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Infinitesimal Methods in Control Theory: Deterministic and Stochastic

NIGEL J. CUTLAND
Department of Pure Mathematics, University of Hull, Hull, England

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Abstract. In Part I, methods of nonstandard analysis are applied to deterministic control theory, extending earlier work of the author. Results established include compactness of relaxed controls, continuity of solution and cost as functions of the controls, and existence of optimal controls. In Part II, the methods are extended to obtain similar results for partially observed stochastic control. Systems considered take the form:

$$\mathrm{d}x_t = f(t, x, y, u(t, y))\,\mathrm{d}t + g(t, x, y, u(t, y))\,\mathrm{d}b_t,$$
$$\mathrm{d}y_t = \bar{f}(t, x, y, u(t, y))\,\mathrm{d}t + \bar{g}(t, y)\,\mathrm{d}\bar{b}_t,$$

where the feedback control u depends on information from a digital read-out of the observation process y. The noise in the state equation is controlled along with the drift. Similar methods are applied to a Markov system in the final section.

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1. Introduction and Preliminaries

1.1. INTRODUCTION

In the papers [6, 7] the author introduced nonstandard methods (particularly Loeb space techniques [17]) into control theory, both deterministic and stochastic. In the stochastic case this built on work of Keisler [13] and others who developed efficient nonstandard techniques for solving stochastic differential equations. The purpose of this paper is two-fold. First, to present a streamlined and complete account of the methods developed in the papers mentioned above, using a slightly different approach that simplifies things somewhat. Second, using this streamlined approach we cover aspects of control theory not discussed in the earlier papers. In the case of partially observed stochastic control systems we establish new *standard* results, as outlined below.

The section headings give a guide to the plan of the paper. In the deterministic theory (Part I) we go further than in [6]; we define a natural topology for controls and show that under general conditions these are compact if we allow relaxed (or generalised) controls. We establish continuity of solutions as a function of the control, and prove various optimality results. In Part I we are not claiming that our results are new; the aim is to give further illustration of the naturalness of nonstandard methods in control theory.

In Part II we build on the methods of Part I and establish new results for partially observed stochastic systems. Elliott and Kohlmann [11] first investigated controlled systems in which observations in a countable observation space were made at a fixed finite sequence of times. The paper [7] used nonstandard methods to extend their results to systems which were past-dependent. The new features in the present paper are (i) a more general information pattern, and (ii) the noise in the state equation is allowed to be controlled. The systems we consider take the form

$$\begin{aligned} \mathrm{d}x_t &= f(t, x, y, u(t, y))\,\mathrm{d}t + g(t, x, y, u(t, y))\,\mathrm{d}b_t, \\ \mathrm{d}y_t &= \bar{f}(t, x, y, u(t, y))\,\mathrm{d}t + \bar{g}(t, y)\,\mathrm{d}\bar{b}_t, \end{aligned}$$

where the control u is allowed to depend at time t on the observation y through a *cumulative digital read-out* $r_t(y)$. Updating of the information $r_t(y)$ can take place at any time.

For such systems we establish compactness of relaxed controls, continuity of both the measure induced by control u and the cost $J(u)$ (as functions of u), and in consequence the existence of optimal controls. The final section of the paper establishes similar results for a Markov controlled system in which all the noise can be controlled.

The main features of the present approach that are different from earlier papers are as follows. First, we do not concern ourselves about the construction of solutions to dynamical equations, but simply content ourselves to quote existence results (which could, of course, be established by nonstandard means, as in [5], for example, for the stochastic systems we consider). This means that instead of dealing with internal ('nonstandard') dynamical systems for discrete time $\mathrm{t} \in T = \{0, \Delta\mathrm{t}, 2\Delta\mathrm{t}, \ldots\}$ and then converting to standard dynamical systems for continuous time, we work with internal solutions to internal equations for (nonstandard) continuous time. These are given without any work by the transfer principle of nonstandard analysis applied to standard existence theorems. Thus, for optimality results, the main idea is the following. Given a nonstandard control V for a dynamical system, take an internal solution X^V to the internal equations; convert this to a standard object x^V, and show that x^V is a solution for an appropriate standard control v. To establish optimality, we may simply take V with cost $J(V) \approx J_0$, the minimum cost; then $J(v) \approx J(V) \approx J_0$, so $J(v) = J_0$.

The second feature that is different here follows from the first: our measure spaces are necessarily hyperinfinite (i.e., not hyperfinite), whereas in earlier work where discrete time was used, it was possible to use entirely hyperfinite spaces – i.e., nonstandard measure spaces that are finite from the nonstandard point of view. Hyperfinite spaces are very safe to deal with internally, so there is perhaps a slight loss here, which is the price of making the rest of the development much smoother. One significant advantage of the present approach is that we do not have to construct any liftings of functions; these are provided by $^*f$ (etc.), and frequent use of Anderson's Lusin Theorem.

1.2. PRELIMINARIES

Our notation throughout is mainly conventional. By $\mathbb{R}^+$ we mean the real half-line $[0, \infty[$; the symbol m is used to denote Lebesgue measure on $\mathbb{R}^n$ for any n, as dictated by the context. We use $C(X)$ to denote the space of continuous real functions on a space X; the Borel probability measures on a metric space M are denoted by $\mathcal{M}(M)$.

We assume familiarity with the basics of nonstandard analysis and the theory of Loeb measure as expounded for example in [8], or [1]. Selected parts of the theory that are important for this paper are recalled below.

We work in a fixed ω_1-saturated and enlarged nonstandard universe $^*V(\mathbb{R})$. On occasions we omit the * on a function or operator: thus, for example λ^V (Lemma 7.13) must mean $^*\lambda^V$ because V is internal (nonstandard).

We use τ as a variable ranging over $^*\mathbb{R}$; when we have a discrete time line $T = \{0, \Delta\mathsf{t}, 2\Delta\mathsf{t}, \ldots\}$ we use sanserif symbols s, t, u to range over T.

The term *bounded* is used to mean finitely (or $\mathcal{S}$-) bounded.

We frequently use the nonstandard criteria for continuity and compactness, recalled in the following Proposition, along with similar criteria for density and lower semi-continuity.

PROPOSITION 1.1. *Let X be a Hausdorff topological space; let f: $X \to \mathbb{R}$, and suppose that $x \in X$ and $y \in {}^*X$.*

(a) *f is continuous at x iff $f(y) \approx f(x)$ whenever $y \approx x$;*

(b) *f is lower semi-continuous at x iff $^\circ f(y) \geq f(x)$ whenever $y \approx x$;*
(Here $^\circ$ denotes standard part).

(c) *X is compact iff every $y \in {}^*X$ is nearstandard (i.e., there is $x \in X$ with $y \approx x$);*

(d) *a set $Y \subseteq X$ is dense in X iff for every $x \in X$ there is $y \in {}^*Y$ with $y \approx x$.*

Recall the criterion for being nearstandard in $C[0, 1]$ (with respect to the uniform topology): $F \in {}^*C[0, 1]$ is nearstandard iff F is *$\mathcal{S}$-continuous*; i.e., $F(\tau_1) \approx F(\tau_2)$ whenever $\tau_1 \approx \tau_2$.

For a product $X = \prod_{i \in I} X_i$ recall that $y \in {}^*X$ is nearstandard if y_i is nearstandard for each *standard* $i \in I$.

As in [8], we use ν_L to denote the Loeb measure obtained from an internal measure ν. Expectation with respect to ν is denoted $\bar{E}$ to distinguish it from the expectation E with respect to ν_L. When considering Loeb measures on $^*\mathbb{R}$ (or $^*\mathbb{R}^+$ etc.) we use the phrases 'for a.a. finite τ' or 'for a.a. $\tau \in \mathrm{ns}({}^*\mathbb{R})$' to mean 'for almost all τ with respect to the σ-finite measure on $\mathrm{ns}({}^*\mathbb{R})$ obtained from $^*m_L$ on $^*[-n, n]$ for each $n \in \mathbb{N}$'. Here $\mathrm{ns}({}^*\mathbb{R})$ denotes the nearstandard (or finite) members of $^*\mathbb{R}$.

We make frequent use of Anderson's Lusin Theorem:

PROPOSITION 1.2 ([3]; [8], Theorem 4.6). *Let $(X, \mathcal{B}, \mu)$ be a Radon probability space, and suppose that $f: X \to \mathbb{R}$ is measurable. Then ${}^*f(x) \approx f({}^\circ x)$ for ${}^*\mu_L$ – a.a. $x \in {}^*X$.*

Recall the following characterisation of weak standard parts of measures:

PROPOSITION 1.3 [4, 18]. *Let X be Hausdorff; an internal Baire probability measure ν on ${}^*X$ is nearstandard (in the weak topology) iff $\nu_L(\mathrm{ns}({}^*X)) = 1$; in which case ${}^\circ\nu(A) = \nu_L(st^{-1}(A))$ for Baire sets A in X.*

For Part II we need the following proposition, which is a variant of results of Anderson and Keisler. Suppose that $\mathbf{\Omega}_0 = (\Omega, \mathcal{A}, (\mathcal{A}_\tau)_{\tau\leq 1}, \nu)$ is an internal filtered space carrying an internal Brownian motion B_τ adapted to $(\mathcal{A}_\tau)$. Let $\mathbf{\Omega} = (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\leq 1}, P)$ be the adapted Loeb space obtained from $\mathbf{\Omega}_0$ as described in [8], p. 572; so $P = \nu_L$.

PROPOSITION 1.4. (a) *$B(\omega)$ is a.s. $\mathcal{S}$-continuous, and the process $b = {}^\circ B$ is a Brownian motion on $\mathbf{\Omega}$.*

(b) (*Keisler's $\mathcal{S}$-continuity*) *If $F: {}^*[0, 1] \times \Omega \to {}^*\mathbb{R}$ is bounded, internal, jointly measurable and adapted to $\mathcal{A}_\tau$, then $G_s = \int_0^s F_\tau \, \mathrm{d}B_\tau$ is a.a. $\mathcal{S}$-continuous.*

(c) *If F in (b) is a lifting of a bounded adapted function $f: [0, 1] \times \Omega \to \mathbb{R}$ then*

$$\int_0^s f_s \, \mathrm{d}b_s = {}^\circ\!\!\int_0^s F_\tau \, \mathrm{d}B_\tau \quad \textit{all } s, \textit{a.a. } \omega.$$

Proof. (a) B is $\mathcal{S}$-continuous on the set $\bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \Omega_{n,m}$, where

$$\omega \in \Omega_{n,m} \leftrightarrow (|\tau - \tau'| < 1/m \to |B_\tau - B_{\tau'}| < 1/n).$$

By transfer of standard facts about Brownian motion, ${}^\circ\nu(\Omega_{n,m}) \to 1$ as $m \to \infty$, so $P(\bigcap_n \bigcup_m \Omega_{n,m}) = 1$. Clearly $b = {}^\circ B$ has the right distributions to make it a Brownian motion, and is adapted.

(b) The easiest proof of Keisler's $\mathcal{S}$-continuity theorem for the present setting is to adapt the proof in the notes [14]. As this proof has not been made public we take the liberty of quoting it here (suitably modified for our situation).

Let

$$\Lambda_{n,m} = \{\omega: |\tau - \tau'| < 1/m \to |G_\tau - G_{\tau'}| < 1/n\};$$

as in (a), it is sufficient to show that for fixed n, ${}^\circ\nu(\Lambda_{n,m}) \to 1$ as $m \to \infty$. For this we show that $\nu(\Lambda_{n,M}) \approx 1$ for infinite M.

Let $\Delta t = M^{-1}$ and let $T = \{0, \Delta t, 2\Delta t, \ldots, 1 - 2\Delta t\}$.

The complement $\bar{\Lambda}_{n,M}$ is given by

$$\bar{\Lambda}_{n,M} = \{\omega: \exists \tau, \tau'(|\tau - \tau'| < 1/M \quad \text{and} \quad |G_\tau - G_{\tau'}| \geq 1/n)\},$$

and we have

$$\bar{\Lambda}_{n,M} \subseteq \left\{ \omega \colon \exists \mathrm{t} \in T \left(\sup_{\mathrm{t} \leq \tau \leq \mathrm{t}+2\Delta \mathrm{t}} |G_\tau - G_\mathrm{t}| \geq 1/2n \right) \right\}. \tag{1.5}$$

Now using Doob's inequality for the internal martingale G we have for each t

$$\nu\left(\sup_{\mathrm{t} \leq \tau \leq \mathrm{t}+2\Delta \mathrm{t}} |G_\tau - G_\mathrm{t}| \geq 1/2n \right) \leq (2n)^4 \bar{E}((G(\mathrm{t}+2\Delta\mathrm{t}) - G(\mathrm{t}))^4). \tag{1.6}$$

Suppose that $|F| \leq \kappa$, finite. An easy calculation using the transfer of Itô's lemma for $(G(\mathrm{t}+\delta) - G(\mathrm{t}))^4$ shows that

$$\bar{E}((G(\mathrm{t}+\delta) - G(\mathrm{t}))^4) \leq 3\kappa^4\delta^2$$

for all δ. Combining this with (1.5) and (1.6) we have

$$\nu(\bar{\Lambda}_{n,M}) \leq \sum_{\mathrm{t} \in T} (2n)^4 3\kappa^4 (2\Delta\mathrm{t})^2 \approx 0,$$

as required.

(c) is proved exactly as in [2], by establishing it first for a sequence of step functions approximating f. □

PART I: DETERMINISTIC CONTROL THEORY

The theory worked out in Sections 2–4 can be found in the standard literature (see [12] for example). The emphasis here is to show how natural the nonstandard approach is for certain aspects of control theory: for example our proof of compactness of generalised controls (Theorem 2.7) should be compared with the lengthy presentation using a standard approach ([12], Theorem 8.1). The results and methods here are developed from the results in [6], where the key ideas were first presented.

2. Deterministic Controls

In this section we define classes $\mathcal{U}_0$ and $\mathcal{V}_0$ of controls appropriate for controlling deterministic dynamical systems; we establish the compactness of $\mathcal{V}_0$ in a suitable topology and the density of the subclass of $\mathcal{U}_0$ consisting of step controls.

Assume that a compact metric space M – the *control space* – is given and fixed; this is the space in which controls take their values.

DEFINITION 2.1. The class $\mathcal{U}_0$ of (*ordinary*, *deterministic*) *controls* is the set of measurable functions $u \colon \mathbb{R}^+ \to M$. (Controls differing on a null set are reckoned to be the same).

A weak* topology is defined on $\mathcal{U}_0$ by means of the set $\mathcal{K}$, defined as follows.

DEFINITION 2.2. (a) $\mathscr{K}$ is the set of bounded measurable functions $g: \mathbb{R}^+ \times M \to \mathbb{R}$ having compact support and with $g(t, \cdot)$ continuous for all t.

(b) For $u \in \mathscr{U}_0$ and $g \in \mathscr{K}$ the action of u on g is defined by

$$u(g) = \int_0^\infty g(t, u_t)\,\mathrm{d}t.$$

(c) The *$\mathscr{K}$-topology* on $\mathscr{U}_0$ has as subbase of open neighbourhoods the sets $\{u : |u(g)| < \epsilon\}_{g \in \mathscr{K}, \epsilon > 0}$.

We shall see that the completion of $\mathscr{U}_0$ is the set $\mathscr{V}_0$ of *relaxed* (or *generalised*) controls, defined as follows.

DEFINITION 2.3. The class $\mathscr{V}_0$ of *relaxed* (*deterministic*) *controls* is the set of measurable functions $v: \mathbb{R}^+ \to \mathscr{M}(M)$, the set of probability measures on M, with the weak* topology. (By identifying each $a \in M$ with the Dirac measure δ_a concentrated at a we have $M \subseteq \mathscr{M}(M)$ and $\mathscr{U}_0 \subseteq \mathscr{V}_0$.)

The topology on $\mathscr{U}_0$ is extended to $\mathscr{V}_0$ by extending each g in $\mathscr{K}$ to $\mathbb{R}^+ \times \mathscr{M}(M)$ with the definition

$$g(t, \nu) = \int_M g(t, a)\,\mathrm{d}\nu(a),$$

for $\nu \in \mathscr{M}(M)$.

In Theorem 2.7, below, we prove that $\mathscr{V}_0$ is compact, using the nonstandard criterion for compactness. This means that for each $V \in {}^*\mathscr{V}_0$ we have to construct $v \in \mathscr{V}_0$ such that $v = {}^\circ V$ (i.e., $V \approx v$), in the $\mathscr{K}$-topology; i.e.,

$$V({}^*g) \approx v(g) \quad (\text{all } g \in \mathscr{K}).$$

2.4. CONSTRUCTION OF THE STANDARD PART

Let $V \in {}^*\mathscr{V}_0$; define an internal *Borel measure Q on ${}^*\mathbb{R}^+ \times {}^*M$ by

$$Q(C \times D) = \int_C V_\tau(D)\,\mathrm{d}\tau$$

for $C \subseteq {}^*\mathbb{R}^+$, $D \subseteq {}^*M$. A standard Borel measure $q(= {}^\circ Q$ in fact) is defined on $\mathbb{R}^+ \times M$ by

$$q(X) = Q_L(st^{-1}(X)).$$

Notice that for a Borel set $A \subseteq \mathbb{R}^+$, $q(A \times M) = m(A)$ (Lebesgue measure), so q can be disintegrated to give $v \in \mathscr{V}$ with the property

$$q(A \times B) = \int_A v_t(B)\,\mathrm{d}t$$

for Borel $A \subseteq \mathbb{R}^+$ and $B \subseteq M$. We will see (Theorem 2.7) that $v \approx V$.

To show that $V \approx v$, we use Lemma 2.6 below, which will play a key role in later applications. First we make a definition.

DEFINITION 2.5. Let $g \in \mathcal{K}$; a *bounded uniform lifting* of g is an internal *measurable function $G: {}^*\mathbb{R}^+ \times {}^*M \to {}^*\mathbb{R}$ such that

(i) $G(\tau, \alpha) = 0$ if τ is infinite;
(ii) G is bounded;
(iii) for a.a. finite $\tau \in {}^*\mathbb{R}^+$, $G(\tau, \alpha) \approx g({}^\circ\tau, {}^\circ\alpha)$ all $\alpha \in {}^*M$.

The following lemma appeared as the main theorem of [6], in a slightly different setting. The simplified proof below was given by Ed Perkins.

LEMMA 2.6. *Suppose that v is constructed from V as in* (2.4) *above, and that $g \in \mathcal{K}$ with bounded uniform lifting G. Then $V(G) \approx v(g)$; i.e.,*

$$\int G(\tau, V_\tau)\, d\tau \approx \int g(t, v_t)\, dt.$$

Proof. There is finite (standard) time s such that g and G are zero outside $[0, s]$. Then

$$\begin{aligned} V(G) &= \int_0^s G(\tau, V_\tau)\, d\tau = \int_0^s \left(\int_{{}^*M} G(\tau, \alpha)\, dV_\tau(\alpha) \right) d\tau \\ &= \int_{{}^*[0,s]\times {}^*M} G(\tau, \alpha)\, dQ(\tau, \alpha) \\ &\approx \int_{{}^*[0,s]\times {}^*M} {}^\circ G(\tau, \alpha)\, dQ_L(\tau, \alpha) \quad \text{(Loeb)} \\ &= \int_{{}^*[0,s]\times {}^*M} g({}^\circ\tau, {}^\circ\alpha)\, dQ_L(\tau, \alpha) \end{aligned}$$

(since $g({}^\circ\tau, {}^\circ\alpha) = {}^\circ G(\tau, \alpha)$ for Q_L-a.a. (τ, α))

$$\begin{aligned} &= \int_{[0,s]\times M} g(t, a)\, dq(t, a) \quad \text{(by definition of } q) \\ &= \int_0^s \left(\int_M g(t, a)\, dv_t(a) \right) \quad \text{(by disintegration theory)} \\ &= \int_0^s g(t, v_t)\, dt = v(g). \end{aligned}$$ □

REMARK. This lemma shows that $V(G) \approx V(G')$ for any two bounded uniform liftings G, G' of $g \in \mathcal{K}$.

Now we have

THEOREM 2.7. *The space of relaxed controls $\mathcal{V}_0$ is compact; in fact if $V \in {}^*\mathcal{V}_0$ then ${}^\circ V = v$, the control constructed as in* (2.4).

Proof. Let $V \in {}^*\mathscr{V}_0$, and let $v \in \mathscr{V}_0$ as given by (2.4). We have to show that $V({}^*g) \approx v(g)$ for every $g \in \mathscr{K}$. Define $\hat{g}: \mathbb{R}^+ \to C(M)$ by $\hat{g}(t) = g(t, \cdot)$. Then from Anderson's Lusin theorem, ${}^*\hat{g}(\tau) \approx \hat{g}({}^\circ\tau)$ from almost all τ, where $\approx$ refers to the supremum norm topology on $C(M)$. Thus for a.a. finite τ,

$$ {}^*g(\tau, \alpha) \approx g({}^\circ\tau, {}^\circ\alpha) \quad \text{for all } \alpha \in {}^*M. $$

So ${}^*g$ is a bounded uniform lifting of g, and $V({}^*g) \approx v(g)$ follows from Lemma 2.6. □

We now show that $\mathscr{U}_0$ is dense in $\mathscr{V}_0$; in fact we have density of the step controls defined by:

DEFINITION 2.8. A control $u \in \mathscr{U}_0$ is a *step control* if there are times $0 = t_0 < t_1 < t_2 < \cdots$, with $t_n \to \infty$, such that u_t is constant on each interval $[t_n, t_{n+1}[$. A step control is *uniform* if the step sizes $t_{n+1} - t_n$ are the same, all n. Let $\mathscr{U}_0^s$ denote the uniform step controls.

The next theorem was given in [6] in a slightly less general form; the proof here is somewhat different.

THEOREM 2.9. *The set $\mathscr{U}_0^s$ is dense in $\mathscr{V}_0$.*

Proof. Let $v \in \mathscr{V}_0$. Take any positive infinitesimal Δt, and let $T = \{0, \Delta t, 2\Delta t, \ldots\}$. It is sufficient to construct a control $U \in {}^*\mathscr{U}_0$ such that ${}^\circ U = v$ and U is constant on $[t, t + \Delta t[$ for each $t \in T$.

Choose infinite $K \in {}^*\mathbb{N}$ such that $K^2\Delta t \approx 0$ (take any infinite $K \leq (\Delta t)^{-1/4}$). Fix an internal sequence of points $(a_i)_{i \leq K}$ in ${}^*M$ such that $M = \{{}^\circ a_i : i \leq K\}$. It is then possible to choose an internal sequence $(D_i)_{i \leq K}$ of disjoint *Borel subsets of ${}^*M$ such that $\bigcup_{1 \leq i \leq K} D_i = {}^*M$ and $a_i \in D_i \subseteq \text{monad}(a_i)$ for all i.

Let $\mathscr{D}$ be the internal algebra on ${}^*M$ generated by $(D_i)_{i \leq K}$. For each $\nu \in {}^*\mathcal{M}(M)$ define $\bar{\nu}$ on $\mathscr{D}$ by

$$ \bar{\nu}(D_i) = [K^2\nu(D_i)]/K^2 \quad (1 \leq i < K) $$

$$ \bar{\nu}(D_K) = 1 - \sum_{1 \leq i < K} \bar{\nu}(D_i). $$

Clearly, the mapping $\nu \to \bar{\nu}$ is internal; notice that $\bar{\nu}(D) \approx \nu(D)$ for $D \in \mathscr{D}$, and $\sum_{i \leq K} \bar{\nu}(D_i) = 1$ (so $\bar{\nu}$ is an internal probability measure on $({}^*M, \mathscr{D})$); moreover $K^2\bar{\nu}(D_i) \in {}^*\mathbb{N}$.

Now we can construct U. Let $\Delta s = K^2\Delta t$ and let $S = \{0, \Delta s, 2\Delta s, \ldots\}$. Define an internal control $V \in {}^*\mathscr{V}_0$ by

$$ V_s = (\Delta s)^{-1} \int_s^{s+\Delta s} {}^*v_\tau \, d\tau \quad (s \in S) $$

$$ V_\tau = V_s \qquad (s \leq \tau < s + \Delta s). $$

Notice that for each s, we have $s < s + \Delta t < s + 2\Delta t < \cdots < s + K^2\Delta t = s + \Delta s$.

Now define from V a control $U \in {}^*\mathcal{U}_0$ with the following properties:

(i) U is constant on $[\mathrm{t}, \mathrm{t}+\Delta\mathrm{t}[$ (all $\mathrm{t} \in T$);

(ii) for each $\mathrm{s} \in S$, U_t is defined for $\mathrm{s} \leq \mathrm{t} < \mathrm{s}+\Delta\mathrm{s}$ so that $U_\mathrm{t} = a_i$ for exactly $K^2 \bar{V}_\mathrm{s}(D_i)$ values of t. This is possible because $\sum_{i\leq K} K^2 \bar{V}_\mathrm{s}(D_i) = K^2$, and there are exactly K^2 values t in $[\mathrm{s}, \mathrm{s}+\Delta\mathrm{s}[$.

The aim now is to show that ${}^\circ U = v$; for this it is sufficient to show that $q^U = q^{*v}$ (where q^U is constructed from U as in (2.4); recall that for Borel $X \subseteq \mathbb{R}^+ \times M$, $q^U(X) = Q_L^U(st^{-1}(X)))$. Note that $st^{-1}(X) \in \sigma(\mathcal{A} \times \mathcal{D})$, where $\mathcal{A}$ is the internal algebra on ${}^*\mathbb{R}^+$ generated by sets of the form $[\mathrm{s}, \mathrm{s}+\Delta\mathrm{s}[$. Thus it is sufficient to show that

$$Q^U(C \times D) \approx Q^{*v}(C \times D)$$

for $C \in \mathcal{A}$, $D \in \mathcal{D}$, with C bounded (finitely).

From the definitions we have

$$\begin{aligned} Q^U([\mathrm{s}, \mathrm{s}+\Delta\mathrm{s}[\times D_i) &= \sum_{\mathrm{s}\leq\mathrm{t}<\mathrm{s}+\Delta\mathrm{s}} U_\mathrm{t}(D_i)\Delta\mathrm{t} \\ &= K^2 \bar{V}_\mathrm{s}(D_i)\Delta\mathrm{t} = \bar{V}_\mathrm{s}(D_i)\Delta\mathrm{s}. \end{aligned}$$

Thus

$$\begin{aligned} Q^U(C \times D) &= \int_C \bar{V}_\tau(D)\,\mathrm{d}\tau \\ &\approx \int_C V_\tau(D)\,\mathrm{d}\tau = \int_C {}^*v_\tau(D)\,\mathrm{d}\tau = Q^{*v}(C \times D), \end{aligned}$$

and the theorem is proved. □

3. Deterministic Control Systems

A deterministic control system in which a state vector $x_t \in \mathbb{R}^d$ is controlled by a control $u \in \mathcal{U}_0$ may be described by a differential equation such as:

$$\begin{aligned} \mathrm{d}x_t &= f(t, x_t, u_t)\,\mathrm{d}t \quad (0 \leq t \leq t_1) \\ x_0 &= c, \end{aligned} \tag{3.1}$$

where $t_1 < \infty$ and $c \in \mathbb{R}^d$. We also allow relaxed controls which operate in f as described in the previous section. We make the following assumptions on the function $f: \mathbb{R}^+ \times \mathbb{R}^d \times M \to \mathbb{R}^d$:

ASSUMPTIONS 3.2. (a) f is measurable, and $f(t, \cdot, \cdot)$ is continuous for each t;

(b) $\|f(t, \xi, a)\| \leq \kappa(1 + \|\xi\|)$ for some constant κ;

(c) Equation (3.1) has a unique solution on $[0, t_1]$ for each control (ordinary *or* relaxed) and all initial conditions c in some prescribed region $D \subseteq \mathbb{R}^d$. For $v \in \mathcal{V}_0$, $c \in D$ denote the solution by $x^{v,c}$.

In the next section we will discuss objectives of control, and questions of cost and optimality. Here we establish continuity of the trajectory as a function of v, c.

THEOREM 3.3. *The solutions* $x^{v,c}$ *are continuous in* (v, c) *with respect to the uniform norm topology on* $C^d[0, t_1]$.

Proof. Let $(V, \gamma) \approx (v, c)$, with $V \in {}^*\mathcal{V}_0$ and $\gamma \in {}^*D$, and let $X = X^{V,\gamma}$ be the internal solution to the equation

$$\mathrm{d}X_\tau = {}^*f(\tau, X_\tau, V_\tau)\,\mathrm{d}\tau \quad (0 \leq \tau \leq t_1);$$
$$X_0 = \gamma.$$

We have to show that $X \approx x^{v,c}$; i.e., X is $\mathcal{S}$-continuous and ${}^\circ X = x^{v,c}$.

The growth condition 3.2(b) on f transfers to ${}^*f$, ensuring that X_τ and hence ${}^*f(\tau, X_\tau, V_\tau)$ are bounded on $[0, t_1]$. So X is $\mathcal{S}$-continuous on $[0, t_1]$; let $x = {}^\circ X$. By Anderson's Lusin theorem, for almost all $\tau \in {}^*[0, t_1]$ we have

$${}^*f(\tau, \xi, \alpha) \approx f({}^\circ\tau, {}^\circ\xi, {}^\circ\alpha) \quad \text{all } \alpha \text{ and finite } \xi,$$

and thus

$${}^*f(\tau, X_\tau, \alpha) \approx f({}^\circ\tau, x_{{}^\circ\tau}, {}^\circ\alpha) \quad \text{all } \alpha.$$

So ${}^*f(\tau, X_\tau, \alpha)$ is a bounded uniform lifting of $f(t, x_t, a)$; hence,

$$x_s = {}^\circ X_s = {}^\circ\gamma + {}^\circ\!\int_0^s {}^*f(\tau, X_\tau, V_\tau)\,\mathrm{d}\tau$$
$$= c + \int_0^s f(t, x_t, v_t)\,\mathrm{d}t \quad \text{(Lemma 2.6)}$$

for $s \leq t_1$. Thus, $x^{v,c} = x \approx X^{V,\gamma}$ as required. □

4. Cost and Optimal Control

Let Γ be a closed region in $\mathbb{R}^+ \times \mathbb{R}^d$. Suppose that the objective of control for the system (3.1) is to steer the state x_t so that $(t, x_t) \in \Gamma$. For a control $v \in \mathcal{V}_0$, let $T^{v,c} = \inf\{t \leq t_1 : (t, x_t^{v,c}) \in \Gamma\}$. To ensure that $T^{v,c}$ is always defined, assume that $\{t_1\} \times \mathbb{R}^d \subseteq \Gamma$. A fixed time-horizon system is modelled by taking $\Gamma = \{t_1\} \times \mathbb{R}^d$, so that $T^{v,c} = t_1$ always.

Now suppose that associated with each control v and $c \in \mathbb{R}^d$ there is a cost $J(v, c)$ taking the form

$$J(v, c) = \int_0^{T^{v,c}} h(t, x_t^{v,c}, v_t)\,\mathrm{d}t + \bar{h}(T^{v,c}, x^{v,c})$$

where h, $\bar{h} \geq 0$. We assume that h satisfies conditions 3.2(a), (b), and $\bar{h}$ is continuous, with the property that $\bar{h}(t, \xi)$ depends only on $\{\xi_s : s \leq t\}$ for $\xi \in C^d[0, t_1]$, and $\bar{h}(\cdot, \xi)$ is nondecreasing for each fixed ξ.

Now we can prove:

THEOREM 4.1. (a) *The functions $T^{v,c}$ and $J(v,c)$ are lower semicontinuous.*

(b) *If $T^{v,c}$ is continuous, then $J(v,c)$ is continuous; in particular, $J(v,c)$ is continuous for a fixed time-horizon control system.*

Proof. (a) Let $v \in \mathcal{V}_0$, $c \in \mathbb{R}^d$ and suppose that $(V,\gamma) \approx (v,c)$. We have to show that $T^{v,c} \leq T^{V,\gamma}$ and $J(v,c) \leq J(V,\gamma)$.

We know that $x^{v,c} = {}^\circ(X^{V,\gamma})$ (Theorem 3.3), so putting $s = {}^\circ T^{V,\gamma}$ we have $(s, x_s^{v,c}) \approx (T^{V,\gamma}, X^{V,\gamma}_{T^{V,\gamma}}) \in {}^*\Gamma$; thus $(s, x_s^{v,c}) \in \Gamma$ since Γ is closed, so $T^{v,c} \leq s = {}^\circ T^{V,\gamma}$.

For lower semicontinuity of J: first notice that ${}^*h(\tau, X_\tau^{V,\gamma}, \alpha)$ is a bounded uniform lifting of $h(t, x_t^{v,c}, a)$, by the reasoning applied to f in the proof of Theorem 3.3. Thus

$$\begin{aligned} J(V,\gamma) &= \int_0^{T^{V,\gamma}} {}^*h(\tau, X_\tau^{V,\gamma}, V_\tau)\,\mathrm{d}\tau + {}^*\bar{h}(T^{V,\gamma}, X^{V,c}) \\ &\approx \int_0^{s} h(t, x_t^{v,c}, v_t)\,\mathrm{d}t + \bar{h}(s, x^{v,c}) \\ &\geq \int_0^{T^{v,c}} h(t, x_t^{v,c}, v_t)\,\mathrm{d}t + \bar{h}(T^{v,c}, x^{v,c}) \\ &= J(v,c) \end{aligned}$$

as required.

(b) If $T^{v,c}$ is continuous, then $s = T^{v,c}$ in the above, and so $J(V,\gamma) \approx J(v,c)$. (Note that here it is not necessary to assume that $\bar{h}$ is nondecreasing in t.) □

We now have some optimality results: first define, for any region $D \subseteq \mathbb{R}^d$,

$$J_0^{\mathcal{U}_0}(D) = \inf_{\substack{c \in D \\ u \in \mathcal{U}_0}} J_0(u,c); \qquad J_0^{\mathcal{V}_0}(D) = \inf_{\substack{c \in D \\ v \in \mathcal{V}_0}} J_0(v,c).$$

It is easy to establish by standard means:

COROLLARY 4.2. (a) *For any compact region $D \subseteq \mathbb{R}^d$ there is a pair $(v,c) \in \mathcal{V}_0 \times D$ that is optimal; i.e., $J(v,c) = J_0^{\mathcal{V}_0}(D)$.*

(b) *If $T^{v,c}$ is continuous (in particular, for fixed time-horizon systems) $J_0^{\mathcal{U}_0}(D) = J_0^{\mathcal{V}_0}(D)$ for any region $D \subseteq \mathbb{R}^d$.*

Proof. (a) Take (v_n, c_n) such that $|J(v_n, c_n) - J_0^{\mathcal{V}_0}(D)| < 1/n$; by Theorem 2.7 and compactness of D, we may assume that $(v_n, c_n) \to (v,c) \in \mathcal{V}_0 \times D$. By Theorem 4.1(a),

$$J(v,c) \leq \lim_{n \to \infty} J(v_n, c_n) = J_0^{\mathcal{V}_0}(D),$$

so $J(v,c) = J_0^{\mathcal{V}_0}(D)$.

(b) is routine using the density of $\mathcal{U}_0$ in $\mathcal{V}_0$ (Theorem 2.9) and the continuity of $J(v, c)$ (Theorem 4.1(b)). □

REMARKS (1) Optimal ordinary controls can be obtained for systems in which the following convexity property holds: for each (t, ξ) the set $\{(f(t, \xi, a), h(t, \xi, a)) : a \in M\}$ is convex. This is achieved by standard measurable selection results, whereby for any relaxed control v, an ordinary control u is constructed to give the same trajectory and running cost.

(2) It is routine to extend the results of Sections 3 and 4 to the case $t_1 = \infty$. In this case, we need to assume that Equation (3.1) has a unique solution on $\mathbb{R}^+$ for each control; in Theorem 3.3 the continuity is with respect to the compact-open topology on $C^d(\mathbb{R}^+)$; and in Section 4 the results all hold if we allow $T^{v,c}$ and $J(v, c)$ to take the value ∞.

PART II: PARTIALLY OBSERVED STOCHASTIC CONTROL THEORY BASED ON A CUMULATIVE DIGITAL READ-OUT

If there are random elements in the evolution of the state x_t of a controlled system, this is conveniently modelled by assuming that x_t is the solution of a stochastic differential equation of the form

$$dx_t = f(t, x, u_t)\, dt + g(t, x, u_t)\, db_t,$$

where b_t is a Brownian motion. Controls can be more efficient if information about the present or past of the trajectory can be fed back to the controller. In general, this will be incomplete or *partial* information – depending only on some observed component $y_t \in \mathbb{R}^m$ of the state. In Section 5 we consider the situation where the information available to the controller is a digital read-out. In this case we can obtain compactness and optimality results (Sections 6–8) similar to those in the deterministic setting. The methods here build on those used in Part II.

5. Cumulative Digital Information for Stochastic Systems

Suppose that $y_t \in \mathbb{R}^m$ is a component of the state of some stochastically evolving system. We begin by setting up our model of an information pattern where the controller has available at time t a cumulative digital read-out $r(t, y)$. For convenience we assume a fixed time horizon $t_1 = 1$, so $y = (y_t)_{t \le 1}$.

DEFINITION 5.1. A cumulative digital read-out is a function $r: [0, 1] \times \mathcal{C}^m \to \mathbb{N}$ (where $\mathcal{C}^m = C^m[0, 1]$) such that

(a) r_t is measurable, each t;
(b) if $y \upharpoonright t = y' \upharpoonright t$ then $r_t(y) = r_t(y')$;
(c) if $r_t(y) = r_t(y')$ then $r_s(y) = r_s(y')$ all $s \le t$.

Conditions (a) and (b) are equivalent to the requirement that r is

$$(\mathcal{F}_t^{(m)})\text{-adapted}, \quad \text{where } \mathcal{F}_t^{(m)} = \sigma\{y_s : y \in \mathscr{C}^m, s \leq t\};$$

i.e., r_t is $\mathcal{F}_t^{(m)}$-measurable for each t. Condition (c) is the accumulation property.

We shall see later that we may assume without any loss of generality that $r_t(y)$ is increasing with t.

Associated with r there is a natural *information filtration* $\mathcal{I}^{(r)} = (\mathcal{I}_t^{(r)})_{t \leq 1}$ defined by $\mathcal{I}_t^{(r)} = \sigma\{r_t\}$. Notice that $\mathcal{I}$ has the properties

(a) $\mathcal{I}_s^{(r)} \subseteq \mathcal{I}_t^{(r)} \subseteq \mathcal{F}_t^{(m)}$; $\quad (s \leq t)$

(b) $\mathcal{I}_1^{(r)}$ is generated by a countable number of atoms (namely the sets $D_i = r_1^{-1}(\{i\})$). (5.2)

We will consider controls $u: [0, 1] \times \mathscr{C}^m \to M$ that are $\mathcal{I}^{(r)}$-adapted – which is equivalent to the requirement that if $r_t(y) = r_t(y')$ then $u(t, y) = u(t, y')$. Before discussing controls, we show that *any* filtration satisfying (5.2) arises from a cumulative digital read-out.

THEOREM 5.3. *Let $\mathcal{I} = (\mathcal{I}_t)_{t \leq 1}$ be a filtration on $\mathscr{C}^m$ satisfying conditions* (5.2). *There is a cumulative digital read-out r such that $\mathcal{I} = \mathcal{I}^{(r)}$, with the additional property that $r_t(y)$ increases with t.*

Proof. Suppose that $\mathcal{I}_1 = \sigma\{D_i : i \in \mathbb{N}\}$, where $D_i \neq \emptyset$, $D_i \cap D_j = \emptyset$ if $i \neq j$, and $\bigcup_{i \in \mathbb{N}} D_i = \mathscr{C}^m$. For each t define a relation $\sim_t$ on $\mathbb{N}$ by

$$i \sim_t j \text{ iff } D_i, D_j \text{ are indistinguishable by sets in } \mathcal{I}_t$$
$$(\text{iff } D_i \subseteq A \leftrightarrow D_j \subseteq A \text{ for all } A \in \mathcal{I}_t).$$

Notice that $i \sim_t j$ implies $i \sim_s j$ for $s \leq t$.

For each i, let $A_i^t = \bigcup_{j \sim_t i} D_j$; we claim that A_i^t is an atom of $\mathcal{I}_t$. To see that $A_i^t \in \mathcal{I}_t$, for each $k \not\sim_t i$ choose $A_k \in \mathcal{I}_t$ with $D_i \subseteq A_k$ and $D_k \cap A_k = \emptyset$. Then $A_i^t = \bigcap_{k \not\sim_t i} A_k \in \mathcal{I}_t$. It is clear that A_i^t is an atom of $\mathcal{I}_t$. Notice that $i \sim_t j \leftrightarrow A_i^t = A_j^t$; and $A_i^t \subseteq A_i^s$ if $s \leq t$.

Now we can define the read-out r by

$$r_t(y) = \text{least } j \text{ such that } y \in A_j^t$$
$$(= \text{least } j \text{ such that } i \sim_t j, \text{ if } y \in D_i).$$

To see that $\sigma\{r_t\} = \mathcal{I}_t$, note that $A_i^t = r_t^{-1}(\{j\})$, where j is least with $j \sim_t i$. For the accumulation property, note that if $s \leq t$ and $y \in D_i$, with $r_t(y) = j$ and $r_s(y) = k$, then

$$r_t^{-1}(\{j\}) = A_i^t \subseteq A_i^s = r_s^{-1}(\{k\}).$$

It is clear from the construction of r that $r_t(y)$ is increasing with t. □

The fact that r is increasing with t means that for each $y \in \mathscr{C}^m$ there are only finitely many updates, as follows.

COROLLARY 5.4. *For each* y *there is* $k = k(y) \in \mathbb{N}$ *and there are times* $0 = t_0(y) < t_1(y) < \cdots < t_k(y) = 1$ *such that* $r(t, y)$ *is constant on each interval* $]t_i(y), t_{i+1}(y)[$.

Proof. Fix y. Use the fact that $r_t(y)$ is increasing, and $r_1(y) \in \mathbb{N}$, so $r_t(y)$ has finite range. □

REMARK. We cannot infer from this Corollary that $\mathcal{I}_t$ increases at only a finite or countable number of times, as shown by the following example. Pick a countable collection of pairwise disjoint sets $(B_i)_{i \in \mathbb{N}}$ from $\mathcal{F}^{(m)}_{1/2}$, with $\bigcup_{i \in \mathbb{N}} B_i = \mathscr{C}^m$. Let $(q_i)_{i \in \mathbb{N}}$ be an enumeration of the rationals in $[0, 1]$, and define

$$\begin{aligned} \mathcal{I}_t &= \{\emptyset, \mathscr{C}^m\} && \text{for } t < \tfrac{1}{2}, \\ &= \sigma\{B_i : q_i \leq t\} && \text{for } t \geq \tfrac{1}{2}. \end{aligned}$$

Clearly $\mathcal{I}_s \subsetneq \mathcal{I}_t$ whenever $\frac{1}{2} \leq s < t$.

We have not assumed that a read-out $r_t(y)$ is jointly measurable, nor that the information filtration $(\mathcal{I}_t)_{t \leq 1}$ is right-continuous. For applications it is necessary to show how to obtain a jointly measurable read-out as follows.

THEOREM 5.5. *Let* $\mathcal{I}$, r *be as in Theorem* 5.3. *Let*

$$r'_t(y) = r_{t-}(y) = \begin{cases} \lim\limits_{s \uparrow t} r_s(y) & \text{if } t > 0 \\ r_0(y) & \text{if } t = 0. \end{cases}$$

Then r' *is jointly measurable,* $\mathcal{I}$*-adapted, and* $r'_t(y)$ *is left continuous in* t *for each* y. *Moreover, for all but countably many values of* t, $r'_t(y) = r_t(y)$ *for all* y.

Proof. r' is definable from the countable collection of functions $(r_q(y))_{q \in \mathbb{Q}}$, so r' is easily seen to be measurable. Left continuity is obvious. For the last part note that if $y, y' \in D_i$ then the jump times $t_p(y)$ and $t_p(y')$ of Corollary 5.4 are the same, and that for $t_p(y) < t < t_{p+1}(y)$ we have $r'_t(y) = r_t(y)$. □

6. Controls for Stochastic Systems

Suppose now that an information filtration $\mathcal{I} = (\mathcal{I}_t)_{t \leq 1}$ is fixed, with readouts r and r' as given by Theorems 5.3 and 5.5. With a fixed control space M as in Section 2 we now define the classes of controls for stochastic systems with information pattern given by $\mathcal{I}$.

DEFINITION 6.1. (a) The class $\mathcal{U}$ of (*admissible ordinary*) *controls* is the class of jointly measurable functions u: $[0, 1] \times \mathscr{C}^m \to M$ such that u is $\mathcal{I}$-adapted.

(b) The class of (*admissible*) *relaxed controls* $\mathcal{V}$ is given by extending M to $\mathcal{M}(M)$ as in Section 2.

Each control $v \in \mathcal{V}$ gives a sequence $(v_i)_{i\in\mathbb{N}}$ of controls in $\mathcal{V}_0$ by setting*

$$v_i(t) = v(t, y) \quad \text{for } y \in D_i.$$

(Recall that $(D_i)_{i\in\mathbb{N}}$ are the atoms of $\mathcal{I}_1$). A natural topology on $\mathcal{V}$ is induced by that on $\mathcal{V}_0$ – namely, the restriction of the product topology on $\mathcal{V}_0^{\mathbb{N}}$. So a subbase of open sets is provided by the sets

$$\{v : |v_i(g)| < \epsilon\}_{i\in\mathbb{N},\, g\in\mathcal{K},\, \epsilon>0}$$

(where $v_i(g) = \int_0^1 g(t, v_i(t))\, dt$ of course).

The compactness of $\mathcal{V}_0$ established in Section 2 is used to show that $\mathcal{V}$ is compact as follows.

THEOREM 6.2. *$\mathcal{V}$ is compact.*

Proof. Let $V \in {}^*\mathcal{V}$. For each $i\in\mathbb{N}$ we have $V_i = V(\cdot, y)$ for $y \in {}^*D_i$. Then $V_i \in {}^*\mathcal{V}_0$; let $v^{(i)} = {}^\circ V_i$ (using Theorem 2.7). Now define v: $[0, 1] \times \mathcal{C}^m \to \mathcal{M}(M)$ by

$$v(t, y) = v^{r'_t(y)}(t).$$

Since r' is jointly measurable and $\mathcal{I}$-adapted, so is v, i.e., $v \in \mathcal{V}$. We now see that $v = {}^\circ V$; i.e., $v_i = {}^\circ V_i$ for each $i\in\mathbb{N}$. Fix $y \in D_i$; then on each interval $]t_p(y), t_{p+1}(y)[$ we have $r'_t(y) = r_t(y) = j$, say. Thus, on this interval,

$$\begin{aligned} v_i(t) = v(t, y) &= v^{(j)}(t) \quad \text{(by definition)} \\ &= ({}^\circ V_j)(t) \\ &= ({}^\circ V_i)(t) \quad \text{a.s.,} \end{aligned}$$

because V is ${}^*\mathcal{I}$-adapted and so $V_i = V_j$ on $]t_p(y), t_{p+1}(y)[$. □

REMARK 6.3. It follows from the proof above and Lemma 2.6 that for $g \in \mathcal{K}$ with bounded uniform lifting G,

$$\int_0^1 G(\tau, V(\tau, Y))\, d\tau \approx \int_0^1 g(t, v(t, y))\, dt$$

whenever $Y \in {}^*D_i$ and $y \in D_i$ (for *finite* i).

Step controls in $\mathcal{U}$ are defined as for $\mathcal{U}_0$, and are now shown to be dense in $\mathcal{V}$.

DEFINITION 6.4. A control $u \in \mathcal{U}$ is a *step control* if there are times $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ with $u(t, y)$ constant on $[t_i, t_{i+1}[$ for each fixed y. Denote by $\mathcal{U}^s$ the *uniform* step controls – i.e., those for which $t_{i+1} - t_i$ is constant.

THEOREM 6.5. *$\mathcal{U}^s$ is dense in $\mathcal{V}$.*

Proof. Let $v \in \mathcal{V}$; fix a positive infinitesimal Δt. From the proof of Theorem 2.9,

*Because of the restriction of the time interval $[0, 1]$ here, whenever we mention $\mathcal{V}_0$ from now on we mean the controls in $\mathcal{V}_0$ restricted to $[0, 1]$.

there are step controls $U^{(i)} \in {}^*\mathcal{U}_0^s$ with $U^{(i)}$ constant on $[\mathrm{t}, \mathrm{t}+\Delta\mathrm{t}[$ for each $\mathrm{t} = k\Delta\mathrm{t} < 1$, such that ${}^\circ U^{(i)} = v_i$ for finite i. Now define $U \in {}^*\mathcal{U}^s$ by

$$U(\tau, Y) = U^{R_\tau(Y)}(\tau)$$

where $R_\tau(Y) = {}^*r_{\mathrm{t}}(Y)$ on $[\mathrm{t}, \mathrm{t}+\Delta\mathrm{t}[$. Then clearly $U \in {}^*\mathcal{U}^s$. We show that ${}^\circ U = v$ (i.e., ${}^\circ U_i = v_i$ for finite i); this will establish the density of $\mathcal{U}^s$ in $\mathcal{V}$.

Fix $y \in D_i$; on an interval $]t_p(y), t_{p+1}(y)[$ we have $r_t(y) = j$, say. Thus, on $[t_p(y) + \Delta\mathrm{t}, t_{p+1}(y)[$ we have $R_\tau(y) = j$ and $U(\tau, y) = U^{(j)}(\tau)$. So on the standard interval $[t_p(y), t_{p+1}(y)[$ we have

$$\begin{aligned}({}^\circ U_i)(t) = ({}^\circ U^{(j)})(t) &= v_j(t)\\ &= v_i(t),\end{aligned}$$

since v is $\mathcal{I}$-adapted.

This proves the result. □

7. Stochastic Control Systems

We now consider a stochastic control system with information structure and controls as described in the previous sections. We model such a system by a state $z_t = (x_t, y_t) \in \mathbb{R}^{d+m} = \mathbb{R}^n$, say, evolving according to past-dependent differential equations of the form:

$$\begin{aligned} dx_t &= f(t, x, y, u(t, y))\,dt + g(t, x, y, u(t, y))\,db_t \quad (t \leq 1)\\ dy_t &= \bar{f}(t, x, y, u(t, y))\,dt + \bar{g}(t, y)\,d\bar{b}_t \end{aligned} \tag{7.1}$$

with x_0, y_0 fixed. The independent Brownian motions b, $\bar{b}$ have dimensions d, m respectively. The component y_t of the state is the *observation*; so controls depend on a cumulative digital read-out of the observation.

ASSUMPTIONS ON COEFFICIENTS

The coefficients f, $\bar{f}$, g, $\bar{g}$ take their values in $\mathbb{R}^d$, $\mathbb{R}^m$, $\mathbb{R}^d \otimes \mathbb{R}^d$, $\mathbb{R}^m \otimes \mathbb{R}^m$ respectively. We make the following assumptions, which are the usual kind of assumptions required to ensure that for each control Equations (7.1) have a solution that is weakly unique (i.e., unique in distribution on $\mathscr{C}^n$):

(a) $f, \bar{f}, g, \bar{g}$ are jointly measurable and adapted in (x, y) to $\mathcal{F}^{(n)}$, where $\mathcal{F}^{(n)} = (\mathcal{F}_t^{(n)}) = (\sigma\{\xi_s : s \leq t, \xi \in \mathscr{C}^n\})$;
(b) for each fixed (t, y), the functions f, $\bar{f}$, g are continuous in (x, u);
(c) g is uniformly Lipschitz in x, $\bar{g}$ is uniformly Lipschitz in y;
(d) g, $\bar{g}$ are positive definite, and g is symmetric; (7.2)
(e) $f, \bar{f}, g, \bar{g}, g^{-1}f, \bar{g}^{-1}\bar{f}$ satisfy linear growth conditions of the form

$$\|\theta(t, x, y, a)\| \leq \kappa(1 + \|(x, y)\|),$$

where $\|g\| = (\sum g_{ij}^2)^{1/2}$ for a matrix g.

CONTROLS

The control u in Equations (7.1) is assumed to belong to the class of admissible controls $\mathcal{U}$ discussed in Section 6. We also allow relaxed controls $v \in \mathcal{V}$; such controls operate in the drift terms f, $\bar{f}$ exactly as for deterministic systems (Section 2). We define the effect of a relaxed control in the diffusion g by defining

$$g(t, x, y, \nu) = \left(\int_M g^2(t, x, y, a)\, \mathrm{d}\nu(a) \right)^{1/2}$$

for $\nu \in \mathcal{M}(M)$. Here $(\cdot)^{1/2}$ denotes the unique nonnegative definite square root matrix. Thus we are defining the covariance $g^2(t, x, y, v_t)$ in the same way that the drift $f(t, x, y, v_t)$ is defined, recognising that it is the covariance g^2 rather than the diffusion g itself that determines the dynamics.

SOLUTIONS TO EQUATIONS (7.1)

Suppose that a control $v \in \mathcal{V}$ is fixed. It is convenient to write $f_v(t, x, y)$ for $f(t, x, y, v(t, y))$, and similarly for $\bar{f}$ and g.

With the given conditions on the coefficients, there is on any space carrying Brownian motions b, $\bar{b}$ a unique solution $\zeta^v = (\xi^v, \eta^v)$ to the equations *without* drift:

$$\begin{aligned} &\text{(a)}\quad \mathrm{d}\xi_t = g_v(t, \xi, \eta)\, \mathrm{d}b_t \quad (\xi_t \in \mathbb{R}^d,\ \xi_0 = x_0) \\ &\text{(b)}\quad \mathrm{d}\eta_t = \bar{g}(t, \eta)\, \mathrm{d}\bar{b}_t \quad (\eta_t \in \mathbb{R}^m,\ \eta_0 = y_0) \end{aligned} \tag{7.3$_v$}$$

for $0 \leq t \leq 1$. This solution induces measures as follows:

DEFINITION 7.4.

(a) λ^v is the measure induced on $\mathcal{C}^n$ by ζ^v;

(b) $\lambda_1^{v,\eta}$ is the measure induced on $\mathcal{C}^n$ by the solution to 7.3_v(a) for each fixed $n \in \mathcal{C}^m$;

(c) λ_2 is the measure induced on $\mathcal{C}^m$ by the solution to 7.3(b). Note that λ_2 is tight.

The measure $\lambda_1^{v,\eta}$ is measurable as a function of η (with respect to the completion of λ_2) and gives the following disintegration of λ^v:

$$\lambda^v(A \times B) = \int_B \lambda_1^{v,\eta}(A)\, \mathrm{d}\lambda_2(\eta) \tag{7.5}$$

for Borel A, B.

The dynamical Equations (7.1) may be solved using the Girsanov measure change technique (or using nonstandard methods as developed in [5]). The solutions obtained are unique in law. Well-known theory tells us the following:

THEOREM 7.6. *Let μ^v be the measure induced on $\mathscr{C}^n$ by any solution to Equations* (7.1) *for control v. Then $\mu^v \ll \lambda^v$, and the density $\rho^v = \mathrm{d}\mu^v/\mathrm{d}\lambda^v$ is given by*

$$\rho^v(\zeta) = \exp\left(\int_0^1 (\gamma^{-1}\phi)'_v(t, \zeta)\,\mathrm{d}\beta(\zeta) - \tfrac{1}{2}\int_0^1 (\gamma^{-1}\phi)_v^2\,\mathrm{d}t\right)$$

where

$$\zeta = (\xi, \eta) \in \mathscr{C}^n, \qquad \phi = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix},$$

and β is the Brownian motion on $(\mathscr{C}^n, \lambda^v)$ given by $\beta = \int \gamma^{-1}\,\mathrm{d}\zeta$.

We also record here for future reference the following standard consequence of the linear growth conditions 7.2(e) on the coefficients. For $v \in \mathscr{V}$ let $z^v = (x^v, y^v)$ be any solution to Equations (7.1).

LEMMA 7.7. *$E(\|z^v\|^2)$ is uniformly bounded for $v \in \mathscr{V}$, where $\|z^v\| = \sup_{t \leq 1}\|z_t^v\|$.*

The main theorem in this section is the following.

THEOREM 7.8. *The measure μ^v is continuous as a function of v (with respect to the weak* topology on $\mathscr{M}(\mathscr{C}^n)$).*

Proof. Let $V \in {}^*\mathscr{V}$ and let $v = {}^\circ V$ as in the previous section. We have to show that $\mu^v \approx \mu^V$; by the Loeb–Anderson–Rashid characterisation of weak standard parts [4, 18], it is sufficient to show that

$$\mu^v = \mu_L^V(st^{-1}(\cdot)). \tag{7.9}$$

The plan is as follows. Fix an internal solution $Z^V = (X^V, Y^V)$ to Equations *(7.1). i.e.,

$$\begin{aligned} \mathrm{d}X_\tau^V &= {}^*f_V(\tau, X^V, Y^V)\,\mathrm{d}\tau + {}^*g_V(\tau, X^V, Y^V)\,\mathrm{d}B_\tau \\ \mathrm{d}Y_\tau^V &= {}^*\bar{f}_V(\tau, X^V, Y^V)\,\mathrm{d}\tau + {}^*\bar{g}(\tau, Y^V)\,\mathrm{d}\bar{B}_\tau, \end{aligned}$$

where $B, \bar{B}$ are (internal) *Brownian motions; these and the solution Z^V live on an internal space $\mathbf{\Omega}_0 = (\Omega, \mathscr{A}, (\mathscr{A}_\tau)_{\tau \leq 1}, \nu)$. Let $\mathbf{\Omega} = (\Omega, \mathscr{G}, (\mathscr{G}_t)_{t \leq 1}, P)$ be the standard filtered Loeb space constructed from $\mathbf{\Omega}_0$ (see Section 1). We will see that almost all paths of Z^V are $\mathscr{S}$-continuous; so we may define a continuous process z^V on $\mathbf{\Omega}$ by

$$z^V = {}^\circ(Z^V) \quad \text{a.s.} \quad (P).$$

We will show that z^V is a solution to Equations (7.1) for control v. If so, then

$$\begin{aligned} \mu^v(A) = P(z^V \in A) &= P(Z^V \in st^{-1}(A)) \\ &= \nu_L(Z^V \in st^{-1}(A)) \\ &= \mu_L^V(st^{-1}(A)), \end{aligned}$$

which establishes (7.9), and we are done.

We proceed by a series of Lemmas.

LEMMA 7.10. *$\bar{E}(\|Z^V\|^2)$ is finite and, hence, $\|Z^V\|$ is finite, a.s.(P).*

Proof. By Lemma 7.7 transferred, $\bar{E}(\|Z^V\|^2)$ is finite; Loeb theory tells us that $E({}^\circ\|Z^V\|^2) \leq {}^\circ\bar{E}(\|Z^V\|^2)$. □

LEMMA 7.11. *Z^V is P-a.s. $\mathcal{S}$-continuous (hence, $\mu_L^V(st^{-1}(\mathscr{C}^n)) = 1$).*

Proof. If the coefficients $^*f$, $^*\bar{f}$, $^*g$, $^*\bar{g}$ are bounded, this follows from Keisler's $\mathcal{S}$-continuity theorem (Proposition 1.4(b)). For coefficients satisfying the linear growth conditions 7.2(e), a routine truncation argument (as in [7], Lemma 7.12(b)) using Lemma 7.10 then shows that Z^V is a.s. $\mathcal{S}$-continuous. □

This lemma allows us to define a standard process z^V on $\mathbf{\Omega}$ by $z^V = {}^\circ Z^V$ P-a.s. The goal now is to show that z^V is a (weak) solution to $(7.1)_v$ on $\mathbf{\Omega}$. We will need to know the following.

LEMMA 7.12. *Let $K \subseteq \mathscr{C}^m$ be compact. The measures $(\lambda_1^{v,\eta})_{v\in\mathscr{V},\eta\in K}$ are uniformly tight; i.e., for each $\epsilon > 0$ there is compact $K_\epsilon \subseteq \mathscr{C}^d$ with $\lambda_1^{v,\eta}(K_\epsilon) \geq 1-\epsilon$ for all $(v, \eta) \in \mathscr{V} \times K$.*

Proof. By Prohorov's Theorem [19], Theorem 6.7, it is sufficient to show that the set of measures $(\lambda_1^{v,\eta})_{v\in\mathscr{V},\eta\in K}$ is relatively compact. Using the nonstandard criterion for compactness, it is enough to show that $\lambda_1^{V,Y}$ is nearstandard for every $V \in {}^*\mathscr{V}$, $Y \in {}^*K$.

Let $\xi^{V,Y}$ be a solution to Equation $^*(7.3)_V$(a) with Y constant; i.e.,

$$d\xi_\tau^{V,Y} = {}^*g_V(\tau, \xi^{V,Y}, Y)\, dB_\tau.$$

Arguing exactly as in the proof of Lemma 7.11 we see that $\xi^{V,Y}$ is a.s. $\mathcal{S}$-continuous. Thus, $(\lambda_1^{V,Y})_L(st^{-1}(\mathscr{C}^d)) = 1$, which shows (as in [4, 17]) that $\lambda_1^{V,Y}$ is nearstandard with standard part $(\lambda_1^{V,Y})_L(st^{-1}(\cdot))$. □

We use this lemma to show that the functions $^*f$ (etc.) are liftings of the functions f (etc.) in the sense of the following lemma.

LEMMA 7.13. *For $\theta = f, g, \bar{f}, \bar{g}$: for almost all $(\tau, Z) \in {}^*[0, 1] \times {}^*\mathscr{C}^n$ (with respect to $({}^*m \times \lambda^V)_L$),*

$${}^*\theta(\tau, Z, \alpha) \approx \theta({}^\circ\tau, {}^\circ Z, {}^\circ\alpha) \quad \textit{for all } \alpha \in {}^*M.$$

Proof. We give the proof for $\theta = f$; the others are similar. Fix $\epsilon > 0$. Since λ_2 is tight, we can take compact $K \subseteq \mathscr{C}^m$ such that $\lambda_2(K) \geq 1-\epsilon$. Take $K_\epsilon \subseteq \mathscr{C}^d$ as given by Lemma 7.12 and define $\hat{f}: [0, 1] \times K \to C(K_\epsilon \times M, \mathbb{R}^d)$ by

$$\hat{f}(t, \eta)(\xi, a) = f(t, \xi, \eta, a).$$

Since $K_\epsilon \times M$ is compact, the space on the right is separable, metric, so by Anderson's Lusin Theorem

$${}^*\hat{f}(\tau, Y) \approx \hat{f}({}^\circ\tau, {}^\circ Y)$$

for almost all $(\tau, Y) \in {}^*[0, 1] \times {}^*K$ (with respect to $({}^*m \times {}^*\lambda_2)_L$). Thus there is internal $B \subseteq {}^*[0, 1] \times {}^*K$ with $({}^*m \times {}^*\lambda_2)(B) \geq 1-\epsilon$, such that

$$^*\hat{f}(\tau, Y) \approx \hat{f}({}^\circ\tau, {}^\circ Y), \quad \text{all } (\tau, Y) \in B.$$

This means that for all $(\tau, Y, X) \in B \times {}^*K_\epsilon$ and $\alpha \in {}^*M$

$$^*f(\tau, X, Y, \alpha) \approx f({}^\circ\tau, {}^\circ X, {}^\circ Y, {}^\circ\alpha).$$

Now using the transfer of the disintegration (7.5) we have that

$$\begin{aligned}({}^*m \times \lambda^V)(B \times {}^*K_\epsilon) &= \int_B {}^*\lambda_1^{V,Y}({}^*K_\epsilon)\, \mathrm{d}({}^*m \times {}^*\lambda_2)(\tau, Y)\\ &\geqslant (1-\epsilon)({}^*m \times {}^*\lambda_2)(B)\\ &\geqslant (1-\epsilon)^2.\end{aligned}$$

Taking ϵ arbitrarily small establishes the result. □

For application we require $^*f$ (etc.) to be a lifting of f (etc.) with respect to $(m^* \times \mu^V)_L$, rather than the measure $({}^*m \times \lambda^V)_L$ as in Lemma 7.13. This is established with the aid of the following lemma and its corollary.

LEMMA 7.14. *The density ρ^V is $\mathcal{S}$-integrable with respect to λ^V.*

Proof. By transfer of standard theory outlined earlier,

$$\mu^V(B) = \int_B \rho^V \,\mathrm{d}\lambda^V \quad \text{for all internal *Borel } B \subseteq {}^*\mathscr{C}^n,$$

so $\bar{E}_{\lambda^V}(\rho^V) = 1$.

Also,

$$\rho^V(Z) = \exp\left(\int_0^1 \Theta_\tau(Z)\,\mathrm{d}\beta_\tau(Z) - \tfrac{1}{2}\int_0^1 \Theta_\tau(Z)^2\,\mathrm{d}\tau\right)$$

where

$$\Theta_\tau(Z) = \left(\begin{pmatrix} {}^*g & 0 \\ 0 & {}^*\bar{g}\end{pmatrix}^{-1} \begin{pmatrix} {}^*f \\ {}^*\bar{f}\end{pmatrix}\right)'_V (\tau, Z)$$

and β is an internal Brownian motion on $({}^*\mathscr{C}^n, \lambda^V)$.

The linear growth conditions (7.2) on the coefficients, together with fact that Z^V is a.s. finite (Lemma 7.10), ensure that

$$^\circ\mu^V(C_n) \to 1 \quad \text{as } n \to \infty,$$

where $C_n = \{Z : \|\Theta(\cdot, Z)\|^2 \leqslant n\}$.

Now consider an internal measurable $A \subseteq {}^*\mathscr{C}^n$ with $\lambda^V(A) \approx 0$. We have to show that $\int_A \rho^V\,\mathrm{d}\lambda^V \approx 0$. Notice that

$$\begin{aligned}(\rho^V)^2 &= \exp\left(\int_0^1 2\Theta\,\mathrm{d}\beta - \tfrac{1}{2}\int_0^1 (2\Theta)^2\,\mathrm{d}\tau\right) \cdot \exp\left(\int_0^1 \Theta^2\,\mathrm{d}\tau\right)\\ &= \sigma \cdot \delta, \quad \text{say,}\end{aligned}$$

where σ is defined like ρ^V using 2Θ in place of Θ. Now $\bar{E}_{\lambda^V}(\sigma) = 1$ (for the same reasons as $\bar{E}_{\lambda^V}(\rho^V)=1$), and $\delta=\exp(\int \Theta^2\,\mathrm{d}\tau)$ is bounded on C_n. Thus, by Hölders inequality

$$\int_{A\cap C_n} \rho^V\,\mathrm{d}\lambda^V \leq \left(\int_{A\cap C_n} \sigma\,\mathrm{d}\lambda^V\right)^{1/2}\left(\int_{A\cap C_n} \delta\,\mathrm{d}\lambda^V\right)^{1/2} \approx 0.$$

So

$$^\circ\!\int_A \rho^V\,\mathrm{d}\lambda^V \leq {}^\circ\!\int_{\bar{C}_n} \rho^V\,\mathrm{d}\lambda^V = {}^\circ\mu^V(\bar{C}_n) \quad \text{(by definition of } \rho^V)$$
$$= 1-{}^\circ\mu^V(C_n), \quad \text{which} \to 0 \text{ as } n\to\infty,$$

and we are done. □

COROLLARY 7.15. (a) *$\mu_L^V \ll \lambda_L^V$ and $\mathrm{d}\mu_L^V/\mathrm{d}\lambda_L^V = {}^\circ\rho$;* (b) *$(\mu_2^V)_L \ll ({}^*\lambda_2)_L$, where μ_2^v is the measure induced on $\mathscr{C}^m$ by a control $v\in\mathscr{V}$.*

Proof. (a) For internal measurable $A\subseteq {}^*\mathscr{C}^n$ we have

$$\mu_L^V(A) = {}^\circ\mu^V(A) = {}^\circ\!\int_A \rho^V\,\mathrm{d}\lambda^V = \int_A {}^\circ\rho^V\,\mathrm{d}\lambda_L^V$$

since ρ^V is $\mathscr{S}$-integrable; (a) follows.

(b) follows from (a) because $\lambda_2(B)=\lambda^v(\mathscr{C}^d\times B)$ and $\mu_2^v(B)=\mu^v(\mathscr{C}^d\times B)$. □

COROLLARY 7.16. (a) *For θ = each of f, g, $\bar{f}$, $\bar{g}$: for almost all (τ, Z) with respect to $({}^*m\times\mu^V)_L$,*

$${}^*\theta(\tau, Z, \alpha) \approx \theta({}^\circ\tau, {}^\circ Z, {}^\circ\alpha), \quad \textit{all } \alpha\in {}^*M.$$

(b) *Hence, for a.a. ω, ${}^*\theta(\tau, Z^V, \alpha)$ is a bounded uniform lifting of $\theta(t, z^V, a)$.*

Proof. (a) Combine Lemmas 7.13 and 7.15(a); (b) follows by Keisler's Fubini theorem. □

Applying the above results to the information structure we have:

LEMMA 7.17. *For a.a. $Y\in {}^*\mathscr{C}^m$ (with respect to $(\mu_2^V)_L$)*

$$Y\in {}^*D_i \quad \textit{iff } {}^\circ Y\in D_i, \quad \textit{for all finite } i.$$

Hence, for a.a. ω,

$$Y^V\in {}^*D_i \quad \textit{iff } y^V\in D_i, \quad \textit{for all finite } i.$$

Proof. From Anderson's Lusin theorem this holds for a.a. Y w.r.t $({}^*\lambda_2)_L$; now apply Corollary 7.15(b). □

Combining this Lemma with Corollary 7.16(b) and Remark 6.3 we obtain:

LEMMA 7.18. *For a.a. ω, for all t,*

(a) $$\int_0^t {}^*f_V(\tau, Z^V)\,\mathrm{d}\tau \approx \int_0^t f_v(s, z^V)\,\mathrm{d}s;$$

(b) $\displaystyle\int_0^t {}^*\bar{f}_V(\tau, Z^V)\,\mathrm{d}\tau \approx \int_0^t \bar{f}_v(s, z^V)\,\mathrm{d}s;$

(c) $\displaystyle\int_0^t {}^*g_V^2(\tau, Z^V)\,\mathrm{d}\tau \approx \int_0^t g_v^2(s, z^V)\,\mathrm{d}s,$

where we mean $(g_v)^2$, *not* $(g^2)_v$, *etc.*

Applying Corollary 7.16(a), to the diffusion coefficient $\bar{g}$, we obtain:

LEMMA 7.19. *The internal function* ${}^*\bar{g}(\tau, Y^V)$ *is a lifting of* $\bar{g}(t, y^V)$ *and is adapted to* $(\mathscr{A}_\tau)_{\tau\leq 1}$. *Hence, for a.a.* ω, *for all* t,

$$\int_0^t {}^*\bar{g}(\tau, Y^V)\,\mathrm{d}\bar{B}_\tau \approx \int_0^t \bar{g}(s, y^V)\,\mathrm{d}\bar{b}_s,$$

where $\bar{b} = {}^\circ\bar{B}$.

Proof. Recall from Section 1 that ${}^\circ\bar{B}$ is a Brownian motion on $\mathbf{\Omega}$. By transfer of standard theory ${}^*\bar{g}(\tau, Y^V)$ is adapted to $(\mathscr{A}_\tau)$; it is a lifting by Corollary 7.16(a). The rest follows by Anderson's Itô integration as discussed in Section 1. □

We are almost ready to show that z^V is a solution to (7.1) for the control $v = {}^\circ V$. From Lemmas 7.18 and 7.19 we have: for a.a. ω, for all t,

(a)
$$\begin{aligned} x_t^V &= {}^\circ X_t^V \\ &= x_0 + {}^\circ\left(\int_0^t {}^*f_V(\tau, Z^V)\,\mathrm{d}\tau + \int_0^t {}^*g_V(\tau, Z^V)\,\mathrm{d}B_\tau\right) \\ &= x_0 + \int_0^t f_v(s, z^V)\,\mathrm{d}s + {}^\circ\left(\int_0^t {}^*g_V(\tau, Z^V)\,\mathrm{d}B_\tau\right); \end{aligned} \tag{7.20}$$

(b)
$$\begin{aligned} y_t^V &= {}^\circ Y_t^V \\ &= y_0 + {}^\circ\left(\int_0^t {}^*\bar{f}_V(\tau, Z^V)\,\mathrm{d}\tau + \int_0^t {}^*\bar{g}(\tau, Y^V)\,\mathrm{d}\bar{B}_\tau\right) \\ &= y_0 + \int_0^t \bar{f}_v(s, z^V)\,\mathrm{d}s + \int_0^t \bar{g}(s, y^V)\,\mathrm{d}\bar{b}_s. \end{aligned}$$

From this it is clear that z^V solves (7.1) provided we can establish the following:

LEMMA 7.21. *There is a d-dimensional Brownian motion* $\tilde{b}$ *on* $\mathbf{\Omega}$, *independent of* $\bar{b}$, *such that a.s. we have*

$${}^\circ\int_0^t {}^*g_V(\tau, Z^V)\,\mathrm{d}B_\tau = \int_0^t g_v(s, z^V)\,\mathrm{d}\tilde{b}_s \quad \text{all } t,$$

Proof. Let

$$M_t = {}^\circ\int_0^t {}^*g_V(\tau, Z^V)\,\mathrm{d}B_\tau = {}^\circ\int_0^t G(\tau, \omega)\,\mathrm{d}B_\tau$$

say. From Lemma 7.7 and the linear growth condition on g, we see that $\|G(\cdot\,\omega)\|$ is finite for almost all ω, and $\bar{E}(\int_0^1 \|G_\tau\|^2\,\mathrm{d}\tau)$ is finite. Thus, setting $G^{(N)} = (G \vee -N) \wedge N$, for suitably small infinite N we have $G^{(N)}$ is $\mathcal{S}L^2$ (i.e., $\|G^{(N)}\|^2$ is $\mathcal{S}$-integrable) and, almost surely, $G_\tau^{(N)} = G_\tau$, all τ. Thus

$$M_t = {}^\circ\!\int_0^t G_\tau^{(N)}\,\mathrm{d}B_\tau \quad \text{all } t, \text{ a.s.}$$

By results of [16], since $G^{(N)}$ is $\mathcal{S}L^2$ then $\int G^{(N)}\,\mathrm{d}B$ is also $\mathcal{S}L^2$, so M is an L^2-martingale. Moreover,

$$\begin{aligned}
[M]_t &= {}^\circ\!\left[\int G^{(N)}\,\mathrm{d}B\right]_t = {}^\circ\!\int_0^t (G_\tau^{(N)})^2\,\mathrm{d}\tau \\
&= {}^\circ\!\int_0^t {}^*g_V^2(\tau, Z^V)\,\mathrm{d}\tau \quad \text{a.s.} \\
&= \int_0^t g_v^2(s, z^V)\,\mathrm{d}s \quad \text{by Lemma 7.18(c).}
\end{aligned}$$

Following [10], Theorem 5.3, p. 449, let

$$\tilde{b}_t = \int_0^t g_v^{-1}(s, z^V)\,\mathrm{d}M_s = \int_0^t \theta_s\,\mathrm{d}M_s, \quad \text{say;}$$

this exists because $E(\int_0^1 \theta_s\,\mathrm{d}[M]_s\theta_s') = I$. Then

$$[\tilde{b}]_t = \int_0^t \theta_s\,\mathrm{d}[M]_s\theta_s' = t \cdot I,$$

so that $\tilde{b}$ is a Brownian motion. Clearly $M_t = \int_0^t g_v(s, z^V)\,\mathrm{d}\tilde{b}_s$, as required. It is routine to see that $(\tilde{b}, \bar{b})$ is a Brownian motion, so that $\tilde{b}$ is independent of $\bar{b}$. □

From (7.20) and Lemma 7.21, we see that z^V is a solution to the Equations (7.1) for control v. As we have seen, this establishes that $\mu^v \approx \mu^V$, and the proof of Theorem 7.8 is complete. □

8. Cost and Optimal Partially Observed Stochastic Control

Let us now suppose that for the stochastic control system discussed in the previous section there is a cost $J(v)$ of implementation of each control $v \in \mathcal{V}$, given by

$$J(v) = E\left(\int_0^1 h(t, z^v, v(t, y^v))\,\mathrm{d}t + \bar{h}(z^v)\right),$$

where $z^v = (x^v, y^v)$ is a solution to Equations (7.1). We assume that $h, \bar{h} \geq 0$, and that h satisfies the same conditions (7.2) (a), (b), (e) as f; the function $\bar{h}$: $\mathcal{C}^n \to \mathbb{R}$ is measurable, with $\bar{h}(\cdot, \eta)$ continuous for each fixed $\eta \in \mathcal{C}^m$, and $|\bar{h}(\zeta)| \leq \kappa(1 + \|\zeta\|)$.

The continuity of the cost function $J(v)$ is established below, by the method used in the proof of continuity of μ^v. As a corollary we obtain optimal controls.

THEOREM 8.1. *The cost function $J(v)$ is continuous.*

Proof. Let $V \in {}^*\mathcal{V}$ and let $v = {}^\circ V$. We have to show that $J(V) \approx J(v)$.

Let $z^v = z^V = {}^\circ Z^V$ be the solution to the dynamics obtained in the proof of Theorem 7.8. The proof of Lemma 7.18 with h in place of f shows that

$$\int_0^1 {}^*h_V(\tau, Z^V)\,d\tau = \int_0^1 h_v(t, z^v)\,dt \quad \text{a.s.},$$

and similarly ${}^*\bar{h}(Z^V) \approx h(z^v)$ a.s. By Lemma 7.10 and the linear growth conditions, we have $\bar{E}(\int_0^1 {}^*h_V^2(\tau, Z^V)\,d\tau)$ and $\bar{E}({}^*\bar{h}^2(Z^V))$ are finite; so $\int_0^1 {}^*h_V(\tau, Z^V)\,d\tau$ and ${}^*\bar{h}(Z^V)$ are $\mathcal{S}$-integrable (by [16, I], Lemma 12).

Thus

$$\begin{aligned} J(V) &= \bar{E}\left(\int_0^1 {}^*h_V(\tau, Z^V)\,d\tau + {}^*\bar{h}(Z^V)\right) \\ &\approx E\left({}^\circ\!\int_0^1 {}^*h_V(\tau, Z^V)\,d\tau + {}^\circ({}^*\bar{h}(Z^V))\right) \\ &= E\left(\int_0^1 h_v(t, z^v)\,dt + h(z^v)\right) \quad \text{(by above)} \\ &= J(v), \end{aligned}$$

as required. □

Now define

$$J_0^{\mathcal{V}} = \inf_{v \in \mathcal{V}} (J(v)) \quad \text{and} \quad J_0^{\mathcal{U}} = \inf_{u \in \mathcal{U}} (J(u)).$$

The following corollary is a routine application of Theorem 8.1, together with the compactness of $\mathcal{V}$, and the density of $\mathcal{U}$ in $\mathcal{V}$.

COROLLARY 8.2. (a) *There is an optimal control $v \in \mathcal{V}$ achieving the minimum cost $J_0^{\mathcal{V}}$;*

(b) $J_0^{\mathcal{U}} = J_0^{\mathcal{V}}$.

REMARKS (1) Optimal ordinary controls can be obtained, as in the deterministic setting, given certain convexity conditions. For example, suppose that for each of the functions $\theta = f, \bar{f}, g^2, h$ we have the decomposition

$$\theta(t, x, y, a) = \theta_0(t, x, y) + \theta_1(t, x, y)\theta_2(t, a),$$

and for each t and the set $(f_2, \bar{f}_2, (g^2)_2, h_2)(t, M)$ is convex. Then standard measurable selection techniques will convert any relaxed control into an ordinary control giving the same trajectories; hence there is an optimal ordinary control.

(2) The methods of this section and Section 7 can be extended in various ways, mostly routine. These include the following.

(a) Let c be the initial point in Equations (7.1); then writing $\mu^{v,c}$ and $J(v, c)$ respectively for the measure on $\mathscr{C}^n$ and the cost for control v, starting at c, we have that $\mu^{v,c}$ and $J(v, c)$ are continuous in (v, c). Thus, for a compact region $D \subseteq \mathbb{R}^n$, there is an optimal pair (v, c) achieving the minimum cost

$$J_0^{\mathscr{V}}(D) = \inf_{c \in D, v \in \mathscr{V}} J(v, c).$$

(b) Vary the running time of the problem, in the following way. Let Γ be a closed region in $[0, 1] \times \mathbb{R}^n$, with $\{1\} \times \mathbb{R}^n \subseteq \Gamma$. Then define, for any solution $z^{v,c} = (x^{v,c}, y^{v,c})$ to (7.1) starting at c,

$$T^{v,c}(\omega) = \inf\{t : (t, z_t^{v,c}) \in \Gamma\}.$$

Now specify a cost for control v of the form

$$J(v, c) = E\left(\int_0^{T^{v,c}} h_v(t, z^{v,c})\, \mathrm{d}t + \bar{h}(T^{v,c}, z^{v,c})\right)$$

where $\bar{h} \geqslant 0$, $\bar{h}(\cdot, \cdot, \eta)$ is continuous for each fixed $\eta \in \mathscr{C}^m$, $\bar{h}(t, \zeta)$ depends only on $(\zeta_s)_{s \leqslant t}$, and $\bar{h}(\cdot, \zeta)$ is nondecreasing for each fixed ζ. (The function h is as before.)

In the framework of Section 7, suppose that $Z^{V,\gamma}$ is an internal solution for $V \in {}^*\mathscr{V}$ and starting point γ. Then ${}^\circ Z^{V,\gamma} = z^{v,c}$, say, is a solution for $v = {}^\circ V$ starting at $c = {}^\circ\gamma$. It is routine to see that $T^{v,c}(\omega) \leqslant {}^\circ T^{V,\gamma}(\omega)$, and so $J(v, c)$ is lower semi-continuous. As in the deterministic setting, this is sufficient to establish the existence of an optimal pair (v, c) for any compact region D.

(c) As in the deterministic setting, the time-horizon can be extended to ∞.

9. Markov Systems and their Control

In this section we discuss Markov controlled systems, showing that results similar to those of Sections 7 and 8 can be obtained in this case with the noise in the observation process also controlled. Thus it is not necessary to treat the state and observation process separately, and we consider systems of the form

$$\mathrm{d}x_t = f(t, x_t, u_t(r_t(x_t)))\, \mathrm{d}t + g(t, x_t, u_t(r_t(x_t)))\, \mathrm{d}b_t \quad (t \leqslant 1), \tag{9.1}$$

where $x_0 \in \mathbb{R}^d$ is fixed and b is a d-dimensional Brownian motion. The control at time t is a function of an instantaneous digital read-out $r_t(x_t)$; we have dropped the cumulation property, and the controls are truly Markov. Details of this system are given below.

Information. We assume a fixed *instantaneous read-out* $r: [0, 1] \times \mathbb{R}^d \to \mathbb{N}$ with r jointly measurable.

Controls. An ordinary *Markov control* is a measurable function $u: [0, 1] \times \mathbb{N} \to M$; a *relaxed Markov control* takes values in $\mathscr{M}(M)$.

Denote by $\mathcal{U}_M$ and $\mathcal{V}_M$ the Markov controls (ordinary and relaxed respectively).

Notice that a control u can be thought of as a sequence of deterministic controls $(u)_{i\in\mathbb{N}}$ given by $u_i(t) = u(t, i)$, so $\mathcal{U}_M = \mathcal{U}_0^{\mathbb{N}}$ and similarly for $\mathcal{V}_M$. A *Markov step control* is one with u_i a step control, all i. A *uniform* step control has uniform step sizes for each i; write $\mathcal{U}_M^s$ for the Markov step controls.

Conditions on coefficients. We begin by making the following assumptions on the coefficients.

$$\left.\begin{array}{ll}\text{(a)} & f, g \text{ are jointly measurable;}\\ \text{(b)} & f(t,\xi,\cdot) \text{ and } g(t,\xi,\cdot) \text{ are continuous, all } (t,\xi);\\ \text{(c)} & g \text{ is positive definite, symmetric;}\\ \text{(d)} & f, g, g^{-1} \text{ are bounded.}\end{array}\right\} \tag{9.2}$$

Notice that we have dropped the condition that g be Lipschitz in the space variable. The above conditions are sufficient to ensure the existence of solutions to Equation (9.1) for any given control (see, for example, Theorem 5.5 of [13]). We do not know of any uniqueness theorem for an equation like (9.1) (even if the Lipschitz condition on g is imposed) so we make the following further special assumption:

ASSUMPTION 9.3. The solutions to Equation (9.1) are unique in distribution.

At the end of this section we indicate a way to avoid making this assumption.

Cost. We assume a cost function J taking the form

$$J(u) = E\left(\int_0^1 h(t, x_t^u, u_t(r_t(x_t^u)))\,dt + \bar{h}(x^u)\right),$$

where h, $\bar{h}$ are nonnegative, bounded, h satisfies the same conditions as f, and $\bar{h}$ is continuous.

Since $\mathcal{U}_M = \mathcal{U}_0^{\mathbb{N}}$ and $\mathcal{V}_M = \mathcal{V}_0^{\mathbb{N}}$, the natural topology on Markov controls is the product of the $\mathcal{K}$-topology on deterministic controls. Then, from the results of Section 2 we have

THEOREM 9.4. (a) *$\mathcal{V}_M$ is compact, and for $V \in {}^*\mathcal{V}_M$ the standard part $v = {}^\circ V$ is given by $v_i = {}^\circ V_i$ for $i \in \mathbb{N}$.*

(b) *$\mathcal{U}_M^s$ is dense in $\mathcal{U}_M$.*

In preparation for the main theorem we make the following observation.

LEMMA 9.5. *Suppose that $n:[0,1]\to\mathbb{N}$ is measurable and $N: {}^*[0,1]\to{}^*\mathbb{N}$ is a lifting of n. Suppose further that $V \in {}^*\mathcal{V}_M$ with $v = {}^\circ V$, and $\theta \in \mathcal{K}$ with uniform lifting Θ. Then*

$$\int_0^1 \Theta(\tau, V(\tau, N_\tau))\,d\tau \approx \int_0^1 \theta(t, v(t, n_t))\,dt.$$

Proof. Let $A_i = n^{-1}\{i\}$ and $B_i = N^{-1}\{i\}$. Since N lifts n, the set $B_i \Delta st^{-1}(A_i)$ is null for each finite i, and $\cup_{i\in\mathbb{N}} B_i$ has measure 1 (both with respect to ${}^*m_L$). So

$$\int_0^1 \Theta(\tau, V(\tau, N_\tau))\,d\tau \approx \sum_{i\in\mathbb{N}} \int_{B_i} \Theta(\tau, V_i(\tau))\,d\tau$$
$$\approx \sum_{i\in\mathbb{N}} \int_{A_i} \theta(t, v_i(t))\,dt \quad \text{(by Lemma 2.6)}$$
$$= \int_0^1 \theta(t, v(t, n_t))\,dt. \qquad \square$$

We shall also need the following important theorem of Krylov [15].

THEOREM 9.6. *For each real number $k > 0$ there is a number $l > 0$ with the following property. Suppose that z_t is a d-dimensional process of the form*

$$z_t = z_0 + \int_0^t \phi_s\,ds + \int_0^t \psi_s\,db_s,$$

where $z_0 \in \mathbb{R}^d$, the functions ϕ, ψ are progressively measurable, and b is a d-dimensional Brownian motion living on an adapted space. If $\|\phi\|$, $\|\psi\|$ and $|\det \psi^{-1}|$ are uniformly bounded by k, and $h: [0, 1]\times\mathbb{R}^d \to \mathbb{R}$ is L^{d+1} and non-negative, then

$$E\left(\int_0^1 h(t, z_t)\,dt\right) \leq l\|h\|_{d+1}.$$

Now we move to the counterpart of Theorem 7.8.

THEOREM 9.7. *Let μ^v be the measure induced on $\mathscr{C}^d$ by solutions to* (9.1) *for control v. Then μ^v is continuous in v.*

Proof. We will mention in detail only those features in the proof that are different from those of Theorem 7.8.

Let $V \in {}^*\mathscr{V}_M$; let X^V be an internal solution to Equation *(9.1):

$$dX_\tau^V = {}^*f(\tau, X_\tau^V, V_\tau({}^*r_\tau(X_\tau^V)))\,d\tau + {}^*g(\tau, X_\tau^V, V_t({}^*r_\tau(X_\tau^V)))\,dB_\tau,$$

with $X_0^V = x_0$. From X^V construct the standard continuous process $x^V = {}^\circ X^V$, living on a filtered Loeb space $\mathbf{\Omega}$. The aim is to show that x^V is a solution to (9.1) for control $v = {}^\circ V$; as before this suffices to prove the theorem. $\square$

LEMMA 9.8. *For a.a. ω, ${}^*f(\tau, X_\tau^V, \alpha)$ is a uniform lifting of $f(t, x_t^V, a)$, and similarly for g.*

Proof. By Anderson's Lusin Theorem

$${}^*f(\tau, Y, \alpha) \approx f({}^\circ\tau, {}^\circ Y, {}^\circ\alpha), \quad \text{all } \alpha,$$

for a.a. $(\tau, Y) \in {}^*[0, 1]\times \text{ns}({}^*\mathbb{R}^d)$. Thus, we can obtain for each finite n an internal set $A_n \subseteq {}^*[0, 1]\times{}^*\mathbb{R}^d$ with ${}^*m(A_n) \leq 1/n$ such that for all $(\tau, Y) \in {}^*[0, 1]\times \text{ns}({}^*\mathbb{R}^d)\backslash A_n$ we have

$$^*f(\tau, Y, \alpha) \approx f(^\circ\tau, ^\circ Y, ^\circ\alpha), \quad \text{all } \alpha.$$

Now apply the transfer of Krylov's theorem to the process X^V, with k given by the condition 9.2(d). Taking $h = \chi_{A_n}$ (the characteristic function of A_n) we see that

$$(\nu \times {}^*m)(\{(\omega, \tau): (\tau, X_\tau^V(\omega)) \in A_n\}) \leq ln^{-1/d+1}.$$

Hence, for a.a. (ω, τ) with respect to $(\nu \times {}^*m)_L$

$$(\tau, X_\tau^V(\omega)) \notin \cap A_n.$$

This means that for a.a. (ω, τ)

$$^*f(\tau, X_\tau^V(\omega), \alpha) \approx f(^\circ\tau, ^\circ X_\tau^V(\omega), ^\circ\alpha), \quad \text{all } \alpha.$$

An application of Keisler's Fubini Theorem now establishes the lemma. □

In a similar way we obtain:

LEMMA 9.9. *For a.a.* ω

$$^*r_\tau(X_\tau^V) = r_{^\circ\tau}(x_{^\circ\tau}^V), \quad \textit{for a.a. } \tau.$$

Proof. By Anderson's Lusin Theorem

$$^*r_\tau(Y) = r_{^\circ\tau}(^\circ Y), \quad \text{a.a. } (\tau, Y) \in {}^*[0, 1] \times \text{ns}(^*\mathbb{R}^d).$$

Now proceed as in the proof of Lemma 9.8. □

Combining these results with Lemma 9.5 we have the counterpart of Lemma 7.18.

LEMMA 9.10. *For a.a.* ω, *for all* t,

(a) $$\int_0^t {}^*f(\tau, X_\tau^V, V_\tau(^*r_\tau(X_\tau^V)))\,d\tau \approx \int_0^t f(s, x_s^V, v_s(r_s(x_s^V)))\,ds,$$

(b) $$\int_0^t {}^*g^2(\tau, X_\tau^V, V_\tau(^*r_\tau(X_\tau^V)))\,d\tau \approx \int_0^t g^2(s, x_s^V, v_s(r_s(x_s^V)))\,ds.$$

Proof. Writing $n_t(\omega) = r_t(x_t^V)$ and $N_\tau(\omega) = {}^*r_\tau(X_\tau^V)$, Lemma 9.9 tells us that for almost all ω, $N_\tau(\omega)$ is a lifting of $n_t(\omega)$. Apply this in Lemma 9.5 with the uniform liftings given by Lemma 9.8, and the result follows. □

The proof that x^V is a solution to Equations (9.1) for control v is now completed in the same way as the proof of Theorem 7.8, using the technique of Lemma 7.21. This concludes the proof of Theorem 9.7. □

The following results for the system (9.1) are established using the ideas in the proof of Theorem 9.7 (just as in Section 8 we called on the methods developed in the proof of Theorem 7.8).

THEOREM 9.11. *The cost function* $J(v)$ *is continuous.*

COROLLARY 9.12. (a) *There is an optimal control* $v \in \mathcal{V}_M$; (b) $J_0^{\mathcal{U}_M} = J_0^{\mathcal{V}_M}$.

Omitting the uniqueness assumption. If we omit the special assumption '9.3', then Equations (9.1) will in general have a *set* of solutions $\mathcal{X}^v$ for each control v, giving rise to a *set* $\mathcal{M}^v$ of probability measures on $\mathcal{C}^d$, and a set $\mathcal{J}^v$ of corresponding costs. If we write μ^x for the measure corresponding to a particular solution x, and let $J(x)$ denote the cost, then $\mathcal{M}^v = \{\mu^x : x \in \mathcal{X}^v\}$ and $\mathcal{J}^v = \{J(x) : x \in \mathcal{X}^v\}$. We also write

$$\mathcal{M}^{\mathcal{V}_M} = \bigcup_{v \in \mathcal{V}_M} \mathcal{M}^v \quad \text{and} \quad \mathcal{J}^{\mathcal{V}_M} = \bigcup_{v \in \mathcal{V}_M} \mathcal{J}^v.$$

It is natural now to define the cost $J(v)$ of a control v as follows.

DEFINITION 9.13. (a) $J(v) = \inf \mathcal{J}^v$ $(= \inf\{J(x) : x \in \mathcal{X}^v\})$; (b) $J_0 = \inf \mathcal{J}^{\mathcal{V}_M}$.

We shall see that there is an optimal control with optimal trajectory. The fundamental result from which everything else follows is stated below.

THEOREM 9.14. *Suppose that* $V \in {}^*\mathcal{V}_M$ *with internal solution* X *to Equation* $^*(9.1)$. *Let* $v = {}^\circ V$ *and* $x = {}^\circ X$. *Then* x *is a solution for control* v; *and* $\mu^x = {}^\circ\mu^X$ *and* $J(x) = {}^\circ J(X)$.

Proof. This is exactly what was established in the proofs of Theorems 9.7 and 9.11. □

We now give a series of elementary applications of this theorem.

THEOREM 9.15. (a) *Let* $v \in \mathcal{V}_M$. *The sets* $\mathcal{M}^v$ *and* $\mathcal{J}^v$ *are compact*; (b) *the sets* $\mathcal{M}^{\mathcal{V}_M}$ *and* $\mathcal{J}^{\mathcal{V}_M}$ *are compact*.

Proof. (a) Let $\mu^X \in {}^*\mathcal{M}^v$; so X is a solution for control $^*v$. Let $x = {}^\circ X$. By Theorem 9.14, ${}^\circ\mu^X = \mu^x \in \mathcal{M}^v$ (since ${}^\circ({}^*v) = v$). Thus $\mathcal{M}^v$ is compact, by the nonstandard criterion for compactness. The other parts of the theorem are proved similarly. □

COROLLARY 9.16. (a) *For any control* v *there is a solution* $x \in \mathcal{X}^v$ *with* $J(x) = J(v)$; (b) *There is an optimal control* v *with optimal trajectory* x *such that* $J(x) = J(v) = J_0$.

Since each control v now gives a *set* of measures and costs, the notion of continuity in v is replaced by *upper semi-continuity*. Recall the definition (as used by Choquet and others).

DEFINITION 9.17. Let A, B be topological spaces and let F: $A \to \mathcal{P}(B)$ be a set valued mapping. Then F is *upper-semi-continuous* if $\{x : F(x) \subseteq G\}$ is open in A whenever G is open in B.

It is easy to establish the following nonstandard criterion for upper-semi-continuity (which we abbreviate u.s.c).

THEOREM 9.18. *F is u.s.c iff whenever $x \approx a$ then*

$$F(x) \subseteq \text{monad}(F(a)), \quad \textit{for } a \in A, x \in {}^*A.$$

(*Here* $\text{monad}(S) = \bigcap\{{}^*G : S \subseteq G, G \text{ open}\}$ *for any* $S \subseteq B$.) □

Now we have

THEOREM 9.19. *The functions $\mathcal{M}^v$ and $\mathcal{J}^v$ are u.s.c. as functions of the control v.*

Proof. Let $V \approx v \in \mathcal{V}_{\mathrm{M}}$, and let $\mu \in \mathcal{M}^V$. By Theorem 9.14, ${}^\circ\mu \in \mathcal{M}^v$; so $\mu \in \text{monad}(\mathcal{M}^v)$. Hence, $\mathcal{M}^V \subseteq \text{monad}(\mathcal{M}^v)$, and $\mathcal{M}^v$ is u.s.c. The proof for $\mathcal{J}^v$ is similar. □

REMARK. Theorem 9.14 actually tells us that if $V \approx v$ then $\mathcal{M}^V \subseteq st^{-1}(\mathcal{M}^v)$; in the proof above we then used the fact that $st^{-1}(\mathcal{M}^v) \subseteq \text{monad}(\mathcal{M}^v)$. In our situation, $\mathcal{M}^v$ is compact, so $st^{-1}(\mathcal{M}^v) = \text{monad}(\mathcal{M}^v)$; similar remarks apply to $\mathcal{J}^v$.

Finally we see that

THEOREM 9.20. *$J(v)$ is lower semi-continuous.*

Proof. As remarked above, if $V \approx v$ then $\mathcal{J}^V \subseteq st^{-1}(\mathcal{J}^v)$ so $st(\mathcal{J}^V) \subseteq \mathcal{J}^v$. Thus

$$\begin{aligned} {}^\circ J(V) &= {}^\circ\inf \mathcal{J}^V \\ &= \inf(st(\mathcal{J}^V)) \\ &\leq \inf \mathcal{J}^v \\ &= J(v). \end{aligned}$$

□

Existence of an optimal control could be alternatively deduced from this last result.

REMARKS (1) The remarks at the end of Section 8 apply also to the Markov situation discussed here.

(2) Boundedness on coefficients could be relaxed in favour of linear growth conditions, using simple truncation arguments.

(3) It is actually only necessary in (9.1) to assume that the diffusion part is Markovian; this is handled in Theorem 9.7 by first considering a solution to

$$\mathrm{d}Z_\tau = {}^*g(\tau, Z_\tau, V_\tau({}^*r_\tau(X_\tau)))\,\mathrm{d}B_\tau$$

and applying the Krylov Theorem. The equation with drift is then handled using the Girsanov formula as in Section 7 (Lemmas 7.14–7.16).

(4) In [9], Section 6, the author announced results similar to the above for Markov-type systems in which the read-out was cumulative, based on instantaneous observations made at a fixed finite number of times. Unfortunately, an error in the proof has been discovered. The proof required the existence of densities at *every* fixed t, and this does not seem to follow from the results of Stroock and Varadhan [20] as had been supposed. However, in a private

communication, T. L. Lindstrøm has shown how to use Krylov's inequality to obtain pointwise densities, and thus it seems that the result of [9], Section 6, is valid. Lindstrøm discusses Markov-type systems in the forthcoming book [1].

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