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Compatibility conditions for a left Cauchy-Green strain field

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Abstract. This paper deals chiefly with various issues pertaining to the existence and uniqueness of a finite deformation that gives rise to a prescribed right or left Cauchy-Green strain-tensor field.

Following a review and discussion of available existence and uniqueness theorems appropriate to a pre-assigned *right* strain field, the extent of uniqueness of a generating deformation is established under minimal smoothness and invertibility assumptions.

The remainder of the paper is devoted to the more involved corresponding existence and uniqueness questions for a given *left* strain-tensor field. These questions are first discussed in a three-dimensional setting and are then resolved for the special class of plane deformations. The results thus obtained stand in marked contrast to their counterparts for a given right strain field.

Introduction

The main aim of this paper is to deal with various questions concerning the existence and uniqueness of a finite deformation that generates a given right or left Cauchy-Green strain-tensor field. These issues are nonlinear analogues of the corresponding questions regarding the existence and uniqueness of a displacement field that gives rise to a pre-assigned infinitesimal strain-tensor field. The familiar results pertaining to infinitesimal strains are of great importance within the linearized theory of elasticity. In particular, the compatibility conditions of the linear theory enable one to formulate the second boundary-value problem of elastostatics in terms of stresses alone. In the special case of plane strain or generalized plane stress for a homogeneous and isotropic elastic solid, this problem is further reducible to a boundary-value problem for the biharmonic equation through the introduction of the Airy stress function. A partial motivation for the present study stems from the possible usefulness of compatibility conditions for finite strain fields in connection with problems involving large deformations of continuous media.

In Section 1, following some relevant preliminaries on finite deformations, we prove the equivalence of length- and distance-preserving deformations on the assumption that the mappings involved are but once continuously differentiable and merely locally invertible. These results are then used to show that a smooth, locally invertible deformation is homogeneous if its right or left strain field is constant.

In Section 2, we recall some known results pertaining to the existence issue for a prescribed *right* strain-tensor field. In this connection, the necessity and sufficiency of the compatibility conditions are discussed in some detail. Further, the extent of uniqueness of a deformation generating a given right strain field is reestablished under minimal smoothness and invertibility assumptions.

In Section 3 the existence and uniqueness questions appropriate to a given *left* strain-tensor field are raised in a three-dimensional setting and certain difficulties attending their resolution are discussed.

The *uniqueness* question posed in Section 3 is dealt with in Section 4 for the special class of plane deformations. This section contains two theorems that establish a relation between plane deformations generating the same left strain field on a simply connected domain. According to these theorems, every deformation producing a given left strain field may be represented in terms of a single such deformation by means of quadratures.

Finally, in Section 5, necessary and sufficient conditions for the *existence* of a plane deformation giving rise to a prescribed left strain field are deduced. If these compatibility conditions hold on a simply connected domain, and barring certain troublesome degeneracies, a generating deformation is shown to admit an explicit integeral representation in terms of the pre-assigned strain field. Furthermore, in the presence of such degeneracies, the problem of constructing a deformation producing a given compatible left strain field is reduced to that of finding an analytic solution of an ordinary linear second-order differential equation in the complex plane. The existence of such a solution is demonstrated.

1. Notation, preliminaries on finite deformations

The symbols E_2 and E_3 are used throughout for Euclidean point spaces of two and three dimensions or – depending on the context – for the associated linear vector spaces. The letter R always denotes a closed region (in either E_2 or E_3) and $\mathring{R}$ is the interior of R; in contrast, D stands for an open region (domain). Lower-case and capital letters in boldface – unless otherwise qualified – designate vectors and (second-order) tensors, respectively. We call L_+ the collection of all tensors with a positive determinant, O_+ the set of all proper-orthogonal tensors, and S^+ the set of all symmetric, positive-definite tensors. Thus,

$$\mathbf{Q} \in O_+ \quad \text{if } \mathbf{Q}\mathbf{Q}^T = \mathbf{1}, \quad \det \mathbf{Q} = 1, \tag{1.1}$$[1]

$$\mathbf{L} \in S^+ \quad \text{if } \mathbf{L} = \mathbf{L}^T \text{ and } \mathbf{v} \cdot \mathbf{L}\mathbf{v} > 0 \quad \text{for every } \mathbf{v} \neq \mathbf{0}. \tag{1.2}$$

The notation $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is used to refer to a rectangular Cartesian coordinate frame for E_3, with origin O and orthonormal base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. If $\mathbf{v}$ is a vector and $\mathbf{L}$ is a tensor in there dimensions, we write v_i and L_{ij} for the components of $\mathbf{v}$ and $\mathbf{L}$ in the underlying coordinate frame; Latin subscripts always having the range $(1, 2, 3)$. Further, summation over repeated indices is taken for granted, and subscripts preceded by a comma indicate partial differentiation with respect to the corresponding Cartesian coordinate. Strictly analogous notation is employed in two dimensions; in this case Greek subscripts with the range $(1, 2)$ are used.

If A and B are sets, we write $f: A \to B$, if f is a function defined on A with values in B, while $f(A)$ stands for the range of f. In particular, if A is a region (open, closed, or neither) in E_2 or E_3, we write $f \in C(A)$ if f is continuous on A, and $f \in C^N(A)$ $(N = 1, 2, \ldots)$ in the event that f is N times continuously differentiable on A. We say that f is *smooth*, provided $f \in C^1(A)$, and write $f \in C^\infty(A)$ to convey that f has continuous derivatives of all orders on A.

We now recall certain prerequisites from the theory of finite deformations that will be needed in what follows. For this purpose, let R be a closed region[2] in E_3. By a *deformation*, we mean a mapping $\hat{\mathbf{y}}: R \to E_3$ described by

$$\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \quad \text{on } R, \tag{1.3}$$

where $\mathbf{x}$ is the position vector of a generic point in R, $\hat{\mathbf{y}}(\mathbf{x})$ is its deformation image in $\overset{*}{R} = \hat{\mathbf{y}}(R)$, and $\mathbf{u}$ is the associated displacement field. Such a mapping will be called a *regular deformation* if

$$\hat{\mathbf{y}} \in C^1(R), \quad J = \det \mathbf{F} > 0, \quad \mathbf{F} = \nabla \hat{y} \quad \text{on } R. \tag{1.4}$$

Here J is the Jacobian determinant of $\hat{\mathbf{y}}$, while $\mathbf{F}$ is its deformation-gradient tensor. Note that a regular deformation need not be globally one-to-one. Although global invertibility of deformations is an essential requirement in continuum mechanics, this restriction will be avoided in dealing with the purely kinematical issues considered in this investigation.

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(1) The superscript T indicates transposition; $\mathbf{1}$ is the identity tensor.

(2) Actually, in all of the definitions and results cited in the present section, R can equally well be replaced by an open region D.

The letters **C** and **G** stand for the right and left Cauchy-Green strain-tensor fields[1] of a regular deformation, whereas **E** denotes its Lagrangian strain-tensor (Green-St. Venant strain-tensor):

$$\mathbf{C}=\mathbf{F}^T\mathbf{F}, \quad \mathbf{G}=\mathbf{F}\mathbf{F}^T, \quad \mathbf{E}=\tfrac{1}{2}(\mathbf{C}-\mathbf{1}) \quad \text{on } R. \tag{1.5}$$

Both **C** and **G** are symmetric, positive-definite tensor fields with common (positive) principal values. According to the polar decomposition theorem, $\mathbf{F}\in L_+$ admits the unique right and left polar resolutions:

$$\mathbf{F}=\mathbf{Q}\mathbf{U}=\mathbf{V}\mathbf{Q}, \quad \mathbf{Q}\in O_+, \quad \mathbf{U}\in S^+, \quad \mathbf{V}\in S^+ \quad \text{on } R; \tag{1.6}$$

furthermore

$$\mathbf{U}=\sqrt{\mathbf{C}}, \quad \mathbf{V}=\sqrt{\mathbf{G}}, \quad \mathbf{Q}=\mathbf{F}\mathbf{U}^{-1} \quad \text{on } R. \qquad (1.7)^{(2)}$$

The tensors **U** and **V**, which are usually called the right and left stretch tensors of the deformation at hand, are thus the unique positive definite square roots of **C** and **G**.

A regular deformation $\hat{\mathbf{y}}: R \to E_3$ is *homogeneous* if its deformation-gradient field is constant. Such a deformation therefore admits the representation:

$$\hat{\mathbf{y}}(\mathbf{x})=\mathbf{F}\mathbf{x}+\mathbf{d} \quad \text{on } R, \quad \mathbf{F}\in L_+; \quad \mathbf{F}, \mathbf{d} \text{ constant.} \tag{1.8}$$

A mapping of this form is a *pure homogeneous deformation* in the event that $\mathbf{F}\in S^+$ and $\mathbf{d}=\mathbf{0}$.

A *rigid deformation* is a distance-preserving regular deformation. We cite the following familiar result concerning rigid deformations.[3]

THEOREM 1.1: *A regular deformation* $\hat{\mathbf{y}}: R \to E_3$ *is rigid if and only if it admits the representation:*

$$\hat{\mathbf{y}}(\mathbf{x})=\mathbf{Q}\mathbf{x}+\mathbf{d} \quad \text{on } R, \quad \mathbf{Q}\in O_+; \quad \mathbf{Q}, \mathbf{d} \text{ constant.} \tag{1.9}$$

Actually, every distance preserving deformation – in the absence of any regularity assumptions – can be shown to admit the representation (1.9) with **Q** merely orthogonal, rather than proper-orthogonal. In view of the polar

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(1) These tensor fields are also commonly referred to as the right and left Cauchy-Green *deformation*-tensor fields.

(2) If **L** is a nonsingular tensor, $\mathbf{L}^{-1}$ stands for its inverse.

(3) See, for example, Gurtin [1], Art. 6.

decompositions (1.6), every homogeneous deformation can be uniquely resolved into a pure homogeneous deformation followed or preceded by a rigid deformation.

A regular deformation $\hat{\mathbf{y}}: R \to E_3$ is said to be *isometric* if the length of every smooth arc in R equals the length of its deformation image. Following Gurtin [1] (p. 49), we now recall certain alternative characterizations of rigid deformations.(1)

THEOREM 1.2: *Let $\hat{\mathbf{y}}: R \to E_3$ be a regular deformation. Then the following are equivalent:*

(*a*) $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{1}$ *on* R, *where* $\mathbf{F} = \nabla\hat{\mathbf{y}}$;
(*b*) $\hat{\mathbf{y}}$ *is isometric;*
(*c*) $\hat{\mathbf{y}}$ *is rigid.*

Proof: What follows is a minor variant of the proof given in [1].(2) We show first that (a) necessitates (b). For this purpose, let $\Lambda \subset R$ be a smooth curve and let s denote the arc length along Λ. Thus Λ admits the parameterization

$$\Lambda: \mathbf{x} = \bar{\mathbf{x}}(s) \quad (0 \leqslant s \leqslant l), \quad \bar{\mathbf{x}} \in C^1([0, l]), \tag{1.10}$$

in which l is the length of Λ. Let $\overset{*}{\Lambda} = \hat{\mathbf{y}}(\Lambda)$, so that

$$\overset{*}{\Lambda}: \mathbf{y} = \bar{\mathbf{y}}(s) \equiv \equiv \hat{\mathbf{y}}(\bar{\mathbf{x}}(s)) \quad (0 \leqslant s \leqslant 1), \quad \hat{\mathbf{y}} \in C^1([0, l]), \tag{1.11}$$

From (1.11), the chain rule, and hypothesis (a) follows

$$|\dot{\bar{y}}(s)| = |\mathbf{F}(\bar{\mathbf{x}}(s))\dot{\bar{\mathbf{x}}}(s)| = |\dot{\bar{\mathbf{x}}}(s)| = 1 \quad (0 \leqslant s \leqslant l). \tag{1.12}$$ (3)

Accordingly, if $\overset{*}{s}(s)$ is the arc length along $\overset{*}{\Lambda}$, measured from $\bar{\mathbf{y}}(0)$, one has

$$\overset{*}{s}(s) = \int_0^s |\dot{\bar{\mathbf{y}}}(\sigma)|\, d\sigma = \int_0^s d\sigma = s \quad (0 \leqslant s \leqslant l). \tag{1.13}$$

Hence, $\hat{\mathbf{y}}$ is isometric.

Next, we verify that (b) implies (c). Let $\mathring{\mathbf{x}}$ be an interior point of R. In view of the regularity of $\hat{\mathbf{y}}$, the inverse-function theorem assures the existence of an

(1) The theorem stated in [1] combines Theorem 1.1 with the result cited below.
(2) Note that although the restriction to globally invertible deformations is introduced in [1], it is not essential to the argument presented there.
(3) Here and in the sequel a superior dot indicates differentiation.

open sphere $N \subset R$, centered at $\mathring{\mathbf{x}}$, on which $\hat{\mathbf{y}}$ is smoothly invertible. Let $\overset{*}{N} = \hat{\mathbf{y}}(N)$ and call $\hat{\mathbf{y}}^{-1}$, defined on $\overset{*}{N}$, the inverse of $\hat{\mathbf{y}}$.

To see that $\hat{\mathbf{y}}$ is rigid on N, let Λ be a straight-line segment lying in N. It evidently suffices to show that $\overset{*}{\Lambda} = \hat{\mathbf{y}}(\Lambda)$ is also a straight-line segment. Suppose this were not the case and let $\overset{*}{\Lambda}' \subset \overset{*}{N}$ be a polygonal approximation to $\overset{*}{\Lambda}$, such that $\overset{*}{\Lambda}'$ and $\overset{*}{\Lambda}$ have common end points and the vertices of $\overset{*}{\Lambda}'$ lie on $\overset{*}{\Lambda}$. The length of $\overset{*}{\Lambda}'$ is thus smaller than that of $\overset{*}{\Lambda}$ and hence $\Lambda' \equiv \hat{\mathbf{y}}^{-1}(\overset{*}{\Lambda}')$ is shorter than the stright-line segment Λ. This, however, is impossible since Λ and Λ' have the same end points. Consequently, $\hat{\mathbf{y}}$ is indeed rigid on N. The rigidity of $\hat{\mathbf{y}}$ on R follows from the rigidity of $\hat{\mathbf{y}}$ in a neighborhood of every interior point on R, Theorem 1.1, and the continuity of $\hat{\mathbf{y}}$ on R.

To complete the proof of the equivalence of (a), (b), and (c), one invokes Theorem 1.1 to see that (c) implies (a). As is at once clear and is observed in [1], $\mathbf{C}$ in (a) may be replaced by $\mathbf{G}$, $\mathbf{U}$, or $\mathbf{V}$.

Our next objective is to show that a regular deformation $\hat{\mathbf{y}}: R \to E_3$ is homogeneous if its right or left strain-tensor field is constant. If $\hat{\mathbf{y}}$ is *twice* continuously differentiable, rather than merely regular, this conclusion – as far as the constancy of the *right* strain-tensor is concerned – follows at once from a well-known relation[(1)] between the second gradient of $\hat{\mathbf{y}}$ and the first gradients of $\hat{\mathbf{y}}$ and $\mathbf{C}$. The theorem below avoids the foregoing additional smoothness hypothesis.

THEOREM 1.3:

(*a*) *Let* $\hat{\mathbf{y}}: R \to E_3$ *be a homogeneous deformation. Then* $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ *and* $\mathbf{G} = \mathbf{F}\mathbf{F}^T$ *are constant on* R.

(*b*) *Suppose* $\hat{\mathbf{y}}: R \to E_3$ *is a regular deformation and either* $\mathbf{C}$ *or* $\mathbf{G}$ *is constant on* R. *Then* $\hat{\mathbf{y}}$ *is a homogeneous deformation.*

Proof: The truth of (a) is immediate from (1.8). With a view towards proving (b), assume first that $\mathbf{C}$ is constant on R and consider the pure homogeneous deformation given by

$$\hat{\mathbf{y}}'(\mathbf{x}) = \mathbf{U}\mathbf{x} \quad \text{on } R, \quad \mathbf{U} = \sqrt{\mathbf{C}}. \tag{1.14}$$

Let $\overset{*}{R} = \hat{\mathbf{y}}(R)$, $\overset{*}{R}' = \hat{\mathbf{y}}'(R)$ and define a deformation $\hat{\mathbf{z}}: \overset{*}{R}' \to E_3$ through

$$\hat{\mathbf{z}}(\mathbf{y}') = \hat{\mathbf{y}}(\mathbf{U}^{-1}\mathbf{y}') \quad \text{for all } \mathbf{y}' \in \overset{*}{R}'. \tag{1.15}$$

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(1) See equation (2.10), which seems to have its origin in a classical paper by Christoffel [2].

Clearly, $\hat{\mathbf{z}}$ is regular and $\hat{\mathbf{z}}(\overset{*}{R}') = \overset{*}{R}$. Moreover

$$(\nabla\hat{\mathbf{z}})^T(\nabla\hat{\mathbf{z}}) = \mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-1} = \mathbf{1} \quad \text{on } \overset{*}{R}'. \tag{1.16}$$

By (1.16) and Theorem 1.2, $\hat{\mathbf{z}}$ is rigid on $\overset{*}{R}$. Hence, Theorem 1.1, in conjunction with (1.14), (1.15), assures the existence of a tensor $\mathbf{Q}$ and a vector $\mathbf{d}$, such that

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{QUx} + \mathbf{d} \quad \text{on } R, \quad \mathbf{Q} \in O_+, \quad \mathbf{U} = \sqrt{\mathbf{C}}, \tag{1.17}$$

whence $\hat{\mathbf{y}}$ is a homogeneous deformation.

Finally, suppose $\mathbf{G}$ is constant on R and let $\hat{\mathbf{z}}: R \to E_3$ be given by

$$\hat{\mathbf{z}}(\mathbf{x}) = \mathbf{V}^{-1}\hat{\mathbf{y}}(\mathbf{x}) \quad \text{on } R, \quad \mathbf{V} = \sqrt{\mathbf{G}}. \tag{1.18}$$

Accordingly, $\hat{\mathbf{z}}$ is a regular deformation and

$$\nabla\hat{\mathbf{z}} = \mathbf{V}^{-1}\mathbf{F} \quad \text{on } R. \tag{1.19}$$

This identity, together with the second of (1.18) yields

$$(\nabla\hat{\mathbf{z}})^T(\nabla\hat{\mathbf{z}}) = \mathbf{F}^T\mathbf{V}^{-1}\mathbf{V}^{-1}\mathbf{F} = \mathbf{F}^T\mathbf{G}^{-1}\mathbf{F} = \mathbf{1} \quad \text{on } R. \tag{1.20}$$

Theorem 1.2 now enables us to infer that $\hat{\mathbf{z}}$ is rigid on R and is therefore of the form (1.9). Hence (1.18) furnishes

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{VQx} + \mathbf{Vd} \quad \text{on } R, \quad \mathbf{Q} \in O_+, \quad \mathbf{V} = \sqrt{\mathbf{G}}, \tag{1.21}$$

and thus $\hat{\mathbf{y}}$ is a homogeneous deformation.

Note that Theorem 1.2, which was used in proving part (b) of Theorem 1.3, in turn follows from the latter theorem.

We turn next to the special class of plane deformations. To this end, let R be a cylindrical region of height h, with the closed cross section Π, and let $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a coordinate frame relative to which R admits the representation

$$R = \{\mathbf{x} \mid (x_1, x_2) \in \Pi, \quad -h/2 \leqslant x_3 \leqslant h/2\}. \tag{1.22}$$

We call $\hat{\mathbf{y}}: R \to E_3$ a *plane deformation* (parallel to the plane $x_3 = 0$) provided

$$y_\alpha = \hat{y}_\alpha(x_1, x_2), \quad y_3 = x_3 \quad \text{for all } (x_1, x_2) \in \Pi; \quad -h/2 \leqslant x_3 \leqslant h/2. \tag{1.23}$$

Such a mapping is a *regular plane deformation* if $\hat{\mathbf{y}}$ additionally obeys (1.4), which in this instance yields

$$[F_{ij}] = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_{\alpha\beta} = \hat{y}_{\alpha,\beta}, \quad J = \det[F_{\alpha\beta}] > 0 \quad \text{on } \Pi. \qquad (1.24)^{(1)}$$

In view of the above, $\hat{\mathbf{y}}: R \to E_3$ is completely characterized by the subsidiary two-dimensional mapping $\hat{y}_\alpha: \Pi \to E_2$. Writing R in place of Π, we shall conveniently speak of the "plane deformation $\hat{\mathbf{y}}: R \to E_2$," with the understanding that R now refers to a closed region in E_2. In this context (1.3), (1.4), (1.5) are to be interpreted as appropriate two-dimensional statements: $\mathbf{x}$ at present is the position vector of a point in R with coordinates x_α, while $\hat{\mathbf{y}}$, $\mathbf{u}$, and $\mathbf{F}$, $\mathbf{C}$, $\mathbf{G}$, $\mathbf{E}$ are the vector and tensor fields with the components $\hat{y}_\alpha$, u_α and $F_{\alpha\beta}$, $C_{\alpha\beta}$, $G_{\alpha\beta}$, $E_{\alpha\beta}$. Further,

$$F_{\alpha\beta} = \hat{y}_{\alpha,\beta}, \quad C_{\alpha\beta} = F_{\gamma\alpha}F_{\gamma\beta}, \quad G_{\alpha\beta} = F_{\alpha\gamma}F_{\beta\gamma}, \quad E_{\alpha\beta} = \tfrac{1}{2}(C_{\alpha\beta} - \delta_{\alpha\beta}). \qquad (1.25)^{(2)}$$

Clearly, all of the definitions and results cited earlier in this section have strict counterparts for plane deformations.

2. The main compatibility issue

In the present section we discuss the questions of the existence and uniqueness of a regular deformation generating a given right strain-tensor field. The first theorem we cite deals with the uniqueness issue and asserts that two regular deformations give rise to the same right strain field if and only if one is obtainable from the other through a rigid deformation. This proposition was stated without proof by E. Cosserat and F. Cosserat [3]; proofs can be found in treatises on differential geometry,(3) where the mathematically identical issue arises in connection with the uniqueness of a curvilinear coordinate system that corresponds to a given metric tensor. The usual proofs demand that the deformations involved be twice continuously differentiable and globally invertible. The proof outlined below employs essentially the argument used by Shield [5], but requires merely that the desired deformation be regular.

(1) Here $[F_{ij}]$ and $[F_{\alpha\beta}]$ denote the appropriate component matrices on $\mathbf{F}$ in the frame $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(2) We use $\delta_{\alpha\beta}$ and δ_{ij} to designate the Kronecker delta in two and three dimensions.

(3) See, for example, Cartan [4], §30. Shield [5] recently presented a proof in the setting of continuum kinematics.

THEOREM 2.1: *Let* $\hat{\mathbf{y}}: R \to E_3$ *and* $\hat{\mathbf{y}}': R \to E_3$ *be regular deformations. Then*

$$(\mathbf{F}')^T\mathbf{F}' = \mathbf{F}^T\mathbf{F} \quad \text{on } R, \text{ where } \mathbf{F} = \nabla\hat{\mathbf{y}},\ \mathbf{F}' = \nabla\hat{\mathbf{y}}' \text{ on } R \tag{2.1}$$

if and only if there is a tensor $\mathbf{Q}$ *and a vector* $\mathbf{d}$, *such that*

$$\hat{\mathbf{y}}'(\mathbf{x}) = \mathbf{Q}\hat{\mathbf{y}}(\mathbf{x}) + \mathbf{d} \quad \text{on } R, \quad \mathbf{Q} \in O_+ . \tag{2.2}$$

Proof: Differentiation of (2.2) leads immediately to (2.1). Conversely, suppose (2.1) holds. Then,

$$\mathbf{F}' = \mathbf{Q}\mathbf{F}, \quad \mathbf{Q} = (\mathbf{F}')^{-T}\mathbf{F}^T, \quad \mathbf{Q} \in O_+ \quad \text{on } R. \tag{2.3}$$

It is sufficient to show that $\mathbf{Q}$ is constant on R since an integration of the first of (2.3) at once confirms (2.2). For this purpose let $\mathring{\mathbf{x}}$ be a point in the interior of R. In view of the regularity of $\hat{\mathbf{y}}$, there is an open sphere $N \subset R$, centered at $\mathring{\mathbf{x}}$, such that $\hat{\mathbf{y}}$ is smoothly invertible on N. We show first that $\mathbf{Q}$ is constant on N. Let $\hat{\mathbf{y}}^{-1}$, defined on $\overset{*}{N} = \hat{\mathbf{y}}(N)$, be the inverse of $\hat{\mathbf{y}}$ and consider the deformation $\hat{\mathbf{z}}$: $\overset{*}{N} \to E_3$ given by

$$\hat{\mathbf{z}}(\mathbf{y}) = \hat{\mathbf{y}}'(\hat{\mathbf{y}}^{-1}(\mathbf{y})) \quad \text{for all } \mathbf{y} \in \overset{*}{N}. \tag{2.4}$$

Clearly, $\hat{\mathbf{z}}$ is regular and (2.4), (2.3) give

$$\nabla\hat{\mathbf{z}}(\mathbf{y}) = \mathbf{F}'(\mathbf{x})\mathbf{F}^{-1}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) \quad \text{for all } \mathbf{y} \in \overset{*}{N},\ \mathbf{x} = \hat{\mathbf{y}}^{-1}(y). \tag{2.5}$$

Accordingly,

$$(\nabla\hat{\mathbf{z}})^T(\nabla\mathbf{z}) = \mathbf{1} \quad \text{on } \overset{*}{N}, \tag{2.6}$$

so that Theorem 1.3 implies that $\nabla\hat{\mathbf{z}}$, and hence also $\mathbf{Q}$, is constant on N. The constancy of $\mathbf{Q}$ on R follows from its constancy in a neighborhood of every interior point on R and the smoothness of $\hat{\mathbf{y}}$ and $\hat{\mathbf{y}}'$ on R.

Note that two deformations producing the same right strain-tensor field also correspond to the same Lagrangian strain field. The analogue of the preceding theorem in the kinematics of infinitesimal deformations is the proposition that two suitably smooth displacement fields give rise to the same infinitesimal strain tensor if and only if they differ from one another by an infinitesimally rigid displacement field.

We recall next conditions necessary and sufficient for the existence of a regular deformation that generates a pre-assigned right strain-tensor field. These "compatibility conditions" were encountered first in the differential geometry literature as necessary and sufficient conditions in order that a Riemannian space be Euclidean.[(1)]

THEOREM 2.2: *Let $D \subset E_3$ be a domain.*

(*a*) *Suppose $\hat{\mathbf{y}}: D \to E_3$ is a regular deformation, let $\hat{\mathbf{y}} \in C^3(D)$, and assume*

$$(\nabla\hat{\mathbf{y}})^T(\nabla\hat{\mathbf{y}}) = \mathbf{C} \quad or \quad \hat{y}_{p,i}\hat{y}_{p,j} = C_{ij} \quad on\ D. \tag{2.7}$$

Then, $C \in S^+$ on D, $\mathbf{C} \in C^2(D)$ and

$$R_{ijkl} \equiv \Gamma_{jli,k} - \Gamma_{jki,l} + C^{-1}_{pq}(\Gamma_{jkp}\Gamma_{ilq} - \Gamma_{jlp}\Gamma_{ikq}) = 0 \quad on\ D,$$

$$\Gamma_{ijk} = \tfrac{1}{2}(C_{jk,i} + C_{ik,j} - C_{ij,k}) \quad \text{on } D. \tag{2.8}$$

(2.8)[(2)]

(*b*) *Conversely, if D is simply connected, $\mathbf{C}: D \to S^+$, and $\mathbf{C} \in C^2(D)$ satisfies* (2.8), *then there exists a regular deformation $\hat{\mathbf{y}}: D \to E_3$, with $\hat{\mathbf{y}} \in C^3(D)$, such that* (2.7) *holds.*

As far as part (a) is concerned, the necessity of (2.8) is readily established by computation. It may be useful to outline an economical procedure for doing so. One verifies easily that every $\hat{\mathbf{y}} \in C^3(D)$ obeying (2.7) satisfies

$$F_{pi}F_{pj,k} = \Gamma_{jki} \quad \text{on } D, \quad F_{ij} = \hat{y}_{i,j}, \tag{2.9}$$

as well as the linear system

$$F_{ij,k} = F_{ip}C^{-1}_{pq}\Gamma_{jkq} \quad \text{on } D, \tag{2.10}$$

Γ_{ijk} being defined by the second of (2.8). If one differentiates (2.9) with respect to x_l and uses (2.10) to eliminate the first gradients of $\mathbf{F}$ from the resulting equation, one is led to

$$F_{pi}F_{pj,kl} = \Gamma_{jki,l} - C^{-1}_{pq}\Gamma_{jkp}\Gamma_{ilq} \quad \text{on } D. \tag{2.11}$$

Since $F_{pj,kl} = F_{pj,lk}$ on D, (2.11) implies (2.8).

An existence proof for a solution $\mathbf{F} \in L_+$ to (2.10) can be found in §46–49 of Cartan's [4] treatise. More recently, Thomas [7] adapted the proof given in [4]

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(1) Christoffel [2] and Cartan [4], §§43–49 are early references; see also Eisenhart [6], §23.

(2) Here C^{-1}_{pq} is the appropriate component of C^{-1}.

to a broader class of systems of partial differential equations that includes (2.10). The argument employed in [4] and [7] relies ultimately on the existence theorem for systems of ordinary differential equations and depends crucially on the simple connectivity of the domain D.

From a tensor field $\mathbf{F} \in L_+$ that is twice continuously differentiable on D and that satisfies the linear system (2.10), one can construct a regular deformation obeying (2.7) as follows. In view of the symmetry of $\mathbf{C}$, the second of (2.8) furnishes

$$\Gamma_{ijk} = \Gamma_{jik} \quad \text{on } D, \tag{2.12}$$

whence

$$F_{ij,k} = F_{ik,j} \quad \text{on } D, \tag{2.13}$$

for every solution $\mathbf{F}$ of (2.10). Accordingly, since D is simply connected, Stokes' theorem implies the existence of a regular deformation $\hat{\mathbf{z}}: D \to E_3$ with $\hat{\mathbf{z}} \in C^3(D)$, such that

$$\nabla \hat{\mathbf{z}} = \mathbf{F} \quad \text{on } D. \tag{2.14}$$

On setting

$$\mathbf{B} = \mathbf{F}^{-T}\mathbf{C}\mathbf{F}^{-1} \quad \text{on } D, \tag{2.15}$$

one finds with the aid of (2.10) and the second of (2.8) that $\mathbf{B}$ is constant on D, while $\mathbf{F} \in L_+$ and $\mathbf{C} \in S^+$ justify that $\mathbf{B} \in S^+$. Finally, one defines a deformation $\hat{\mathbf{y}}$ through

$$\hat{\mathbf{y}}(\mathbf{x}) = \sqrt{\mathbf{B}}\hat{\mathbf{z}}(\mathbf{x}) \quad \text{on } D, \tag{2.16}$$

and invokes (2.15), (2.14) to confirm that $\hat{\mathbf{y}}$ is a regular deformation complying with (2.7).

While (2.8) guarantees the existence of a regular deformation satisfying (2.7), the compatibility conditions are not sufficient to guarantee the global invertibility of such a deformation.

Note that (2.8) is equivalent to

$$R_{ijkl} \equiv \tfrac{1}{2}(C_{jk,il} + C_{il,jk} - C_{jl,ik} - C_{ik,jl})$$
$$+ C^{-1}_{pq}(\Gamma_{jkp}\Gamma_{ilq} - \Gamma_{jlp}\Gamma_{ikq}) = 0 \quad \text{on } D, \tag{2.17}$$

so that (2.12) and the symmetry **C** yield

$$R_{ijkl} = -R_{ijlk} = R_{klij} \quad \text{on } D. \tag{2.18}$$

In view of these relations, (2.17) constitutes a system of six independent scalar equations.

On account of the last of (1.5), the compatibility equations (2.17) may alternatively be cast in terms of the Lagrangian strain components E_{ij}. Although the latter linearize to the infinitesimal strains under the assumption of infinitesimal deformations, (2.17) do not reduce to the familiar compatibility conditions of the linear theory unless products of the *second* displacement gradients are neglected as well.

In the special case of plane deformations all but one of the scalar compatibility equations are trivially satisfied, and thus (2.8) degenerates into the single equation

$$R_{1212} = 0 \quad \text{on } D, \tag{2.19}$$

provided $D \subset E_2$ is now the open cross section of the relevant cylindrical region. Alternative *constructive* proofs of the sufficiency of (2.19) in the presence of plane deformations were given by Fosdick and Schuler [8] and by Shield [5]. The proof in [8] is confined to locally volume-preserving plane deformations.

3. Remarks on deformations generating a given left strain-tensor field in three dimensions

We now turn to the existence and uniqueness issues for a regular deformation that gives rise to a prescribed left strain-tensor field in three dimensions. Proceeding first to the *uniqueness* issue, let D be an open region in E_3 and suppose $\hat{\mathbf{y}}: D \to E_3$ and $\hat{\mathbf{y}}': D \to E_3$ are regular deformations in $C^2(D)$ with the same left strain field. Thus

$$\mathbf{G} = \mathbf{F}\mathbf{F}^T = \mathbf{F}'(\mathbf{F}')^T \quad \text{on } D, \text{ where} \quad \mathbf{F} = \nabla\hat{\mathbf{y}}, \quad \mathbf{F}' = \nabla\hat{\mathbf{y}}'. \tag{3.1}$$

If one sets

$$\mathbf{Q} = \mathbf{F}^T(\mathbf{F}')^{-T} \quad \text{on } D, \tag{3.2}$$

it follows from (3.1) and the smoothness of $\hat{\mathbf{y}}$, $\hat{\mathbf{y}}'$ that

$$\mathbf{F}' = \mathbf{F}\mathbf{Q}, \quad \mathbf{Q} \in C^1(D), \quad \mathbf{Q} \in O_+. \tag{3.3}$$

Further, because $\mathbf{F}' = \nabla\hat{\mathbf{y}}'$, $\mathbf{Q}$ obeys

$$\operatorname{curl}(\mathbf{F}\mathbf{Q}) = \mathbf{0} \quad \text{or} \quad (F_{ip}Q_{pj})_{,k} = (F_{ip}Q_{pk})_{,j} \quad \text{on } D. \tag{3.4}$$

Conversely, suppose $\mathbf{F} = \nabla\hat{\mathbf{y}}$ satisfies the first of (3.1) and $\mathbf{Q}\in C^1(D)$ is any proper-orthogonal tensor field that conforms to (3.4). Then one can define a regular deformation $\hat{\mathbf{y}}': D \to E_3$, $\hat{\mathbf{y}}'\in C^2(D)$ that generates the same left strain-tensor field as $\hat{\mathbf{y}}$ through the path-independent line integral

$$\hat{\mathbf{y}}'(\mathbf{x}) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \mathbf{F}(\boldsymbol{\xi})\mathbf{Q}(\boldsymbol{\xi})\, d\xi \quad \text{or} \quad \hat{y}_i'(x) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} F_{ip}(\boldsymbol{\xi})Q_{pj}(\boldsymbol{\xi})\, d\xi_j \quad \text{on } D, \tag{3.5}$$

provided D is simply connected and $\mathring{\mathbf{x}}\in D$ is fixed.

Next, consider the homogeneous deformation

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \mathbf{d} \quad \text{on } D, \tag{3.6}$$

where $\mathbf{F}$ and $\mathbf{d}$ are constant, $\mathbf{F}\in L_+$. In this case, $\mathbf{G} = \mathbf{F}\mathbf{F}^T$ is constant, and, according to Theorem 1.3, every regular deformation that generates this $\mathbf{G}$ is necessarily homogeneous. Thus, if (3.1) and (3.6) hold, (3.3) requires that

$$\hat{\mathbf{y}}'(\mathbf{x}) = \mathbf{F}\mathbf{Q}\mathbf{x} + \mathbf{d}' \quad \text{on } D, \tag{3.7}$$

in which $\mathbf{Q}$ and $\mathbf{d}'$ are constant, $\mathbf{Q}\in O_+$. If $\hat{\mathbf{y}}$ is now understood to be the unique extension to E_3 of the mapping (3.6) originally defined on D, one has, on account of (3.7),

$$\hat{\mathbf{y}}'(\mathbf{x}) = \hat{\mathbf{y}}(\hat{\mathbf{z}}(\mathbf{x})) \quad \text{on } D, \tag{3.8}$$

where $\hat{\mathbf{z}}: D \to E_3$ is the rigid deformation given by

$$\hat{\mathbf{z}}(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{F}^{-1}(\mathbf{d}' - \mathbf{d}) \quad \text{on } D. \tag{3.9}$$

The mapping $\hat{\mathbf{y}}'$ thus corresponds to the rigid deformation $\hat{\mathbf{z}}$ followed by the extended homogeneous deformation $\hat{\mathbf{y}}$.

As is evident from the preceding remarks, if $\hat{\mathbf{y}}$ and $\hat{\mathbf{y}}'$ satisfy (3.1) and hence (3.3), $\hat{\mathbf{y}}'$ may be interpreted *locally* as a rigid mapping followed by the deformation $\hat{\mathbf{y}}$.

It appears difficult to determine the totality of regular deformations that generate the left strain-tensor field associated with an arbitrarily assigned regular deformation $\hat{\mathbf{y}}\in C^2(D)$. In order to accomplish this purpose, it would suffice to find the set of all proper-orthogonal tensor fields $\mathbf{Q}\in C^1(D)$ obeying

(3.4) for the given $\mathbf{F} = \nabla\hat{\mathbf{y}}$. As is clear already, when $\hat{\mathbf{y}}$ is a homogeneous deformation, $\mathbf{Q}$ satisfies (3.4) if and only if $\mathbf{Q}$ is constant.

In contrast to the uniqueness issue for the left strain field discussed here, the corresponding issue for the *right* strain field presents no particular difficulties, as is evident from the proof of Theorem 2.1.

The *existence* issue for $\mathbf{G}$ – like the corresponding uniqueness issue – seems to be difficult to resolve in three dimensions. The object here is to establish necessary and sufficient conditions for the existence of a regular deformation generating a pre-assigned left strain-tensor field.

With a view toward *necessary* conditions, suppose $\hat{\mathbf{y}}: D \to E_3$ is regular, $\hat{\mathbf{y}} \in C^2(D)$, and

$$\mathbf{F}\mathbf{F}^T = \mathbf{G} \quad \text{on } D, \quad \text{where} \quad \mathbf{F} = \nabla\hat{\mathbf{y}}. \tag{3.10}$$

Clearly, $\mathbf{G} \in C^1(D)$ and $\mathbf{G} \in S^+$. Further, (3.10), in conjunction with the left polar decomposition for $\mathbf{F}$, guarantees the existence of a tensor field[(1)] $\mathbf{Q} \in C^1(D)$, such that

$$\mathbf{F} = \mathbf{V}\mathbf{Q} \quad \text{or} \quad F_{ij} = V_{ip}Q_{pj}, \quad \mathbf{V} = \sqrt{\mathbf{G}}, \quad \mathbf{Q} \in O_+ \quad \text{on } D. \tag{3.11}$$

Because $\mathbf{F} = \nabla\hat{\mathbf{y}}$ on D, $\mathbf{Q}$ satisfies

$$\operatorname{curl}(\mathbf{V}\mathbf{Q}) = \mathbf{0} \quad \text{or} \quad (V_{ip}Q_{pj})_{,k} = (V_{ip}Q_{pk})_{,j} \quad \text{on } D. \tag{3.12}$$

Equations (3.12) are found to be equivalent to the following quasi-linear system of partial differential equations

$$Q_{ij,k} = Q_{ip}[Q_{qp}\Omega_{qjk} - Q_{qj}\Omega_{qpk} - Q_{qk}\Omega_{qpj}] \quad \text{on } D, \qquad (3.13)^{(2)}$$

provided

$$\Omega_{qjk} = \tfrac{1}{2}V_{qi}^{-1}(V_{ip,j}Q_{pk} - V_{ip,k}Q_{pj}) = -\Omega_{qkj} \quad \text{on } D. \tag{3.14}$$

Conversely, if D is simply connected and there is a tensor field $\mathbf{Q} \in C^1(D)$, $\mathbf{Q} \in O_+$ satisfying (3.13), there exists a regular deformation $\hat{\mathbf{y}} \in C^2(D)$ that

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(1) The smoothness of $\mathbf{Q}$ follows from the fact that $\mathbf{Q} = \mathbf{V}^{-1}\mathbf{F}$ on D and from the differentiability of $\mathbf{F}$ and $\mathbf{V} = \sqrt{\mathbf{G}}$. See, for example, Gurtin [1], Art. 3, for a proof that the components of $\mathbf{V}$ depend smoothly on the components of $\mathbf{G}$.

(2) A system of equations analogous to (3.13), satisfied by $\mathbf{Q} = \mathbf{F}\mathbf{U}^{-1}$, $\mathbf{U} = \sqrt{\mathbf{C}}$ for a given *right* strain-tensor field $\mathbf{C}$, was deduced by Shield [5], §3. In contrast to (3.13), the system obtained by Shield is a *linear* system for $\mathbf{Q}$.

obeys (3.10). Indeed, such a deformation is given by the path-independent line integral

$$\hat{\mathbf{y}}(\mathbf{x}) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \mathbf{V}(\boldsymbol{\xi})\mathbf{Q}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \quad \text{or} \quad \hat{y}_i(\mathbf{x}) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} V_{ip}(\boldsymbol{\xi})Q_{pj}(\boldsymbol{\xi})\, d\xi_j \quad \text{on } D. \tag{3.15}$$

Thus, conditions necessary and sufficient for the existence of a smooth proper-orthogonal tensor field $\mathbf{Q}$ conforming to (3.13) are at the same time necessary and sufficient for the existence of a regular deformation $\hat{\mathbf{y}} \in C^2(D)$ satisfying (3.10). As will emerge in Section 5, it is possible to obtain explicit necessary and sufficient conditions for the existence of a solution to (3.13) in the special case of *plane* deformations. Further, in these circumstances, the integration of (3.13) is either reducible to a quadrature or to the integration of an ordinary, linear differential equation of the second order.

4. Uniqueness of plane deformations generating a given left strain-tensor field

The uniqueness question discussed in the previous section in the context of general *three*-dimensional deformations is dealt with at present for *plane* deformations. We thus establish two theorems concerning the extent of uniqueness of plane deformations producing a prescribed left strain-tensor field. From here on D stands for a domain in E_2, all vectors and tensors being two-dimensional.

THEOREM 4.1: *Let* $\hat{\mathbf{y}}: D \to E_2$, $\hat{\mathbf{y}}': D \to E_2$ *be regular deformations in* $C^3(D)$, *such that*

$$\mathbf{F}\mathbf{F}^T = \mathbf{F}'(\mathbf{F}')^T \quad \text{on } D, \quad \mathbf{F} = \nabla\hat{\mathbf{y}}, \quad \mathbf{F}' = \nabla\hat{\mathbf{y}}'. \tag{4.1}$$

Let

$$\mathbf{Q} = \mathbf{F}^T(\mathbf{F}')^{-T} \quad \text{on } D, \tag{4.2}$$

so that

$$\mathbf{F}' = \mathbf{F}\mathbf{Q} \quad \text{on } D, \quad \mathbf{Q} \in O_+, \quad \mathbf{Q} \in C^2(D). \tag{4.3}$$

Assume

$$\mathbf{Q} \not\equiv \pm\mathbf{1} \quad \text{on } D \tag{4.4}$$ [(1)]

[(1)] If $\mathbf{Q} = \pm\mathbf{1}$ on D one has $\hat{\mathbf{y}}'(\mathbf{x}) = \pm\hat{\mathbf{y}}(\mathbf{x}) + \mathbf{d}$ on D, with $\mathbf{d}$ constant. Thus, in this trivial case, $\hat{\mathbf{y}}$ and $\hat{\mathbf{y}}'$ differ either by a translation or by an in-plane rotation through π about the origin and a translation.

and set

$$\gamma_\alpha = -F^{-1}_{\alpha\beta}F_{\beta\lambda,\lambda}, \quad a = \gamma_{\alpha,\alpha} + \gamma_\alpha\gamma_\alpha, \quad b = \varepsilon_{\alpha\beta}\gamma_{\alpha,\beta} \quad \text{on } D. \tag{4.5}$$[1]

If $a^2 + b^2 \neq 0$ *on* D *and* b *has at most isolated zeros on* D, *then either*

$$Q_{\alpha\beta} = (\delta_{\alpha\beta}a + \varepsilon_{\alpha\beta}b)/\sqrt{a^2+b^2} \quad \textit{on } D$$

or (4.6)

$$Q_{\alpha\beta} = -(\delta_{\alpha\beta}a + \varepsilon_{\alpha\beta}b)/\sqrt{a^2+b^2} \quad \textit{on } D.$$

If $a = b = 0$ *on* d *and* D *is simply connected,* (4.6) *holds again with* a *and* b *replaced by* $\mu + c$ *and* η, *respectively, where*

$$\eta(\mathbf{x}) = \exp\left\{\int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \gamma_\alpha(\boldsymbol{\xi})\, \mathrm{d}\xi_\alpha\right\}, \quad \mu(\mathbf{x}) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \varepsilon_{\alpha\beta}\eta_{,\beta}(\boldsymbol{\xi})\, \mathrm{d}\xi_\alpha \quad \textbf{on } D, \tag{4.7}$$

$\mathring{\mathbf{x}} \in D$ *is an arbitrary fixed point, while* $c = Q_{\alpha\alpha}(\mathring{\mathbf{x}})/\varepsilon_{\beta\lambda}Q_{\beta\lambda}(\mathring{\mathbf{x}})$.

Proof: Since $\mathbf{F}' = \nabla\hat{\mathbf{y}}'$ and $\mathbf{F}' \in C^2(D)$, (4.3) demands that

$$\operatorname{curl}(\mathbf{FQ}) = \mathbf{0} \quad \text{or} \quad \varepsilon_{\lambda\mu}(F_{\alpha\beta}Q_{\beta\lambda})_{,\mu} = 0 \quad \text{on} \quad D. \tag{4.8}$$

Further, because $\mathbf{Q} \in O_+$ on D, $\mathbf{Q}$ admits the representation

$$[Q_{\alpha\beta}] = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \quad \text{or}$$

$$\mathbf{Q}_{\alpha\beta} = \delta_{\alpha\beta}\cos\omega - \varepsilon_{\alpha\beta}\sin\omega \quad \text{on } D, \quad \omega \in [0, 2\pi). \tag{4.9}$$

While the functions $\cos\omega$ and $\sin\omega$ are in $C^2(D)$, ω itself need not be continuous on D. It is clear, however, that ω is twice continuously differentiable on any subdomain of D on which it is continuous.

Let

$$\chi_\alpha = \cos 2\omega(\sin 2\omega)_{,\alpha} - \sin 2\omega(\cos 2\omega)_{,\alpha} \quad \text{on } D. \tag{4.10}$$

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[1] Here and in the sequel $\varepsilon_{\alpha\beta}$ is the two-dimensional alternating symbol: $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$.

On any domain of continuity of ω, (4.10) reduces to $\chi_\alpha = 2\omega_{,\alpha}$. Substitution from (4.9) into (4.8), after some manipulation and an appeal to (4.10), (4.5), yields

$$\chi_\alpha = [\delta_{\alpha\beta} \sin 2\omega + \varepsilon_{\alpha\beta}(1 - \cos 2\omega)]\gamma_\beta \quad \text{on } D. \tag{4.11}$$

On differentiating (4.11) with respect to x_λ and using (4.10), (4.5), one eventually arrives at

$$\varepsilon_{\alpha\lambda}\chi_{\alpha,\lambda} = a(1 - \cos 2\omega) + b \sin 2\omega = 0 \quad \text{on } D, \tag{4.12}$$

from which follows

$$(a^2 + b^2)\cos^2 2\omega - 2a^2 \cos 2\omega + a^2 - b^2 = 0 \quad \text{on } D. \tag{4.13}$$

Assume first $a^2 + b^2 \neq 0$ on D. Then, by (4.12), (4.13), at each point on D either

$$\cos 2\omega = 1, \quad \sin 2\omega = 0 \tag{4.14}$$

or

$$\cos 2\omega = \frac{a^2 - b^2}{a^2 + b^2}, \quad \sin 2\omega = \frac{-2ab}{a^2 + b^2}. \tag{4.15}$$

Note that (4.14), (4.15) coalesce only at zeros of b and that $b = 0$ throughout D is excluded by (4.4), in view of (4.9). Now assume further that b has at most isolated zeros[1] on D and let D_0 be the subdomain of D on which $b \neq 0$. Suppose there is a point in D_0 at which (4.14) holds. Then, the continuity of $\cos 2\omega$ demands that (4.14) hold at *all* points of D, which is precluded by (4.4). Hence (4.15) holds on D. Consequently, either

$$\cos \omega = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin \omega = \frac{-b}{\sqrt{a^2 + b^2}} \quad \text{on } D$$

or (4.16)

$$\cos \omega = \frac{-a}{\sqrt{a^2 + b^2}}, \quad \sin \omega = \frac{b}{\sqrt{a^2 + b^2}} \quad \text{on } D,$$

and (4.16), (4.9) now confirm (4.6).

[1] Actually, it is sufficient to assume that the subset of D on which b fails to vanish is connected.

Next, suppose $a = b = 0$ on D and D is simply connected. In this case, the last two of (4.5) become

$$\gamma_{\alpha,\alpha} + \gamma_\alpha \gamma_\alpha = 0, \quad \varepsilon_{\alpha\beta}\gamma_{\alpha,\beta} = 0 \quad \text{on } D. \tag{4.17}$$

On account of the second of (4.17), one may define a function $\eta \in C^2(D)$ through the first of (4.7), the line integral here involved being path-independent. Accordingly, one has

$$\eta_{,\alpha} = \eta\gamma_\alpha \quad \text{on } D, \tag{4.18}$$

which – together with the first of (4.17) – leads to

$$\eta_{,\alpha\alpha} = \eta(\gamma_{\alpha,\alpha} + \gamma_\alpha\gamma_\alpha) = 0 \quad \text{on } D, \tag{4.19}$$

so that η is harmonic and hence in $C^\infty(D)$. One may thus define another function, $\mu \in C^\infty(D)$, through the path-independent line integral appearing in the second of (4.7). The latter implies the Cauchy-Riemann equations

$$\mu_{,\alpha} = \varepsilon_{\alpha\beta}\eta_{,\beta} \quad \text{on } D. \tag{4.20}$$

By virtue of (4.20), the complex-valued function determined by

$$g(z) = \mu(\mathbf{x}) + i\eta(\mathbf{x}) \quad \text{on } D, \quad z = x_1 + ix_2, \tag{4.21}$$

is analytic on D; further, because of (4.18), one has

$$\frac{\mathrm{d}g}{\mathrm{d}z} = \eta(\gamma_2 + i\gamma_1) \quad \text{on } D. \tag{4.22}$$

A direct computation based on (4.11), and involving (4.10), (4.17), justifies

$$\chi_{\alpha,\alpha} = 0 \quad \text{on } D. \tag{4.23}$$

Let ρ be the scalar field defined by

$$\rho(\mathbf{x}) = \exp\left\{\frac{1}{2}\int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \varepsilon_{\alpha\beta}\chi_\alpha(\boldsymbol{\xi})\, \mathrm{d}\xi_\beta\right\} \quad \text{on } D, \tag{4.24}$$

noting that the foregoing line integral is path-independent owing to (4.23). With the aid of (4.24), (4.10) one readily verifies that the complex-valued function f given by

$$f(z) = \rho(\mathbf{x})\, \mathrm{e}^{-i\omega(\mathbf{x})} \quad \text{on } D, \quad z = x_1 + ix_2, \tag{4.25}$$

is analytic and nonzero on D, as well as that

$$-\frac{2}{f}\frac{\mathrm{d}f}{\mathrm{d}z} = \chi_2 + i\chi_1 \quad \text{on } D. \tag{4.26}$$

From (4.26), (4.11) one draws

$$\frac{2}{f}\frac{\mathrm{d}f}{\mathrm{d}z} = i(\mathrm{e}^{2i\omega} - 1)(\gamma_2 + i\gamma_i) \quad \text{on } D, \tag{4.27}$$

or, invoking (4.22) and $\eta \neq 0$ on D,

$$2\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{i}{\eta}(\mathrm{e}^{2i\omega} - 1)f\frac{\mathrm{d}g}{\mathrm{d}z} \quad \text{on } D. \tag{4.28}$$

Substitution from (4.25) into (4.28), in turn, leads to

$$\eta\frac{\mathrm{d}f}{\mathrm{d}z} = -\rho \sin\omega \frac{\mathrm{d}g}{\mathrm{d}z} \quad \text{on } D. \tag{4.29}$$

Thus, if one sets

$$v = -\rho \sin\omega = \mathrm{Im}\{f\} \quad \text{on } D, \qquad (4.30)^{(1)}$$

(4.29) becomes

$$\eta\frac{\mathrm{d}f}{\mathrm{d}z} = v\frac{\mathrm{d}g}{\mathrm{d}z} \quad \text{on } D \tag{4.31}$$

or, by (4.30) and (4.21),

$$\eta(v_{,2} + iv_{,1}) = v(\eta_{,2} + i\eta_{,1}) \quad \text{on } D. \tag{4.32}$$

Equating real and imaginary parts of this identity, one infers that

$$v(\mathbf{x}) = c_1\eta(\mathbf{x}) \quad \text{on } D, \tag{4.33}$$

provided c_1 is a real constant. From (4.30), (4.21), (4.33), as well as the analyticity of the functions f and g, follows

$$f(z) = c_1 g(z) + c_2 \quad \text{on } D, \tag{4.34}$$

(1) Recall (4.25).

in which c_2 is another real constant. Since $f \neq 0$ on D, $c_1^2 + c_2^2 \neq 0$. Moreover, because $\eta = \text{Im}\{g\} \neq 0$ on D, (4.34), (4.25) give

$$\omega(\mathbf{x}) = -\arg\{c_1 g(z) + c_2\} \quad \text{on } D, \quad \omega \in [0, 2\pi). \tag{4.35}$$ [(1)]

If $c_1 = 0$, then $c_2 \neq 0$, $\omega = -\arg\{c_2\}$ on D, and hence

$$\omega = 0 \quad \text{if } c_2 > 0, \quad \omega = \pi \quad \text{if } c_2 < 0. \tag{4.36}$$

As is clear from (4.9) and (4.36), $c_1 = 0$ leads to $Q_{\alpha\beta} = \pm\delta_{\alpha\beta}$, so that $c_1 \neq 0$ by (4.4).

Thus, set

$$c = c_2/c_1, \tag{4.37}$$

whence (4.35) may be written as

$$\omega(\mathbf{x}) = -\arg\{c_1[g(z) + c]\} \quad \text{on } D. \tag{4.38}$$

Since $\text{Im}\{g\} > 0$, one has

$$\left.\begin{aligned} \omega(\mathbf{x}) &= -\arg\{g(z) + c\} \quad \text{on } D \text{ if } c_1 > 0, \\ \omega(\mathbf{x}) &= -\arg\{g(z) + c\} + \pi \quad \text{on } D \text{ if } c_1 < 0. \end{aligned}\right\} \tag{4.39}$$

This, together with (4.21), (4.9), furnishes

$$Q_{\alpha\beta} = \pm[\delta_{\alpha\beta}(\mu + c) + \varepsilon_{\alpha\beta}\eta]/\sqrt{(\mu + c)^2 + \eta^2} \quad \text{on } D. \tag{4.40}$$

Finally, observe that $\eta(\mathring{\mathbf{x}}) = 1$, $\mu(\mathring{\mathbf{x}}) = 0$ by (4.7); consequently, the evaluation of the constant c in (4.40) leads to $c = Q_{\alpha\alpha}(\mathring{\mathbf{x}})/\varepsilon_{\beta\lambda}Q_{\beta\lambda}(\mathring{\mathbf{x}})$. This completes the proof.

Before turning to the next theorem, we add a remark concerning the case in which $a^2 + b^2 \neq 0$. Thus, let $\hat{\mathbf{y}}$, $\hat{\mathbf{y}}'$ satisfy the hypotheses of Theorem 4.1 and suppose in particular that $a^2 + b^2 \neq 0$ on D, while b has at most isolated zeros on D. If one assumes that $\hat{\mathbf{y}} \in C^4(D)$, rather than merely in $C^3(D)$, substitution

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[(1)] From here on we take for granted that $\arg\{c_1 g(z) + c_2\} \in (-2\pi, 0]$.

from (4.15)[1] into (4.10) and (4.11), upon equating the ensuing right-hand members, yields

$$ab_{,\alpha} - ba_{,\alpha} - ab\gamma_\alpha + b^2\varepsilon_{\alpha\beta}\gamma_\beta = 0 \quad \text{on } D. \tag{4.41}$$

Therefore, in the present circumstances (4.41) is a necessary condition for the existence of a deformation $\hat{\mathbf{y}}'(\mathbf{x}) \not\equiv \pm\hat{\mathbf{y}}(\mathbf{x}) + \mathbf{d}$ on D with the same left strain field as $\hat{\mathbf{y}}$. One can show by example that there are plane deformations in $C^4(D)$ for which (4.41) fails to hold, although a and b have no common zeros and b has at most isolated zeros on D. If $\hat{\mathbf{y}}: D \to E_2$ is such a deformation, it follows that $\hat{\mathbf{y}}': D \to E_2$ is regular, in $C^3(D)$, and satisfies (4.1) only if $\hat{\mathbf{y}}'(\mathbf{x}) = \pm\hat{\mathbf{y}}(\mathbf{x}) + \mathbf{d}$ on D.

The following theorem is a converse of Theorem 4.1 and supplies an algorithm for the construction of a deformation $\hat{\mathbf{y}}'$ that generates the same left strain field as a given deformation $\hat{\mathbf{y}}$.

THEOREM 4.2: *Let D be simply connected and $\hat{\mathbf{y}}: D \to E_2$ be a regular deformation in $C^4(D)$, $\mathbf{F} = \nabla\hat{\mathbf{y}}$. Suppose the fields a, b, and γ_α, defined through* (4.5), *satisfy $a^2 + b^2 \neq 0$ on D and obey* (4.41). *Then the deformation $\hat{\mathbf{y}}': D \to E_2$ determined by*

$$\hat{\mathbf{y}}'(\mathbf{x}) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \mathbf{F}(\boldsymbol{\xi})\mathbf{Q}(\boldsymbol{\xi})\,d\boldsymbol{\xi} \quad \text{or} \quad \hat{y}'_\alpha(\mathbf{x}) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} F_{\alpha\beta}(\boldsymbol{\xi})Q_{\beta\gamma}(\boldsymbol{\xi})\,d\xi_\gamma \quad \text{on } D, \tag{4.42}$$

in which $\mathbf{Q}$ is given by either of (4.6), *is regular, $\hat{\mathbf{y}}' \in C^4(D)$, and $\hat{\mathbf{y}}'$ conforms to* (4.1).

If $a = b = 0$ on D, define functions μ and η on D through (4.7). *Let $\mathbf{Q}$ be given by either of* (4.6) *but with a and b replaced by μ and η, respectively. Then $\hat{\mathbf{y}}'$ determined by* (4.42) *is regular, $\hat{\mathbf{y}}' \in C^4(D)$, and $\hat{\mathbf{y}}'$ obeys* (4.1).

Proof: Suppose first (4.41) holds with $a^2 + b^2 \neq 0$ on D, and let $\mathbf{Q}$ be given by either of (4.6). In order to reach the desired conclusion, it is sufficient to show that $\mathbf{Q} \in O_+$, $\mathbf{Q} \in C^3(D)$, and that $\mathbf{Q}$ satisfies (4.8). The latter assures the path independence of the line integral in (4.42), while (4.42) implies $\mathbf{F}' = \mathbf{FQ}$, and thus (4.1), since $\mathbf{Q}$ is orthogonal.

The proper orthogonality of $\mathbf{Q}$ follows directly from (4.6). Further, because $\mathbf{F} \in C^3(D)$, (4.5), (4.6) imply that $\mathbf{Q}$ is at least in $C^1(D)$, and (4.6) results in

$$Q_{\alpha\beta,\lambda} = \varepsilon_{\mu\beta}Q_{\alpha\mu}(ab_{,\lambda} - ba_{,\lambda})/(a^2 + b^2) \quad \text{on } D. \tag{4.43}$$

---

[1] Recall that (4.15) was shown to hold at every point of D.

With the aid of (4.41) and (4.6), Equation (4.43) may now be written as

$$Q_{\alpha\beta,\lambda} = \tfrac{1}{2}\varepsilon_{\mu\beta}\varepsilon_{\sigma\tau}Q_{\sigma\tau}Q_{\alpha\mu}Q_{\rho\lambda}\gamma_\rho \quad \text{on } D. \tag{4.44}$$

One infers from (4.44) and $\gamma_\alpha \in C^2(D)$ that $\mathbf{Q}$ is actually in $C^3(D)$, rather than merely once continuously differentiable.

It remains to show that (4.8) holds. To this end, note from (4.44) and the orthogonality of $\mathbf{Q}$ that

$$\varepsilon_{\beta\lambda}Q_{\alpha\beta,\lambda} = -\tfrac{1}{2}\varepsilon_{\sigma\tau}Q_{\sigma\tau}\gamma_\alpha \quad \text{on } D. \tag{4.45}$$

Substitution for γ_α from (4.5) now gives

$$\varepsilon_{\beta\lambda}F_{\mu\alpha}Q_{\alpha\beta,\lambda} = \tfrac{1}{2}\varepsilon_{\sigma\tau}Q_{\sigma\tau}F_{\mu\rho,\rho} \quad \text{on } D. \tag{4.46}$$

In view of (4.6) and because $\varepsilon_{\alpha\lambda}F_{\mu\alpha,\lambda} = 0$ on D, (4.46) is found to be equivalent to

$$\varepsilon_{\beta\lambda}F_{\mu\alpha}Q_{\alpha\beta,\lambda} + \varepsilon_{\beta\lambda}F_{\mu\alpha,\lambda}Q_{\alpha\beta} = 0 \quad \text{on } D, \tag{4.47}$$

which confirms (4.8).

Next, assume $a = b = 0$ on D, so that the last two of (4.5) reduce to (4.17). Let η and μ be defined through (4.7). One now argues as in the proof of the last theorem that the required line integrals are path-independent and that μ, η are both in $C^\infty(D)$. On defining $\mathbf{Q}$ by

$$Q_{\alpha\beta} = \pm(\delta_{\alpha\beta}\mu + \varepsilon_{\alpha\beta}\eta)/\sqrt{\mu^2+\eta^2} \quad \text{on } D, \tag{4.48}$$

one infers $\mathbf{Q} \in O_+$, $\mathbf{Q} \in C^\infty(D)$. Moreover, (4.48) together with (4.18), (4.20) yield

$$Q_{\alpha\beta,\lambda} = \eta\varepsilon_{\mu\beta}Q_{\alpha\mu}(\delta_{\rho\lambda}\mu + \varepsilon_{\rho\lambda}\eta)\gamma_\rho/(\mu^2+\eta^2) \quad \text{on } D. \tag{4.49}$$

But (4.49), (4.48) entitle us to assert (4.44) also in the present instance, which once again enables us to justify (4.8). The line integral in (4.42) is thus path-independent, and $\hat{\mathbf{y}}'$ so determined is in $C^4(D)$ with $\det \mathbf{F}' > 0$ on D; further, $\hat{\mathbf{y}}'$ satisfies (4.1). The proof is now complete.

As is evident from (4.5), any deformation $\hat{\mathbf{y}}$ that is harmonic on D gives rise to the case in which $a = b = 0$ on D. This class of deformations includes two-dimensional conformal mappings and homogeneous deformations. The subsequent example shows that there are *nonharmonic* mappings for which $a = b = 0$ on D.

Let $D = \{\mathbf{x} \mid x_1 > 0, x_2 > 0\}$ and consider the deformation $\hat{\mathbf{y}}: D \to E_2$ given by

$$\hat{y}_1(\mathbf{x}) = \mathrm{Log}\, x_1, \quad \hat{y}_2(\mathbf{x}) = \mathrm{Log}\, x_2 \quad \text{on } D. \tag{4.50}$$

Then, clearly, $\hat{\mathbf{y}} \in C^\infty(D)$,

$$[F_{\alpha\beta}] = \begin{bmatrix} 1/x_1 & 0 \\ 0 & 1/x_2 \end{bmatrix}, \quad [G_{\alpha\beta}] = [F_{\alpha\lambda}F_{\beta\lambda}] = \begin{bmatrix} 1/x_1^2 & 0 \\ 0 & 1/x_2^2 \end{bmatrix}, \tag{4.51}$$

whence $\hat{\mathbf{y}}$ is a regular deformation. Here, (4.51), (4.5) give

$$\gamma_1(\mathbf{x}) = 1/x_1, \quad \gamma_2(\mathbf{x}) = 1/x_2, \quad a = b = 0 \quad \text{on } D. \tag{4.52}$$

Choosing $(\mathring{x}_1, \mathring{x}_2) = (1, 1)$, one finds from (4.52), (4.7) that

$$\eta(\mathbf{x}) = x_1 x_2, \quad \mu(\mathbf{x}) = \tfrac{1}{2}(x_1^2 - x_2^2) \quad \text{on } D, \tag{4.53}$$

and (4.48) with the upper sign yields

$$[Q_{\alpha\beta}] = \frac{1}{x_1^2 + x_2^2}\begin{bmatrix} x_1^2 - x_2^2 & 2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{bmatrix} \quad \text{on } D. \tag{4.54}$$

Finally, from (4.54), the first of (4.51), and (4.42) follows

$$\hat{y}_1'(\mathbf{x}) = -\mathrm{Log}[2x_1/(x_1^2 + x_2^2)], \quad \hat{y}_2'(\mathbf{x}) = \mathrm{Log}[2x_2/(x_1^2 + x_2^2)] \quad \text{on } D. \tag{4.55}$$

5. Existence of plane deformations generating a given left strain-tensor field

In this section we prove two theorems concerning the existence of a plane deformation that gives rise to a pre-assigned left strain-tensor field. The first of these two theorems establishes *necessary* existence conditions; the second theorem asserts that the latter are also *sufficient*.

THEOREM 5.1: *Let* $\hat{\mathbf{y}}: D \to E_2$ *be a regular deformation in* $C^4(D)$ *and*

$$(\nabla\hat{\mathbf{y}})(\nabla\hat{\mathbf{y}})^T = \mathbf{G} \quad \text{or} \quad \hat{y}_{\alpha,\gamma}\hat{y}_{\beta,\gamma} = G_{\alpha\beta} \quad \text{on } D, \tag{5.1}$$

so that $\mathbf{G} \in C^3(D) \cap S^+$. *Let*

$$\mathbf{V} = \sqrt{\mathbf{G}} \quad \text{on } D, \tag{5.2}$$

$$m_\alpha = \varepsilon_{\lambda\mu} V^{-1}_{\alpha\beta} V_{\beta\mu,\lambda} - \varepsilon_{\alpha\beta} V^{-1}_{\beta\lambda} V_{\lambda\mu,\mu}, \tag{5.3}$$

$$n_\alpha = \varepsilon_{\lambda\mu} V^{-1}_{\alpha\beta} V_{\beta\mu,\lambda} + \varepsilon_{\alpha\beta} V^{-1}_{\beta\lambda} V_{\lambda\mu,\mu} \quad on\ D,$$

$$p = n_{\alpha,\alpha} + \varepsilon_{\alpha\beta} m_\alpha n_\beta,\ q = \varepsilon_{\alpha\beta} n_{\alpha,\beta} + m_\alpha n_\alpha,\ r = \varepsilon_{\alpha\beta} m_{\beta,\alpha} - n_\alpha n_\alpha \quad on\ D. \tag{5.4}$$

Then

$$p^2 + q^2 - r^2 \geqslant 0 \quad on\ D \tag{5.5}$$

and the scalar field Δ, *given by*

$$\Delta = \sqrt{p^2 + q^2 - r^2} \quad on\ D, \tag{5.6}$$

is in $C^1(D)$. *Moreover, if either* $\Delta = 0$ *on* D *or* Δ *has at most isolated zeros on* D, *the following equation holds throughout with at least one of the two sign alternatives:*

$$(p^2 + q^2)m_\alpha + r(\delta_{\alpha\beta} q + \varepsilon_{\alpha\beta} p) n_\beta + q_{,\alpha} p - p_{,\alpha} q$$
$$\pm [\Delta(\delta_{\alpha\beta} p - \varepsilon_{\alpha\beta} q) n_\beta + \Delta_{,\alpha} r - r_{,\alpha} \Delta] = 0 \quad on\ D \tag{5.7}$$

Proof: The smoothness, symmetry, and positive definiteness of $\mathbf{G}$ are immediate from (5.1) together with the regularity and assumed additional smoothness of $\hat{\mathbf{y}}$. Further, by (5.1) and the left polar decomposition of $\nabla\hat{\mathbf{y}}$, there is a tensor field $\mathbf{Q} \in C^3(D)$, such that

$$\nabla\hat{\mathbf{y}} = \mathbf{VQ} \quad \text{or} \quad \hat{y}_{\alpha,\lambda} = V_{\alpha\beta} Q_{\beta\lambda} \quad \text{on } D, \quad \mathbf{V} = \sqrt{\mathbf{G}}, \quad \mathbf{Q} \in O_+ \quad \text{on } D. \tag{5.8}$$

Accordingly, $\mathbf{Q}$ obeys

$$\text{curl}(\mathbf{VQ}) = \mathbf{0} \quad \text{or} \quad \varepsilon_{\lambda\mu}(V_{\alpha\beta} Q_{\beta\lambda})_{,\mu} = 0 \quad \text{on } D. \tag{5.9}$$

The above system constitutes a pair of scalar equations, which is found to be equivalent to the quasilinear system of partial differential equations

$$Q_{\alpha\beta,\gamma} = Q_{\alpha\lambda}[Q_{\rho\lambda}\Omega_{\rho\beta\gamma} - Q_{\rho\beta}\Omega_{\rho\lambda\gamma} - Q_{\rho\gamma}\Omega_{\rho\lambda\beta}] \quad \text{on } D, \tag{5.10}$$

where

$$\Omega_{\rho\beta\gamma} = -\Omega_{\rho\gamma\beta} = \tfrac{1}{2} V^{-1}_{\rho\lambda}(V_{\lambda\mu,\beta} Q_{\mu\gamma} - V_{\lambda\mu,\gamma} Q_{\mu\beta}) \quad \text{on } D. \tag{5.11}$$

Since $\mathbf{Q} \in O_+$ on D, $\mathbf{Q}$ admits the representation

$$Q_{\alpha\beta} = \delta_{\alpha\beta} \cos \omega - \varepsilon_{\alpha\beta} \sin \omega \quad \text{on } D, \quad \omega \in [0, 2\pi). \tag{5.12}$$

As is observed in the proof of Theorem 4.1, the scalar field ω may not be continuous on D. However, it is apparent from (5.12) and the smoothness of $\mathbf{Q}$ that the functions $\cos \omega$ and $\sin \omega$ are in $C^3(D)$, and thus ω is three times continuously differentiable on any subdomain of D on which it is continuous.

Let

$$\chi_\alpha = \cos 2\omega (\sin 2\omega)_{,\alpha} - \sin 2\omega (\cos 2\omega)_{,\alpha} \quad \text{on } D. \tag{5.13}$$

On any domain of continuity of ω, (5.13) reduces to $\chi_\alpha = 2\omega_{,\alpha}$. Multiplication of (5.10) by $\varepsilon_{\beta\gamma} Q_{\alpha\sigma}$ and a subsequent appeal to (5.11), (5.12), in view of (5.13), yields

$$\chi_\alpha = m_\alpha + (\delta_{\alpha\beta} \cos 2\omega + \varepsilon_{\alpha\beta} \sin 2\omega) n_\beta \quad \text{on } D, \tag{5.14}$$

with m_α and n_β defined through (5.3). Differentiating (5.14), using (5.13), and invoking (5.14) once again, one eventually arrives at

$$\varepsilon_{\alpha\lambda} \chi_{\alpha,\lambda} = p \sin 2\omega + q \cos 2\omega - r = 0 \quad \text{on } D, \tag{5.15}$$

where p, q, r are given by (5.4) and are in $C^1(D)$. From (5.15) follows the quadratic equation for $\cos 2\omega$:

$$(p^2 + q^2) \cos^2 2\omega - 2qr \cos 2\omega + r^2 - p^2 = 0 \quad \text{on } D. \tag{5.16}$$

The latter has real roots only if (5.5) holds. Moreover, (5.15), (5.16), (5.6) demand that either

$$(p^2 + q^2) \cos 2\omega = qr + p\Delta, \quad (p^2 + q^2) \sin 2\omega = pr - q\Delta$$

or (5.17)

$$(p^2 + q^2) \cos 2\omega = qr - p\Delta, \quad (p^2 - q^2) \sin 2\omega = pr + q\Delta$$

at *each* point of D. Note that the two alternatives in (5.17) coalesce only at zeros of Δ. From (5.17) one draws

$$(p^2 + q^2)[p \cos 2\omega - q \sin 2\omega] = (p^2 + q^2)\Delta \quad \text{on } D \tag{5.18}$$

and, since (5.5), (5.6) imply that $\Delta = 0$ at a common zero of p and q,

$$p \cos 2\omega - q \sin 2\omega = \Delta \quad \text{on } D. \tag{5.19}$$

The smoothness of p, q, $\cos 2\omega$, $\sin 2\omega$ now assures that $\Delta \in C^1(D)$.

Suppose first that Δ has at most isolated zeros[1] on D. In this event a continuity argument confirms that either the first two or the last two of (5.17) hold at *all* points of D.

Next, if Δ vanishes identically on D, the two alternatives in (5.17) coalesce and hold on D. Thus in this case, as well as if Δ has at most isolated zeros on D, one infers from (5.17), (5.13),

$$(p^2 + q^2)\chi_\alpha = qp_{,\alpha} - pq_{,\alpha} \pm (\Delta r_{,\alpha} - r\Delta_{,\alpha}) \quad \text{on } D. \tag{5.20}$$

The upper or lower sign in (5.20) holds throughout D according as the first or second possibility in (5.17) is valid on D. Combining (5.17), (5.20), (5.14), one arrives at (5.7). This completes the derivation of the necessary conditions (5.5) and (5.7).

We now proceed to a theorem that establishes the sufficiency of (5.5) and (5.7).

THEOREM 5.2: *Let D be bounded and simply connected. Let $\mathbf{G}: D \to S^+$ be in $C^3(D)$ and continuously differentiable on the closure $\bar{D}$ of D, $\mathbf{V} = \sqrt{\mathbf{G}}$. Suppose the fields $\mathbf{m}$, $\mathbf{n}$, p, q, and r, defined through* (5.3), (5.4), *obey* (5.5). *Further, let Δ, given by* (5.6), *be in $C^1(D)$ and satisfy* (5.7) *throughout D with at least one of the two sign alternatives. Then if either $p^2 + q^2 \neq 0$ on D or $p = q = 0$ on D, there exists a regular deformation $\hat{\mathbf{y}}: D \to E_2$, $\hat{y} \in C^4(D)$, such that* (5.1) *holds.*

Proof: In order to show that (5.5), (5.7) guarantee the existence of a generating deformation $\hat{\mathbf{y}}$, it is sufficient to show that these "compatibility conditions" for $\mathbf{G}$ imply the existence of a proper-orthogonal tensor field $\mathbf{Q} \in C^3(D)$ that satisfies (5.9). To see this, note that if $\mathbf{Q}$ is such a tensor field, one may define $\hat{\mathbf{y}}$ through

$$\hat{\mathbf{y}}(\mathbf{x}) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \mathbf{V}(\boldsymbol{\xi})\mathbf{Q}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \quad \text{or} \quad \hat{y}_\alpha(x) = \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} V_{\alpha\beta}(\boldsymbol{\xi})Q_{\beta\gamma}(\boldsymbol{\xi})\, d\xi_\gamma \quad \text{on } D, \tag{5.21}$$

provided $\mathring{\mathbf{x}} \in D$ is fixed. The above line integral is path-independent by virtue of (5.9); the deformation $\hat{\mathbf{y}}$ so determined is regular, in $C^4(D)$, and – on account of (5.2) and the orthogonality of $\mathbf{Q}$ – obeys (5.1).

[1] It is actually sufficient to assume that the subset of D on which Δ fails to vanish is connected.

Assume first that p and q have no common zeros on D. Set

$$c = \frac{qr \pm p\Delta}{p^2+q^2}, \quad s = \frac{pr \mp q\Delta}{p^2+q^2} \quad \text{on } D, \tag{5.22}$$

where Δ is given by (5.6) and the upper or lower sign is used according as (5.7) holds with the upper or lower sign. The functions c and s are at least once continuously differentiable on D, and one verifies with the aid of (5.6) that

$$c^2 + s^2 = 1 \quad \text{on } D. \tag{5.23}$$

Differentiation of (5.22) leads to

$$\begin{aligned} c_{,\alpha} &= \frac{-s}{p^2+q^2}[qp_{,\alpha} - pq_{,\alpha} \pm (\Delta r_{,\alpha} - r\Delta_{,\alpha})], \\ s_{,\alpha} &= \frac{c}{p^2+q^2}[qp_{,\alpha} - pq_{,\alpha} \pm (\Delta r_{,\alpha} - r\Delta_{,\alpha})] \quad \text{on} \quad D. \end{aligned} \tag{5.24}$$

In view of (5.7), (5.22), one may write (5.24) as

$$\begin{aligned} c_{,\alpha} &= -s[m_\alpha + (\delta_{\alpha\beta}c + \varepsilon_{\alpha\beta}s)n_\beta], \\ s_{,\alpha} &= c[m_\alpha + (\delta_{\alpha\beta}c + \varepsilon_{\alpha\beta}s)n_\beta] \quad \text{on } D. \end{aligned} \tag{5.25}$$

Because of the smoothness of $\mathbf{m}$ and $\mathbf{n}$, (5.25) is easily seen to imply that c and s are actually in $C^3(D)$. Let $\mathring{\mathbf{x}}$ be a fixed point in D and define $\omega \in C^3(D)$ through

$$\omega(\mathbf{x}) = \frac{1}{2}\int_{\mathring{\mathbf{x}}}^{\mathbf{x}} [c(\boldsymbol{\xi})s_{,\alpha}(\boldsymbol{\xi}) - s(\boldsymbol{\xi})c_{,\alpha}(\boldsymbol{\xi})]\, \mathrm{d}\xi_\alpha + \mathring{\omega} \quad \text{on } D, \tag{5.26}$$

in which $\mathring{\omega}$ is a constant determined by

$$\cos 2\mathring{\omega} = c(\mathring{\mathbf{x}}), \quad \sin 2\mathring{\omega} = s(\mathring{\mathbf{x}}), \quad \mathring{\omega} \in [0, \pi). \tag{5.27}$$

The line integral in (5.26) is path-independent in view of (5.23). Further, one finds that (5.26), (5.27) require

$$\cos 2\omega = c, \quad \sin 2\omega = s \quad \text{on } D, \tag{5.28}$$

which, in conjunction with (5.26), (5.25), (5.23), justifies

$$\omega_{,\alpha} = \tfrac{1}{2}[m_\alpha + (\delta_{\alpha\beta} \cos 2\omega + \varepsilon_{\alpha\beta} \sin 2\omega) n_\beta] \quad \text{on } D. \tag{5.29}$$

Next, define $\mathbf{Q} \in C^3(D)$ through

$$Q_{\alpha\beta} = \delta_{\alpha\beta} \cos \omega - \varepsilon_{\alpha\beta} \sin \omega \quad \text{on } D. \tag{5.30}$$

Evidently, $\mathbf{Q} \in O_+$ on D and, because of (5.29), one has

$$Q_{\alpha\beta,\lambda} = \tfrac{1}{2}\varepsilon_{\beta\rho} Q_{\alpha\rho}(m_\lambda + Q_{\tau\lambda} Q_{\sigma\tau} n_\sigma) \quad \text{on } D, \tag{5.31}$$

whence

$$\varepsilon_{\beta\lambda} Q_{\alpha\beta,\lambda} = \tfrac{1}{2}(Q_{\alpha\lambda} m_\lambda + Q_{\lambda\alpha} n_\lambda) \quad \text{on } D. \tag{5.32}$$

Substituting into (5.32) for $Q_{\alpha\lambda}$, m_λ, and n_λ from (5.30) and (5.3), one arrives at

$$\varepsilon_{\beta\lambda} V_{\rho\alpha} Q_{\alpha\beta,\lambda} = -\varepsilon_{\beta\lambda} V_{\rho\beta,\lambda} \cos \omega - V_{\rho\beta,\beta} \sin \omega \quad \text{on } D, \tag{5.33}$$

or, on appealing to (5.30) once again,

$$\varepsilon_{\beta\lambda} V_{\rho\alpha} Q_{\alpha\beta,\lambda} = -\varepsilon_{\beta\lambda} V_{\rho\alpha,\lambda} Q_{\alpha\beta} \quad \text{on } D. \tag{5.34}$$

Thus, (5.9) holds.

Next, suppose $p = q = 0$ on D. In this case (5.5) requires $r = 0$ on D and (5.7) holds trivially. Moreover, (5.4) reduces to

$$n_{\alpha,\alpha} + \varepsilon_{\alpha\beta} m_\alpha n_\beta = 0, \quad \varepsilon_{\alpha\beta} n_{\alpha,\beta} + m_\alpha n_\alpha = 0, \quad \varepsilon_{\alpha\beta} m_{\beta,\alpha} - n_\alpha n_\alpha = 0 \quad \text{on } D. \tag{5.35}$$

If, in addition, $\mathbf{n} = \mathbf{0}$ on D, (5.35) becomes

$$\varepsilon_{\alpha\beta} m_{\alpha,\beta} = 0 \quad \text{on } D, \tag{5.36}$$

so that one may define a scalar field $\omega \in C^3(D)$ through the path-independent line integral

$$\omega(\mathbf{x}) = \frac{1}{2} \int_{\mathring{\mathbf{x}}}^{\mathbf{x}} m_\alpha(\boldsymbol{\xi})\, \mathrm{d}\xi_\alpha \quad \text{on } D, \tag{5.37}$$

where $\mathring{\mathbf{x}} \in D$ is an arbitrary fixed point. Let $\mathbf{Q} \in C^3(D)$ be given by (5.30), whence $\mathbf{Q} \in O_+$ on D. In the present instance one finds from (5.37), (5.30) that

(5.31) holds once again. This, as before, entitles one to assert that $\mathbf{Q}$ satisfies (5.9).

We now turn to the case in which $p = q = 0$ on D, but $\mathbf{n}$ fails to vanish indentically on D. We shall show that in these circumstances, the problem of finding a tensor field $\mathbf{Q} \in O_+ \cap C^3(D)$ obeying (5.9) is reducible to the integration of an ordinary linear differential equation of the second order.

Suppose for the time being that there *exists* a $\mathbf{Q} \in O_+ \cap C^3(D)$ satisfying (5.9), and hence also (5.10). Such a tensor field admits the representation (5.12), and, because of (5.10), ω obeys (5.14) with χ_α given by (5.13).

The assumed smoothness of $\mathbf{G}$, along with (5.2), implies that the vector field $\mathbf{m}$ defined in (5.3) can be continuously extended onto $\bar{D}$. If $\mathbf{m}$ now stands for this extension, $\mathbf{m} \in C^2(D) \cap C(\bar{D})$ and admits a Helmholtz resolution

$$m_\alpha = \theta_{,\alpha} - \varepsilon_{\alpha\beta}\phi_{,\beta} \quad \text{on } D, \tag{5.38}$$

where the scalar potentials θ and ϕ are both in $C^2(D)$. Next, define functions $f_\alpha \in C^2(D)$ through

$$f_\alpha = \mathrm{e}^{-\phi}(\delta_{\alpha\beta} \cos\theta + \varepsilon_{\alpha\beta} \sin\theta) n_\beta \quad \text{on } D. \tag{5.39}$$

Equations (5.39), (5.38), and the first two of (5.35), by direct calculation, are found to yield

$$f_{\alpha,\alpha} = 0, \quad \varepsilon_{\alpha\beta} f_{\alpha,\beta} = 0, \quad \text{or} \quad f_{2,\lambda} = \varepsilon_{\lambda\mu} f_{1,\mu} \quad \text{on } D. \tag{5.40}$$

Accordingly, f_2, f_1 are conjugate harmonic functions and are thus in $C^\infty(D)$. Further, from (5.39),

$$f_\alpha f_\alpha = \mathrm{e}^{-2\phi} n_\alpha n_\alpha \quad \text{on } D. \tag{5.41}$$

The zeros of $\mathbf{n}$ therefore coincide with those of $\mathbf{f}$ and are necessarily isolated since f_2, f_1 are harmonic conjugates and $\mathbf{n}$ is assumed not to vanish identically on D. Combining (5.38), (5.41), and the third of (5.35), one obtains

$$\phi_{,\alpha\alpha} = f_\alpha f_\alpha \, \mathrm{e}^{2\phi} \quad \text{on } D. \tag{5.42}$$

The harmonicity of $\mathbf{f}$ on D now implies[1] that $\phi \in C^\infty(D)$.

Substitution from (5.38) into (5.14) yields

$$\chi_\alpha - \theta_{,\alpha} = -\varepsilon_{\alpha\beta}\phi_{,\beta} + (\delta_{\alpha\beta} \cos 2\omega + \varepsilon_{\alpha\beta} \sin 2\omega) n_\beta \quad \text{on } D. \tag{5.43}$$

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[1] See, for example, Courant and Hilbert [9], p. 502.

If one sets

$$\sigma = 2\omega - \theta \quad \text{on } D, \tag{5.44}$$

the smoothness of $\cos\omega$, $\sin\omega$, ω, and θ implies that σ is twice continuously differentiable on any domain of continuity of ω, while $\cos\sigma$ and $\sin\sigma$ are both in $C^2(D)$. Next, let $\Phi_\alpha \in C^1(D)$ be given by

$$\Phi_\alpha = \cos\sigma(\sin\sigma)_{,\alpha} - \sin\sigma(\cos\sigma)_{,\alpha} = \chi_\alpha - \theta_{,\alpha} \quad \text{on } D. \qquad (5.45)^{(1)}$$

On any subdomain of D on which ω is continuous (5.45) reduces to $\Phi_\alpha = \sigma_{,\alpha}$. On account of (5.45), (5.44), and (5.39), Equation (5.43) can be written as

$$\Phi_\alpha = -\varepsilon_{\alpha\beta}\phi_{,\beta} + \mathrm{e}^{\phi}(\delta_{\alpha\beta}\cos\sigma + \varepsilon_{\alpha\beta}\sin\sigma)f_\beta \quad \text{on } D. \tag{5.46}$$

A direct calculation based on (5.46), (5.45), and (5.40) gives

$$\Phi_{\alpha,\alpha} = 0 \quad \text{on } D. \tag{5.47}$$

One may thus define a scalar field $\rho \in C^2(D)$ through

$$\rho(\mathbf{x}) = \exp\left\{\int_{\mathring{\mathbf{x}}}^{\mathbf{x}} \varepsilon_{\alpha\beta}\Phi_\alpha(\boldsymbol{\xi})\,\mathrm{d}\xi_\beta\right\} \quad \text{on } D, \tag{5.48}$$

where $\mathring{\mathbf{x}} \in D$ is fixed, the line integral here involved being path-independent by virtue of (5.47).

Next, let $z = x_1 + ix_2$. One verifies with the aid of (5.48), (5.45) that the complex-valued function **h** defined by

$$h(z) = \rho(\mathbf{x})\,\mathrm{e}^{-i\sigma(\mathbf{x})/2} \quad \text{on } D, \quad z = x_1 + ix_2, \tag{5.49}$$

is analytic and nonzero on D and that

$$h'(z) \equiv \mathrm{d}h(z)/\mathrm{d}z = -\tfrac{1}{2}h(z)[\Phi_2(\mathbf{x}) + i\Phi_1(\mathbf{x})] \quad \text{on } D. \tag{5.50}$$

Further, if f_α is the pair of functions defined in (5.39), let

$$f(z) = f_2(\mathbf{x}) + if_1(\mathbf{x}) \quad \text{on } D, \quad z = x_1 + ix_2. \tag{5.51}$$

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(1) Recall (5.13).

In view of (5.40), the function f is analytic on D. Finally, let $\hat{\phi}(z,\bar{z})$ be determined by

$$\hat{\phi}(z,\bar{z}) = \phi(\mathbf{x}) \quad \text{on } D, \quad z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2. \tag{5.52}$$ [1]

Clearly, $\hat{\phi}$ is real-valued, $\hat{\phi} \in C^\infty(D)$, and

$$\begin{aligned} &\hat{\phi}_z(z,\bar{z}) = \tfrac{1}{2}[\phi_{,1}(\mathbf{x}) - i\phi_{,2}(\mathbf{x})],\\ &\hat{\phi}_{\bar{z}}(z,\bar{z}) = \overline{\hat{\phi}_z(z,\bar{z})}, \quad \hat{\phi}_{z\bar{z}}(z,\bar{z}) = \tfrac{1}{4}\phi_{,\alpha\alpha}(\mathbf{x}),\\ &\hat{\phi}_{zz}(z,\bar{z}) = \tfrac{1}{4}[\phi_{,11}(\mathbf{x}) - \phi_{,22}(\mathbf{x}) - 2i\phi_{,12}(\mathbf{x})], \quad \hat{\phi}_{\bar{z}\bar{z}}(z,\bar{z}) = \overline{\hat{\phi}_{zz}(z,\bar{z})} \quad \text{on } D. \end{aligned} \tag{5.53}$$ [2]

Combining (5.50), (5.46) and using (5.51), (5.52), (5.53), one arrives at

$$-2h'(z) = 2h(z)\hat{\phi}_z + h(z)f(z)\,\mathrm{e}^{\hat{\phi}+i\sigma(\mathbf{x})} \quad \text{on } D, \quad \hat{\phi} = \hat{\phi}(z,\bar{z}), \tag{5.54}$$

so that, because of (5.49)

$$-2\,\mathrm{e}^{-\hat{\phi}}[h'(z) + h(z)\hat{\phi}_z] = f(z)\overline{h(z)} \quad \text{on } D. \tag{5.55}$$

Partial differentiation of (5.55) with respect to z yields[3]

$$-2\,\mathrm{e}^{-\hat{\phi}}\{h''(z) + h(z)[\hat{\phi}_{zz} - \hat{\phi}_z^2]\} = f'(z)\overline{h(z)} \quad \text{on } D, \tag{5.56}$$

and on eliminating $\overline{h(z)}$ between (5.55), (5.56), one is led to

$$f(z)h''(z) - f'(z)h'(z) + \hat{g}(z,\bar{z})h(z) = 0 \quad \text{on } D, \tag{5.57}$$

provided

$$\hat{g}(z,\bar{z}) = f(z)[\hat{\phi}_{zz} - \hat{\phi}_z^2] - f'(z)\hat{\phi}_z \quad \text{on } D, \quad \hat{\phi} = \hat{\phi}(z,\bar{z}). \tag{5.58}$$

Since f and $\hat{\phi}$ are determined by $\mathbf{G}$, the same is true of $\hat{g}$, which is in $C^\infty(D)$.

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[1] Here and in the sequel, a superior bar indicates complex conjunction. Moreover, we say that $\hat{\phi}(z,\bar{z})$ is defined for every pair of complex conjugate numbers $(z,\bar{z})$ such that z is in D.

[2] We write $\hat{\phi}_z$, $\hat{\phi}_{z\bar{z}}$ in place of $\partial\hat{\phi}/\partial z$, $\partial^2\hat{\phi}/\partial z\,\partial\bar{z}$, respectively. The subscripts z and $\bar{z}$ are thus exempt from the usual range and summation convention in what follows.

[3] Observe that if $h(z) = h_1(\mathbf{x}) + ih_2(\mathbf{x})$ is analytic on D and $\hat{H}(z,\bar{z}) = \overline{h(z)}$ on D, then $\hat{H}_z(z,\bar{z}) \equiv \{h_{1,1}(\mathbf{x}) - h_{2,2}(\mathbf{x}) - i[h_{1,2}(\mathbf{x}) + h_{2,1}(\mathbf{x})]\}/2 = 0$ by the Cauchy-Riemann equations.

Moreover, it is clear from (5.57) that $\hat{g}(z, \bar{z})$ is independent of $\bar{z}$ and hence analytic[1] on D.

The preceding considerations which *presupposed* the existence of the requisite proper orthogonal tensor field $\mathbf{Q}$, serve to motivate the following *construction* of such a tensor field. For this purpose, guided by (5.57), we start by considering the ordinary differential equation

$$f(z)w''(z) - f'(z)w'(z) + \hat{g}(z, \bar{z})w(z) = 0 \quad \text{on } D, \tag{5.59}$$

where f is the analytic[2] function defined by (5.51), (5.39), (5.38), and $\hat{g}$ is given by (5.58), (5.52), (5.38). The analyticity of $\hat{g}$ on D may be deduced from its definition and known properties of $\hat{\phi}$ and f as follows. Observe first from (5.51), (5.52), (5.53) that (5.42) may be written as

$$4\hat{\phi}_{z\bar{z}} = f(z)\overline{f(z)}\, e^{2\hat{\phi}} \quad \text{on } D, \quad \hat{\phi} = \hat{\phi}(z, \bar{z}), \tag{5.60}$$

so that

$$4\hat{\phi}_{zz\bar{z}} = \overline{f(z)}\, e^{2\hat{\phi}}[f'(z) + 2f(z)\hat{\phi}_z] \quad \text{on } D. \tag{5.61}$$

Multiplication of (5.61) by $f(z)$, in view of (5.60), yields

$$f(z)\hat{\phi}_{zz\bar{z}} - [f'(z) + 2f(z)\hat{\phi}_z]\hat{\phi}_{z\bar{z}} = 0 \quad \text{on } D, \tag{5.62}$$

which, on account of (5.58), implies

$$\hat{g}_{\bar{z}}(z, \bar{z}) = 0, \quad \hat{g}(z, \bar{z}) = g(z), \quad g \text{ analytic on } D. \tag{5.63}$$

Accordingly, the differential equation (5.59) becomes

$$f(z)w''(z) - f'(z)w'(z) + g(z)w(z) = 0 \quad \text{on } D \tag{5.64}$$

and thus has analytic coefficients.

We now show that (5.64) admits a nontrivial solution, analytic on D, which may be used to construct a proper-orthogonal tensor field $\mathbf{Q} \in C^3(D)$ obeying (5.9).

The existence of two linearly independent analytic solutions of (5.64) in a neighborhood of any point z_* of D at which $f(z_*) \neq 0$ (ordinary point) is

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[1] If $\hat{g}(z, \bar{z}) \equiv g_1(\mathbf{x}) + g_2(\mathbf{x})$ on D, $\hat{g}_{\bar{z}}(z, \bar{z}) \equiv \{g_{1,1}(\mathbf{x}) - g_{2,2}(\mathbf{x}) + i[g_{1,2}(\mathbf{x}) + g_{2,1}(\mathbf{x})]\}/2 = 0$ on D implies the Cauchy-Riemann equations.

[2] See equation (5.40).

guaranteed by the theory of linear ordinary differential equations.[1] We confirm next that the general solution of (5.64) is also analytic in a neighborhood of a zero of f. Let $z_* \in D$ be a (necessarily isolated) zero of f. Thus there is a neighborhood N of z_*, such that

$$f(z) = (z - z_*)^M \Lambda(z), \quad \Lambda \neq 0 \quad \text{on } N, \tag{5.65}$$

where M is a positive integer and Λ is analytic on N. From (5.65), (5.58), (5.63), one infers that z_* is a regular singular point of (5.64) and that the roots of the corresponding indicial equation are zero and $M + 1$. Therefore,[2] the differential equation (5.64) has a solution on N of the form

$$w_1(z) = (z - z_*)^{M+1}\Psi(z), \quad \Psi \text{ analytic on } N, \quad \Psi(z_*) \neq 0. \tag{5.66}$$

The existence of a second analytic solution of (5.64) on N that is linearly independent of w_1 is established next. Let

$$\hat{w}_2(z, \bar{z}) = e^{-\phi}[\overline{w_1'(z)} + \overline{w_1(z)}\phi_{\bar{z}}]/\overline{f(z)} \quad \text{on } N, \quad z \neq z_*. \tag{5.67}$$

Clearly, $\hat{w}_2$ is in C^∞ on its domain of definition. Differentiating[3] (5.67) and bearing in mind (5.53), (5.58), (5.63), and that w_1 satisfies (5.64), one arrives at

$$\partial \hat{w}_2(z, \bar{z})/\partial \bar{z} = 0, \quad \hat{w}_2(z, \bar{z}) = w_2(z) \quad \text{on } N, \quad z \neq z_*, \tag{5.68}$$

with w_2 analytic on the deleted neighborhood of z_* at hand. To see that w_2 has a removable singularity at z_*, substitute from (5.65), (5.66) into (5.67) to obtain

$$\lim_{z \to z_*} w_2(z) = (M + 1) \exp[-\hat{\phi}(z_*, \bar{z}_*)]\overline{\Psi(z_*)}/\overline{\Lambda(z_*)}. \tag{5.69}$$

A direct calculation based on (5.67) and involving (5.58), (5.63), (5.62) enables one to conclude that w_2 satisfies (5.64) on N. Further, since $\Psi(z_*) \neq 0$, (5.69) implies $w_2(z_*) \neq 0$. But $w_1(z_*) = 0$ by (5.66), so that w_1 and w_2 are linearly independent on N.

Thus, there are two linearly independent, analytic solutions of (5.64) in a neighborhood of *each* point on D. Accordingly, either of these two solutions can be continued analytically along any path in the simply connected domain

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(1) See, for example, Copson [10], Section 10.11.

(2) See Copson [10], Section 10.15.

(3) Since w_1 is analytic on N, one has $\mathrm{d}W_1(\bar{z})/\mathrm{d}\bar{z} = \overline{\mathrm{d}w_1(z)/\mathrm{d}z}$ on N, provided $W_1(\bar{z}) = \overline{w_1(z)}$ on N.

D by the usual circle-chain argument. Moreover, it follows from the Monodromy Theorem[1] that these analytic continuations give rise to a function that is analytic on the entire domain D. Finally, since the function so generated satisfies (5.64) in a neighborhood of a point of D, it follows from the Identity Theorem that this function satisfies the differential equation throughout D.

In view of what preceded, one is assured of the existence of a non-trivial solution w of (5.64) that is analytic on D. Let D_0 be the subdomain of D on which $f \neq 0$. Define a complex-valued function $\hat{h} \in C^\infty(D_0)$ through

$$\hat{h}(z, \bar{z}) = \kappa w(z) - 2\bar{\kappa}\, \mathrm{e}^{-\hat{\phi}}[\overline{w'(z)} + \overline{w(z)}\hat{\phi}_{\bar{z}}]/\overline{f(z)} \quad \text{on } D_0, \quad \hat{\phi} = \hat{\phi}(z, \bar{z}), \quad (5.70)$$

where κ is a complex constant, chosen so that $\hat{h}$ fails to vanish identically on D_0. One now infers with the aid of (5.58), (5.63), (5.64), and (5.53) that

$$\hat{h}_{\bar{z}}(z, \bar{z}) = 0, \quad \hat{h}(z, \bar{z}) = h(z), \quad h \text{ analytic on } D_0. \quad (5.71)$$

We show next that h has a removable singularity at each zero of f. To this end, let z_* be a zero of f and N be the neighborhood of z_* introduced earlier. Then

$$w(z) = a w_1(z) + b w_2(z) \quad \text{on } N. \quad (5.72)$$

where a, b are complex constants and w_1, w_2 are the two linearly independent solutions of (5.64) established previously. One confirms readily by recourse to (5.72), (5.67), (5.70), and (5.60) that

$$h(z) = \tfrac{1}{2}(2\kappa a - \bar{\kappa}\bar{b}) w_1(z) - (2\bar{\kappa}\bar{a} - \kappa b) w_2(z) \quad \text{on } N. \quad (5.73)$$

Thus, h admits an analytic continuation onto D.

Next, we show that $h \neq 0$ on D. A direct calculation starting from (5.70) and making use of (5.60) confirms that h satisfies (5.55). Suppose now that there were a $z_0 \in D$, such that $h(z_0) = 0$. Then $h'(z_0) = 0$ by (5.55) and successive differentiations of this equation would require the derivatives of h of *all* orders to vanish at z_0. This, in turn, would necessitate h to vanish identically on D which is a contradiction[2]. Hence, $h \neq 0$ on D.

Let

$$h_1(\mathbf{x}) = \mathrm{Re}\{h(z)\}, \quad h_2(\mathbf{x}) = \mathrm{Im}\{h(z)\} \quad \text{on } D, \quad (5.74)$$

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[1] See, for example, Knopp [11], §25.

[2] See the remark following (5.70).

and define $\sigma \in C^{\infty}(D)$ through the path-independent line integral

$$\sigma(\mathbf{x}) = 2\int_{\mathring{\mathbf{x}}}^{\mathbf{x}} [h_2(\boldsymbol{\xi})h_{1,\alpha}(\boldsymbol{\xi}) - h_1(\boldsymbol{\xi})h_{2,\alpha}(\boldsymbol{\xi})]/[h_1^2(\boldsymbol{\xi}) + h_2^2(\boldsymbol{\xi})]\, \mathrm{d}\xi_\alpha + \mathring{\sigma} \quad \text{on } D, \tag{5.75}$$

where $\mathring{\mathbf{x}} \in D$ and

$$\mathrm{e}^{-i\mathring{\sigma}/2} = h(\mathring{z})/|h(\mathring{z})| \quad \text{on } D, \quad \mathring{z} = \mathring{x}_1 + i\mathring{x}_2, \quad \mathring{\sigma} \in [0, 4\pi). \tag{5.76}$$

From (5.74), (5.75), (5.76) follows

$$\mathrm{e}^{-i\sigma(\mathbf{x})/2} = h(z)/|h(z)| \quad \text{on } D, \quad z = x_1 + ix_2, \tag{5.77}$$

which implies

$$-2h'(z)/h(z) = \sigma_{,2}(\mathbf{x}) + i\sigma_{,1}(\mathbf{x}) \quad \text{on } D. \tag{5.78}$$

Substitution from (5.77), (5.78) and (5.52), (5.53) into (5.55), in view of (5.51), eventually leads to

$$\sigma_{,\alpha} = -\varepsilon_{\alpha\beta}\phi_{,\beta} + \mathrm{e}^{\phi}(\delta_{\alpha\beta}\cos\sigma + \varepsilon_{\alpha\beta}\sin\sigma)f_\beta \quad \text{on } D. \tag{5.79}$$

If θ is the scalar potential introduced in (5.38), define $\omega \in C^2(D)$ through

$$\omega = \tfrac{1}{2}(\theta + \sigma) \quad \text{on } D. \tag{5.80}$$

Eliminating σ between (5.79), (5.80) and using (5.38), (5.39), one verifies that ω satisfies (5.29) also in the present circumstances. Moreover, $\omega \in C^3(D)$ because $\mathbf{m}$ and $\mathbf{n}$ are in $C^2(D)$. Finally, let $\mathbf{Q}$ be defined by (5.30), so that $\mathbf{Q} \in C^3(D) \cap O_+$. As seen before, (5.29), (5.30) imply that (5.31) holds, and the latter, along with (5.3), enables one to conclude that $\mathbf{Q}$ obeys (5.9). The line integral (5.21) is thus path-independent and $\hat{\mathbf{y}}$ so determined is regular, in $C^4(D)$, and satisfies (5.1). This completes the proof.

In order to illustrate the construction of a regular deformation generating a given left strain-tensor field for which $p = q = 0$ on D, let $D \subset \{\mathbf{x} \mid x_1 > 0\}$ and suppose

$$[G_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/x_1^2 \end{bmatrix} \quad \text{on } D. \tag{5.81}$$

Evidently, $\mathbf{G}\in C^{\infty}(D)$, $\mathbf{G}\in S^{+}$, and (5.2) gives

$$[V_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/x_1 \end{bmatrix} \quad \text{on } D. \tag{5.82}$$

Further, (5.3), (5.4) now yield

$$m_1 = n_1 = 0, \quad m_2 = n_2 = -1/x_1 \quad p = q = r = 0 \quad \text{on } D. \tag{5.83}$$

Thus, (5.5), (5.7) hold. One confirms that the scalar potentials given by

$$\theta = 0, \quad \phi = -\operatorname{Log} x_1 \quad \text{on } D \tag{5.84}$$

satisfy (5.38) in the present circumstances and (5.39), (5.51), (5.52), (5.58) lead to

$$f(z) = -1, \quad \hat{\phi}(z, \bar{z}) = -\operatorname{Log}[(z + \bar{z})/2], \quad g(z) = 0 \quad \text{on } D. \tag{5.85}$$

A nontrivial analytic solution of (5.64) in this case is supplied by $w(z) = 1$ on D. Choosing $\kappa = i$ in (5.70), one finds that

$$h(z) = 2i \quad \text{on } D, \tag{5.86}$$

and (5.75), (5.80), (5.84), (5.30) give

$$Q_{\alpha\beta} = \varepsilon_{\alpha\beta} \quad \text{on } D. \tag{5.87}$$

Finally taking $(\mathring{x}_1, \mathring{x}_2) = (1, 0)$ in (5.21), one arrives at

$$\hat{y}_1(\mathbf{x}) = x_2, \quad \hat{y}_2(\mathbf{x}) = -\operatorname{Log} x_1 \quad \text{on } D. \tag{5.88}$$

Theorems 4.1 and 4.2 may be used to infer that every regular deformation $\hat{\mathbf{y}}'(\mathbf{x}) \not\equiv \pm\hat{\mathbf{y}}(\mathbf{x}) + \mathbf{d}$ on D ($\mathbf{d}$ constant) giving rise to the left strain-tensor field (5.81) admits the representation

$$\begin{aligned} \hat{y}_1'(\mathbf{x}) &= \pm\sqrt{x_1^2 + (x_2 - k)^2} + d_1 \\ \hat{y}_2'(\mathbf{x}) &= \pm\operatorname{Log}\{[x_2 - k + \sqrt{x_1^2 + (x_2 - k)^2}]/x_1\} + d_2 \quad \text{on } D, \end{aligned} \tag{5.89}$$

with k and $\mathbf{d}$ constant.

For the purpose of relating the foregoing existence theorems to the uniqueness theorems of the previous section, let $\mathbf{G}$ be a tensor field that satisfies the hypotheses of Theorem 5.2, so that there is a regular deformation $\hat{y} \in C^4(D)$ obeying (5.1). If $\mathbf{Q}$ is the tensor field introduced in (5.8), and γ_α, a, b are the functions defined through (4.5), one confirms by means of (5.3), (5.12), (5.31) that

$$\gamma_\alpha = \varepsilon_{\alpha\beta} Q_{\lambda\beta} Q_{\mu\lambda} n_\mu \quad \text{on } D, \tag{5.90}$$

which, in conjunction with (5.31), (5.12), (5.15), is found to yield

$$a^2 + b^2 = p^2 + q^2, \quad a = -r, \quad b^2 = \Delta^2 \quad \text{on } D, \tag{5.91}$$

with p, q, r, Δ given by (5.4) and (5.5). It is clear from (5.90) that the zeros of $\boldsymbol{\gamma}$ and $\mathbf{n}$ coincide. Thus, one draws with the aid of (4.5) that $\mathbf{n} = \mathbf{0}$ on D is equivalent to

$$F_{\alpha\beta,\beta} = \hat{y}_{\alpha,\beta\beta} = 0 \quad \text{on } D, \tag{5.92}$$

whence $\hat{\mathbf{y}}$ is harmonic on D. Also, because of (5.91), the fields a and b vanish jointly if and only if the same is true of p and q. Further, one can show that γ_α, a, b obey (4.41) if and only if $\mathbf{G}$ satisfies (5.7) for *both* choices of sign. Accordingly, if $p^2 + q^2 \neq 0$ and Δ has at most isolated zeros on D, there is a second deformation $\hat{\mathbf{y}}'(\mathbf{x}) \not\equiv \pm\hat{\mathbf{y}}(\mathbf{x}) + \mathbf{d}$ on D ($\mathbf{d}$ constant) that gives rise to the left strain-tensor field at hand if and only if (5.7) holds for both sign alternatives.[1]

Note that Theorem 5.2 does not cover the possibility that p, q satisfy (5.5), (5.7) and have common zeros, but fail to vanish identically on D. One can show that if $p^2 + q^2$ have joint *isolated* zeros, the functions c and s defined in (5.22) have removable singularities at these zeros. Moreover, c and s may be used to construct the desired deformation by following the procedure adopted for the case in which $p^2 + q^2 \neq 0$ on D.

Finally, we remark that the results obtained in this section stand in marked contrast to their counterparts for plane deformations generating a given *right* strain-tensor field: in the latter case, the compatibility conditions require merely the vanishing of a single scalar field.

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[1] Recall Theorem 4.2 and the remarks immediately preceding it.

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