t, C^*(t_1 x'_1 + (1 - t_1) x'_2)\right] dt_1 C^* z',$$

then using (5.7) with $\eta = C^* z'$, we get that z' and ξ satisfy the inequality (5.11) and, hence, for these z' and ξ (5.10) is true, i.e.

$$\begin{aligned} &(x'_1 - x'_2)^* H\big\{(A + \gamma I)(x'_1 - x'_2) \\ &+ B\big[\Phi(t, C^* x'_1) - \Phi(t, C^* x'_2)\big]\big\} \leq 0. \end{aligned} \tag{5.12}$$

Setting $f(t, x') = Ax' + B\Phi(t, C^* x')$ and taking into account the relation

$$J(t, x')v = \lim_{h \to 0} h^{-1} \left[f(t, x' + hv) - f(t, x')\right] \quad \forall v \in R^n,$$

we get from (5.12), with $x'_1 = x' + hv$ and $x'_2 = x'$, where x' is an arbitrary point of Ω, that

$$v^* \left[J^*(t, x')H + HJ(t, x') + 2\gamma H\right] v \leq 0.$$

It follows from (5.9) that for $\xi = 0$, the matrix H is negative definite [13, Lemma 1.2.4]. □

Using Corollary 4.1 and Lemma 5.1 with $\Omega = G$, we get the assertion of Theorem 5.1.

The validity of Theorem 5.2 follows from Corollary 4.2 and Lemma 5.1 with $\Omega = D$.

We shall demonstrate now the applications of Theorems 5.1 and 5.2 by means of the generalized Lorenz system. Rewrite the system (2.1) in the form (5.1). To do this we put $x' = \mathrm{col}(x, y, z)$, $B = \mathrm{diag}(-\kappa_3, -1, -1)$, $C = \mathrm{diag}(\kappa_2, 1, 1)$,

$$A = \begin{pmatrix} -d & d_1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix},$$

$\Phi(\sigma) = \kappa_2^{-1}\mathrm{col}\,(-\kappa_1 v + \kappa_2\kappa_3^{-1}avw, u(w-r), -uv)$ and $\sigma = \mathrm{col}\,(u, v, w)$.

Here $\kappa_1 \geq 0$, $\kappa_2 > 0$, $\kappa_3 > 0$ are varied parameters, $d_1 = d - \kappa_1\kappa_2^{-1}\kappa_3$.

Denote $a_0 = \kappa_2\kappa_3^{-1}a$. Introduce the positive number λ^* defined as follows:

for $a = 0$

$$\lambda^* = \frac{1}{2\kappa_2}\sqrt{\ell^2 r^2 + 2\kappa_1 r + \kappa_1^2}; \tag{5.13}$$

for $a > 0$

$$\lambda^* = \frac{1}{2\kappa_2}[(1 + a_0)\ell r + |a_0 r - \kappa_1|]; \tag{5.14}$$

for $a < 0$

$$\lambda^* = \begin{cases} \frac{1-a_0}{4\kappa_2}\sqrt{4\ell^2 r^2 - \frac{1}{a_0}(a_0 r - \kappa_1)^2}, \\ \qquad\qquad \text{if } |1 + a_0|(a_0 r - \kappa_1) - 4a_0\ell r \geq 0, \\ \frac{1}{2\kappa_2}(|1 + a_0|\ell r - a_0 r + \kappa_1), \\ \qquad\qquad \text{if } |1 + a_0|(a_0 r - \kappa_1) - 4a_0\ell r \leq 0. \end{cases} \tag{5.15}$$

Let us denote by $\mu_0(y, z)$ the largest eigenvalue of the matrix $\frac{1}{2}(\partial\Phi^*/\partial\sigma + \partial\Phi/\partial\sigma)$. Let D^* be a dissipativity region of the system (2.1) which satisfies the inclusion (2.4) for $a \neq 0$ and the inclusion (2.6) for $a = 0$.

Lemma 5.2. *We have the estimate*

$$\mu_0(y, z) \leq \lambda^* \quad \forall (x, y, z) \in D^*. \tag{5.16}$$

Proof. Since

$$\frac{1}{2}\left(\frac{\partial\Phi^*}{\partial\sigma} + \frac{\partial\Phi}{\partial\sigma}\right) = \frac{1}{\kappa_2}\begin{pmatrix} 0 & w & -\frac{1}{2}(1 - a_0)y \\ w & 0 & 0 \\ -\frac{1}{2}(1 - a_0)y & 0 & 0 \end{pmatrix}, \tag{5.17}$$

where $w = \frac{1}{2}[(1 + a_0)z - r - \kappa_1]$, then the characteristic equation for this matrix

$$\det\left[\frac{1}{2}\left(\frac{\partial\Phi^*}{\partial\sigma} + \frac{\partial\Phi}{\partial\sigma}\right) - \mu I\right] = 0$$

reduces to

$$(-\mu\kappa_2)^3 + \frac{1}{4}\mu\kappa_2\{(1-a_0)^2y^2 + [(1+a_0)z - r - \kappa_1]^2\} = 0.$$

Therefore

$$\mu_0 = \frac{1}{2\kappa_2}\sqrt{(1-a_0)^2y^2 + [(1+a_0)z - r - \kappa_1]^2}.$$

From the inclusion (2.4) it follows

$$\mu_0 \le \frac{1}{2\kappa_2}\sqrt{c_1z^2 + c_2z + c_3}, \tag{5.18}$$

where

$$\begin{aligned} c_1 =& 4a_0, \quad c_2 = 2[(1-a_0)^2r - (1+a_0)(\kappa_1 + r)], \\ c_3 =& (1-a_0)^2(\ell^2 - 1)r^2 + (\kappa_1 + r)^2. \end{aligned}$$

Consider separately the cases $a = 0$, $a > 0$ and $a < 0$.

Case $a = 0$. We have $c_1 = 0$, $c_2 = -2\kappa_1$, $c_3 = \ell^2r^2 + 2\kappa_1 r + \kappa_1^2$. Due to the inclusion (2.6) in D^*, we have $z \ge 0$. Hence, it follows from (5.18)

$$\mu_0 \le \frac{1}{2\kappa_2}\sqrt{\ell^2r^2 + 2\kappa_1r + \kappa_1^2}.$$

Before starting the consideration of the cases $a > 0$ and $a < 0$, we introduce the following notations. We let $\varphi_1(z) = c_1z^2 + c_2z$ and denote by M the maximum $\varphi_1(z)$ in the segment $[-\Delta_1, \Delta_2]$, where $\Delta_1 = (\ell - 1)r$ and $\Delta_2 = (\ell + 1)r$. Function $\varphi_1(z)$ is equal to zero for $z = 0$ and for $z = z_1$, where $z_1 = -c_2/c_1$.

Case $a > 0$. We have

$$M = \begin{cases} \max\{\varphi_1(-\Delta_1), \varphi_1(\Delta_2)\} & \text{if } c_2 \le 0; \\ \varphi_1(\Delta_2) & \text{if } c_2 \ge 0. \end{cases}$$

Moreover,

$$\varphi_1(\Delta_2) - \varphi_1(-\Delta_1) = 4\ell r(1+a_0)(a_0r - \kappa_1).$$

Therefore, if $a_0r - \kappa_1 \le 0$, then $\varphi_1(\Delta_2) - \varphi_1(-\Delta_1) \le 0$. But

$$\frac{1}{2}c_2 = -3a_0r + a_0^2r - \kappa_1 - a_0\kappa_1$$

and, consequently, if $a_0 r - \kappa_1 \le 0$, then

$$\frac{1}{2}c_2 \le -4a_0 r + a_0(a_0 r - \kappa_1) < 0.$$

Thus

$$M = \begin{cases} \varphi_1(-\Delta_1), & \text{if } a_0 r - \kappa_1 \le 0; \\ \varphi_1(\Delta_2), & \text{if } a_0 r - \kappa_1 \ge 0. \end{cases}$$

We have

$$\varphi_1(-\Delta_1) + c_3 = [(1+a_0)lr - (a_0 r - \kappa_1)]^2,$$
$$\varphi_1(\Delta_2) + c_3 = [(1+a_0)lr + a_0 r - \kappa_1]^2.$$

From this, (5.16) follows.

Case $a < 0$. We have

$$M = \begin{cases} \varphi_1(z_1/2) & \text{if } c_2 \le 0,\ z_1/2 \ge -\Delta_1; \\ \varphi_1(-\Delta_1) & \text{if } z_1/2 \le -\Delta_1; \\ \varphi_1(z_1/2) & \text{if } c_2 \ge 0,\ z_1/2 \le \Delta_2; \\ \varphi_1(\Delta_2) & \text{if } z_1/2 \ge \Delta_2. \end{cases} \tag{5.19}$$

The inequalities $z_1/2 \ge -\Delta_1$ and $z_1/2 \le \Delta_2$ are equivalent to the inequalities

$$(1+a_0)(a_0 r - \kappa_1) \ge -4|a_0|\ell r$$

and

$$(1+a_0)(a_0 r - \kappa_1) \le 4|a_0|\ell r,$$

respectively. Therefore (5.19) may be rewritten in the form

$$M = \begin{cases} \varphi_1(z_1/2) & \text{if } |(1+a_0)(a_0 r - \kappa_1)| \le 4|a_0|\ell r; \\ \varphi_1(-\Delta_1) & \text{if } (1+a_0)(a_0 r - \kappa_1) \le -4|a_0|\ell r; \\ \varphi_1(\Delta_2) & \text{if } (1+a_0)(a_0 r - \kappa_1) \ge 4|a_0|\ell r. \end{cases} \tag{5.20}$$

We have for the square root in (5.18)

$$\begin{aligned} &\sqrt{\varphi_1(-\Delta_1) + c_3} \\ =&(1+a_0)\ell r - (a_0 r - \kappa_1) \text{ if } (1+a_0)(a_0 r - \kappa_1) \le -4|a_0|\ell r, \\ &\sqrt{\varphi_1(\Delta_2) + c_3} \\ =&-(1+a_0)\ell r - (a_0 r - \kappa_1) \text{ if } (1+a_0)(a_0 r - \kappa_1) \ge 4|a_0|\ell r \end{aligned}$$

or

$$\begin{aligned}&\sqrt{\varphi_1(-\Delta_1)+c_3}\\ &=|1+a_0|\ell r-a_0 r+\kappa_1 \text{ if } (1+a_0)(a_0 r-\kappa_1)\le -4|a_0|\ell r,\end{aligned} \tag{5.21}$$

$$\begin{aligned}&\sqrt{\varphi_1(\Delta_2)+c_3}\\ &=|1+a_0|\ell r-a_0 r+\kappa_1 \text{ if } (1+a_0)(a_0 r-\kappa_1)\ge 4|a_0|\ell r.\end{aligned} \tag{5.22}$$

Moreover

$$\varphi_1(z_1/2)+c_3=\frac{(1-a_0)^2}{4}\left[4\ell^2 r^2-\frac{1}{a_0}(a_0 r-\kappa_1)^2\right]. \tag{5.23}$$

From (5.20)–(5.23) the estimate (5.16) follows. □

The following lemma yields the conditions ensuring the fulfillment of the hypotheses of Theorem 5.1 and 5.2 concerning the linear part of a system.

Lemma 5.3. *If for arbitrary numbers $\kappa_1 \ge 0$, $\kappa_2>0$, $\kappa_3>0$, $\lambda>0$ and $\gamma>0$, the inequalities*

$$d+\lambda\kappa_2\kappa_3-\gamma<0,\quad b+\lambda-\gamma<0, \tag{5.24}$$

$$4\kappa_2(d+\lambda\kappa_2\kappa_3-\gamma)(1+\lambda-\gamma)-\kappa_2(d-\kappa_1\kappa_3/\kappa_2)^2>0 \tag{5.25}$$

hold, then for the system (2.1) *written in the form* (5.1), *the hypotheses* (2)–(4) *of Theorems 5.1 and 5.2 are fulfilled.*

Proof. The validity of the condition concerning the spectrum of the matrix $A+\gamma I$ follows directly from (5.24), (5.25). Controllability and observability of corresponding pairs are evident. Let us check the validity of the frequency condition. For any complex number p, we have

$$\chi(p)=\begin{pmatrix} -\frac{\kappa_2\kappa_3}{p+d} & -\frac{\kappa_2 d_1}{(p+d)(p+1)} & 0\\ 0 & -\frac{1}{p+1} & 0\\ 0 & 0 & -\frac{1}{p+b}\end{pmatrix}.$$

Therefore

$$\begin{aligned}&\frac{1}{2}[\chi^*(p)+\chi(p)]-\lambda\chi^*(p)\chi(p)\\ &=\begin{pmatrix} -\kappa_2\kappa_3\frac{Re\, p+d+\lambda\kappa_2\kappa_3}{|p+d|^2} & -\frac{\kappa_2 d_1(\overline{p}+d+2\lambda\kappa_2\kappa_3)}{2|p+d|^2(p+1)} & 0\\ -\frac{\kappa_2 d_1(p+d+2\lambda\kappa_2\kappa_3)}{2|p+d|^2(\overline{p}+1)} & -\frac{(Re\, p+1)|p+d|^2+\lambda\kappa_2^2 d_1^2+\lambda|p+d|^2}{|p+d|^2|p+1|^2} & 0\\ 0 & 0 & -\frac{Re\, p+b+\lambda}{|p+b|^2}\end{pmatrix}.\end{aligned}$$

Setting $p = i\omega - \gamma$ and using Sylvester's criterion, we conclude that the frequency condition is true if the following inequalities hold

$$d + \lambda\kappa_2\kappa_3 - \gamma < 0, \quad b + \lambda - \gamma < 0,$$
$$[4\kappa_3(d + \lambda\kappa_2\kappa_3 - \gamma)(1 + \lambda - \gamma) - \kappa_2 d_1^2][(d - \gamma)^2 + \omega^2] > 0.$$

□

Let us now proceed to upper bounds of the Hausdorff dimension of the attractors of the generalized Lorenz equations.

Let us introduce the following square equation with respect to γ

$$4\kappa_3(d + \lambda^*\kappa_2\kappa_3 - \gamma)(1 + \lambda^* - \gamma) - \kappa_2(d - \kappa_1\kappa_3/\kappa_2)^2 = 0.$$

Denote by γ^* the number which is equal to the largest real root of this equation if it exists. Otherwise, let $\gamma^* = 0$.

Theorem 5.3. *Let K be an attractor of the system* (2.1). *Then*

$$\dim K \leq 3 - (d + b + 1)/k_1, \tag{5.26}$$

where

$$k_1 = \inf_{\kappa_1 \geq 0, \kappa_2 > 0, \kappa_3 > 0} \max\{d + \lambda^*\kappa_2\kappa_3, b + \lambda^*, \gamma^*\}.$$

Proof. Take an arbitrary number $\gamma' > \max\{d + \lambda^*\kappa_2\kappa_3, b + \lambda^*, \gamma^*\}$. Then for $\gamma = \gamma'$ and $\lambda = \lambda^*$, the inequalities (5.24), (5.25) hold. Obviously, there exist a number $\lambda' > \lambda^*$ such that for $\gamma = \gamma'$ and $\lambda = \lambda'$ the inequalities (5.24), (5.25) remain valid. Due to Lemma 5.3, these values γ and λ ensure the validity of the hypotheses (2)–(4) of Theorem 5.1. For $\varepsilon > 0$, we denote by D_ε^* the ε–neighborhood of the set $\overline{D^*}$. Due to (5.16), we can choose ε so small that the inequality

$$\frac{1}{2}\eta^* \left[\frac{\partial\Phi^*}{\partial\sigma}(C^*x') + \frac{\partial\Phi}{\partial\sigma}(C^*x')\right] \eta \leq \lambda'|\eta|^2$$

is true for all $\eta \in R^3$ and $x' \in D_\varepsilon^*$. Hence, if $\lambda = \lambda'$ and $G = D_\varepsilon^*$, then hypothesis (1) of Theorem 5.1 holds. Setting $D = D_\varepsilon^*$ in the latter and taking into account the inclusion $K \subset \overline{D^*} \subset D$, we conclude from Theorem 5.1 that

$$\dim K < 3 - (d + b + 1)/\gamma'.$$

Since γ' is an arbitrary number we get the estimate (5.26). □

The next result slightly deteriorates the one of Theorem 5.3, but gives a rather simple bound of dimension for the attractors of the Lorenz system.

Corollary 5.1. *Let* $a = 0$, $b > 1$. *Let* K *be an attractor of the system* (2.1). *Then*

$$\dim K \leq 3 - (d + b + 1)/k_2, \tag{5.27}$$

where $k_2 = \frac{1}{2}\left[d + b + \sqrt{(d-b)^2 + (b/\sqrt{b-1} + 2)dr}\right]$.

Proof. Denote

$$s = \frac{(b-d)\kappa_1}{d\sqrt{\frac{1}{4}b^2r^2/(b-1) + 2\kappa_1 r + \kappa_1^2}}.$$

Choose the following values for varied parameters

$$\kappa_1 = \frac{1}{2}br/\sqrt{b-1}, \quad \kappa_2 = s + \sqrt{s^2 + \kappa_1/d}, \quad \kappa_3 = d\kappa_2/\kappa_1.$$

Due to the choice of κ_3, we have $\gamma^* = \max\{d + \lambda^*\kappa_2^2 d/\kappa_1, 1 + \lambda^*\}$. Due to the choice of κ_2, we have $d + \lambda^*\kappa_2^2 d/\kappa_1 = b + \lambda^*$. Therefore, in view of $b > 1$, we get

$$\max\{b + \lambda^*, \gamma^*\} = b + \lambda^*. \tag{5.28}$$

Substituting the expressions for κ_1 and κ_2 to the right–hand side of (5.28) we obtain from Theorem 5.3 that (5.27) is true. □

Let us compare the estimate given by Corollary 5.1 with that of R. Smith [29, Theorem 9]:

$$\dim K \leq 3 - (d + b + 1)/k_3,$$

where

$$k_3 = \frac{1}{2}\left[d + b + \frac{1}{2}br/\sqrt{b-1} + \sqrt{\left(d + b + \frac{1}{2}br/\sqrt{b-1}\right)^2 - 4db}\right].$$

Let us prove that

$$k_2 - k_3 < 0, \tag{5.29}$$

i.e. that estimate (5.27) is better than Smith's result.
We have

$$4(k_2 - k_3) = -\frac{br}{\sqrt{b-1}} + \frac{4dr - 2b^2r/\sqrt{b-1} - \frac{1}{2}b^2r^2/(b-1)}{\sqrt{(d-b)^2 + (\frac{b}{\sqrt{b-1}} + 2)dr} + \sqrt{(d-b)^2 + \frac{(d+b)br}{\sqrt{b-1}} + \frac{1}{4}\frac{b^2r^2}{(b-1)}}}$$

Therefore, (5.29) is equivalent to

$$-\frac{br}{\sqrt{b-1}}[\sqrt{(d-b)^2 + (b/\sqrt{b-1} + 2)dr} + \sqrt{(d-b)^2 + (d+b)br/\sqrt{b-1} + \frac{1}{4}b^2r^2/(b-1)}] + 4dr - 2b^2r/\sqrt{b-1} - \frac{1}{2}b^2r^2/(b-1) < 0.$$

Since $b/\sqrt{b-1} \geq 2$ and, without loss of generality, we can assume that $r > 1$, then it is sufficient to check the following inequality

$$\Delta \overset{\text{def}}{=} -\frac{br}{\sqrt{b-1}}[\sqrt{(d-b)^2 + 4d} + \sqrt{(d-b)^2 + 2(d+b) + 1}] + 4dr - 2b^2r/\sqrt{b-1} - \frac{1}{2}b^2r^2/(b-1) < 0.$$

Consider separately the case $d - b \geq 0$ and $d - b < 0$.
Let $d - b \geq 0$. Then

$$\sqrt{(d-b)^2 + 4d} + \sqrt{(d-b)^2 + 2(d+b) + 1} > 2(d-b).$$

Therefore

$$\Delta \leq -\frac{br}{\sqrt{b-1}}2(d-b) + 4dr - 2b^2r/\sqrt{b-1} - \frac{1}{2}b^2r^2/(b-1) \leq -\frac{1}{2}b^2r^2/(b-1) < 0.$$

Let $d - b < 0$. Then

$$\Delta < -\frac{br}{\sqrt{b-1}}\left[\sqrt{(d-b)^2 + 4d} + \sqrt{(d-b)^2 + 2(d+b) + 1}\right] - \frac{1}{2}b^2r^2/(b-1) < 0.$$

The inequality (5.29) is proved.

Let us also compare the estimate given by Corollary 5.1 with the one stated by R. Temam [34, Theorem 6.1.1]

$$\dim K \leq 2 + \frac{k_4}{d+b+1+k_4}, \tag{5.30}$$

where

$$k_4 = -(d+b+1) + m_1 + \frac{b(r+d)}{4\sqrt{m_2(b-1)}},$$
$$m_1 = \max(b,d), \quad m_2 = \min(1,d).$$

Let us prove that the estimate (5.27) is better than (5.30). To do this we must check that

$$\frac{d+b+1}{k_2} + \frac{k_4}{d+b+1+k_4} > 1. \tag{5.31}$$

The inequality (5.31) is equivalent to

$$m_1 + \frac{b(r+d)}{4\sqrt{m_2(b-1)}} - \frac{1}{2}(d+b) - \frac{1}{2}\sqrt{(d-b)^2 + (b/\sqrt{b-1}+2)dr} > 0.$$

The latter will be true if

$$\left[m_1 - \frac{1}{2}(d+b)\right]^2 + \frac{1}{2}\left[m_1 - \frac{1}{2}(d+b)\right]\frac{b(r+d)}{\sqrt{b-1}} + \frac{b^2(r+d)^2}{16(b-1)}$$
$$-\frac{1}{4}(d-b)^2 - \frac{1}{4}(b/\sqrt{b-1}+2)dr > 0. \tag{5.32}$$

But the validity of (5.32) is obvious since we have

$$\left[m_1 - \frac{1}{2}(d+b)\right]^2 = \frac{1}{4}(d-b)^2,$$
$$\frac{b^2(r+d)^2}{4(b-1)} - (b/\sqrt{b-1}+2)dr$$
$$= \left(\frac{br}{2\sqrt{b-1}} - d\right)^2 + \frac{(b-2)^2}{4(b-1)}(d^2+2dr).$$

Consequently, the inequality (5.32) is true except when $d = b = r = 2$. In this case, the right–hand side of (5.32) is equal to zero. However, without loss of generality, we can ignore this case because the Lorenz system is completely stable for these values of the parameters (see Corollary 5.2 below).

In particular, for $d = 10, b = 8/3, r = 28$ Corollary 5.1 infers $\dim K \leq 2.421$, Smith's estimate $\dim K \leq 2.666$ and Temam's estimate $\dim K \leq 2.538$.

From Theorem 5.3, we have $\dim K \leq 2.405$ ($\kappa_1 = 24$, $\kappa_2 = 1.3$, $\kappa_3 = 0.45$). Note also that A.Eden, C.Foias, and R.Temam recently obtained a new estimate of the Hausdorff dimension of attractors of the Lorenz system. For the considered values of the parameters, this estimate infers $\dim K \leq 2.4081$ [35, 36].

On the basis of Theorem 5.2, we now obtain the conditions of the complete stability of the generalized Lorenz equations.

Theorem 5.4. *Assume, that for $r = r'$, there exist numbers $\kappa_1 \geq 0$, $\kappa_2 > 0$, $\kappa_3 > 0$ such that*

$$\lambda^* \kappa_2 \kappa_3 - b - 1 \leq 0, \quad \lambda^* - d - b \leq 0, \quad \lambda^* - d - 1 \leq 0, \tag{5.33}$$

$$4\kappa_3(\lambda^* \kappa_2 \kappa_3 - b - 1)(\lambda^* - d - b) - \kappa_2(d - \kappa_1 \kappa_3/\kappa_2)^2 \geq 0. \tag{5.34}$$

Then system (2.1) is completely stable for $r < r'$.

Proof. It is easy to see the following. The number λ^*, defined by equalities (5.13)–(5.15) can be regarded as a function of variable r when all the other parameters fixed. This function is strictly increasing. Then it follows from (5.33), (5.34) that for $r < r'$ there exists the number $\gamma < d + b + 1$ such that

$$\lambda^* \kappa_2 \kappa_3 + d - \gamma < 0, \quad \lambda^* + b - \gamma < 0,$$
$$4\kappa_3(\lambda^* \kappa_2 \kappa_3 + d - \gamma)(1 + \lambda^* - \gamma) - \kappa_2(d - \kappa_1 \kappa_3/\kappa_2)^2 > 0.$$

Hence, for $r < r'$, due to Lemma 5.3, the hypotheses (2)–(4) of Theorem 5.2 hold with $\lambda = \lambda^*$. Setting $D = D^*$ and taking into account (5.16) we deduce that for $\lambda = \lambda^*$, hypothesis (1) of Theorem 5.2 is also true. Since D^* is a dissipativity region of the system (2.1), then the validity of Theorem 5.4 follows from Theorem 5.2. □

The next result slightly deteriorates the one of Theorem 5.4 but gives a rather simple condition for the complete stability of the Lorenz system.

Corollary 5.2. *Let $a = 0$, $b > 1$. If*

$$r < \frac{4\sqrt{b-1}(b+1)(d+1)}{d(b + 2\sqrt{b-1})} \tag{5.35}$$

then system (2.1) *is completely stable.*

Proof . We shall show that the conditions of Theorem 5.4 will be fulfilled if we take

$$r' = \frac{4\sqrt{b-1}(b+1)(d+1)}{d(b+2\sqrt{b-1})}, \quad \kappa_1 = \frac{br'}{2\sqrt{b-1}},$$
$$\kappa_2 = \sqrt{\frac{(b+1)\kappa_1}{d(d+1)}}, \quad \kappa_3 = d\kappa_2/\kappa_1.$$

Indeed, due to this choice of varied parameters, the inequalities (5.33), (5.34) are reduced to the unique inequality

$$\lambda^* \leq d+1. \tag{5.36}$$

Substituting λ^* defined by (5.13) into (5.36) we get that (5.36) is equivalent to

$$\frac{b^2 r'^2}{4(b-1)} + 2r'\kappa_1 + \kappa_1^2 - 4(d+1)^2\kappa_2^2 \leq 0.$$

The validity of the last inequality is verified by the direct substitution of the chosen values of r', κ_1, κ_2 and κ_3. □

R.Smith proved [29, Theorem 10] that the Lorenz system is completely stable if

$$r < \frac{2\sqrt{b^2-1}}{b} \min\left\{\frac{\sqrt{b+1}(d+b)}{d+b+1}, \frac{d+1}{\sqrt{d+b+1}}\right\}. \tag{5.37}$$

Let us show that the inequality (5.35) supplies a larger domain than (5.37) in the space of parameters of the Lorenz system. In order to verify this fact, it is sufficient to rewrite the right–hand side of (5.35) in the form

$$\frac{2\sqrt{b^2-1}}{b}\frac{2b\sqrt{b+1}(d+1)}{d(b+2\sqrt{b-1})}$$

and to utilize the inequality

$$\frac{2b(d+1)}{d(b+2\sqrt{b-1})} > \frac{d+b}{d+b+1}.$$

In particular, for $d = 10$, $b = 8/3$ (5.35) infers $r < 3.96$ while from (5.37) we get $r < 3.29$.

From Theorem 5.4, we receive the complete stability of the Lorenz system for $r < 4.5$ ($\kappa_1 = 0.4$, $\kappa_2 = 0.37$, $\kappa_3 = 0.8$).

We shall state now another effectively verified condition of the complete stability of the generalized Lorenz equations. Its proof is based on the following trivial consequence from Theorem 5.2.

Rewrite the system

$$\frac{dx}{dt} = f(x) \tag{5.38}$$

in the form of (5.4) with $A = 0$, $B = -I$, $C = I$ and $\Phi(x) = -f(x)$. It follows from Theorem 5.2 that if, for certain $\lambda > 0$, the inequalities

$$\frac{1}{2}[J^*(x) + J(x)] + \lambda I \geq 0,$$
$$(n-2)\lambda + tr\, J(x) < 0$$

are true for all $x \in \overline{D}$, then each trajectory of the system (5.38) in D converges to an equilibrium state.

Of course, this statement follows from Corollary 4.2 but it is important here that it is embraced by the frequency Theorem 5.2.

For an arbitrary $\alpha > 0$ we put

$$\begin{aligned} p(\alpha) =& (d+b)\left(\alpha - \frac{a}{\alpha}\right)^2 - (d+1)\left(\alpha + \frac{a}{\alpha}\right)^2, \\ q(\alpha) =& \frac{1}{\alpha}(d+1)\left(\alpha + \frac{a}{\alpha}\right)(d - ar). \end{aligned}$$

Theorem 5.5. *If for a certain* α *one of the two hypotheses:*

$$\begin{aligned} (1)\quad & \ell r p(\alpha) \leq |q(\alpha)|, \\ & 4(d+b)(b+1) - \left[\ell r|\alpha + \frac{a}{\alpha}| + \frac{1}{\alpha}|d - ar|\right]^2 > 0; \\ (2)\quad & \ell r p(\alpha) > |q(\alpha)|, \\ & 4(d+1)(b+1)p(\alpha) - \left(\alpha - \frac{a}{\alpha}\right)^2 \ell^2 r^2 p(\alpha) \\ & -\frac{1}{\alpha^2}(d+1)\left(\alpha - \frac{a}{\alpha}\right)^2 (d - ar)^2 > 0 \end{aligned}$$

is fulfilled then the system (2.1) *is completely stable.*

Recall that the number ℓ is defined by (2.3).

Before beginning the proof of Theorem 5.5, we shall introduce the necessary notations and formulate an auxiliary statement. Let

$$\varphi_2(z) = c_4 z^2 + 2c_5 z,$$

where c_4, c_5 are arbitrary real numbers.

The following simple lemma, the proof of which we omit, gives the value of minimum of $\varphi_2(z)$ in the segment $[-\Delta_1, \Delta_2]$, the numbers $\Delta_1 = (\ell - 1)r$ and $\Delta_2 = (\ell + 1)r$ being introduced above.

Lemma 5.4. *Let*

$$m = \min_{[-\Delta_1, \Delta_2]} \varphi_2(z).$$

Then

(1) *if* $c_4 \le 0$, $c_5 + c_4 r \le 0$ *then* $m = \varphi_2(\Delta_2)$;
(2) *if* $c_4 \le 0$, $c_5 + c_4 r \ge 0$ *then* $m = \varphi_2(-\Delta_1)$;
(3) *if* $c_4 > 0$, $c_5 + c_4\Delta_2 \le 0$ *then* $m = \varphi_2(\Delta_2)$;
(4) *if* $c_4 > 0$, $c_5 - c_4\Delta_1 \ge 0$ *then* $m = \varphi_2(-\Delta_1)$;
(5) *if* $c_4 > 0$, $-c_4\Delta_2 \le c_5 \le c_4\Delta_1$ *then* $m = \varphi_2(-c_5/c_4)$.

Proof. Let us carry out the change of variables in (2.1): $x \mapsto \alpha x$, $y \mapsto y$, $z \mapsto z$. We get

$$\begin{aligned} \frac{dx}{dt} &= -dx + \frac{d}{\alpha}y - \frac{a}{\alpha}yz, \\ \frac{dy}{dt} &= \alpha x(r - z) - y, \\ \frac{dz}{dt} &= -bz + \alpha xy. \end{aligned} \tag{5.39}$$

Since the inclusion (2.4) takes place, it is sufficient to show that the matrix

$$\frac{1}{2}(J^* + J) + \lambda I$$
$$= \begin{pmatrix} \lambda - d & \frac{1}{2}[\frac{d-az}{\alpha} + \alpha(r - z)] & \frac{1}{2}(\alpha - \frac{a}{\alpha})y \\ \frac{1}{2}[\frac{d-az}{\alpha} + \alpha(r - z)] & \lambda - 1 & 0 \\ \frac{1}{2}(\alpha - \frac{a}{\alpha})y & 0 & \lambda - b \end{pmatrix}$$

is positive difinite for $\lambda = \lambda_0 = -\mathrm{tr}\, J$ on the set $\overline{D}_1$. Due to Sylvester's criterion, it is sufficient to state that

$$\psi(y, z) \overset{\text{def}}{=} \det\, [\frac{1}{2}(J^* + J) + \lambda_0 I] > 0 \tag{5.40}$$

for all $(y,z) \in \overline{D}_1$. We have

$$\begin{aligned}\psi(y,z) =&\lambda_0^3 - (d+b+1)\lambda_0^2 + [b + d(b+1)]\lambda_0 - bd \\ &-\frac{1}{4}(\lambda_0 - 1)\left(\alpha - \frac{a}{\alpha}\right)^2 y^2 - \frac{1}{4}(\lambda_0 - b)\left[\frac{d - az}{\alpha} + \alpha(r - z)\right]^2.\end{aligned}$$

Denote $u = (d+b)(d+1)(b+1)$. Since $\lambda_0 = d + b + 1$ and $(y,z) \in \overline{D}_1$, we have

$$\begin{aligned}\psi(y,z) \geq u &- \frac{1}{4}(d+b)\left(\alpha - \frac{a}{\alpha}\right)^2 \ell^2 r^2 + \frac{1}{4}(d+b)\left(\alpha - \frac{a}{\alpha}\right)^2 (z-r)^2 \\ &- \frac{1}{4}(d+1)\left[\frac{d - az}{\alpha} + \alpha(r - z)\right]^2 = \frac{1}{4}[\varphi_2(z) + c_6],\end{aligned}$$

where the coefficients of the polynomial $\varphi_2(z)$ and the constant c_6 have the following values:

$$\begin{aligned}c_4 &= (d+b)\left(\alpha - \frac{a}{\alpha}\right)^2 - (d+1)\left(\alpha + \frac{a}{\alpha}\right)^2, \\ c_5 &= (d+1)\left(\alpha + \frac{a}{\alpha}\right)\left(\frac{d}{\alpha} + \alpha r\right) - (d+b)\left(\alpha - \frac{a}{\alpha}\right)^2 r, \\ c_6 &= 4u - (d+b)\left(\alpha - \frac{a}{\alpha}\right)^2 (\ell^2 - 1) r^2 - (d+1)\left(\frac{d}{\alpha} + \alpha r\right)^2.\end{aligned}$$

To estimate $\varphi_2(z)$ from below, we shall use Lemma 5.4. We first note that the following obvious relations hold

$$c_5 + c_4 r = q(\alpha), \tag{5.41}$$

$$c_5 + c_4 \Delta_2 = q(\alpha) + \ell r c_4, \tag{5.42}$$

$$c_5 - c_4 \Delta_1 = q(\alpha) - \ell r c_4. \tag{5.43}$$

Moreover

$$\varphi_2(\Delta_2) + c_6 = 4u - (d+1)\left[\ell r\left(\alpha + \frac{a}{\alpha}\right) - \frac{1}{\alpha}(d - ar)\right]^2, \tag{5.44}$$

$$\varphi_2(-\Delta_1) + c_6 = 4u - (d+1)\left[\ell r\left(\alpha + \frac{a}{\alpha}\right) + \frac{1}{\alpha}(d - ar)\right]^2. \tag{5.45}$$

Therefore, if $q(\alpha) \leq 0$, then it follows from (5.44) that

$$\varphi_2(\Delta_2) + c_6 = 4u - (d+1)\left[\ell r|\alpha + \frac{a}{\alpha}| + \frac{1}{\alpha}|d - ar|\right]^2 . \qquad (5.46)$$

But if $q(\alpha) \geq 0$, then it follows from (5.45) that

$$\varphi_2(-\Delta_1) + c_6 = 4u - (d+1)\left[\ell r|\alpha + \frac{a}{\alpha}| + \frac{1}{\alpha}|d - ar|\right]^2 . \qquad (5.47)$$

Finally,

$$\begin{aligned}
&\varphi_2(-c_5/c_4) + c_6 \\
&= \frac{1}{c_4}\left[4uc_4 - (d+b)\left(\alpha - \frac{a}{\alpha}\right)^2 \ell^2 r^2 c_4\right. \\
&\left. - \frac{1}{\alpha^2}(d+b)(d+1)\left(\alpha - \frac{a}{\alpha}\right)^2 (d - ar)^2\right]. \qquad (5.48)
\end{aligned}$$

Suppose that hypothesis (1) of the theorem holds, i.e. $\ell r c_4 \leq |q(\alpha)|$.

Assume that $c_4 \leq 0$. If $q(\alpha) \leq 0$, then the validity of the theorem follows from (5.41), (5.46) and Lemma 5.4 (case 1). If $q(\alpha) \geq 0$, then the validity of the theorem follows from (5.41), (5.47) and Lemma 5.4 (case 2).

Assume that $c_4 > 0$. If $q(\alpha) \leq -\ell r c_4$, then the validity of the theorem follows from (5.42), (5.46) and Lemma 5.4 (case 3). If $q(\alpha) \geq \ell r c_4$, then the validity of the theorem follows from (5.43), (5.47) and Lemma 5.4 (case 4).

Thus, if the hypothesis (1) holds, then the theorem is proved.

Suppose now that hypothesis (2) of the theorem holds, i.e. $\ell r c_4 > |q(\alpha)|$. Then the validity of the theorem follows from Lemma 5.4 (case 5) and (5.41), (5.48). □

From Theorem 5.5 we get the following simple condition of the complete stability of the Lorenz system.

Corollary 5.3. *Let* $a = 0$. *If*

$$r < \frac{(d+b)(b+1)}{d}, \quad b \leq 2,$$

or

$$r < \frac{2\sqrt{b-1}(d+b)(d+1)(b+1)}{bd} \min\left\{\frac{1}{d+1}, \frac{1}{b-1}\right\}, \quad b \geq 2,$$

then (2.1) *is completely stable.*

Proof. We have $p(\alpha) = (b-1)\alpha^2$, $q(\alpha) = d(d+1)$. The hypothesis (1) of Theorem 5.5 is reduced to the inequalities

$$\ell r(b-1)\alpha^2 \leq d(d+1), \quad \alpha\ell r + \frac{d}{\alpha} < 2\sqrt{(d+b)(b+1)}.$$

Choose

$$\alpha = \frac{d}{\sqrt{(d+b)(b+1)}}.$$

Then these inequalities become

$$\ell r(b-1)d \leq (d+b)(d+1)(b+1), \quad r < \frac{(d+b)(b+1)}{\ell d}.$$

Hence, we have the complete stability if

$$r < (d+b)(b+1)/d, \quad b \leq 1,$$

or

$$r < \frac{(d+b)(d+1)(b+1)}{\ell d} \min\left\{\frac{1}{d+1}, \frac{1}{b-1}\right\}, \quad b > 1.$$

Recalling the notation of ℓ, we obtain the conditions which are formulated in Corollary 5.3. □

For $d = 10$, $b = 8/3$, Corollary 5.3 gives the complete stability of the Lorenz system for $r < 4.4$.

The consequence from Theorem 5.5 formulated below turns out to be useful in the investigations of some concrete systems. This will be clear by means of examples considered in the last section.

Corollary 5.4. *Assume that* $d = ar$. *If*

$$r < (b+1)(b/d+1)/\ell^2, \tag{5.49}$$

then the system (2.1) *is completely stable.*

Proof. We have $q(\alpha) = 0$. The hypothesis (1) of Theorem 5.5 is reduced to the inequalities

$$p(\alpha) \leq 0, \quad 4(d+b)(b+1) - \left(\alpha + \frac{a}{\alpha}\right)^2 \ell^2 r^2 > 0. \tag{5.50}$$

Choose $\alpha = \sqrt{a}$. Then $p(\alpha) = -4(d+1)a < 0$ and the second inequality of (5.50) is written in the form

$$r^2 < \frac{(d+b)(b+1)}{\ell^2 a}.$$

Substituting in the last inequality $a = d/r$ we obtain

$$r < \frac{(d+b)(b+1)}{\ell^2 d}.$$

□

In order to estimate the dissipativity region of the generalized Lorenz equations, in Theorem 5.5 we usedthe inclusion (2.4). Further improvement of the complete stability conditions of the (2.1) is possible by means of improving this inclusion. In some cases, it may be useful to use Lemma 2.2. As an example, we formulate here the following result for the Lorenz system.

Theorem 5.6, [33]. *Let* $a = 0,\ b > 1$. *If*

$$r < \frac{4(d+b)(b+1)}{d\left[2 + \sqrt{\frac{(d+b)b^2 - 4(b-1)^2}{(d+1)(b-1)}}\right]}$$

then system (2.1) *is completely stable.*

For $d = 10,\ b = 8/3$, Theorem 5.6 gives the complete stability of the Lorenz system for $r < 4.5$, i.e. the same result as that of Theorem 5.4.

The form (5.4) of the system (5.38) allows us to obtain a simple estimate of the Hausdorff dimension of the attractors of the Lorenz system from Theorem 5.1. Its proof is analogous to the one of Theorem 5.5.

Let $\alpha > 0$ be an arbitrary number. Let us take into consideration the cubic equation

$$\nu^3 + a_1\nu^2 + a_2\nu + a_3 = 0 \tag{5.51}$$

with the coefficients

$$a_1 = -(d+b+1),$$
$$a_2 = d + b(d+1) - \frac{1}{16}\alpha^2 b^2 r^2/(b-1) - \frac{1}{4}d^2/\alpha^2 - \frac{1}{2}dr,$$
$$a_3 = -db + \frac{1}{16}\alpha^2 b^2 r^2/(b-1) + \frac{1}{4}d^2 b/\alpha^2 + \frac{1}{2}dbr + \frac{1}{4}\alpha^2 r^2 (b-1).$$

Denote by ν_0 the largest real root of (5.51). Let

$$k_5 = \inf_{\alpha} \max\{d + b + 1, b + \alpha^2 r(b-1)/d, \nu_0\}.$$

Theorem 5.7, [3]. *Let* $a = 0$, $b > 1$. *Let* K *be an attractor of system* (2.1). *Then*

$$\dim K \leq 3 - (d + b + 1)/k_5. \tag{5.52}$$

Note that the version of Theorem 5.7 without varied parameter α ($\alpha = 1$) was first obtained in [30].

For $d = 10$, $b = 8/3$, $r = 28$ the estimate (5.52) implies the same result as Theorem 5.3, $\dim K \leq 2.405$ ($\alpha = 0.6$).

6. Applications to Concrete Systems

In this section, applications of the above obtained results concerning the generalized Lorenz system (2.1) are illustrated by examples of concrete physical systems: rigid body rotation in a resisting medium, convection of a fluid contained within an ellipsoidal rotating cavity, interaction between waves in plasma.

We could also consider the following: the forced motion of gyrostat [72, 73], the convection of a horizontal layer of fluid making the harmonic oscillations [74], the model of Kolmogorov flow [75]. However, the three examples presented below allow us to fully demonstrate applications of the main theorems proved in the preceding sections for (2.1).

6.1. RIGID BODY ROTATION IN RESISTING MEDIUM

Consider the rotation of rigid body about the center of mass in a linearly resisting medium when there is a constant moment with direction along one of its main axes. The Euler equations describing such a rotation of the body read

$$\begin{aligned}
A_1 \frac{d\omega_1}{dt} &= (A_2 - A_3)\omega_2\omega_3 - s_1\omega_1 + m, \\
A_2 \frac{d\omega_2}{dt} &= (A_3 - A_1)\omega_1\omega_3 - s_2\omega_2, \\
A_3 \frac{d\omega_3}{dt} &= (A_1 - A_2)\omega_1\omega_2 - s_3\omega_3,
\end{aligned} \tag{6.1}$$

where A_i are moments of inertia of the body, ω_i are components of the angular velocity vector, m is a constant moment of external forces, and s_i are coefficients of resistance.

Many published works are devoted to different types of stability of equilibrium states of the system (6.1) (see, for example, [76] and references contained there). However, complete stability of (6.1) was apparently first investigated in the paper [33].

Further, we shall suppose that the inequality $(A_1 - A_2)(A_3 - A_1)$ is true. Indeed if the opposite inequality holds, then (6.1) has a unique equilibrium state for any value of m but this case isn't of any interest for a complete stability study.

The following change of variables [72] reduces the system (6.1) to (2.1)

$$\omega_1 \mapsto \frac{m}{s_1} - s_2 s_3 (A_3 - A_1)^{-1} T^{-1} z, \qquad \omega_2 \mapsto s_2 s_3 S^{-1} T^{-1} y,$$
$$\omega_3 \mapsto s_2 S^{-1} x, \qquad t \mapsto \frac{1}{s_2} A_2 t.$$

Here

$$S = \left[A_1^{-1} A_2 |(A_3 - A_1)(A_2 - A_3)|\right]^{\frac{1}{2}}, \qquad T = \frac{m}{s_1}(A_1 - A_2).$$

The parameters of (2.1), by means of such a change, take the values:

$$d = \frac{s_3}{s_2} A_2 A_3^{-1}, \quad b = \frac{s_1}{s_2} A_1^{-1} A_2, \quad r = \frac{m}{s_1 s_2 s_3}(A_3 - A_1) T,$$
$$a = s_3^2 A_2 A_3^{-1} (A_1 - A_2)(A_3 - A_1)^{-1} T^{-2}.$$

Using Theorem 2.1 and Corollary 5.4, we immediatley get the following result.

Theorem 6.1, [33]. *Suppose that* $(A_1 - A_2)(A_3 - A_1) > 0$. *System* (6.1) *is globally asymptotically stable if*

$$m^2 (A_1 - A_2)(A_3 - A_1) < s_1^2 s_2 s_3.$$

System (6.1) *is completely stable if*

$$m^2 (A_1 - A_2)(A_3 - A_1) < s_1^2 A_1^{-2} (s_1 A_2 + s_2 A_1)(s_1 A_3 + s_3 A_1),$$
$$s_1 A_2 \leq 2 s_2 A_1$$

or

$$m^2 (A_1 - A_2)(A_3 - A_1) < 4 s_2 A_1^{-1} A_2^{-2} (s_1^2 A_2^2 - s_2^2 A_1^2)(s_1 A_3 + s_3 A_1),$$
$$s_1 A_2 \geq 2 s_2 A_1.$$

6.2. CONVECTION OF THE ROTATING FLUID

In [46], the convection of the fluid contained within ellipsoidal rotating cavity is considered. The axis of rotation coincides with one of the main axes of ellipsoid and composes an angle unequal to zero with the vector of gravity. Convection is excited by the external horizontal heating.

The system of differential equations arising in this model read

$$\begin{aligned} \frac{dx}{dt} &= d(y-x) - \frac{a_0 d^2}{(a_0 R + 1)^2} yz, \\ \frac{dy}{dt} &= \frac{R}{d}(a_0 R + 1)x - y - xz, \\ \frac{dz}{dt} &= -z + xy, \end{aligned} \tag{6.2}$$

where d, R, a_0 are positive numbers.
System (6.2) coincides with (2.1) if we set

$$b = 1, \qquad a = \frac{a_0 d^2}{(a_0 R + 1)^2}, \qquad r = \frac{R}{d}(a_0 R + 1).$$

Theorem 6.2, [33]. *System* (6.2) *is completely stable if*

$$R < \frac{1}{2a_0}\left(\sqrt{8a_0(d+1)+1} - 1\right). \tag{6.3}$$

Proof. We shall use Theorem 5.5 to prove the theorem. We have used the notations of Theorem 5.5 $\ell = 1$, $p(\alpha) = -4(d+1)a$. Hypotheses (1) of Theorem 5.5 is reduced to the unique inequality

$$8(d+1) - \left[\frac{R}{d}(a_0 R + 1)\alpha + \frac{d}{\alpha}\right]^2 > 0,$$

which is true for

$$R < \frac{1}{2a_0}\left[\sqrt{4\frac{a_0 d}{\alpha}\sqrt{8(d+1)} - 4\frac{a_0 d^2}{\alpha^2} + 1} - 1\right].$$

Taking

$$\alpha = \frac{d}{\sqrt{2(d+1)}},$$

we get condition (6.3). □

In [46], for $d = 4$, $a_0 = 0.04$, $R = 250$ the existence of strange attractors was discovered by means of numerical experiments. It follows from Theorem 5.3 that for these values of parameters, the upper bound of the Hausdorff dimension is 2.895 ($\kappa_1 = 400$, $\kappa_2 = 10$, $\kappa_3 = 0.09$).

It follows from Theorem 2.1 that if $d = 4$ and $a_0 = 0.04$, then (6.2) is globally asymptotically stable for $R < 3.5$. Theorem 5.4 ensures complete stability for $R < 7.6$ ($\kappa_1 = 0.9$, $\kappa_2 = 0.5$, $\kappa_3 = 0.8$). We get immediately the same result from Theorem 6.2.

6.3. INTERACTION BETWEEN WAVES IN PLASMA

In [21] (see also [77]), on the example of the interaction of waves in plasma, the following equations

$$\begin{aligned} \frac{dx}{dt} &= hy - \nu_1 x - yz, \\ \frac{dy}{dt} &= hx - \nu_2 y + xz, \\ \frac{dz}{dt} &= -z + xy \end{aligned} \tag{6.4}$$

are deduced. This system describes the interaction of three resonantly coupled waves two of them being parametrically excited. Here, the parameter h is proportional to the pumping amplitude and the parameters ν_1 and ν_2 are normalized dumping decrements.

The change of variables

$$x \mapsto \nu_1\nu_2 h^{-1} y, \quad y \mapsto \nu_1 x, \quad z \mapsto \nu_1\nu_2 h^{-1} z, \quad t \mapsto \nu_1^{-1} t$$

reduces system (6.4) to the form of (2.1) with the parameters

$$d = \nu_1^{-1}\nu_2, \quad b = \nu_1^{-1}, \quad a = -\nu_2^2 h^{-2}, \quad r = \nu_1^{-1}\nu_2^{-1}h^2.$$

Numerical computations [21] showed that if $\nu_1 = 1$, $\nu_2 = 4$, $h = 4.92$, then there exists a strange attractor for the system (6.4). For these values of parameters, it follows from Theorem 5.3 that an upper bound of the Hausdorff dimension of this attractor is 2.339 ($\kappa_1 = 3.1$, $\kappa_2 = 0.7$, $\kappa_3 = 0.9$). For these values of ν_1 and ν_2, hypothesis (2) of Theorem 2.1 reduces to the inequality $h < 2$ which ensures the global asymptotic stability of (6.4). We infer from Theorem 5.4 the complete stability in the case of $h < 2.4$ ($\kappa_1 = 0.6$, $\kappa_2 = 0.25$, $\kappa_3 = 1.6$).

We should like to draw the reader's attention to the fact that in Section 4, by means of introducing the Lyapunov function into the estimates of the Hausdorff dimension, we succeeded in getting better results for the system (6.4), with $\nu_1 = 1$, (see Theorems 4.6 and 4.7) than those given in the present subsection.

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