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Tolerance Space Theory and Some Applications

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Abstract. The paper deals with the notion of tolerance space (introduced by E. C. Zeeman, but discerned earlier by H. Poincaré), which formalizes the idea of resemblance. The category of tolerance spaces is described, their homology and homotopy theories developed. Applications include almost-fixed point theorems, almost-solution existence theorems for difference schemes and the three-channel principle (a general theorem on multichannel data transmission).

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A tolerance on a set is a mathematical structure formalizing the idea of resemblance or, to be more precise, the idea of being the same up to a small (or allowable) error, specified in advance.

Basic definitions, as well as some advertising of the concept of tolerance space, are contained in Part I, which is introductory. Part II, where some of the (mildly sophisticated) techniques are developed, will primary interest the theoretically-minded mathematician. Part III is for applied mathematicians and other users. Its main theorems and ideas can be understood without reading Part II (but the proofs require the techniques presented in Sections 6–10).

In writing Part III, I did not attempt to give as many ready-to-use specific applications as possible. Actually, my aim was to present a few simple model examples from various useful branches of mathematics, in the hope that the reader will work out the detailed result he needs himself, using the techniques developed here as tools and the examples for inspiration.

PART I. PHILOSOPHY

1. What is a Tolerance Space?

1.1. *A tolerance space* is a set X supplied with a binary relation ξ (i.e., a subset $\xi \subset X \times X$) which is reflexive $(\forall x, (x, x) \in \xi)$ and symmetric

$(\forall x, y \ (x, y) \in \xi \Rightarrow (y, x) \in \xi)$. We write briefly X_ξ and abbreviate $(x, y) \in \xi$ to $x\xi y$. Note that we do not require the transitivity of ξ; if we did, we would get an equivalence – a particular case of tolerance which does not interest us.

Here are some examples of tolerance spaces.

1.2. X is a metric space, $\epsilon > 0$ a fixed number; the tolerance relation (also denoted ϵ) is 'the distance between the points x and y is less than ϵ', written $x\epsilon y$ (if 'less than' is replaced by 'less than or equal to', you still get a tolerance space on X, denoted $X_{\bar{\epsilon}}$, which may differ from X_ϵ).

1.3. X is a topological space with a fixed covering ω; the relation (also denoted by ω) is 'the points x and y are both contained in one element of the covering ω', written $x\omega y$.

1.4. X is the set of vertices of simplicial complex (or a simplicial set); the relation is 'the vertices x and y are in the same simplex'.

1.5. X is the set of nodes of a difference scheme (used for the approximate solution of a differential equation); the relation is 'the nodes x and y are next to each other' (or 'are in the same group of nodes used to calculate the recurrent value at some node').

1.6. X is the set of vertices of a (nonoriented) graph; the relation is 'x and y are vertices of the same edge of the graph'. (Of course, oriented graphs do not possess a natural tolerance space structure, because the symmetry property does not hold.)

1.7. $X = C^\infty[0, 1]$ is the set of smooth functions on the closed interval $[0, 1]$; $\epsilon > 0$ is a fixed real number; the relation is 'the functions f and g and all their derivatives at all points of $[0, 1]$ differ by less than ϵ' (or 'by a value less than or equal to ϵ').

1.8. X is the set of finite (n-term) binary sequences, the relation is 'the two sequences differ by only one (or, say, two, or three) binary digits'. More generally, different types of finite codes with a specified level of error have a natural tolerance space structure.

1.9. X is the set of all left cosets of a group G with respect to any of the elements H_α of a family $\{H_\alpha : \alpha \in \mathfrak{A}\}$ of subgroups of G. The relation is 'two cosets $p = g_1 H_\infty$, $q = g_2 H_\beta$ $(g_1, g_2 \in G, \alpha, \beta \in \mathfrak{A})$ have a nonempty intersection'.

Note that tolerance space structures appear just as naturally in sets with continuous structures (1.2, 1.3, 1.7) as in discrete structured sets (1.4–1.6, 1.8). In fact, the role of tolerance is often to bridge the gap between discrete and continuous structures; the bridge appears as soon as we specify what error we are willing to tolerate (this will be explained in Section 3.5 below).

2. Why Tolerance?

The range of examples presented in Section 1 is hardly exhaustive. Taken from very varied branches of mathematics, the examples show that the exact idea of closeness or of 'resembling', or of 'being within tolerance') is universal enough to appear quite naturally in almost any mathematical setting. It is especially natural in mathematical applications: practical problems, more often than not, deal with approximate input data and only require viable results – results with a tolerable level of error, guaranteed in advance.

At first glance it seems dubious that a definition as simple and general as that of tolerance can give rise to a meaningful theory. But it does (see, for example, Sections 7, 8, 11, developed in the most general case), and I, for one, must admit that I haven't stopped being surprised by this strange circumstance.

I think one of the reasons Definition 1.1 works is that it expresses the idea of resemblance succinctly and precisely. The absence of transitivity in the definition is the crux of the matter. 'John resembles Joe' and 'Joe resembles Moe' does not necessarily imply 'John resembles Moe'. Another reason is that mathematicians in pre-tolerance times had devised theories for evaluating the 'degree of nontransitivity' (homological algebra is one), and these theories seem to feel quite at home in the tolerance setting. In fact, they tend to become simpler (e.g., Sections 7–9). A further reason is that tolerance, in a way, is a trick for avoiding the specific hazards of infinite-dimensional function spaces, e.g., their local noncompactness (see Sections 8, 12); moreover, in a certain sense in tolerance space theory you can't even have large finite dimensions (see Section 10).

To summarize the answer to the question in the title of this section, let us say that:

(i) tolerance spaces appear quite naturally in the most varied branches of mathematics;
(ii) the tolerance setting is very convenient for the use of many existing powerful mathematical tools;
(iii) only results 'within tolerance' are usually required in practical applications.

Arguing philosophically, one can say that equality (unlike tolerance) is meaningless in the physical world, since it can never be verified, neither in practice (because of measurement errors and the like), nor in theory (by Heisenberg's indeterminacy principle). The next step in the argument is to declare the classical mathematics of equality inadequate and replace them by 'tolerance mathematics'. Think of all the PhD's that would result from rewriting all of mathematics from the tolerance standpoint, as it was rewritten from the 'intuitionist' and 'constructivist' point of view in bygone days and is now being rewritten from 'nonstandard' and 'fuzzy' positions! This article is definitely not a rallying cry for such a ridiculous endeavor. I have only tried to point out situations where the tolerance formalism gives rise to interesting mathematics and to potentially useful applications.

3. The Category of Tolerance Spaces

3.1. Let X_ξ and X_η be tolerance spaces. A *morphism* or *map of tolerance spaces* $f \in \mathrm{Mor}(X_\xi, Y_\eta)$ is any* mapping $f\colon X \to Y$ preserving tolerance, i.e., possessing the property

$$\forall x, x' \in X, \quad x\xi x' \Rightarrow f(x)\eta f(x').$$

The identical morphism id_X and the composition of morphisms $\circ$ is defined in the usual way (as in set theory). Thus, we obtain the *category of tolerance spaces*, denoted by Tol.

A morphism $f\colon X_\xi \to Y_\eta$ is *injective*, if

$$\forall x, x' \in X, \quad x \neq x' \Rightarrow f(x) \neq f(x'), \quad f(x)\eta f(x') \Rightarrow x\xi x',$$

surjective, if $\forall y \in Y, \ \exists x \in X, \ f(x) = y$ and *bijective* if both conditions hold. Bijectivity is also called *isomorphism* and denoted by $\simeq$; we do not distinguish isomorphic tolerance spaces.

3.2. The set of all morphisms $\mathrm{Mor}(X_\xi, Y_\eta)$ possesses the *canonical tolerance* $\xi\eta$, defined by the rule

$$f(\xi\eta)g \Leftrightarrow \forall x \in X f(x)\eta g(x).$$

One can also consider the *graphical tolerance* $\xi \times \eta$

$$f(\xi \times \eta)g \Leftrightarrow (\forall x \in X, \exists x' \in X, x\xi x', f(x)\eta g(x')) \quad \text{and}$$
$$(\forall \bar{x} \in X, \exists \bar{x}' \in X, \bar{x}\xi\bar{x}', g(\bar{x})\eta f(\bar{x}')$$

If f is a map of a set X into a tolerance space Y_η, the *induced* tolerance, denoted $f^*\eta$, arises in X in accordance to the rule

$$x(f^*\eta)x' \Leftrightarrow f(x)\eta f(x').$$

In particular, for an inclusion $i\colon B \hookrightarrow Y$ the induced tolerance or *subtolerance* $i^*\eta$ on B is simply denoted by B_η (this does not lead to misunderstandings). Together with the category Tol, we consider the *category of pairs* $\mathrm{Tol}^2 \ni (X_\xi, A_\xi)$ *of tolerance spaces*.

* We consider only single-valued mappings here, although multi-valued mappings preserving tolerance are also interesting.

3.3. If $A \subset X_\xi$, then the (1-fold) *widening* of A is the set $\xi A = \{x \in X : \exists a \in A x\xi a\}$. By induction, we define the *k-fold widening* of $A : k\xi A = \xi(k-1)\xi A$. The set $\mathrm{Bd}\, A = \xi A - A$ is called the *boundary* of A. Note that in general $\xi\xi A \neq \xi A$ and $\mathrm{Bd}\, \xi A \neq \mathrm{Bd}\, A \neq \mathrm{Bd}\,\mathrm{Bd}\, A \neq \emptyset$.

The *doubled* tolerance of X_ξ, denoted $X_{2\xi}$, is given by the rule

$$x(2\xi)x' \Leftrightarrow \exists y \in X x\xi y,\ y\xi x'.$$

The *tripled* tolerance of X_ξ, denoted $X_{3\xi}$, is

$$x(3\xi)x' \Leftrightarrow \exists y, y' \in X,\ x\xi y,\ y\xi y',\ y'\xi x'.$$

In a similar way, one defines the *k-fold* tolerance $k\xi$ of X_ξ. For $A \subset X_\xi$, the notation $k\xi A$ may be understood as the 1-fold widening of A in $X_{k\xi}$ or as the k-fold widening of A in X_ξ – this is the same thing.

If, for the given tolerance X_ξ, there is a tolerance η on X such that $X_\xi = X_{k\eta}$, it is natural to write $\eta = \xi/k$. But beware: ξ/k is not unique in general, although it is often unique up to isomorphism. For any rational $m/n \in \mathbb{Q}$, we can consider (if it exists) the tolerance $(m/n)\xi$.

Note that if f: $X_\xi \to Y_\eta$ is a morphism, then so is f: $X_{k\xi} \to Y_{l\eta}$ for any l, $k \in \mathbb{N}$, $k \leq l$.

3.4. A *skeleton* of the tolerance X_ξ is a subtolerance i: $A_\xi \hookrightarrow X_\xi$ for which there exists a morphism (in the doubled tolerance) r: $X_\xi \to A_{2\xi}$ such that $r \circ i = \mathrm{id}_A$ and $\forall x \in X x\xi r(x)$. If A_ξ is a skeleton of X_ξ, then $\xi A = X$.

3.5. EXAMPLE. Suppose $X \subset \mathbb{R}^2$ is the unit disk in the plane, A is the set of vertices of a square lattice of mesh h contained in X. Then, for $\epsilon > \sqrt{2}\, h$ the set A is a skeleton of the tolerance X_ϵ (where the metric tolerance ϵ is defined as in Example 1.2, namely $\mathbf{x}\epsilon\mathbf{y} \Leftrightarrow \|\mathbf{x} - \mathbf{y}\| < \epsilon$. This is a typical example: it shows how easily you can pass from a continuous object (the disk) to a discrete one (the set of vertices A) which approximates it, if you are willing to pay the price (double the tolerance).

3.6. *The Cartesian product* of two tolerance spaces X_ξ and Y_η is the tolerance space $(X \times Y)_{\xi \times \eta}$ defined by the rule

$$(x, y)\xi \times \eta(x', y') \Leftrightarrow x\xi x',\ y\eta y'.$$

(No confusion with the notation $\xi \times \eta$ in Section 3.2 arises – we use the latter for morphism only.)

3.7. A tolerance space X_ξ is called *finite*, if the set X is finite, of *finite type*, if X_ξ has a finite skeleton, *connected*, if $\forall x, x' \in X\ \exists k\ (k\xi)(x) \ni x'$, *discrete*, if $\forall x, x' \in X$ $x\xi x' \Rightarrow x = x'$ and *trivial*, if $\forall x, x' \in X$, $x\xi x'$.

The reader should pause at this point to imagine examples possessing various combinations of these properties. In particular, examples of tolerance spaces of finite type, which play the role of compact sets (or sometimes, of bounded sets). However, the analogy with topology and functional analysis is often misleading – for example, the *open* disk, with any metric tolerance ϵ on it, is of finite type, but Euclidean space $\mathbb{R}^n_\epsilon$ is not (whatever the choice of $\epsilon > 0$). Note that two Euclidean metric tolerances $\mathbb{R}^n_\epsilon$ and $\mathbb{R}^n_{\epsilon'}$ are always isomorphic, whatever the choice of ϵ and ϵ', but, of course, two unit disks D^n_ϵ and $D^n_{\epsilon'}$ or two unit spheres S^n_ϵ, $S^n_{\epsilon'}$ are different if $\epsilon \neq \epsilon'$ and $\epsilon, \epsilon' < 1$.

In general, the difference between open and closed sets disappears when one introduces a natural tolerance structure. Continuity of functions is also irrelevant – the relevant objects are functions possessing only small (tolerable!) discontinuities and variations (incidentally, such functions are always bounded on tolerance spaces of finite type).

The following general statement may be helpful in developing an intuition of tolerance. Tolerance is pragmatic – it refuses to understand those concepts of pure mathematics (products of the human mind or objects of Plato's world of ideas, as you like) which are not verifiable in practice.

4. Toleomorphism

4.1. The isomorphism relation in the class of all tolerance spaces is very harsh – it means two tolerances are exactly the same. One would like to have a weaker relation expressing the idea that two tolerance spaces (not two elements of tolerance space, but the spaces themselves) are alike. In accordance to the tolerance philosophy, such a relation cannot be transitive (otherwise we get an equivalence), but must be reflexive and symmetric. The appropriate relation for tolerance spaces is called toleomorphism and is defined below (see Section 4.2). This relation is also meaningful for metric spaces, topological spaces, and other objects not possessing a fixed tolerance structure. The definition in this case is presented in Section 4.4.

4.2. Two tolerance spaces X_ξ and Y_η will be called *toleomorphic* if there exist two morphisms $f\colon X_\xi \to Y_\eta$, $g\colon Y_\eta \to X_\xi$ which are *almost injective* (i.e., $\forall x, x' \in X\ f(x) = f(x') \Rightarrow x 2\xi x'$ and similarly for g), *almost surjective* (i.e., $\forall y \in Y\ \exists x \in X f(x) \eta y$ and similarly for g) and *almost inverse to each other* (i.e., $\forall x \in X\ \exists y \in Y f(x) \eta y$, $g(y) 2\xi x$ and similarly $\forall y \in Y\ \exists x \in X\ g(y) \xi x$, $f(x) 2\eta y$. In this case, the pair of maps (f, g) constitutes a *toleomorphism of* X_ξ and Y_η; we write this as $(f, g)\colon X_\xi \rightleftarrows Y_\eta$.

4.3. EXAMPLES. (i) The lattice of points $L_\epsilon = \{(k/n, l/n)\colon k, l = 0, 1, \ldots, n\}$ (in the unit square $Q = [0, 1] \times [0, 1]$) supplied with the tolerance $\epsilon = \sqrt{2}/n$ is toleomorphic to the unit square Q itself with the same tolerance. In general, the skeleton A_ξ of a tolerance X_ξ of finite type is toleomorphic to X_ξ.

(ii) The circle $S^1 = \{(x, y), x^2 + y^2 = 1\}$ with tolerance ϵ is toleomorphic to the annulus $A = \{(x, y), 1 - \epsilon/4 < x^2 + y^2 < 1 + \epsilon/4\}$ with the same tolerance.

(iii) The ball $B^n = \{(x_i): \Sigma x_i^2 < 1\}$ is not toleomorphic to the sphere $S^n = \{(x_i): \Sigma x_i^2 = 1\}$ nor to the ball $2B^n = \{(x_i): \Sigma x_i^2 < 2\}$ if the metric tolerance ϵ satisfies $\epsilon < \frac{1}{4}$.

4.4. Two metric spaces X and Y are toleomorphic, if for any $\epsilon > 0$ there exist two positive numbers δ_1, δ_2 such that the tolerance spaces X_{δ_1} and X_{δ_2} are toleomorphic (here, as usual, X_{δ_1} denotes the metric tolerance $x\delta_1 x' \Leftrightarrow \mathrm{d}(x, x') < \delta_1$). Two topological spaces X and Y are toleomorphic, if for any two coverings ω_1 and ω_2 of X and Y there exist finer coverings $\omega_1' < \omega_1$, $\omega_2' < \omega_2$ such that the tolerance spaces $X_{\omega_1'}$ and $Y_{\omega_2'}$ are toleomorphic (for any covering ω of the topological space X, the tolerance space structure X_ω is defined in Section 3.2).

4.5. EXAMPLES. (i) Isometric and similar metric spaces are toleomorphic.

(ii) Homeomorphic topological spaces are toleomorphic.

(iii) The circle (as a metric or topological space) is not toleomorphic to the annulus.

4.6 Toleomorphism of tolerance spaces is not an equivalence relation (no transitivity!), it is a tolerance on the class of all tolerance spaces. If X_ξ is toleomorphic to Y_η and Y_η to Z_ζ, it often follows that $X_{2\xi}$ is toleomorphic to $Z_{2\zeta}$, but this is not a general theorem.

5. Tolerance is Crisp, not Fuzzy

The ideas underlying tolerance space theory are not far removed from the ones of fuzzy mathematics. Perhaps for this reason T. Poston [14] uses various derivatives of the word 'fuzz' in his work on tolerance space. In any case, a juxtaposition of these two approaches is inevitable; we will be quite brief here.

Mathematically, tolerance and fuzzy structures are in 'general position' – neither one implies the other in any reasonable sense. A prodigious amount of publications on fuzzy mathematics has appeared (for the basic ideas, see [23]), whereas very little has been published on tolerance spaces.

Rather than express my antifuzz bias in my own words, I prefer quoting Saunders MacLane (see [12]):

> The original idea was an attractive one – instead of saying that an element x is or is not in the set A, let us measure the likelihood that x is in A. Someone then recalled that all mathematics can be based on set theory; it followed at once that all mathematics could be rewritten so as to be based on fuzzy sets. Moreover, it could be based on fuzzy sets in more than one way, so this turned out to be a fine blueprint for the publication of lots and lots of newly based mathematics. This has been duly done, complete with extravagant claims for applications (e.g., 'fuzzy decision theory'). Most of those intended do not seem to have materialized. New ideas are nice, but promotional gimmicks are not.

Leaving emotions aside, I would like to point out once again that tolerance space theory is a setting where existing powerful mathematical tools fit neatly, usually with simplifying modifications. Apparently the army of authors of fuzzy mathematics has found it difficult to adapt existing crisp mathematics to a fuzzy setting, and this setting, to my knowledge, has not produced, as of now, any powerful methods.

6. Historical Note

The main notions of tolerance space theory are due to E. C. Zeeman, who coined the term (a rather unfortunate one, I think), first thought of defining and using homology groups of tolerance spaces and applied this to theoretical biology in the well-known 1961 paper 'Topology of the brain and visual perception' ([21], see also [22]).

Nevertheless, like many other beautiful mathematical ideas, the main one underlying tolerance space theory goes back to Henri Poincaré. Poincaré contrasts the 'mathematical continuum' (the real numbers) and the 'physical continuum' (measurable magnitudes in the real world). In the latter, unlike the former, $a = b$ and $b = c$ does not necessarily imply $a = c$, because the (inevitable) errors in the first two relations may add, and eventually accumulate if the use of the transitivity rule is iterated. Poincaré did not work out the mathematics of the physical continuum (at least in written form), but was undoubtedly aware that this should be done. He repeatedly returned to this topic in his nonmathematical books (in particular, see his *Last Essays*, p. 37 of the standard Dover edition in English). Tolerance, in the framework of Poincaré's pragmatic philosophy of science, distinguishes the meaningful (i.e., verifiable) part of mathematics, as applied to the physical world, from ideal (mathematical) mathematics.

Another important development in the history of the subject was C. H. Dowker's remarkable (but rarely quoted) 1952 paper [4], where he constructed the homology theory of arbitrary binary relations. In the early 60s, the construction of a homology theory of symmetric binary relations (i.e., tolerance spaces) were undertaken by E. C. Zeeman, but his book [20], which has been 'to-appearing' since 1961, is still, to my knowledge, unpublished.

In 1972, the first rigorous exposition of tolerance space homology theory appeared in [16], followed by [17, 18] where almost fixed-point theorems and applications to difference schemes were presented. At approximately the same time (1973), I. M. Lapitski wrote a long paper on the homotopy of tolerance spaces [11] and A. V. Černavski noticed that the L. V. Keldysh embedding of the 3-cube into the 4-cube has meaningful applications (the 3-channel theorem, Section 15) to coding theory (also unpublished, except for a brief mention in a joint paper [1]).

More recent developments include T. Poston's fundamental thesis [14] (also unpublished), A. M. Vinogradov's work on group crystals (in this issue of *Acta*

Applicandae Mathematicae, pp. 169–180), the notion of toleomorphism (see Section 4) and the tolerance interpretation (the 3-cube theorem, Section 10) of L. V. Keldysh's embedding theorem.

A striking aspect of this little history (which makes no claim to completeness) is the amount of unpublished work in the field. In this connection, I hope that this article, where the basic facts of the theory (which have been 'in the air' for over two decades) are finally set to paper, will facilitate further publications.

PART II. TECHNIQUES

7. Homotopy

7.1. Two morphisms $f, g: X_\xi \to Y_\eta$ of tolerance spaces are called *homotopic* (notation: $f \simeq g$) if there exists a number $\epsilon > 0$ and a morphism $F: X_\xi \times I_\epsilon \to Y_\eta$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$; here I_ϵ denotes the closed interval $[0, 1]$, ϵ is the metric tolerance (see Section 1.1) and X the Cartesian product (see Section 3.6). Two morphisms f and g are *elementary homotopic* (notation $f \simeq_e g$) if for all $x, x' \in X$, $x\xi x' \Rightarrow f(x)\eta g(x')$. Let us (temporarily) say that two morphisms are homotopic in the second sense, if there exists a finite sequence $f_0, f_1, \ldots, f_k$ of pairwise elementary homotopic morphisms $f = f_0 \simeq_e f_1, f_1 \simeq_e f_2, \ldots, f_{k-1} \simeq_e f_k = g$ linking f and g (intuitively, this means f can be transformed into g by a sequence of 'little pushes').

7.2. PROPOSITION. *The two definitions of homotopy in Section* 7.1 *are equivalent.*

Proof. If $f \simeq g$ in the first sense, we put $k =$ (integer-part $(1/\epsilon)$) and $f_i(x) = F(x, i/k)$, $i = 0, 1, \ldots, k$, obtaining a sequence of pairwise elementary homotopic morphisms f_i joining f and g. Conversely, if we have a sequence $f = f_0 \simeq_e f_1 \simeq_e \cdots \simeq_e f_k = g$, then the required morphism $F: X_\xi \times I_\epsilon = Y_\eta$ may be constructed by putting $\epsilon = 1/k$ and

$$F(x, t) = f_i(x),\ i - \frac{i}{2k} \leq t < i + \frac{i}{2k}.$$

7.3. Two tolerance spaces X_ξ and Y_η are *homotopy equivalent* (notation $X_\xi \simeq Y_\eta$), if there exist morphisms $f: X_\xi \to Y_\eta$ and $g: Y_\eta \to X_\xi$, such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. By Proposition 7.2, this definition has a simple geometric meaning: for example, $X_\xi \simeq X_\eta$ means that there exists a finite sequence of 'little pushes' $f_i: X_\xi \to X_\eta$, beginning with the identity, deforming X_ξ along itself into X_η.

7.4. EXAMPLES. For any $\epsilon > 0$ and any n, we have $D^n_\epsilon \simeq \mathrm{pt}$ (where pt denotes the point with the unique tolerance on it). But $\mathbb{R}^n_\epsilon \not\simeq \mathrm{pt}$!

7.5. PROPOSITION. *Isomorphic tolerance spaces are homotopy equivalent.*

The proof is obvious: if $f\colon X_\xi \to Y_\eta$ is an isomorphism, then the map $g = f^{-1}$ is a morphism and $f \circ g = \mathrm{id}_X$, $g \circ f = \mathrm{id}_Y$. The converse statement to the proposition is false (e.g., $I_{1/10} \neq \mathrm{pt}$).

7.6. A tolerance space X_ξ is called *contractible*, if $X_\xi \simeq \mathrm{pt}$. By Proposition 7.2, this means X_ξ can indeed be contracted to a point by a finite sequence of 'little pushes'.

7.7. An ordered *n-simplex* in a tolerance space X_ξ is a sequence of $n+1$ points $x_0, x_1, \ldots, x_n$ all within tolerance of each other (repetitions are allowed); notation: $\sigma_n = \xi_n[x_0, \ldots, x_n]$. The points x_i are the *vertices* of the simplex. The assignment

$$\partial_n^i\colon \xi_n[x_0, \ldots, x_n] \mapsto \xi_{n-1}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \quad (i = 0, 1, \ldots, n)$$

is the ith *face operator*, and $\xi_{n-1}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ is the ith *face* of σ_n which of course is an $(n-1)$-simplex. The assignment

$$\delta_n^i\colon \xi_n[x_0, \ldots, x_n] \mapsto \xi_{n+1}[x_0, \ldots, x_i, x_i, \ldots, x_n]$$

is the ith *degeneracy operator*; of course $\xi_{n+1}[x_0, \ldots, x_i, x_i, \ldots, x_n]$ is an $(n+1)$-simplex.

7.8. If $f\colon X_\xi \to Y_\eta$ is a morphism of tolerance spaces, then the image of any n-simplex $\xi_n[x_0, \ldots, x_n]$ in X_ξ, $f_n(\sigma_n) = \eta_n[f(x_0), \ldots, f(x_n)]$, is also a simplex (in Y_η).

7.9. PROPOSITION. *The definitions of Sections* 7.7 *and* 7.8 *determine a functor* Φ *from the category of tolerance spaces Tol to the category of semi-simplicial complexes SSC (see* [13]). *If* $X_\xi \in$ *Tol, then* $\Phi(X_\xi) = (\{K_n\}, \{\partial_n^i\}, \{\delta_n^i\})$ *is called the semi-simplicial complex associated to* X_ξ. *If* X_ξ *is finite, the associated complex is a finite simplicial complex.*

Proof. Define K_n, $n \geq 0$, as the free Abelian group generated by the set of all n-simplices in X_ξ. Extend by linearity the operators ∂_n^i and δ_n^i defined in Section 7.7 to homomorphisms $\partial_n^i\colon K_n \to K_{n-1}$ and $\delta_n^i\colon K_n \to K_{n+1}$ and extend the operator f_n to a homomorphism $f_n\colon K_n(X_\xi) \to K_n(Y_\eta)$. It is easy to check that the required commutation relations (see [13]) hold, so that $(\{K_n\}, \{\partial_n^i\}, \{\delta_n^i\})$ is indeed a semi-simplicial complex and $\Phi(f) = (\{f_n\})$ is a morphism in SSC.

7.10. The category of *pointed tolerance spaces* $\mathrm{Tol}_{\mathrm{pt}}$, by definition, consists of *connected* tolerance spaces with fixed *base point*, i.e., pairs (X_ξ, pt), $\mathrm{pt} \in X$ with morphisms sending base point to base point. Homotopy (and homotopy equivalences) are base point preserving, when we deal with $\mathrm{Tol}_{\mathrm{pt}}$.

7.11. Using the functor defined in Proposition 7.9, we can construct the homotopy theory of tolerance spaces automatically, by copying the existing results of semi-simplicial theory (e.g., from [13]). In particular, there is a *homotopy group functor* from the category $\mathrm{Tol}_{\mathrm{pt}}$ to the category of groups (Abelian if $n \geqslant 2$) with the usual properties.

Since no applications to this theory are presented in this paper, we omit the details here.

7.12. The homotopy theory of tolerance spaces was first developed by I. Lapitski in a long unpublished paper (1975) [11] and, independently (1983), by T. Poston (who uses loop spaces) in his (also unpublished) thesis [14]. The approach of these two authors is different, neither uses semi-simplicial complexes.

8. Homology

8.1. Denote by C_nX_ξ the free Abelian group (called nth *chain group*) generated by the set of all ordered n-simplices (see Section 7.7) of the tolerance space X_ξ. Define the *differential* ∂_n: $C_nX_\xi \to C_{n-1}X_\xi$ in the standard way, by putting

$$\partial_n\xi_n[x_0, \ldots, x_n] = \sum_{i=0}^{n} (-1)^i \xi_{n-1}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \qquad (*)$$

on each n-simplex and extending by linearity to C_nX_ξ. An automatic verification shows that $\partial_{n-1} \circ \partial_n = 0$, so that we obtain a chain complex $\{CX_\xi = \bigoplus_{n=1}^{\infty} C_nX_\xi, \partial\}$ called the *ordered chain complex* of the tolerance space X_ξ.

8.2. If f: $X_\xi \to Y_\eta$ is a morphism of tolerance spaces, then it sends simplices into simplices (see Section 7.8) and by extending we obtain the *induced chain group homomorphism* f_n: $C_nX_\xi \to C_nY_\eta$. It is easy to check that it commutes with the differential, i.e., $\partial_n \circ f_n = \partial_n \circ f_{n-1}$.

8.3. The sets $\mathrm{Ker}\, \partial_n$ and $\mathrm{Im}\, \partial_{n+1}$ are subgroups of C_nX_ξ and $\mathrm{Ker}\, \partial_n \supset \mathrm{Im}\, \partial_{n+1}$ (since $\partial_n \circ \partial_{n+1} = 0$). The quotient group

$$H_nX_\xi = \mathrm{Ker}\, \partial_n / \mathrm{Im}\, \partial_{n+1}$$

is called the nth *homology group* of the tolerance space X_ξ.

8.4. If f: $X_\xi \to Y_\eta$ is a morphism, the induced homomorphism f_n is constant on cosets mod $\mathrm{Im}\, \partial_n$ (because of the commutativity relation 8.2), so that we have a (well-defined) *induced homomorphism* of homology groups (also denoted f_n),

$$f_n\colon H_nX_\xi \to H_nY_\eta .$$

8.5. In a standard way (for details see Section 10) we can now define *relative homology groups* $H_n(X_\xi, A_\xi)$ for pairs of tolerance spaces $X_\xi \supset A_\xi$, the differential $d_n\colon H_n(X_\xi, A_\xi) \to H_nA_\xi$, induced homomorphisms and other standard ingredients of homology theory. The result may be summarized as follows.

8.6. THEOREM. *The homology theory* $(H_*, (\ )_*, d_*)$ *outlined above is a covariant functor from the category* Tol^2 *of pairs of tolerance spaces to the category of graded Abelian groups, satisfying the analogues of all the Steenrod–Eilenberg axioms* [5].

In particular, there is an exact homology sequence for pairs (of tolerance spaces) with a natural differential, and any homotopy equivalence induces an isomorphism of all homology groups.

For the proof (which is quite standard), see Section 10 below or my paper [16].

8.7. The fact that the above homology theory satisfies Steenrod–Eilenberg axioms means that most of the apparatus of homological topology is ready to use without any new constructions or modifications (e.g., cohomology, exact sequences, Kunneth and universal coefficients formulae, homology dimension, duality theorems, etc.)

8.8. EXAMPLES. For finite tolerance spaces, the homology groups are simply those of the associated simplicial complex. For $\epsilon < 1$, the unit sphere S^n_ϵ has the usual homology $H_kS^n_\epsilon = \mathbb{Z}$ for $k = 0$, n and $= 0$ otherwise. For any $\epsilon > 0$, $H_k\mathbb{R}^n_\epsilon = 0$ for all k except 0 ($H_0\mathbb{R}^n_\epsilon = \mathbb{Z}$). More generally, $H_0X_\xi = \mathbb{Z}$ if and only if X_ξ is connected.

8.9. REMARK. According to Zeeman's definition of isomorphism ([22], p. 283), the disconnected pair of points and three points within tolerance of each other are isomorphic tolerance spaces. However, they have different homology groups H_0 (namely $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}$, respectively). It should also be noted that homology groups *are not* toleomorphism invariants.

8.10. Suppose A_ξ is a skeleton of the tolerance space (see Section 3.4), i.e., there are morphisms $i\colon A_\xi \subset X_\xi$, $r\colon X_\xi \to A_{2\xi}$ satisfying $r \circ i = \mathrm{id}_A$ and $\forall x \in X$, $r(x)\xi x$. Then the embedding i is a homotopy equivalence (since $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq_e \mathrm{id}_X$), hence the homology of A and X is the same (in the *doubled* tolerance).

Further, any morphism $f\colon A_\xi \to A_\xi$ can be canonically extended to the morphism $\bar{f}\colon X_\xi \to X_\xi$ by putting $\bar{f} = i \circ f \circ r$. Then the diagram

$$\begin{array}{ccc} H_nA & \xrightarrow{f_n} & H_nA \\ {\scriptstyle r_n^{-1}}\downarrow & & \downarrow{\scriptstyle \cong\, i_n} \\ H_nX & \xrightarrow{\bar{f}_n} & H_nX \end{array}$$

is commutative. Identifying H_nA and H_nX by the isomorphism $i_n = r_n^{-1}$ we may view f_n and $\bar{f}_n$ as the same homomorphism.

Conversely, any morphism $g\colon X_\xi \to X_\xi$ canonically induces the morphism $\tilde{g}\colon A_\xi \to A_\xi$ via the formula $\tilde{g} = r \circ g \circ i$ and the corresponding square diagram is also commutative.

8.11. Thus, from the homological point of view, instead of any tolerance space X_ξ of finite type, we may consider its (finite!) skeleton A_ξ, as long as we are willing to pay the price – take homology in the doubled tolerance.

9. Almost Fixed Points

9.1. An *almost fixed point* of the morphism $f\colon X_\xi \to X_\xi$ is a point $x_0 \in X$ such that $x_0 \xi f(x_0)$. The tolerance space X_ξ possesses the *almost fixed point property*, if any morphism $f\colon X_\xi \to X_\xi$ has a fixed point.

9.2. For any n, the morphism $f\colon X_\xi \to X_\xi$ induces the endomorphism $f_n\colon H_nX_\xi \to H_nX_\xi$. The alternated sum* of traces** of these endomorphisms is known as the Lifschitz number of f:

$$\lambda(f) = \sum_{i=1}^{\infty} (-1)^i \operatorname{Tr} f_i$$

9.3. LEMMA. *If X_ξ is a finite tolerance space, then the Lifschitz number of any morphism $f\colon X_\xi \to X_\xi$ is defined.*

The proof is obvious: since X_ξ is finite, H_nX_ξ is always finitely generated and vanishes for sufficiently large n. Hence, the sum (1) defined.

9.4. THEOREM. *If the morphism $f\colon X_\xi \to X_\xi$ of a finite tolerance space has a nonzero Lifschitz number $\lambda(f) \neq 0$, then f has an almost fixed point.*

The proof is given in Section 10 or in [17].

* This sum is defined (finite) if $\operatorname{Tr} f_n = 0$ for all sufficiently large n.

** The trace is only defined for free finitely generated Abelian groups; if H_nX_ξ is not free, we take its quotient group mod torsion and consider the trace of the quotient endomorphism f_n on the quotient group H_nX_ξ mod torsion.

9.5. COROLLARY ([17]). *All finite contractible spaces possess the almost fixed point property.*

Proof. If $X_\xi \simeq \text{pt}$, then by Theorem 8.6 $H_n X_\xi \simeq H_n(\text{pt})$; the latter group is $\mathbb{Z}$ if $n=0$ and trivial otherwise. Hence, for any morphism $f\colon X_\xi \to X_\xi$ we have $\lambda(f) = \Sigma(-1)^n \operatorname{Tr} f_n = \operatorname{Tr} f_0 = 1$, so that Corollary 9.5 follows from Theorem 9.4.

9.6. REMARKS. (i) The condition $\lambda(f) \neq 0$ is not sufficient for f to have a real fixed point. The simplest example is the map interchanging the two points of the trivial two-point tolerance.

(ii) Further, the condition $\lambda(f) \neq 0$ is not necessary for the existence of an almost fixed point. Thus, the identity map of the torus $T_\epsilon \subset \mathbb{R}^3_\epsilon$ (ϵ is some small number) even has many ordinary fixed points, although $\lambda(\text{id}_T) = 0$.

9.7. The Lifschitz number of the identity map of any tolerance space X_ξ is called the *Euler characteristic* of X_ξ and denoted $\chi(X_\xi)$.

9.8. For the classical approach to almost fixed points, see [7] and its bibliography.

10. Dimension

10.1. In the 16th century, the German cossist (=algebraist) Michael Stifel, discussing dimension, wrote that one cannot "go beyond the limits of the cube as if there were more than three dimensions, since this would be unnatural" (see [15], pp. 148 ff). From the point of view of classical 20th century mathematics, this statement sounds naive and/or erroneous. However, it happens to be a mathematical truth from the tolerance point of view. We will show in this section that manifolds (locally) are either 0, 1, 2 or 3-dimensional. More precisely, that the n-cube I^n is toleomorphic (as a metric space, see Section 4.4) to I^3 whenever $n \geq 3$, but $I^0 = \text{pt}$, $I = [0, 1]$, I^2 and I^3 are nontoleomorphic to each other. We do not give a general definition of tolerance dimension here (or else the reader will suspect that the 3-cube Theorem 10.6 below is not a general principle, but the result of cleverly doctored definitions).

10.2. *The L. V. Keldysh embedding* $k\colon I^3 \to I^4$. This construction, undertaken in a classical framework for a totally different purpose (see [8, 9]) can be modified to show that the 3-cube is toleomorphic to the 4-cube, and, more generally, to the n-cube. The main point is that, for any $\epsilon > 0$, I^3 embeds in I^4 so that the ϵ-neighborhood of the image $k(I^3)$ covers I^4 and any two points of $k(I^3)$ whose distance in I^4 is less than ϵ can be joined by an arc (in $k(I^3)$ of diameter less than 2ϵ.

Let $\epsilon > 0$ be given. Take $\eta = \epsilon/5$ and choose $m \in \mathbb{N}$ so that $1/m < 2\eta$. Partition I^3 into m^3 little cubes with edges (of length $1/m$) parallel to the coordinate axes.

Let K_1 be the union of all the edges of the little cubes. Denote by U_1 the interior of the union of all cubes (with edges of length $1/2m$ parallel to the axes) whose centres are at the vertices and the midpoints of all the edges in K_1; then U_1 is a neighborhood of K_1. The set $U_2 = I^3 - \bar{U}_1$ (the bar denotes closure) is a similar sum of cubes with edges $1/2m$ while $B = I^3 - (U_1 \cup U_2)$ is the (connected) common boundary of the domains U_1 and U_2.

In each U_i, $i = 1, 2$, take a decreasing sequence of m polyhedra

$$U_i \supset U_{i1} \supset \cdots \supset U_{im}$$

obtained by uniformly shrinking U_1 towards K_1 (when $i = 1$) and carrying out a similar construction when $i = 2$. Denote by B_{ir} the boundary of U_{ir} in $U_{i,r-1}$.

Define the continuous function φ: $I^3 \to I$ by putting

$$\varphi(x) = \begin{cases} \frac{1}{2} & \text{if } x \in B \\ \dfrac{1}{2} + \dfrac{r}{2m+1} & \text{if } x \in B_{1r} \\ \dfrac{1}{2} - \dfrac{r}{2m+1} & \text{if } x \in B_{2r} \end{cases}$$

and extending φ within $A_{ir} = U_{ir} - \bar{U}_{i,r+1}$ so that $\varphi|_{A_{ir}}$ is bounded by its values on the boundary of A_{ir}, within U_1 so that $\varphi|_{U_1}$ is bounded by $(\frac{1}{2}) + m/(2m+1)$ and 1, and within U_2 so that $\varphi|_{U_2}$ is bounded by $\frac{1}{2} - m/(2m+1)$ and 0. Then $0 \leq \varphi(x) \leq 1$. The graph of the function φ is the required embedding.

10.3. In Section 10.2 above we have presented the Keldysh embedding $I^3 \to I^4$ as it appears in [8] (the original construction [9] was more complicated). Another construction still can be obtained by using the engulfing lemma (see [19]). We can use it to obtain Keldysh-type embeddings $I^k \to I^n$ directly whenever $k \geq 3$ and $n \geq k$. It is easy to verify that the engulfing lemma fails to work when $k < 3$.

10.4. A map $f: X \to Y$ in the category of metric spaces (in particular, an embedding) is called ϵ-*dense* gf the ϵ-neighborhood of the image $f(X)$ covers Y. An ϵ-dense embedding has the 2ϵ *small arc property* if for all $x, y \in f(X)$, $d(x, y) < \epsilon$ implies the existence of an arc $l \subset f(X)$ joining x and y such that $\text{diam}(l) < 2\epsilon$.

10.5. THEOREM [9]. *When $\epsilon < \frac{1}{2}$, any ϵ-dense embedding $I \to I^2$ cannot have the small arc property. When $\epsilon < \frac{1}{4}$ any ϵ-dense embedding $I^2 \to I^3$ or $I^1 \to I^3$ cannot have the small arc property either.*

The proofs are a rather unpleasant technical exercise (especially for the embeddings in I^3) and are omitted here. The reader should have no trouble with the case $I \to I^2$, where the Peano curve embeddings (see the figure in Section 15) provide an excellent illustration. (Moving from the boundary of I^2 inward, it is easy to obtain a contradiction with the Jordan curve theorem.) The values of ϵ in the statement of the theorem ($\frac{1}{2}$ and $\frac{1}{4}$) are not the best possible; the calculation of their maximal values ('Keldysh constants') is a mildly interesting unsolved problem.

10.6. The main result of this section is the following:

THREE-CUBE THEOREM. *The* 3-*cube is toleomorphic* (*as a metric space, see Section* 2.1) *to the n-cube whenever* $n \geq 3$ *and is not toleomorphic to the point, the line segment and the square.*

Sketch of the proof. To prove the first part of the theorem when $n = 4$, take any $\epsilon > 0$ and choose N so that $2^{-N} < \epsilon$. Let ϵ be the (metric) tolerance on the 4-cube $I^4 \subset \mathbb{R}^4$ and δ be the tolerance on $I^3 = \{(x_i) : x_4 = 0\}$. The direct toleomorphism map $f: I^3_\delta \to I^4_\epsilon$ is the Keldysh embedding 10.2, which is a morphism for an appropriate choice of δ. The opposite morphism $g: I^4_\epsilon \to I^3_\delta$ is defined by sending each point of I^4 into the nearest point of $f(I^3)$ (or, if there are more than one such points, the one which has the smallest x_4 (eventually x_3, x_2, x_1) coordinate). The fact that $f, g: I^3 \rightleftarrows I^4$ is a toleomorphism readily follows from the definition of the Keldysh embedding.

In the case $n > 4$ a similar construction works. Or one can use engulfing techniques (see Section 10.2).

The other statements of the 3-cube theorem are essentially re-wordings of Theorem 10.5 and are proved in the same way.

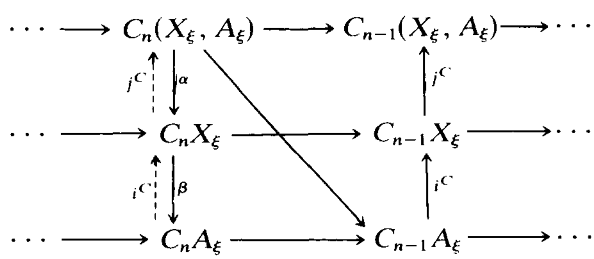
10.7. Intuitively, the 3-cube theorem says the following. If we look at things with some (even every small) tolerance for error (as in real life we must), we cannot perceive any dimension above three, although moderately sharp vision distinguishes the line, the plane and 3-space. In this connection, see A. M. Vinogradov's article in this issue.

11. Homology Theory

11.1. This section fills in the details missing in Sections 8–9, thus making our exposition of the homology theory of tolerance spaces practically self-contained. In general, here we follow [16].

11.2. The nth chain group $C_n X_\xi$ of a tolerance space X_ξ is defined in Section 8.1 and consists of finite linear combinations $c = \Sigma z_i \sigma^i_n$, $z_i \in \mathbb{Z}$ of simplices $\sigma^i_n = \xi_n[x^i_0, \ldots, x^i_n]$ (see Section 7.7) of X_ξ. The differential $\partial_n: C_n X_\xi \to C_{n-1} X_\xi$ is defined in the standard way (Equation 8.1 (*)) and possesses the property $\partial_{n-1} \circ \partial_n = 0$. Indeed, it suffices to prove this property for one simplex $\sigma_n = \xi_n[x_0, \ldots, x_n]$, but in this case it follows from the fact that $(\partial_{n-1} \circ \partial_n)(\sigma_n)$ is the sum of $n(n-1)$ summands which may be grouped in pairs of identical $(n-2)$-simplices not containing the vertices x_i, x_j and appearing with opposite signs $(-1)^j(-1)^i$ and $(-1)^i(-1)^{j+1}$. Thus, $\{C_nX_\xi, \partial_n\}$ is a chain complex, and this allows us to define the homology groups H_nX_ξ of a tolerance space (see Section 8.3) as the ones of this chain complex.

11.3. By definition, the *chain complex of the pair* (X_ξ, A_ξ) of tolerance spaces (see Section 3.2) is the sub-chain complex $C_n(X_\xi, A_\xi)$ of C_nX_ξ consisting of all n-chains with zero coefficients for all simplexes from A_ξ (these chains are sometimes called *chains mod* A). We have obvious maps: $j_n^C\colon C_nX_\xi \to C_n(X_\xi, A_\xi)$ (setting the coefficients of simplices of A_ξ equal to zero), $\beta\colon C_nX_\xi \to C_nA_\xi$ (setting the coefficients of simplices not belonging to A_ξ equal to zero) and the inclusions $\alpha\colon C_n(X_\xi, A_\xi) \to C_nX_\xi$ and $i_n^C\colon C_nA_\xi \to C_nX_\xi$. Denote by ∂ and ∂^0 the boundary operators of the chain complexes $\{C_nX_\xi\}$ and $\{C_nA_\xi\}$ respectively (from now on we omit the dimension subscript in our notations for operators) and define $\hat{\partial} = j^C\partial^0\alpha$ and $d = \partial^0\beta\alpha$. All this information appears in the diagram



(1)

11.4. LEMMA. (i) $\hat{\partial}\circ\hat{\partial} = 0$ (*in other words* $\{C_n(X_\xi, A_\xi), \hat{\partial}\}$ *is a chain complex.*

(ii) $\partial^0 d = d\hat{\partial}$ (*in other words, d is a chain map of degree* (-1) *of the chain complex* $\{C_n(X_\xi, A_\xi), \hat{\partial}\}$ *into the chain complex* $\{C_nA_\xi, \partial^0\}$).

(iii) $\hat{\partial}j^C = j^C\partial$ (*in other words, j^C is a chain map of degree* 0 *of the chain complex* $\{C_nX_\xi, \partial\}$ *into* $\{C_n(X_\xi, A_\xi), \hat{\partial}\}$).

The three statements follow from straightforward calculations using the (obvious) relations $j^C\alpha = i^C\beta = \mathrm{id}$, $j^Ci^C = \beta\alpha = 0$, $\alpha j^C + i^C\beta = \mathrm{id}$. For example

$$\hat{\partial}\hat{\partial} = j^C\,\partial\alpha j^C\,\partial\alpha = j^C\,\mathrm{d}(\mathrm{id} - i\beta) = i^C\partial\,\partial\alpha - j^C\,\partial i^C\beta\,\partial\alpha$$
$$= 0 - j^Ci^C\partial^0\beta\,\partial\alpha = 0.$$

11.5. Lemma 11.4 implies that we can define *relative homology groups* $H_n(X_\xi, A_\xi)$ (for pairs of tolerance spaces), as promised in Section 8.5, as quotient groups of $\operatorname{Ker}\hat{\partial}$ by its subgroup $\operatorname{Im}\hat{\partial}$.

We leave to the reader the (straightforward) construction of the *homomorphism* $\varphi\colon H_n(X_\xi, A_\xi) \to H_n(Y_\eta, B_\eta)$ *induced* by a morphism of pairs $\varphi\colon (X_\xi, A_\xi) \to (Y_\eta, B_\eta)$.

11.6. Lemma 11.4 also implies that the *homology sequence of the pair* (X_ξ, A_ξ)

$$\cdots \xrightarrow{\mathrm{d}} H_n A_\xi \xrightarrow{i} H_n X_\xi \xrightarrow{j} H_n(X_\xi, A_\xi) \xrightarrow{\mathrm{d}} H_{n-1} A_\xi \to \cdots \tag{2}$$

is well defined and is exact. The exactness of (2) follows from diagram chasing in (1) or from the (obvious) exactness of the short sequence

$$\begin{array}{ccc} & CX_\xi & \\ {}^{i^C}\nearrow & & \searrow{}^{j^C} \\ CA_\xi & \xleftarrow{\ \mathrm{d}\ } & C(X_\xi, A_\xi) \end{array} \tag{3}$$

where C without any subscript denotes the direct sum of the corresponding groups with subscripts, e.g., $CA_\xi = \bigoplus_{n=0}^{\infty} C_n A_\xi$.

11.7. It is just as easy to show that the differential d of the homology sequence (2) is natural, i.e., that we have the commutative diagram

$$\begin{array}{ccc} H_n(X_\xi, A_\xi) & \xrightarrow{\mathrm{d}} & H_{n-1} A_\xi \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle (f|_A)_{n-1}} \\ H_n(Y_\eta, B_\eta) & \xrightarrow{\mathrm{d}} & H_{n-1} B_\eta \end{array}$$

for any morphism $f: (X_\xi, A_\xi) \to (Y_\eta, B_\eta)$.

11.8. The *excision axiom* in the homology theory of tolerance spaces can be stated as follows: if $X_\xi \supset A \supset A_0$ and A_0 is contained in the interior of A (i.e., $\xi A_0 \subset A$, see Section 3.3), then the inclusion $i: (X - A_0, A - A_0) \hookrightarrow (X, A)$ induces an isomorphism in homology:

$$i_n: H_n(X - A_0, A - A_0) \xrightarrow{\simeq} H_n(X, A).$$

The proof immediately follows from the fact that the chain groups $C_n(X - A_0, A - A_0)$ and $C_n(X, A)$ are naturally isomorphic and their boundary operators coincide.

11.9. The *dimension axiom* says that $H_n(\mathrm{pt}) = 0$, when $n \geqslant 1$ and $H_0(\mathrm{pt}) \simeq \mathbb{Z}$; its proof is trivial. Thus, we have established all the analogues of Steenrod–Eilenberg axioms of homology theory, except the homotopy axiom, which we now consider. Note that its proof below is quite elementary (unlike that in classical homology theory) and does not involve acyclic models (see [5]); it follows easily from the next lemma.

11.10. LEMMA. *If two morphisms f, g: $X_\xi \to Y_\eta$ are elementary homotopic, then the induced homomorphisms f_n, g_n in homology coincide.*

Proof. To prove $f_n = g_n$, it suffices (see [10]) to establish the existence of a chain homotopy joining f_n^C and g_n^C, i.e., a homomorphism D: $C_nX_\xi \to C_{n+1}Y_\eta$ such that $f_n^C - g_n^C = \partial D + D\partial$; but D can be constructed by defining it on each simplex as follows

$$D(\xi_n[x_0, \ldots, x_n]) = \sum_{q=0}^{n} (-1)^q \eta_{n+1}[f(x_0), \ldots, f(x_q), g(x_q), \ldots, g(x_n)]$$

and then extending to chains by linearity. The verification of the formula $f_n^C - g_n^C = \partial D + D\partial$ is straightforward.

11.11. It now follows immediately from Lemma 11.10 and Proposition 7.2 that homotopic morphisms f, g: $X_\xi \to Y_\eta$ induce the same homomorphism $f_n = g_n$ in homology.

Further, if f: $X_\xi \to Y_\eta$ and g: $Y_\eta \to X_\xi$ are homotopy equivalences, inverse to each other, i.e., $g \circ f \simeq \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, using the functorial character of homology theory, we get $g_n \circ f_n = (g \circ f)_n = (\mathrm{id}_X)_n = \mathrm{id}_{H_nX}$ and $f_n \circ g_n = (f \circ g)_n = (\mathrm{id}_Y)_n = \mathrm{id}_{H_nY}$, which implies that f_n and g_n are mutually inverse isomorphisms of H_nX_ξ and H_nY_η. This establishes the homotopy axiom in the homology theory of tolerance spaces and concludes our proof of Theorem 8.6.

11.12. Our next goal is to present the missing details of Section 9.

Recall that the *trace* $\mathrm{Tr}\,\varphi$ of the endomorphism φ: $G \to G$ of a finitely generated free Abelian group G is the trace of the matrix $((a_{ij}))$, where $f(e_i) = \Sigma a_{ij}e_j$ for some free basis $\{e_i\}$ of G ($\mathrm{Tr}\,\varphi$ is, of course, independent of its choice). If G is not free, then its quotient mod torsion $G' = G/\mathrm{Tor}\,G$ is free, and for $\mathrm{Tr}\,\varphi$ we take the trace $\mathrm{Tr}\,\varphi'$ of the quotient endomorphism φ': $G' \to G'$.

11.13. Suppose f: $X_\xi \to X_\xi$ is a morphism of the tolerance space X_ξ. Assume that the homology groups H_nX_ξ are finitely generated for all n. Then (see Section 9.2) the number $\lambda(f) = \Sigma(-1)^i \mathrm{Tr}\, f_i$ (where f_i: $H_iX_\xi \to H_iX_\xi$ is the homomorphism in the homology of dimension i induced by f) is called the *Lifschitz number* of the morphism f. It is certainly defined if the homology of X_ξ is trivial in dimensions greater than some fixed number n_0. By Theorem 8.6 $\lambda(f)$ is a homotopy class invariant of f. The Lifschitz number may be defined on the chain level by putting $\lambda^C(f) = \Sigma(-1)^i \mathrm{Tr}\, f_i^C$ (where f_i^C: $C_iX_\xi \to C_iX_\xi$ is the induced homomorphism of the chain group).

11.14. LEMMA. *The two definitions of Lifschitz number given in Section* 11.13 *are equivalent in the sense that $\lambda^C(f) = \lambda(f)$ when both are defined.*

Proof. This immediately follows from the following purely algebraic fact, known as the *Hopf lemma*:

Suppose φ_i: $C_i \to C_i$ is a chain complex morphism of finitely generated free Abelian group chains and $(\varphi_i)_*$: $H_iC \to H_iC$ is the induced homomorphism in homology; then

$$\sum_{i=0}^{\infty} (-1)^i \operatorname{Tr} \varphi_i = \sum_{i=1}^{\infty} (-1)^i \operatorname{Tr} (\varphi_i)_* .$$

The proof of this relation may be found in [10].

11.15. Our last step is the proof of Theorem 9.4, which states that morphisms of finite tolerance spaces with nonzero Lifschitz number have almost fixed points. Suppose the converse: f: $X_\xi \to X_\xi$ is a morphism, X_ξ is finite, $\lambda(f) \neq 0$ but for any $x \in X_\xi$ it is false that $x\xi f(x)$. Then no simplex $\xi_n[x_0, \ldots, x_n]$ is mapped into itself. For the free basis of C_nX_ξ take the set of all n-simplices and consider the matrix $((a_{ij}))$ of the endomorphism f_n^C in this basis. All the diagonal elements of this matrix are zero (no simplex is mapped into itself), hence $\operatorname{Tr} f_n^C = 0$ and $\lambda^C(f) = 0$. But then, by Lemma 11.14, $\lambda(f) = 0$, contradicting the assumption of Theorem 9.4, which is therefore proved.

PART III. APPLICATIONS

12. Almost Solutions

12.1. Roughly speaking, an almost solution of an equation (or a system of equations) is an object which, when substituted into the equation, transforms it into a numerical 'almost identity', i.e., a relation between numbers which is true only approximately (within a prescribed tolerance). The formal definition is given below (12.3). It applies to a wide and disparate class of equations (algebraic, integral, differential, linear, nonlinear, etc.); the existence theorem for almost solutions (12.6) are applicable to totally unrelated branches of mathematics.

12.2. One should not confuse almost solutions with *approximate solutions*. The latter term is used by mathematicians to mean 'approximation of an exact solution', whereas an almost solution may exist in cases when the given equation has no exact solutions whatsoever (see Figure 1 for a simple example). There is nothing wrong with this in practice, although most mathematicians are so conditioned that they dislike such a situation. From the pragmatic point of view, however, exact solutions are ideal abstractions which do not actually exist and, moreover, in most nontrivial cases arising in the applications, cannot be expressed in a finite number of symbols. In most practical cases, the existence of an exact solution is unnecessary: in the case of Figure 1, say, if $f(x)$ is electric current flowing through a conductor, x is time and ϵ is small enough (less than

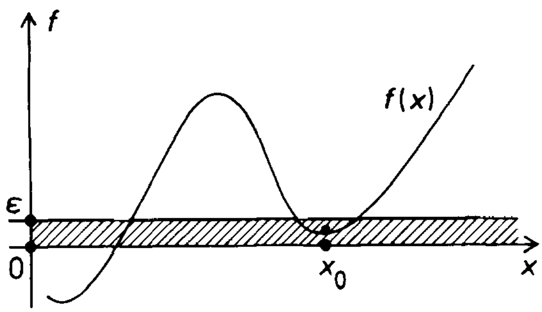


Fig. 1.

the sensitivity of our ammeter), then the almost solution x_0 gives the time when the ammeter reading will *actually* be zero.

12.3. Suppose we are given a tolerance space F_α* (the 'function space'), a morphism (the 'operator') A: $F_\xi \to F_\xi$ and a map (the 'norm') $\|\cdot\|$ from F to $\mathbb{R}_\epsilon$). An *almost solution* of the equation $A(f) = 0$ is any $f_0 \in F_\xi$ such that $\|A(f_0)\| < \epsilon$ (or, in other notations, $\|A(f_0)\| \epsilon 0$).

12.4. Suppose the space F_ξ is supplied with a binary operation + with a zero and inverse elements. Denote by A_1 the operator in F_ξ defined by the relation $A_1(f) = A(f) + f$. Then the existence of an almost solution of the equation $A(f) = 0$ is equivalent to the existence of an almost fixed point of the operator A_1: $E \to F$ in the tolerance ϵ' on F induced by the norm $\|\ \ \|$ in $\mathbb{R}_\epsilon$, i.e., an element f such that $A_1(f)\epsilon' f$; here $\epsilon' = \|\ \ \|^*(\epsilon)$ (see the notation in Section 3.2); this is because $A(f) = 0$ if and only if $f = A_1(f)$.

This simple consideration can be used to convert the almost fixed point theorem (Section 9) into a general theorem on almost solutions. We leave this general reformulation to the reader, and consider instead some specific situations when this theorem applies.

12.5. Suppose F_ξ is the space of real-valued functions on a fixed tolerance space X_ξ, i.e., $F_\xi = \mathrm{Mor}(X_\zeta, \mathbb{R}_\lambda)$, where $\xi = \zeta\lambda$ is the canonical tolerance on F (see Section 3.2); the sum of functions in F is the usual one $(f + g)(x) = f(x) + g(x)$; the norm $\|\ \ \|$ is the least upper bound of a function's values ($\|f\| = \sup\{|f(x)| : x \in X\}$; the number $\epsilon > 0$ (the 'tolerance of error') is fixed; for an operator A defined on a part of F_ξ (i.e., $A \in \mathrm{Mor}(G_\xi, G_\xi)$ for some subtolerance

* We do not require the norm to be defined on the entire space F_ξ. We need only know what functions are 'within tolerance of zero'.

$G_\xi \subset F_\xi$) let A_1 denote the map* $A_1 = A + \text{id } G_\xi$ defined by $A_1(f) = A(f) + f$.

The problem is to find sufficient conditions for the existence of an almost solution of the equation $A(f) = 0$ in the tolerance $\epsilon' = \|\ \ \|^*(\epsilon)$. We will assume that the tolerance ϵ' is greater than ξ (i.e., $\xi \subset \epsilon'$). The main result is the following.

12.6. THEOREM. *An almost solution of the equation $A(f) = 0$ exists if we can find a contractible** finite subtolerance (see Section 3.2) $K_\xi \subset F_\xi$ which is mapped into itself by the operator $A + \text{id}_F = A_1$ (i.e., $A_1 \in \text{Mor}(K_\xi, K_\xi)$).*

This statement immediately follows from Corollary 9.5.

12.7. For the reader who has read through Section 8 in Part II, let us mention that the contractibility condition in Theorem 12.6 may be replaced by either of the following two weaker assumptions:

(i) K_ξ is acyclic (i.e., $H_0K_\xi = \mathbb{Z}$, $H_nK_\xi = 0$ when $n \geq 1$)
(ii) the Lifschitz number $\lambda(A_1)$ is non-zero.

12.8. If we replace the contractability condition (or (i), (ii)) by a convexity assumption (which is stronger), the conclusion of the theorem obviously holds; we then obtain an analogue of the classical Schauder theorem.

12.9. Theorem 12.6 is also valid if we require the contractability of $K_{\epsilon'}$ (instead of that of K_ξ).

12.10. It is impossible to omit the finiteness requirement imposed on K_ξ in Theorem 12.6: it is easy to construct appropriate counterexamples ('parallel translations' in an infinite space $K_\xi = F_\xi$). It is also easy to find examples of finite spaces K_ξ (e.g., with one-dimensional homology isomorphic to $\mathbb{Z}$) and an operator A_1 without almost fixed points ('rotation around the generator' of the group H_1K_ξ), so that the contractability condition cannot be omitted either.

12.11. The statement of Theorem 12.6 is simplified if we require that the space F_ξ itself be finite. This condition will be satisfied automatically if we consider only uniformly-bounded functions from X_ξ into $\mathbb{R}_\lambda$. Denote

$$\mathbb{R}^{(M)} = \{x \in \mathbb{R} : |x| < M\}.$$

Then the following statement holds.

* This will not necessarily be a morphism in the tolerance ξ – we may have to double the tolerance.
** This means (see Section 7 for details) that there exists a sequence of morphisms g_i: $K_\xi \to K_\xi$, $i = 1, \ldots, n$, of K_ξ 'collapsing K to a point', i.e., such that $g_1 = \text{id}_K$, $g_i(x)\xi g_{i+1}(x)$ for all $x \in K$, $i = 1, \ldots, n-1$ and $g_n(K) = \text{pt} \in K$.

12.12. LEMMA. *If X_ξ is a finite tolerance space, then $F' = \mathrm{Mor}(X_\xi, \mathbb{R}_\lambda^{(M)})$ is also finite.*

The proof is a straightforward verification of definitions and is left to the reader. Note that this lemma has no analogues in classical functional analysis (usually function spaces are not even locally compact).

13. The Dirichlet Problem

13.1. In this section we consider, as an illustration, a specific difference scheme boundary-value problem where Theorem 12.6 works. It should be noted that the theory of difference schemes and its computer implementation is the most natural field for the application of tolerance space theory, but little has been done to date here.

13.2. Suppose X_ζ is a tolerance space with a fixed nonempty subtolerance $\Gamma_\zeta \subset X_\zeta$ (the 'boundary') and a morphism f_0: $\Gamma_\zeta \to \mathbb{R}_\lambda$ (the 'boundary condition'). Denote by K_ξ the set of all morphisms f: $X_\zeta \to \mathbb{R}_\lambda$ which coincide with f_0 on Γ_ζ, ξ being the canonical tolerance ($\xi = \zeta\lambda$). Assume that $K_\xi \neq \emptyset$ (i.e., there exists at least one extension of f_0 from Γ_ζ to $X_\zeta - \bar{f}_0$: $X_\zeta \to \mathbb{R}_\lambda$). In this situation we say that the functions $f \in K_\xi$ satisfy the (generalized) *Dirichlet condition* $(X_\zeta, \Gamma_\zeta, f_0)$.

13.3. THEOREM. *If K_ξ satisfies the generalized Dirichlet condition $(X_\zeta, \Gamma_\xi, f_0)$ and x_ζ is finite, then K_ξ is contractible and finite.*

The proof is presented below (see Section 13.10).

13.4. Now suppose that under the assumptions of Section 13.2 we are given an equation $A(f) = 0$ such that the operator $A_1 = A + \mathrm{id}_K$ is a morphism from K_ξ to K_ξ (we then say that the operator A_1 *preserves the Dirichlet condition*). The main result of this section is the following.

13.5. THEOREM. *The equation $A(f) = 0$ always has an almost solution in the class of functions K_ξ satisfying the generalized Dirichlet conditions $(X_\zeta, \Gamma_\zeta, f_0)$ if X_ζ is finite and the operator $A_1 = A + \mathrm{id}_K$ preserves the Dirichlet condition.*

(By an almost solution here we understand an almost solution in the tolerance ξ containing ϵ': here, as in Section 12.5, we assume that $\xi \supset \epsilon'$, where $\epsilon' = \| \quad \|^*(\epsilon)$ and $\epsilon > 0$ is the desired precision for the almost solution.)

This theorem follows immediately from Theorems 13.3 and 12.6.

13.6. In the theory of difference schemes [6], X_ζ is the lattice on which the approximating functions are defined, ζ is the mesh of the lattice, Γ is the set of boundary points of X, f_0 the boundary-value function, λ the precision with which the computer works (i.e., the order of last digit appearing in the presentation of real numbers) and A is the finite-difference operator approximating the given differential operator.

13.7. The reader should not think that Theorem 13.5 means that the Dirichlet problem *always* has an almost solution. It does if the following three conditions are met:

(i) X_ζ is finite;
(ii) $A_1(f_0)\xi f_0$:
(iii) $f\xi g \Rightarrow A_1(f)\xi A_1(g)$.

In computer applications conditions (i) and (ii) are easy to satisfy. However, condition (iii) is far from automatic, and the user of the theorem must play around with ζ, ϵ, λ to make it work in the most efficient way.

Similarly, in applying the more general Theorem 12.6, the difficult part is finding a finite K_ξ such that A_1 restricted to it is a morphism.

13.8. Conditions (i)–(iii) are always satisfied in the case of the discrete harmonic operator (see [2]) so that the Dirichlet problem always has an almost solution in this case; but this is well known anyway. A general theory ('harmonic analysis on tolerance spaces') suggested by E. C. Zeeman is discussed in T. Poston's thesis [14].

On the other hand, it is easy to construct examples of parabolic equations for which (iii) fails and the finite-difference method diverges [2].

13.9. Theorem 13.5, when it is applicable to a Dirichlet problem in partial differential equations being processed by computer, guarantees the existence of an almost solution of the finite difference scheme approximating the given differential equation, as well as an almost solution of the differential equation itself, independently of the existence of an exact solution of the latter.

The point of this theory is to tackle the question of existence by applying homological algebra *directly* to the real problem processed by computer. The classical approach to difference schemes, in contrast, is very roundabout: first use existence theorems for the exact solution of the *continuous* differential equation (proved by very delicate applications of homology theory in an infinite-dimensional setting), then find a sequence of *finite approximations* of the equation (the difference schemes) and their approximate solutions and, finally, prove that these approximate solutions *converge* to the existing exact solution. It is not surprising that in serious problems this program cannot be carried out entirely and computer processing of difference schemes has become an experimental science. For another direct approach to the existence problem for difference schemes, see the work of A. A. Dezin [3].

13.10. *Proof of Theorem* 13.3. Recall that $K_\xi \subset \mathrm{Mor}(X_\zeta, \mathbb{R}_\lambda)$, where X_ζ is a finite type and K_ξ satisfies the Dirichlet condition, i.e., $f \in K_\xi \Rightarrow f|_\Gamma = f_0$, where $\Gamma \subset X$ and the boundary condition $f_0\colon \Gamma_\zeta \to \mathbb{R}_\lambda$ can be extended to a morphism $\bar{f}_0\colon X_\zeta \to \mathbb{R}_\lambda$. Suppose N_0 is the number of points in X and $x_0 \in \Gamma$.

It is easy to see that the functions are uniformly bounded (e.g., by the constant $M = |f(x_0)| + |N_0 + 2|\lambda$. Hence, by Lemma 12.12, K_ξ is finite.

Now let us collapse K_ξ to the point $\bar{f}_0$ (thus showing its contractability), i.e., construct the following sequence of morphisms of K_ξ into itself:

$$\mathrm{id}_K = g_0 \simeq_e g_1 \simeq_e \cdots \simeq_e g_N = \mathrm{const}\colon K_\xi \to \{\bar{f}_0\}.$$

To do this, take an integer N so large that $N\lambda > M$. Suppose (by induction) that we have constructed a morphism $g_k\colon K_\xi \to K_\xi$ satisfying the condition

$$Q(k)\colon \forall x \in X\ \forall f \in K_\xi\ |f_0(x) - g_k(f)(x)| \leqslant (N-k)\lambda.$$

In order to construct g_{k+1} (satisfying $Q(k+1)$ and elementary homotopic to g_k) it suffices to put $g_{k+1} = s_{k+1} \circ g_k$, where s_{k+1} 'slices off' the largest and smallest values of all functions $g_k(f)$, making them nearer by λ to the values of $\bar{f}_0$ at all points of $x \in X$ such that

$$|f_0(x) - g_k(f)(x)| = (N-k)\lambda.$$

To be more precise, we put

$$(s_{k+1} \circ g_k)(f)(x) = \begin{cases} g_k(f)(x), & \text{if } |\bar{f}_0(x) - g_k(f)(x)| < (N-k)\lambda, \\ g_k(f)(x) - \lambda, & \text{if } f_0(x) - g_k(f)(x) = (N-k)\lambda, \\ g_k(f)(x) + \lambda, & \text{if } f_0(x) - g_k(f)(x) = -(N-k)\lambda, \end{cases}$$

and inductively define $g_{k+1} = s_{k+1} \circ g_k$. It is easy to check that g_{k+1} is hereby well defined (using $Q(k)$), that g_{k+1} satisfies $Q(k+1)$ and $g_k \simeq_e g_{k+1}$. Obviously g_N is the constant map to $\bar{f}_0$ and the theorem is proved.

14. Linear Algebraic Systems

14.1. The theory as developed in Part II does not yield any meaningful results concerning the approximate solution of linear algebraic systems, although the statement of the problem, as it really stands in computer practice, sounds very natural in terms of tolerance spaces. Here we very briefly state a version of the problem and explain why Theorem 12.6 does not work in this case.

14.2. Let F_ξ denote the *truncated n-dimensional vector space*, consisting of vectors all of whose coordinates are less than a certain number M (the upper bound of real numbers which the computer is designed to process), where $\xi > 0$ is a fixed number of our choice (not less than the precision of the computer), A is a given linear algebraic operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose matrix consists of elements not greater than a certain given number N (where $N \leqslant M$) and b is a fixed element of F_ξ. The problem is to find sufficient conditions for the existence of an almost solution of the equation $A(x) = b$ with respect to a certain tolerance ϵ on F_ξ ($\epsilon > 0$ being as small as possible for the given M, N, A, b).

14.3. Theorem 12.6 does not work here because the operator A_1 (defined by $A_1(x) = A(x) + x - b$) is not a morphism of F_ξ into F_ξ in the general case (it is not defined on all of F_ξ, especially if $\det\|A\|$ is small). When A_1 is defined, existence follows from elementary considerations such as the contracting operator principle.

15. The Three-Channel Principle

15.1. Roughly speaking, the principle in question says that *three* continuous channels are enough to transmit information coming in on any number n of continuous channels. More precisely, if $3 \leqslant k \leqslant n$, then the input information from n continuous channels can be coded continuously into k continuous channels, transmitted and then uncoded back into n output channels with error within any prescribed tolerance. If $k \leqslant 2$ and $n \geqslant k$, this cannot be done. The exact formulation appears below (15.6) The principle is essentially another way of stating the three-cube theorem (10). Based on the work of L. V. Keldysh [8, 9], it was first noticed [1] by A. V. Chernavski (or Černavskiĭ, in another transcription).

15.2. We begin with an informal discussion of the simplest case $k = 1$, $n = 2$. Let us try to use the Peano curve to transmit information coming in via two input channels $x_1(t)$, $x_2(t)$ by means of one transmission channel $y(t)$ which will then be decoded into two output channels $\tilde{x}_1(t)$, $\tilde{x}(t)$ with error less than a small fixed $\epsilon > 0$.

Recall that the Peano curve is a continuous surjective map $p: I \to Q$ (where $I = [0, 1]$ is the segment and $Q = I \times I$ the square) obtained as the limit of a sequence of embeddings $p_n: I \to Q$. Each curve $L_k = p_k(I)$ is a polygonal line without self-intersections passing through 2^{2k} little squares filling up the square; for $k = 3$, this line is shown on Figure 2. Take k such that $1/2^{2k} < \epsilon$ (e.g., $k = 3$, we can then use the figure). Choose the parameter s on the curve L_3 proportional to distance along the curve, normed by the conditions $s(A) = 0$, $s(B) = 1$.

The input signal $x_1(t)$, $x_2(t)$ may be viewed as moving point $P(t) = (x_1(t), x_2(t)) \in Q$. Finding the point $\tilde{P}(t)$ on the line L_3 nearest* to $P(t)$, let us transmit its coordinate via the transmission channel: $y(t) = s(\tilde{P}(t))$. The number $s = y(t)$ allows us to recover the point $\tilde{P}(t)$ (using Figure 2 again), whose

* If there is more than one such point, choose the one with the smallest coordinate $s(\tilde{P}(t))$ on L_3.

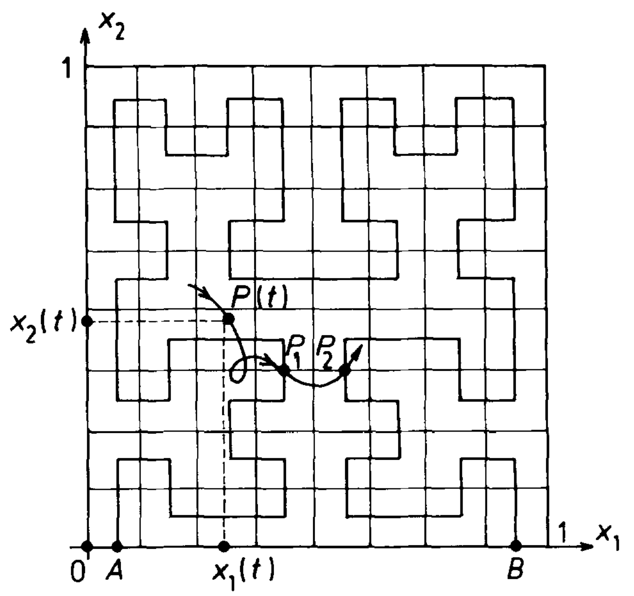


Fig. 2.

coordinates $\tilde{x}_1(t)$, $\tilde{x}_2(t)$ can be sent to the two output channels. Then, obviously, we have

$$|\tilde{x}_i(t) - x_i(t)| \leq \tfrac{1}{16} < \frac{\epsilon}{2} \quad (i = 1, 2),$$

so that the transmission error is less than the prescribed tolerance ϵ.

Unfortunately, this method of transmitting continuous information is no good at all. This is because even a very small change in the position of the point $P(t)$ (e.g., from P_1 to P_2, see Figure 2) can lead to a large discontinuous jump in the value of the coordinate s of the nearest point on the polygonal line (thus in our case $s(\tilde{P}_1) = 10/64$, but $s(\tilde{P}_2) = 55/64$. Small variations in the input data can bring about huge jumps in the data sent along the transmission channel, so that the transmission algorithm described above fails to be effective.

15.3. The reason for the failure of the approach described in Section 15.2 is that points within tolerance in Q are not within tolerance on L (with respect to a natural metric on L). An effective continuous transmission algorithm of two channels via one channel would be ensured by an embedding p': $L \hookrightarrow Q$ such that points are within tolerance on L and Q simultaneously (i.e., a toleomorphism, see Section 4). The Peano embeddings do not possess this property and, in fact, no embeddings of the segment into the square with this property exist. We give an exact formulation of this fact in the next subsection.

15.4. An embedding of metric spaces $h: M \to N$ is called *ϵ-dense*, if the ϵ-neighborhood of the image $h(M)$ covers N.

THEOREM. *For any $\epsilon < \frac{1}{2}$, all ϵ-dense embeddings of the line segment into the square $p: I \to Q$ have the following property: there exist two points $P_1, P_2 \in L = p(I)$ whose distance in Q is less than ϵ but whose distance along L is greater than $\frac{1}{2}$.*

This theorem is essentially a restatement of Theorem 10.5.

15.5. For the same reasons as in the case $k = 1$, $n = 2$, effective continuous transmission of n channels via k channels is impossible when $n \geq 3$ and $k \leq 2$ (one or two channels are not enough). But three channels are always enough, precisely because embeddings of the 3-cube into the n-cube with the required property do exist (Section 10).

15.6. We now state the three-channel principle in classical terms (avoiding tolerance terminology). To do this, we need the following definition: a real-valued function on a metric space with distance d is called *ϵ-almost-continuous* (where $\epsilon > 0$ is a fixed number), if

$$(\exists \epsilon' > 0\, \exists x_0\, \forall \delta > 0\, \exists x\; d(x_0, x) < \delta \;\&\; |f(x_0) - f(x)| \geq \epsilon') \Rightarrow \epsilon' < \epsilon.$$

THE THREE-CHANNEL THEOREM. *When $k \geq 3$, for any $\epsilon > 0$ and arbitrary n, there exists a family of k ϵ-almost-continuous functions*

$$y_1(x_1, \ldots, x_n), \ldots, y_k(x_1, \ldots, x_n) \tag{1}$$

with values in $[0, 1]$, defined for all $x_i \in [0, 1]$, $i = 1, \ldots, n$, and a family of n continuous functions

$$\tilde{x}_1(y_1, \ldots, y_k), \ldots, \tilde{x}_n(y_1, \ldots, y_k) \tag{2}$$

with values in $[0, 1]$ defined for all $y_j \in [0, 1]$, $j = 1, \ldots, k$, such that for all $i = 1, \ldots, n$ and any sequence of numbers

$$x_1^0, \ldots, x_n^0, 0 \leq x_i^0 \leq 1 \quad (i = 1, \ldots, n)$$

we have

$$|x_i^0 - \tilde{x}_i(y_1(x_1^0, \ldots, x_n^0), \ldots, y_k(x_1^0, \ldots, x_n^0))| < \epsilon.$$

When $k = 1$ and $n \geq 2$, as well as when $k = 2$ and $n \geq 3$, families of functions (1) and (2) with the above properties no konger exist if ϵ is taken less than $\frac{1}{4}$.

15.7. The proof is approximately the same as that of the 3-cube theorem in Section 10; it is an effective construction, yielding an algorithm for actually constructing the 'coding functions' $y_1, \ldots, y_k$ and the 'uncoding functions'

$\tilde{x}_1, \ldots, \tilde{x}_n$ so that the three channel principle is not only a theoretical theorem, but (eventually) a practicably applicable one.

15.8. In the statement of the three-channel theorem, the words 'ϵ-almost-continuous' can be replaced by 'continuous', but then the proof becomes much more difficult. For practical purposes, this change in the formulation is useless, however.

16. Applications to Biology

16.1. It is perhaps appropriate that we conclude our list of applications of tolerance space theory with theoretical biology, with which it all began (in E. C. Zeeman's work). We will not summarize Zeeman's work here, referring the reader to the original paper 'The topology of the brain and visual perception' [21] and its sequel [22]. I will only mention that [21] contains a simple model of the brain's neuron network as a certain tolerance space (called the 'thought cube'), an informal sketch of tolerance space homology, including the fixed point theorem and applications of this theorem to brain functioning (visual perception in particular) in which thoughts or images are fixed points.

16.2. Another way to model visual perception, in which the image on the retina is endowed with a tolerance space structure, has been proposed by A. M. Vinogradov, whose paper on the subject appears in this issue. The main point of the first part of his paper is that the three-channel theorem explains why we perceive the world as three-dimensional.

16.3. Various biological networks (and not only nerve cells) may be modelled as tolerance spaces. In particular, the various structures of the DNA double spiral can be viewed as points of a tolerance space, two spirals being within tolerance if they resemble each other in a certain specified sense. Then, perhaps, evolution and/or mutation may be viewed as certain tolerance space maps. No serious research, to my knowledge, has been done in this direction, however.

17. Questions

In this section we mention a few open questions which might be the topic of further research.

17.1. There are at least two alternative approaches to the foundations of tolerance space theory, based on different definitions of morphism (map). The first involves multivalued maps, the second uses what may be called the 'almost' approach: maps $f: X_\xi \to Y_\eta$ are not necessarily defined for all $x \in X$ but only within tolerance of any point of X, surjective means 'almost surjective' (i.e., the image $f(X)$ is within tolerance of any point $y \in Y_\eta$ (compare 4.2), etc. Are these approaches worth looking into? The second is very natural, but does not yield a category. But should it?

17.2. Is there a meaningful abstract definition of tolerance dimension?

17.3. Are almost fixed point theorems (Section 9) possible (a) in the case of morphisms $f: K_\xi \to K_\eta$ with $\xi \neq \eta$ (compare 9.12)? (b) when the underlying space is not of finite type?

17.4. Say that two natural numbers are within tolerance if there is a short algorithm (e.g., using less than 1000 operations on your personal computer) transforming one into the other. (Then 0 and 10^{100} are within tolerance, while 0 and certain fairly large primes p, $p \leqslant 10^{100}$ are not). Can the investigation of this tolerance space structure on $\mathbb{N}$ shed some light on current problems involving large numbers, e.g., coding?

17.5. Are there meaningful tolerance space models of physical situations, with tolerance implied by the Heisenberg indeterminacy principle?

17.6. Can a differential and integral *calculus* based on tolerance spaces be developed and applied, in particular, to difference schemes?

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