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On superposed small deformations on a large deformation of an elastic Cosserat surface

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ABSTRACT

Within the scope of the theory of a Cosserat surface, this paper is concerned with small deformations superposed on a large deformation in elastic shells and plates together with some related aspects of the subject. Special attention is given to problems of stability and vibrations of initially stressed isotropic plates.

RÉSUMÉ

Dans le cadre de la théorie d'une surface de Cosserat, le présent rapport étudie, pour des plaques et coques élastiques, de petites déformations superposées à une grande déformation, ainsi que certains aspects connexes du sujet. Une attention particulière est accordée aux problèmes de stabilité et de vibrations de plaques isotropiques initialement sous tension.

1. Introduction

Starting with the nonlinear theory of a Cosserat surface, our main purpose here is to consider small deformations superposed on a large deformation of an elastic Cosserat surface and then investigate, in particular, the related problem of stability. We recall that various aspects of the theory of a Cosserat surface have been developed in several papers [1–4]. The relationship of this theory to one resulting from the three-dimensional equations of classical continuum mechanics for thin shells, as well as the relevance and applicability of the theory of an elastic Cosserat surface to elastic shells and plates, has been studied and explored in [4–6]. Additional references on the subject can be found in the papers already cited.

In the present paper, following some preliminaries in sec. 2, we present a general theory of small deformations superposed on a large deformation of an elastic Cosserat surface (sec. 3)*. Next we apply the theory of sec. 3 to an isotropic Cosserat plate in sec. 4, followed by a discussion of special problems in which the plate is deformed by large extensions in two perpendicular directions (secs. 5–6).

As the development in sec. 3 is obtained from a nonlinear theory which is exact, it provides a suitable basis for discussing stability of shells and plates and related problems. With this in mind and using concepts similar to those employed by Zubov [7] and Movchan [8], we discuss in sec. 7 the stability of an elastic Cosserat plate subjected to small motions superposed on a large deformation of the type considered in sec. 6. A more detailed discussion of stability of a plate under all around compression is given in sec. 8 which also contains comparison with corresponding results obtained with the use of classical plate theory. We close this paper with a brief discussion of the small transverse vibrations of an initially stressed isotropic plate (sec. 9).

* Results similar or equivalent to the contents of sec. 3 were also obtained independently by R. C. Koeller (while a Visiting Scholar at the University of California, Berkeley during 1966–67) and R. B. Osborn (while a NATO Post-doctoral Fellow at the University of Newcastle upon Tyne during 1967).

When the present paper was completed our attention was drawn to a paper on the theory of stability of an elastic Cosserat surface by Popelar [9]. The overlap between his paper and ours is slight so that no detailed comparison is necessary *. However, some comment appears to be appropriate concerning the notion of stability used by Popelar [9] who takes over the energy approach, stating that Koiter [10] has shown that a necessary and sufficient condition for stability of an equilibrium configuration in the sense of Liapunov is the existence of a proper minimum of the potential energy in this configuration.

Koiter's proof depends on the inclusion of arbitrary small, but non-zero, strain gradient terms in the strain energy function. Such a procedure is open to criticism. If thermodynamical concepts and procedures currently used in continuum mechanics are accepted, then it is impossible for the strain energy to depend on strain gradients in the absence of couple-stresses. On the other hand, if couple-stresses are admitted in the theory, then we no longer have a problem within the scope of classical elasticity. Even if thermodynamics is disregarded and the problem is viewed in the context of a purely mechanical theory the same difficulty remains, since (in a mechanical theory) the constitutive equations for stresses may be obtained from the assumption that the rate of work of the stresses is equal to the rate of change of an elastic potential. Hence, it again follows that the elastic potential (and the stresses) cannot depend on strain gradients. We emphasize that we do not appeal, in the present paper, to the energy criterion for stability of the type used in [9, 10].

2. Preliminaries. Basic equations for a Cosserat surface

Consider a Cosserat surface, i.e., a surface embedded in an Euclidean 3-space to every point of which a single deformable director is assigned. Let x^α, $(\alpha = 1, 2)$, be convected coordinates on the surface which surface will be referred to as $\mathfrak{s}$ in the deformed configuration at time t.

Let $\boldsymbol{r}$, a function of x^α and t, be the position vector of a typical point of $\mathfrak{s}$ and let the director displacement at the same point be designated by $\boldsymbol{d}$. Further, let $\boldsymbol{a}_\alpha$ denote the covariant base vectors along the x^α-curves on $\mathfrak{s}$ and let $\boldsymbol{a}_3$ be the unit normal to $\mathfrak{s}$. Then,

$$\begin{aligned}
&\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}, \quad a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad \boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta, \quad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta, \\
&\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3 = 0, \boldsymbol{a}_3 = \boldsymbol{a}^3, \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 1, a = \det a_{\alpha\beta}, \\
&\boldsymbol{a}_{\alpha|\beta} = b_{\alpha\beta}\boldsymbol{a}_3, \quad \boldsymbol{a}_{3,\beta} = -b^\alpha_\beta \boldsymbol{a}_\alpha,
\end{aligned} \tag{2.1}$$

where δ^α_β is the Kronecker delta, a comma denotes partial differentiation with respect to x^α, $\boldsymbol{a}^\alpha$ are the contravariant base vectors and $a_{\alpha\beta}$, $a^{\alpha\beta}$ are the metric tensors for $\mathfrak{s}$. Also, $b_{\alpha\beta}$ is the second fundamental form of $\mathfrak{s}$ and a vertical bar stands for covariant differentiation with respect to $a_{\alpha\beta}$.

The motion of a Cosserat surface may be characterized by

$$\boldsymbol{r} = \boldsymbol{r}(x^\alpha, t), \boldsymbol{d} = \boldsymbol{d}(x^\alpha, t), [\boldsymbol{a}_1\, \boldsymbol{a}_2\, \boldsymbol{d}] > 0, \tag{2.2}$$

where

$$\boldsymbol{d} = d^i \boldsymbol{a}_i = d_i \boldsymbol{a}^i. \tag{2.3}$$

In (2.1)–(2.3) and throughout the paper, all Latin indices take the values 1, 2, 3 and Greek indices the values 1, 2 only. We denote the initial values of $\boldsymbol{r}, \boldsymbol{d}$ at time $t = 0$ by $\boldsymbol{R} = \boldsymbol{R}(x^\alpha)$, $\boldsymbol{D} = \boldsymbol{D}(x^\alpha)$ and refer to the initial (undeformed) surface as $\mathfrak{S}$. Also, the initial values of

* For example, the contents of sec. 3 of [9] correspond to our sec. 3 only up to (3.14) and his stability analysis is based on an approach which is different from that given here.

$\boldsymbol{a}_i$, $\boldsymbol{a}^\alpha$, $a_{\alpha\beta}$, $a^{\alpha\beta}$, $b_{\alpha\beta}$, b^α_β will be denoted by $\boldsymbol{A}_i$, $\boldsymbol{A}^\alpha$, $A_{\alpha\beta}$, $A^{\alpha\beta}$, $B_{\alpha\beta}$, B^α_β, respectively. The velocity and the director velocity vectors will be denoted by $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively, where

$$\boldsymbol{v} = \dot{\boldsymbol{r}}, \ \boldsymbol{w} = \boldsymbol{d} \tag{2.4}$$

and a superposed dot denotes differentiation with respect to t holding x^α fixed. The kinematic variables for a Cosserat surface may be specified by

$$2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}, \ \kappa_{i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha}, \ \delta_i = d_i - D_i, \tag{2.5}$$

where

$$\lambda_{i\alpha} = \boldsymbol{a}_i \cdot \boldsymbol{d}_{,\alpha}, \ \Lambda_{i\alpha} = \boldsymbol{A}_i \cdot \boldsymbol{D}_{,\alpha}, \tag{2.6}$$

and

$$\begin{aligned} \lambda_{\beta\alpha} &= d_{\beta|\alpha} - b_{\alpha\beta} d_3, \quad \lambda^{\beta}_{.\alpha} = a^{\beta\gamma}\lambda_{\gamma\alpha}, \\ \lambda_{3\alpha} &= d_{3,\alpha} + b^\beta_\alpha d_\beta, \quad \lambda^{3}_{.\alpha} = \lambda_{3\alpha}, \end{aligned} \tag{2.7}$$

together with corresponding expressions for $\Lambda_{\beta\alpha}$, $\Lambda_{3\alpha}$.

Let σ be an area of $\mathfrak{s}$ bounded by a closed curve c, let $\boldsymbol{\nu}$ be the outward unit normal to c lying in the surface and let ρ denote the mass density per unit area of $\mathfrak{s}$. Then, the equation for conservation of mass is

$$\rho a^{\frac{1}{2}} = \rho_0 A^{\frac{1}{2}} = k, \tag{2.8}$$

where ρ_0 is the initial mass density, $A = \det A_{\alpha\beta}$ and k is a function of x^α, independent of t. We define the curve force vector $\boldsymbol{N}$ and the director force vector $\boldsymbol{M}$, each per unit length of c, by

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{N}^\alpha \nu_\alpha = \boldsymbol{n}^\alpha \nu_\alpha / a^{\frac{1}{2}}, \\ \boldsymbol{M} &= \boldsymbol{M}^\alpha \nu_\alpha = \boldsymbol{m}^\alpha \nu_\alpha / a^{\frac{1}{2}}, \end{aligned} \tag{2.9}$$

where *

$$\begin{aligned} \boldsymbol{N}^\alpha &= \boldsymbol{n}^\alpha / a^{\frac{1}{2}} = \bar{\boldsymbol{n}}^\alpha (a^{\alpha\alpha})^{\frac{1}{2}} \quad \text{(no sum on } \alpha), \\ \boldsymbol{M}^\alpha &= \boldsymbol{m}^\alpha / a^{\frac{1}{2}} = \bar{\boldsymbol{m}}^\alpha (a^{\alpha\alpha})^{\frac{1}{2}} \quad \text{(no sum on } \alpha) \end{aligned} \tag{2.10}$$

and

$$\boldsymbol{\nu} = \nu_\alpha \boldsymbol{a}^\alpha. \tag{2.11}$$

Let $\boldsymbol{F}$ and $\boldsymbol{L}$, each per unit mass of $\mathfrak{s}$, denote the assigned surface force vector and the assigned surface director force, respectively. We put

$$\bar{\boldsymbol{L}} = \boldsymbol{L} - j\dot{\boldsymbol{w}}, \tag{2.12}$$

where the director inertia coefficient j is a function of x^α but independent of t. The equations of motion for a Cosserat surface are **

$$\boldsymbol{n}^\alpha_{,\alpha} + k\boldsymbol{F} = k\dot{\boldsymbol{v}}, \tag{2.13}$$

$$\boldsymbol{m}^\alpha_{,\alpha} + k\bar{\boldsymbol{L}} = \boldsymbol{\pi} a^{\frac{1}{2}}, \tag{2.14}$$

and

* The notation here differs slightly from that in [1].

** The notation in (2.13)–(2.15) again differs slightly from that in [1].

$$\boldsymbol{n}^\alpha \times \boldsymbol{a}_\alpha + (\boldsymbol{m}^\alpha \times \boldsymbol{d})_{,\alpha} + k\bar{\boldsymbol{L}} \times \boldsymbol{d} = 0, \tag{2.15}$$

where $-\boldsymbol{\pi}$ is the intrinsic surface director force per unit area of $\mathcal{S}$. In view of (2.10) and (2.14), (2.15) alternatively may be written as

$$\boldsymbol{N}^\alpha \times \boldsymbol{a}_\alpha + \boldsymbol{\pi} \times \boldsymbol{d} + \boldsymbol{M}^\alpha \times \boldsymbol{d}_{,\alpha} = 0. \tag{2.16}$$

Referred to the base vectors $\boldsymbol{a}_i$, the various vector fields occurring in (2.13)–(2.15) can be expressed as *

$$\boldsymbol{N}^\alpha = N^{\alpha i}\boldsymbol{a}_i,\ \boldsymbol{M}^\alpha = M^{\alpha i}\boldsymbol{a}_i,\ \boldsymbol{\pi} = \pi^i \boldsymbol{a}_i, \tag{2.17}$$

$$\boldsymbol{F} = F^i\boldsymbol{a}_i,\ \bar{\boldsymbol{L}} = \bar{L}^i\boldsymbol{a}_i,\ \dot{\boldsymbol{v}} = c^i\boldsymbol{a}_i. \tag{2.18}$$

Then, the equations of motion in component forms are

$$N^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha N^{\alpha 3} + \rho F^\beta = \rho c^\beta, \quad N^{\alpha 3}{}_{|\alpha} + b_{\alpha\lambda} N^{\alpha\lambda} + \rho F^3 = \rho c^3, \tag{2.19}$$

$$M^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha M^{\alpha 3} + \rho \bar{L}^\beta = \pi^\beta, \quad M^{\alpha 3}{}_{|\alpha} + b_{\alpha\lambda} M^{\alpha\lambda} + \rho \bar{L}^3 = \pi^3, \tag{2.20}$$

and

$$\begin{aligned} \bar{N}^{\alpha\beta} &= \bar{N}^{\beta\alpha} = N^{\alpha\beta} - \pi^\alpha d^\beta - M^{\gamma\alpha}\lambda^\beta_{.\gamma}, \\ N^{\alpha 3} &+ \pi^3 d^\alpha - \pi^\alpha d^3 + M^{\gamma 3}\lambda^\alpha_{.\gamma} - M^{\gamma\alpha}\lambda^3_{.\gamma} = 0. \end{aligned} \tag{2.21}$$

We also record here the constitutive equations for an isothermal elastic Cosserat surface. The isothermal Helmholtz free energy (or the strain energy) function is specified by

$$A = A(e_{\alpha\beta}, \kappa_{i\alpha}, \delta_i, A_{\alpha\beta}, A_{i\alpha}, D_i) \tag{2.22}$$

and the remaining constitutive relations have the forms

$$\bar{N}^{\alpha\beta} = \tfrac{1}{2}\rho\left(\frac{\partial A}{\partial e_{\alpha\beta}} + \frac{\partial A}{\partial e_{\beta\alpha}}\right), \quad M^{\alpha i} = \rho\frac{\partial A}{\partial \kappa_{i\alpha}}, \quad \pi^i = \rho\frac{\partial A}{\partial \delta_i}. \tag{2.23}$$

3. Small deformation superposed on a large deformation

We consider here three configurations of the Cosserat surface corresponding to isothermal deformations as follows: First, we specify the initial (undeformed) configuration by a position vector $\boldsymbol{R} = \boldsymbol{R}(x^\alpha)$ and director $\boldsymbol{D} = \boldsymbol{D}(x^\alpha)$. Second, we consider a deformed configuration which we assume to be one of equilibrium and specified by a position vector $\boldsymbol{r} = \boldsymbol{r}(x^\alpha)$ and director $\boldsymbol{d} = \boldsymbol{d}(x^\alpha)$. Finally, we consider a configuration specified by a position vector

$$\boldsymbol{r}(x^\alpha) + \varepsilon\boldsymbol{u}(x^\alpha, t) \tag{3.1}$$

and a director displacement

$$\boldsymbol{d}(x^\alpha) + \varepsilon\boldsymbol{b}(x^\alpha, t), \tag{3.2}$$

where ε is a small non-dimensional parameter, (3.1) is the position vector of a surface which we refer to as $\mathcal{S}'$ and the differences of the ordinary and director displacements from the first deformed configuration to the second deformed configuration are $\varepsilon\boldsymbol{u}$ and $\varepsilon\boldsymbol{b}$, respectively.

* The order of indices αi in $N^{\alpha i}$ and $M^{\alpha i}$ differs from those in [1] and corresponds to the customary notation in shell theory.

In the following development, powers of ε above the first will be neglected except in the free energy A. Thus, the base vectors, the unit normal vector, metric tensors and the second fundamental form of $\mathscr{S}'$ will be denoted by

$$\begin{aligned} &\mathbf{a}_\alpha+\varepsilon\mathbf{a}'_\alpha, \quad \mathbf{a}^\alpha+\varepsilon\mathbf{a}'^\alpha, \quad \mathbf{a}_3+\varepsilon\mathbf{a}'_3, \\ &a_{\alpha\beta}+\varepsilon a'_{\alpha\beta}, \quad a^{\alpha\beta}+\varepsilon a'^{\alpha\beta}, \quad b_{\alpha\beta}+\varepsilon b'_{\alpha\beta}. \end{aligned} \tag{3.3}$$

Let

$$\mathbf{u} = u^i\mathbf{a}_i = u_i\mathbf{a}^i. \tag{3.4}$$

Then, it follows that

$$\mathbf{a}'_\alpha = \mathbf{u}_{,\alpha} = \phi^i_{\cdot\alpha}\mathbf{a}_i = \phi_{i\alpha}\mathbf{a}^i, \tag{3.5}$$

where

$$\begin{aligned} &\phi_{\lambda\alpha} = \mathbf{a}_\lambda\cdot\mathbf{u}_{,\alpha} = u_{\lambda|\alpha}-b_{\lambda\alpha}u_3, \quad \phi^\lambda_{\cdot\alpha} = a^{\lambda\nu}\phi_{\nu\alpha}, \\ &\phi_{3\alpha} = \mathbf{a}_3\cdot\mathbf{u}_{,\alpha} = u_{3,\alpha}+b^\lambda_\alpha u_\lambda, \quad \phi^3_{\cdot\alpha} = \phi_{3\alpha}. \end{aligned} \tag{3.6}$$

Also,

$$\mathbf{a}'_3 = -\phi_{3\alpha}\mathbf{a}^\alpha, \quad \mathbf{a}'^\alpha = -\phi^\alpha_{\cdot\lambda}\mathbf{a}^\lambda+a^{\alpha\lambda}\phi_{3\lambda}\mathbf{a}_3, \tag{3.7}$$

$$\begin{aligned} b'_{\alpha\beta} &= -\mathbf{a}_\alpha\cdot\mathbf{a}'_{3,\beta}+b^\lambda_\beta\mathbf{a}_\lambda\cdot\mathbf{a}'_\alpha = \phi_{3\alpha|\beta}+b^\lambda_\beta\phi_{\lambda\alpha} \\ &= u_{3|\alpha\beta}-b^\lambda_\beta b_{\lambda\alpha}u_3+b^\lambda_{\alpha|\beta}u_\lambda+b^\lambda_\alpha u_{\lambda|\beta}+b^\lambda_\beta u_{\lambda|\alpha} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &a'_{\alpha\beta} = \mathbf{a}_\alpha\cdot\mathbf{a}'_\beta+\mathbf{a}_\beta\cdot\mathbf{a}'_\alpha = \phi_{\alpha\beta}+\phi_{\beta\alpha}, \\ &a'^{\alpha\beta} = \mathbf{a}^\alpha\cdot\mathbf{a}'^\beta+\mathbf{a}^\beta\cdot\mathbf{a}'^\alpha = -a^{\alpha\lambda}a^{\beta\mu}a'_{\lambda\mu}. \end{aligned} \tag{3.9}$$

In line with the preceding notations, we also introduce

$$\begin{aligned} &a+\varepsilon a' = \det(a_{\alpha\beta}+\varepsilon a'_{\alpha\beta}), \quad \mathbf{b} = b^i\mathbf{a}_i = b_i\mathbf{a}^i, \quad \mathbf{b}_{,\alpha} = \mu^i_{\cdot\alpha}\mathbf{a}_i = \mu_{i\alpha}\mathbf{a}^i, \\ &d_i+\varepsilon d'_i = (\mathbf{a}_i+\varepsilon\mathbf{a}'_i)\cdot(\mathbf{d}+\varepsilon\mathbf{b}), \quad \lambda_{i\alpha}+\varepsilon\lambda'_{i\alpha} = (\mathbf{a}_i+\varepsilon\mathbf{a}'_i)\cdot(\mathbf{d}_{,\alpha}+\varepsilon\mathbf{b}_{,\alpha}). \end{aligned} \tag{3.10}$$

Then, to the first order in ε, we have

$$\begin{aligned} &a' = aa^{\alpha\beta}a'_{\alpha\beta} = 2a\phi^\lambda_{\cdot\lambda}, \\ &d'_\alpha = b_\alpha+d_\beta\phi^\beta_{\cdot\alpha}+d_3\phi_{3\alpha}, \quad d'_3 = b_3-d^\alpha\phi_{3\alpha}, \\ &\lambda'_{\beta\alpha} = \mu_{\beta\alpha}+\lambda_{\gamma\alpha}\phi^\gamma_{\cdot\beta}+\lambda_{3\alpha}\phi_{3\beta}, \quad \lambda'_{3\alpha} = \mu_{3\alpha}-\lambda^\beta_{\cdot\alpha}\phi_{3\beta}, \\ &\mu_{\beta\alpha} = b_{\beta|\alpha}-b_{\beta\alpha}b_3, \quad \mu_{3\alpha} = b_{3,\alpha}+b^\beta_\alpha b_\beta. \end{aligned} \tag{3.11}$$

Let $e_{\alpha\beta}+\varepsilon e'_{\alpha\beta}$, $\kappa_{i\alpha}+\varepsilon\kappa'_{i\alpha}$, $\delta_i+\varepsilon\delta'_i$ be the values of the kinematic variables corresponding to those in (2.5) for the configuration of the Cosserat surface specified by (3.1)–(3.2). It then follows that

$$\begin{aligned} &2e'_{\alpha\beta} = a'_{\alpha\beta} = \phi_{\alpha\beta}+\phi_{\beta\alpha} = u_{\alpha|\beta}+u_{\beta|\alpha}-2b_{\alpha\beta}u_3, \\ &\kappa'_{i\alpha} = \lambda'_{i\alpha}, \quad \delta'_i = d'_i. \end{aligned} \tag{3.12}$$

Corresponding to the configuration specified by (3.1)–(3.2), we need additional kinematical results. Put

$$d^\alpha+\varepsilon d'^\alpha = (a^{\alpha\lambda}+\varepsilon a'^{\alpha\lambda})(d_\lambda+\varepsilon d'_\lambda), \quad d^3+\varepsilon d'^3 = d_3+\varepsilon d'_3,$$
$$\lambda^\alpha_{.\beta}+\varepsilon\lambda'^\alpha_{.\beta} = (a^{\alpha\nu}+\varepsilon a'^{\alpha\nu})(\lambda_{\nu\beta}+\varepsilon\lambda'_{\nu\beta}), \quad \lambda^3_{.\alpha}+\varepsilon\lambda'^3_{.\alpha} = \lambda_{3\alpha}+\varepsilon\lambda'_{3\alpha}. \tag{3.13}$$

Then, to the first order in ε,

$$d'^\alpha = b^\alpha+d_3 a^{\alpha\nu}\phi_{3\nu}-d^\nu\phi^\alpha_{.\nu}, \quad d'^3 = d'_3$$
$$\lambda'^\alpha_{.\beta} = \mu^\alpha_{.\beta}+a^{\alpha\nu}\lambda_{3\beta}\phi_{3\nu}-\lambda^\nu_{.\beta}\phi^\alpha_{.\nu}, \quad \lambda'^3_{.\alpha} = \lambda'_{3\alpha}. \tag{3.14}$$

We now turn our attention to the equations of motion which must hold in the deformed configuration corresponding to (3.1)–(3.2). In this configuration, the area σ bounded by a closed curve c on the surface $\mathfrak{s}$ becomes an area σ' bounded by a closed curve c' on the surface $\mathfrak{s}'$. The force vector and the director force vector on $\mathfrak{s}'$, acting on the curve c' and measured per unit length of c, are respectively $\boldsymbol{N}+\varepsilon\boldsymbol{N}'$ and $\boldsymbol{M}+\varepsilon\boldsymbol{M}'$, where

$$\boldsymbol{N}+\varepsilon\boldsymbol{N}' = v_\alpha(\boldsymbol{n}^\alpha+\varepsilon\boldsymbol{n}'^\alpha)/a^{\frac{1}{2}},$$
$$\boldsymbol{M}+\varepsilon\boldsymbol{M}' = v_\alpha(\boldsymbol{m}^\alpha+\varepsilon\boldsymbol{m}'^\alpha)/a^{\frac{1}{2}}, \tag{3.15}$$

while the quantities $\boldsymbol{n}^\alpha$, $\boldsymbol{m}^\alpha$ become $\boldsymbol{n}^\alpha+\varepsilon\boldsymbol{n}'^\alpha$, $\boldsymbol{m}^\alpha+\varepsilon\boldsymbol{m}'^\alpha$. Also, let quantities corresponding to $N^{\alpha i}$, $M^{\alpha i}$ be denoted by $N^{\alpha i}+\varepsilon N'^{\alpha i}$, $M^{\alpha i}+\varepsilon M'^{\alpha i}$ in the final configuration specified by (3.1)–(3.2). Then

$$\boldsymbol{n}^\alpha+\varepsilon\boldsymbol{n}'^\alpha = a^{\frac{1}{2}}\left(1+\varepsilon\frac{a'}{2a}\right)(N^{\alpha i}+\varepsilon N'^{\alpha i})(\boldsymbol{a}_i+\varepsilon\boldsymbol{a}'_i) = a^{\frac{1}{2}}(T^{\alpha i}+\varepsilon T'^{\alpha i})\boldsymbol{a}_i, \tag{3.16}$$

$$\boldsymbol{m}^\alpha+\varepsilon\boldsymbol{m}'^\alpha = a^{\frac{1}{2}}\left(1+\varepsilon\frac{a'}{2a}\right)(M^{\alpha i}+\varepsilon M'^{\alpha i})(\boldsymbol{a}_i+\varepsilon\boldsymbol{a}'_i) = a^{\frac{1}{2}}(C^{\alpha i}+\varepsilon C'^{\alpha i})\boldsymbol{a}_i, \tag{3.17}$$

to the first order in ε, where

$$T^{\alpha i} = N^{\alpha i}, \; C^{\alpha i} = M^{\alpha i}, \tag{3.18}$$

$$T'^{\alpha\beta} = N'^{\alpha\beta}+N^{\alpha\lambda}\phi^\beta_{.\lambda}-N^{\alpha 3}a^{\lambda\beta}\phi_{3\lambda}+\frac{a'}{2a}N^{\alpha\beta},$$
$$T'^{\alpha 3} = N'^{\alpha 3}+N^{\alpha\lambda}\phi_{3\lambda}+\frac{a'}{2a}N^{\alpha 3}, \tag{3.19}$$

$$C'^{\alpha\beta} = M'^{\alpha\beta}+M^{\alpha\lambda}\phi^\beta_{.\lambda}-M^{\alpha 3}a^{\lambda\beta}\phi_{3\lambda}+\frac{a'}{2a}M^{\alpha\beta},$$
$$C'^{\alpha 3} = M'^{\alpha 3}+M^{\alpha\lambda}\phi_{3\lambda}+\frac{a'}{2a}M^{\alpha 3}. \tag{3.20}$$

Next, let $-(\boldsymbol{\pi}+\varepsilon\boldsymbol{\pi}')$ denote the intrinsic surface director force vector in the configuration $\mathfrak{s}'$. Then,

$$\boldsymbol{\pi}+\varepsilon\boldsymbol{\pi}' = (\pi^i+\varepsilon\pi'^i)(\boldsymbol{a}_i+\varepsilon\boldsymbol{a}'_i) = (\pi^i+\varepsilon\omega^i)\boldsymbol{a}_i \tag{3.21}$$

to the first order in ε, where

$$\omega^\alpha = \pi'^\alpha+\pi^\nu\phi^\alpha_{.\nu}-\pi^3 a^{\alpha\nu}\phi_{3\nu}, \quad \omega^3 = \pi'^3+\pi^\alpha\phi_{3\alpha}. \tag{3.22}$$

Let the assigned surface force vector and the assigned surface director force vector, each per unit mass of $\mathfrak{s}'$ in the final configuration, be denoted by $\boldsymbol{F}+\varepsilon\boldsymbol{F}'$ and $\boldsymbol{L}+\varepsilon\boldsymbol{L}'$, respectively. Since the first deformed configuration is assumed to be one of equilibrium, recalling the equations of motion (2.13), (2.14) and (2.16), we have

$$n'^{\alpha}{}_{,\alpha}+kF' = k\ddot{u}, \tag{3.23}$$

$$m'^{\alpha}{}_{,\alpha}+k(L'-j\ddot{b}) = \left(\pi'+\frac{a'\pi}{2a}\right)a^{\frac{1}{2}} \tag{3.24}$$

and

$$n'^{\alpha}\times a_{\alpha}+n^{\alpha}\times a'_{\alpha}+(\pi'\times d+\pi\times b)a^{\frac{1}{2}}+\frac{a'}{2a^{\frac{1}{2}}}\pi\times d+m'^{\alpha}\times d_{,\alpha}+m^{\alpha}\times b_{,\alpha} = 0. \tag{3.25}$$

With the help of (3.16)-(3.17) and (3.21), the equations of motion (3.23)-(3.25) can be written in component forms as follows:

$$T'^{\alpha\beta}{}_{|\alpha}-b^{\beta}_{\alpha}T'^{\alpha 3}+\rho F'^{\beta} = \rho\ddot{u}^{\beta}, \quad T'^{\alpha 3}{}_{|\alpha}+b_{\alpha\lambda}T'^{\alpha\lambda}+\rho F'^{3} = \rho\ddot{u}^{3}, \tag{3.26}$$

$$\begin{aligned} C'^{\alpha\beta}{}_{|\alpha}-b^{\beta}_{\alpha}C'^{\alpha 3}+\rho(L'^{\beta}-j\ddot{b}^{\beta}) &= \omega^{\beta}+\frac{a'}{2a}\pi^{\beta}, \\ C'^{\alpha 3}{}_{|\alpha}+b_{\alpha\lambda}C'^{\alpha\lambda}+\rho(L'^{3}-j\ddot{b}^{3}) &= \omega^{3}+\frac{a'}{2a}\pi^{3} \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &\bar{N}'^{\alpha\beta} = \bar{N}'^{\beta\alpha} = N'^{\alpha\beta}-\pi^{\alpha}d'^{\beta}-\pi'^{\alpha}d^{\beta}-M^{v\alpha}\lambda'^{\beta}_{.v}-M'^{v\alpha}\lambda^{\beta}_{.v}, \\ &N'^{\alpha 3}+\pi'^{3}d^{\alpha}+\pi^{3}d'^{\alpha}-\pi'^{\alpha}d^{3}-\pi^{\alpha}d'^{3}+M'^{v3}\lambda^{\alpha}_{.v}+M^{v3}\lambda'^{\alpha}_{.v}-M'^{v\alpha}\lambda_{3v}-M^{v\alpha}\lambda'_{3v} = 0, \end{aligned} \tag{3.28}$$

where $\bar{N}^{\alpha\beta}+\varepsilon\bar{N}'^{\alpha\beta}$ is the value of $\bar{N}^{\alpha\beta}$ in the final configuration.

Finally, with the help of (2.8) and (2.23), we record the constitutive relations for $\bar{N}'^{\alpha\beta}$, $M'^{\alpha i}$ and π'^{i} as follows:

$$\begin{aligned} \frac{\bar{N}'^{\alpha\beta}}{\rho} = &-\frac{a'}{4a}\left(\frac{\partial A}{\partial e_{\alpha\beta}}+\frac{\partial A}{\partial e_{\beta\alpha}}\right)+\frac{1}{4}\left(\frac{\partial}{\partial e_{\alpha\beta}}+\frac{\partial}{\partial e_{\beta\alpha}}\right)\left(\frac{\partial A}{\partial e_{\lambda v}}+\frac{\partial A}{\partial e_{v\lambda}}\right)e'_{\lambda v} \\ &+\frac{1}{2}\left(\frac{\partial}{\partial e_{\alpha\beta}}+\frac{\partial}{\partial e_{\beta\alpha}}\right)\frac{\partial A}{\partial \kappa_{i\alpha}}\kappa'_{i\alpha}+\frac{1}{2}\left(\frac{\partial}{\partial e_{\alpha\beta}}+\frac{\partial}{\partial e_{\beta\alpha}}\right)\frac{\partial A}{\partial \delta_{i}}\delta'_{i}, \end{aligned} \tag{3.29}$$

$$\frac{M'^{\alpha i}}{\rho} = -\frac{a'}{2a}\frac{\partial A}{\partial \kappa_{i\alpha}}+\frac{1}{2}\left(\frac{\partial}{\partial e_{\lambda v}}+\frac{\partial}{\partial e_{v\lambda}}\right)\frac{\partial A}{\partial \kappa_{i\alpha}}e'_{\lambda v}+\frac{\partial^{2} A}{\partial \kappa_{j\beta}\partial \kappa_{i\alpha}}\kappa'_{j\beta}+\frac{\partial^{2} A}{\partial \kappa_{i\alpha}\partial \delta_{j}}\delta'_{j}, \tag{3.30}$$

and

$$\frac{\pi'^{i}}{\rho} = -\frac{a'}{2a}\frac{\partial A}{\partial \delta_{i}}+\frac{1}{2}\left(\frac{\partial}{\partial e_{\alpha\beta}}+\frac{\partial}{\partial e_{\beta\alpha}}\right)\frac{\partial A}{\partial \delta_{i}}e'_{\alpha\beta}+\frac{\partial^{2} A}{\partial \kappa_{j\alpha}\partial \delta_{i}}\kappa'_{j\alpha}+\frac{\partial^{2} A}{\partial \delta_{i}\partial \delta_{j}}\delta'_{j}. \tag{3.31}$$

In the formulae (3.29)-(3.31), the free energy function A is of the form (2.22) appropriate to the first deformed (equilibrium) configuration.

4. Isotropic plates

We assume here that the Cosserat surface in its initial configuration is plane and that the initial director $\boldsymbol{D}$ is coincident with the unit normal $\boldsymbol{A}_3$, i.e.,

$$\boldsymbol{D} = \boldsymbol{A}_3 \tag{4.1}$$

where now $\boldsymbol{A}_3$ is a constant unit vector normal to the plane surface defined by the position vector $\boldsymbol{R}(x^{\alpha})$. It follows that

$$B_{\alpha\beta} = 0,\ D_\alpha = 0,\ D_3 = 1,\ \Lambda_{i\alpha} = 0. \tag{4.2}$$

We further imitate the properties of a three-dimensional isotropic flat plate of uniform thickness h_0 and assume that the free energy A in (2.22), an isotropic function of $e_{\alpha\beta}$, $\kappa_{\beta\alpha}$, $\kappa_{3\alpha}$, δ_β and δ_3, is invariant under the transformations

$$\begin{aligned} &e_{\alpha\beta} \to e_{\alpha\beta},\ \kappa_{\beta\alpha} \to -\kappa_{\beta\alpha},\ \kappa_{3\alpha} \to \kappa_{3\alpha}, \\ &\delta_\beta \to -\delta_\beta,\ \delta_3 \to \delta_3. \end{aligned} \tag{4.3}$$

In order to obtain an explicit representation for A, we first define the following 2×2 matrices:

$$\begin{aligned} I &= [A^{\alpha\gamma} e_{\gamma\beta}] = [e^{\alpha}_{\beta}] = I^T, \\ J &= [A^{\alpha\gamma} \kappa_{\gamma\beta}] = [\kappa^{\alpha}_{\cdot\beta}], \\ J^T &= [A^{\alpha\gamma} \kappa_{\beta\gamma}] = [\kappa_{\beta}^{\cdot\alpha}], \\ K &= uu^T,\ P = uv^T,\ P^T = vu^T,\ Q = vv^T, \end{aligned} \tag{4.4}$$

where $(\ )^T$ denotes transpose. Also u, v are 2×1 vectors and u^T, v^T are 1×2 vectors given by

$$\begin{aligned} u &= A^{\alpha\gamma}\kappa_{3\gamma} = \kappa_3^{\cdot\alpha},\quad u^T = \kappa_{3\alpha}, \\ v &= A^{\alpha\gamma}\delta_\gamma = \delta^\alpha,\ v^T = \delta_\alpha. \end{aligned} \tag{4.5}$$

Assuming that A is a polynomial function, it can then be shown that A is a polynomial in the list of invariants given below:

List of Invariants

δ_3, tr I, tr K, tr Q, tr I^2, tr J^2, tr JJ^T, tr IK, tr IQ, tr JP, tr JP^T, tr IJJ^T, tr IJP, tr IJP^T, tr JJ^TK, tr JJ^TQ,

(tr $J)^2$, tr J tr P, tr J tr IJ, tr J tr IP, tr J tr JK, tr J tr JQ, tr J tr IJK, tr J tr IJQ, tr J tr JJ^TP,

(tr $P)^2$, tr P tr IJ, tr P tr IP, tr P tr JK, tr P tr JQ, tr P tr IJK, tr P tr IJQ, tr P tr IJ^TP,

(tr $IJ)^2$, tr IJ tr IP, tr IJ tr JK, tr IJ tr JQ, tr IJ tr IJK, tr IJ tr IJQ, tr IJ tr JJ^TP,

(tr $IP)^2$, tr IP tr JK, tr IP tr JQ, tr IP tr IJK, tr IP tr IJQ, tr IP tr JJ^TP,

(tr $JK)^2$, tr JK tr JQ, tr JK tr IJK, tr JK tr IJQ, tr JK tr JJ^TP,

(tr $JQ)^2$, tr JQ tr IJK, tr JQ tr IJQ, tr JQ tr JJ^TP,

(tr $IJK)^2$, tr IJK tr IJQ, tr IJK tr JJ^TP,

(tr $IJQ)^2$, tr IJQ tr JJ^TP, (tr $JJ^TP)^2$.

In the above List of Invariants, tr denotes the trace operator. If the plate is initially unstressed, then A will not contain terms linear in δ_3, tr I.

5. Large extension of isotropic plate

We suppose that the initially unstressed plate is subjected to large extensions in two perpendicular directions (in the plane of the plate) with extension ratios k_1, k_2. We choose

$x^\alpha (= x_\alpha)$ to be rectangular Cartesian coordinates in the first deformed configuration and these are related to rectangular Cartesian coordinates X_α of the initial configuration, referred to the same axes, by

$$x_1 = k_1 X_1, x_2 = k_2 X_2. \tag{5.1}$$

Then,

$$\begin{aligned} &a_{\alpha\beta} = \delta_{\alpha\beta}, a = 1, b_{\alpha\beta} = 0, \boldsymbol{a}^\alpha = \boldsymbol{a}_\alpha, \\ &A_{11} = \frac{1}{k_1^2}, \quad A_{22} = \frac{1}{k_2^2}, \quad A_{12} = 0, \quad B_{\alpha\beta} = 0, \\ &2e_{11} = 1 - \frac{1}{k_1^2}, \quad 2e_{22} = 1 - \frac{1}{k_2^2}, \quad e_{12} = 0. \end{aligned} \tag{5.2}$$

Recalling (4.2) and assuming that the director $\boldsymbol{d}$ in the first deformed configuration is specified by

$$\boldsymbol{d} = d_3 \boldsymbol{a}_3 = d \boldsymbol{a}_3, \tag{5.3}$$

where d is a constant, it follows that

$$\begin{aligned} &d_3 = d, d_\alpha = 0, \delta_3 = d - 1, \delta_\alpha = 0, \\ &\lambda_{i\beta} = 0, \kappa_{i\alpha} = 0. \end{aligned} \tag{5.4}$$

In view of (5.4), the only nonvanishing invariants in the Lists of Invariants in Sec. 4 are

$$\begin{aligned} &L_1 = \operatorname{tr} I = \tfrac{1}{2}(k_1^2 + k_2^2 - 2), \quad \delta_3, \\ &L_2 = \operatorname{tr} I^2 = \tfrac{1}{4}[(k_1^2 - 1)^2 + (k_2^2 - 1)^2], \end{aligned} \tag{5.5}$$

since

$$\begin{aligned} &u = v = 0, \\ &J = K = P = Q = 0. \end{aligned} \tag{5.6}$$

Keeping the above in mind, with reference to the first deformed configuration, we have

$$\rho = \rho_0/(k_1 k_2) = k, \tag{5.7}$$

which is obtained from (2.8). Also, by (2.23) and (2.21), we now have

$$\begin{aligned} &\bar{N}^{\alpha\beta} = \rho \frac{\partial A}{\partial L_1} A^{\alpha\beta} + \rho \frac{\partial A}{\partial L_2} (A^{\alpha\lambda} A^{\beta\mu} + A^{\alpha\mu} A^{\beta\lambda}) e_{\lambda\mu}, \\ &\pi^3 = \rho \frac{\partial A}{\partial \delta_3}, \quad \pi^\alpha = 0, \quad M^{\alpha i} = 0 \end{aligned} \tag{5.8}$$

and

$$\bar{N}^{\alpha\beta} = N^{\alpha\beta}, N^{\alpha 3} = 0. \tag{5.9}$$

It follows that the equations of equilibrium (2.19)–(2.20), with $c^i = 0$, $F^i = 0$, $\bar{L}^i = 0$, are satisfied provided

$$\pi^3 = 0 \quad \text{or} \quad \frac{\partial A}{\partial \delta_3} = 0. \tag{5.10}$$

The above equation defines δ_3 in terms of k_1, k_2 so that the deformation is prescribed. We observe that the deformation can be maintained by forces N^{11}, N^{22} with $N^{12} = 0$, in view of (5.8)–(5.9) *.

6. Small deformations superposed on extensions

We consider now a second deformed configuration resulting from *small* deformations superposed on large extensions in the first deformed configuration. Recalling (5.2) it follows that covariant differentiation using the metric $a_{\alpha\beta}$ becomes partial differentiation and in the formulae of section 3

$$u^i = u_i, \; b^i = b_i. \tag{6.1}$$

Also, the kinematic variables in Sec. 3 reduce to

$$\begin{aligned} &\phi_{\beta\alpha} = u_{\beta,\alpha}, \quad \phi_{3\alpha} = u_{3,\alpha}, \\ &2e'_{\alpha\beta} = a'_{\alpha\beta} = -a'^{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha}, \\ &a' = 2u_{\lambda,\lambda}, \quad b'_{\alpha\beta} = u_{3,\alpha\beta} \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} &\delta'_\alpha = d'^\alpha = d'_\alpha = b_\alpha + du_{3,\alpha}, \quad \delta'_3 = d'_3 = b_3, \\ &\kappa'_{\beta\alpha} = \lambda'_{\beta\alpha} = \mu_{\beta\alpha} = b_{\beta,\alpha}, \\ &\kappa'_{3\alpha} = \lambda'_{3\alpha} = \mu_{3\alpha} = b_{3,\alpha}. \end{aligned} \tag{6.3}$$

From (3.19)–(3.22) and (5.8)–(5.10), we have

$$\begin{aligned} &T'^{\alpha\beta} = N'^{\alpha\beta} + N^{\alpha\lambda} u_{\beta,\lambda} + N^{\alpha\beta} u_{\lambda,\lambda}, \\ &T'^{\alpha 3} = N'^{\alpha 3} + N^{\alpha\lambda} u_{3,\lambda}, \end{aligned} \tag{6.4}$$

$$\begin{aligned} &C'^{\alpha\beta} = M'^{\alpha\beta}, \; C'^{\alpha 3} = M'^{\alpha 3}, \\ &\omega^\alpha = \pi'^\alpha, \; \omega^3 = \pi'^3 \end{aligned} \tag{6.5}$$

and equations (3.28) yield

$$\overline{\mathrm{N}}'^{\alpha\beta} = \overline{\mathrm{N}}'^{\beta\alpha} = N'^{\alpha\beta}, \quad N'^{\alpha 3} = d\pi'^\alpha. \tag{6.6}$$

Further, the equations of motion (3.26) and (3.27) reduce to

$$\begin{aligned} &T'^{\alpha\beta}{}_{,\alpha} + \rho F'^\beta = \rho \ddot{u}^\beta, \\ &C'^{\alpha 3}{}_{,\alpha} + \rho(L'^3 - j\ddot{b}^3) = \omega^3 \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} &C'^{\alpha\beta}{}_{,\alpha} + \rho(L'^\beta - j\ddot{b}^\beta) = \omega^\beta, \\ &T'^{\alpha 3}{}_{,\alpha} + \rho F'^3 = \rho \ddot{u}^3. \end{aligned} \tag{6.8}$$

Constitutive equations for $\overline{N}'^{\alpha\beta}$, $M'^{\alpha i}$ and π'^i may now be found from (3.29)–(3.31) with the help of the List of Invariants in sec. 4 and the results of section 5. We observe that the equations of motion separate into two sets. One set, namely (6.7), is concerned with stretching

* More general results of this type for large deformation of an isotropic Cosserat surface are discussed in [11].

of the plate while the other given by (6.8) is associated with the transverse flexure of the plate. In what follows, we confine our attention to transverse flexural deformation and therefore only need to record the constitutive equations for $M'^{\alpha\beta}$ and π'^{α}. Thus,

$$M'^{\alpha\beta} = \rho \frac{\partial^2 A}{\partial \kappa_{\mu\lambda} \partial \kappa_{\beta\alpha}} \kappa'_{\mu\lambda}, \quad \pi'^{\alpha} = \rho \frac{\partial^2 A}{\partial \delta_\alpha \partial \delta_\beta} \delta'_\beta. \tag{6.9}$$

With further help from (4.6), the above constitutive equations can be expressed in a more explicit form. For this purpose, we introduce the notations

$$\begin{aligned}
&L_3 = \operatorname{tr} Q = A^{\alpha\beta}\delta_\alpha\delta_\beta, \quad L_5 = (\operatorname{tr} J)^2 = A^{\alpha\beta}A^{\lambda\mu}\kappa_{\beta\alpha}\kappa_{\mu\lambda},\\
&L_6 = \operatorname{tr} J\, J^T = A^{\alpha\lambda}A^{\beta\mu}\kappa_{\mu\lambda}\kappa_{\beta\alpha}, \quad L_7 = \operatorname{tr} J^2 = A^{\alpha\mu}A^{\beta\lambda}\kappa_{\mu\lambda}\kappa_{\beta\alpha},\\
&L_{10} = \operatorname{tr} I\, J\, J^T = A^{\alpha\lambda}e^{\beta\mu}\kappa_{\mu\lambda}\kappa_{\beta\alpha}, \quad L_{11} = 2 \operatorname{tr} J \operatorname{tr} I\, J = 2A^{\alpha\beta}e^{\lambda\mu}\kappa_{\mu\lambda}\kappa_{\beta\alpha},\\
&L_{12} = (\operatorname{tr} I\, J)^2 = e^{\alpha\beta}e^{\lambda\mu}\kappa_{\mu\lambda}\kappa_{\beta\alpha}, \quad L_{13} = \operatorname{tr} I\, Q = e^{\alpha\beta}\delta_\alpha\delta_\beta,\\
&e^{\alpha\beta} = A^{\alpha\lambda}A^{\beta\mu}e_{\lambda\mu}
\end{aligned} \tag{6.10}$$

and also write

$$\alpha_k = 2\rho_0 \frac{\partial A}{\partial L_k}, \qquad (\text{for } k = 3, 5, 6, 7, 10, 11, 12, 13). \tag{6.11}$$

Then, (6.9) becomes

$$\begin{aligned}
M'^{\alpha\beta} &= k_1 k_2 h^{\alpha\beta\lambda\mu} b_{\mu,\lambda},\\
\pi'^{\alpha} &= k_1 k_2 h^{\alpha\beta}(b_\beta + du_{3,\beta}),
\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}
h^{\alpha\beta\lambda\mu} = h^{\lambda\mu\alpha\beta} = {}& \alpha_5 A^{\alpha\beta}A^{\lambda\mu} + \alpha_6 A^{\alpha\lambda}A^{\beta\mu} + \alpha_7 A^{\alpha\mu}A^{\beta\lambda}\\
&+ \alpha_{10} A^{\alpha\lambda}e^{\beta\mu} + \alpha_{11}(A^{\alpha\beta}e^{\lambda\mu} + A^{\lambda\mu}e^{\alpha\beta}) + \alpha_{12} e^{\alpha\beta}e^{\lambda\mu},
\end{aligned} \tag{6.13}$$

$$h^{\alpha\beta} = h^{\beta\alpha} = \alpha_3 A^{\alpha\beta} + \alpha_{13} e^{\alpha\beta}. \tag{6.14}$$

From (6.4), (6.6) and (6.12) it follows that

$$T'^{\alpha 3} = k_1 k_2 [d h^{\alpha\beta}(b_\beta + du_{3,\beta}) - \xi^2 f^{\alpha\beta} u_{3,\beta}], \tag{6.15}$$

where

$$-\xi^2 f^{\alpha\beta} = N^{\alpha\beta}/(k_1 k_2) = \rho_0 \frac{\partial A}{\partial L_1} A^{\alpha\beta} + \rho_0 \frac{\partial A}{\partial L_2}(A^{\alpha\lambda}A^{\beta\mu} + A^{\alpha\mu}A^{\beta\lambda})e_{\lambda\mu} \tag{6.16}$$

and we note that $f^{12} = f^{21} = 0$.

The displacement equations of motion can be obtained by substitution of (6.12) and (6.15) into (6.8). In the absence of assigned force and director force F'^3, L'^β, these displacement equations of motion are

$$\begin{aligned}
&h^{\alpha\beta\lambda\mu} b_{\mu,\lambda\alpha} - h^{\alpha\beta}(b_\alpha + du_{3,\alpha}) = \rho_0\, j \ddot{b}_\beta,\\
&d h^{\alpha\beta}(b_{\beta,\alpha} + du_{3,\beta\alpha}) - \xi^2 f^{\alpha\beta} u_{3,\alpha\beta} = \rho_0 \ddot{u}_3.
\end{aligned} \tag{6.17}$$

We consider two types of boundary conditions which can be associated with equations (6.17). In the first the edge of the plate is simply supported after the plate has been subjected to a prior finite deformation, and in the other the edges are clamped after the finite deformation. In the case of a clamped edge, we have

$$b_\alpha = 0,\ u_3 = 0 \tag{6.18}$$

while the simply supported edge is specified by

$$\boldsymbol{M}' \cdot \boldsymbol{v} = 0,\ b^\alpha \boldsymbol{a}_\alpha - \boldsymbol{v}(b^\beta v_\beta) = 0,\ u_3 = 0$$

or equivalently

$$M'^{\alpha\beta} v_\alpha v_\beta = 0,\ b_\beta - v_\beta(b^\gamma v_\gamma) = 0,\ u_3 = 0. \tag{6.19}$$

7. Stability

The stability of a plate whose flexural motion is characterized by the system of equations (6.17) subject to boundary conditions of the type (6.18)–(6.19), as well as suitable initial conditions, can be discussed by methods used by Movchan [8] or other writers *. Here we limit our discussions to some relatively simple aspects of the problem in which the stability criteria depend upon variables integrated over the plate. Pointwise stability criteria could also be obtained in a manner similar to that of Shield [12].

We assume that the constants $h^{\alpha\beta}, f^{\alpha\beta}$ satisfy the conditions

$$\begin{gathered} h^{11} > 0,\ h^{22} > 0,\ h^{12} = 0, \\ f^{11} \geqq 0, f^{22} > 0 \text{ or } f^{11} > 0, f^{22} \geqq 0, f^{12} = 0 \end{gathered} \tag{7.1}$$

and that

$$h^{\alpha\beta\lambda\mu} \kappa'_{\beta\alpha} \kappa'_{\mu\lambda} > 0, \tag{7.2}$$

for all non-zero values of $\kappa'_{\beta\alpha}$. We also assume that $\rho_0 > 0, j > 0$.

Corresponding to equations (6.17) we define an energy function $E(t)$ by

$$E(t) = \tfrac{1}{2} \iint_{\mathfrak{s}} [\rho_0(\dot{u}^2 + j\dot{b}_\beta^2) + h^{\alpha\beta\lambda\mu} b_{\mu,\lambda} b_{\beta,\alpha} + h^{\alpha\beta}(b_\alpha + du_{,\alpha})(b_\beta + du_{,\beta}) - \xi^2 f^{\alpha\beta} u_{,\alpha} u_{,\beta}] d\sigma \tag{7.3}$$

where u is written for u_3 and the integration is over a finite connected area $\mathfrak{s}$ of the plane of the plate. Using (6.17) and either set of the boundary conditions (6.18) or (6.19) specified at the edge of the plate, it can be shown that

$$\dot{E}(t) = 0 \text{ or } E(t) = E(0). \tag{7.4}$$

Let b_β, u be any set of functions ** which satisfy either set of the boundary conditions (6.18)–(6.19) but not necessarily the equations of motion (6.17). Then,

$$\iint_{\mathfrak{s}} [h^{\alpha\beta\lambda\mu} b_{\beta,\alpha} b_{\mu,\lambda} + h^{\alpha\beta}(b_\alpha + du_{,\alpha})(b_\beta + du_{,\beta})] d\sigma \geqq \theta_1^2 \iint_{\mathfrak{s}} f^{\alpha\beta} u_{,\alpha} u_{,\beta}\, d\sigma, \tag{7.5}$$

where θ_1^2 is the smallest eigenvaue of the equations

$$\begin{gathered} h^{\alpha\beta\lambda\mu} b_{\mu,\lambda\alpha} - h^{\alpha\beta}(b_\alpha + du_{,\alpha}) = 0, \\ dh^{\alpha\beta}(b_{\beta,\alpha} + du_{,\beta\alpha}) - \theta^2 f^{\alpha\beta} u_{,\beta\alpha} = 0, \end{gathered} \tag{7.6}$$

subject to boundary conditions (6.18) or (6.19). Further, if u is any function which vanishes on the boundary of $\mathfrak{s}$, then

* See, for example, Shield [12], Hsu [13] and Knops and Wilkes [14] and references contained therein.

* Throughout this section any functions b_β, u are assumed to be twice continuously differentiable throughout $\mathfrak{s}$.

$$\iint_s f^{\alpha\beta} u_{,\alpha} u_{,\beta} \, d\sigma \geqq \theta_2^2 \iint_s u^2 d\sigma, \tag{7.7}$$

where θ_2^2 is the smallest eigenvalue of the equations

$$f^{\alpha\beta} u_{,\alpha\beta} + \theta^2 u = 0, \tag{7.8}$$

subject to $u = 0$ on the boundary. Similarly, if b_β are functions which satisfy the first two of (6.18) or (6.19) on the boundary, then

$$\iint_s h^{\alpha\beta\lambda\mu} b_{\beta,\alpha} b_{\mu,\lambda} \, d\sigma \geqq \theta_3^2 \iint_s b_\beta^2 \, d\sigma, \tag{7.9}$$

where θ_3^2 is the smallest eigenvalue of the equations

$$h^{\alpha\beta\lambda\mu} b_{\mu,\lambda\alpha} + \theta^2 b_\beta = 0, \tag{7.10}$$

subject to the first two conditions in (6.18) or in (6.19).

Using the inequalities (7.5), (7.7) and (7.9) in (7.3), we have

$$E(t) \geqq \tfrac{1}{2}(\theta_1^2 - \xi^2) \iint_s f^{\alpha\beta} u_{,\alpha} u_{,\beta} \, d\sigma \geqq \theta_2^2(\theta_1^2 - \xi^2) \iint_s u^2 d\sigma \tag{7.11}$$

provided $\xi^2 \leqq \theta_1^2$. Also

$$\begin{aligned} E(t) &\geqq \frac{1}{2} \frac{\theta_1^2 - \xi^2}{\theta_1^2} \iint_s [h^{\alpha\beta\lambda\mu} b_{\beta,\alpha} b_{\mu,\lambda} + h^{\alpha\beta}(b_\alpha + d u_{,\alpha})(b_\beta + d u_{,\beta})] d\sigma \\ &\geqq \frac{1}{2} \frac{\theta_1^2 - \xi^2}{\theta_1^2} \iint_s h^{\alpha\beta\lambda\mu} b_{\beta,\alpha} b_{\mu,\lambda} \, d\sigma \geqq \frac{\theta_3^2(\theta_1^2 - \xi^2)}{2\theta_1^2} \iint_s b_\beta^2 \, d\sigma \end{aligned} \tag{7.12}$$

if $\xi^2 \leqq \theta_1^2$. Moreover, under the same condition

$$E(t) \geqq \tfrac{1}{2} \iint_s \rho_0 (\dot{u}^2 + j \dot{b}_\beta^2) d\sigma. \tag{7.13}$$

If $\theta_1^2, \theta_2^2, \theta_3^2$ are all real and positive and if $\xi^2 < \theta_1^2$, then all the integrals

$$\iint_s u^2 d\sigma, \quad \iint_s b_\beta^2 \, d\sigma, \quad \iint_s \rho_0(\dot{u}^2 + j\dot{b}_\beta^2) d\sigma, \quad \iint_s f^{\alpha\beta} u_{,\alpha} u_{,\beta} \, d\sigma, \quad \iint_s h^{\alpha\beta\lambda\mu} b_{\beta,\alpha} b_{\mu,\lambda} \, d\sigma, \tag{7.14}$$

can be made as small as we please provided $E(0)$, the initial energy, is small; and, in this sense, the plate will be stable with respect to the intial energy $E(0)$ and the quantities in (7.14).

As remarked at the beginning of this section, pointwise stability criteria could also be obtained by the methods used by Shield [12] but this requires more restrictive assumptions on the initial disturbance.

8. Plate under all around compression

In order to illustrate the nature of the above results and make them more explicit, we study the special case of an isotropic plate under all around compression, the faces of the plate being free from assigned force. For this purpose, in the formulae of section 5, we put $k_2 = k_1$ so that

$$A_{\alpha\beta} = \frac{1}{k_1^2} \delta_{\alpha\beta}, \quad A^{\alpha\beta} = k_1^2 \delta_{\alpha\beta}, \quad e^{\alpha\beta} = \tfrac{1}{2} k_1^2 (k_1^2 - 1) \delta_{\alpha\beta}. \tag{8.1}$$

It follows from (6.13)–(6.14) and (6.16) that

$$h^{\alpha\beta\lambda\mu} = \beta_5 \delta_{\alpha\beta} \delta_{\lambda\mu} + \beta_6 \delta_{\alpha\lambda} \delta_{\beta\mu} + \beta_7 \delta_{\alpha\mu} \delta_{\beta\lambda}, \quad h^{\alpha\beta} = \beta_3 \delta_{\alpha\beta}, \quad f^{\alpha\beta} = f \delta_{\alpha\beta}, \tag{8.2}$$

where

$$\begin{aligned} \beta_5 &= [\alpha_5 + \alpha_{11}(k_1^2 - 1) + \tfrac{1}{4}\alpha_{12}(k_1^2-1)^2]k_1^4, \\ \beta_6 &= [\alpha_6 + \tfrac{1}{2}\alpha_{10}(k_1^2-1)]k_1^4, \quad \beta_7 = \alpha_7 k_1^4, \quad \beta_3 = [\alpha_3 + \tfrac{1}{2}\alpha_{13}(k_1^2-1)]k_1^2, \\ -\xi^2 f &= \rho_0 k_1^2 \left[\frac{\partial A}{\partial L_1} + (k_1^2 - 1)\frac{\partial A}{\partial L_2}\right] = \frac{N^{11}}{k_1^2} = \frac{N^{22}}{k_1^2}. \end{aligned} \tag{8.3}$$

Then, the conditions (7.1)–(7.2) reduce to

$$f > 0, \quad \beta_6 > 0, \quad \beta_6 \pm \beta_7 > 0, \quad \beta_6 + \beta_7 + 2\beta_5 > 0, \quad \beta_3 > 0. \tag{8.4}$$

We now examine the special case of a square plate under all around compression, the edges being of length 2 a and simply supported after the compression. Under these boundary conditions, it is straightforward to verify that the smallest eigenvalues θ_2^2 and θ_3^2 of equations (7.8) and (7.10), respectively, are real and positive. The stability conditions of section 7, reduce to

$$\xi^2 < \theta_1^2, \tag{8.5}$$

where θ_1^2 is the smallest eigenvalue of equations (7.6) subject to simply supported boundary conditions. By a usual type of analysis we find that

$$f\theta_1^2 = \frac{2d^2 \dfrac{\pi^2}{a^2} \beta_3(\beta_5+\beta_6+\beta_7)}{\beta_3 + 2\dfrac{\pi^2}{a^2}(\beta_5+\beta_6+\beta_7)}, \tag{8.6}$$

where d (or δ_3) is determined by (5.10). Apart from the linearization procedure used for obtaining the stability equation, the result (8.6) is based on a complete nonlinear theory of a Cosserat plate which imitates properties of a three-dimensional isotropic plate. Further understanding of this result can be obtained by assuming that the (nonlinear) strain measures $e_{\alpha\beta}$, $\kappa_{i\alpha}$, δ_i are all small and that the strain energy function can be adequately represented by the quadratic form *

$$\rho_0 A = \tfrac{1}{2}\alpha_1 L_1^2 + \alpha_2 L_2 + \tfrac{1}{2}\alpha_3 L_3 + \tfrac{1}{2}\alpha_4 \delta_3^2 + \tfrac{1}{2}\alpha_5 L_5 + \tfrac{1}{2}\alpha_6 L_6 + \tfrac{1}{2}\alpha_7 L_7 + \tfrac{1}{2}\alpha_8 L_8 + \alpha_9 \delta_3 L_1, \tag{8.7}$$

where the coefficients α_1, α_2, α_4, α_9 are different from those defined by (6.11), the invariants L_1 to L_7 were defined previously and

$$L_8 = \operatorname{tr} K. \tag{8.8}$$

The coefficients α_1, α_2, α_4, α_9 in the above quadratic form will be the same as those for the *linear* theory of an isotropic Cosserat plate, with appropriate values suggested in a number of papers (e.g., [5, 6]), so that (8.7) imitates the properties of a three dimensional isotropic plate of initial thickness h_0. Thus **

* Alternatively we may regard A in (8.7) as an exact special form of strain energy density.

** A value for α_8 is not required here. The approximate value for α_3 in (8.9) is that obtained in [6] from comparison of an elastostatic solution for torsion of a Cosserat plate with the corresponding result in the Saint-Venant theory of torsion and agrees with that in Reissner's [15] plate theory. A different value for α_3 may be employed for dynamical problems of plates and this ($\alpha_3 = (\pi^2/24)(1-\eta)D$) can be determined in a manner similar to that discussed by Mindlin [16]. The two values for α_3, however, are very close and here we retain that given in (8.9).

$$\alpha_1 = \alpha_9 = \frac{\eta(1-\eta)D}{1-2\eta}, \quad \alpha_2 = \tfrac{1}{2}(1-\eta)D, \quad \alpha_4 = \frac{(1-\eta)^2 D}{1-2\eta},$$
$$\alpha_5 = \eta B, \quad \alpha_6 = \alpha_7 = \tfrac{1}{2}(1-\eta)B, \quad \alpha_3 = \tfrac{5}{12}(1-\eta)D, \tag{8.9}$$
$$\beta_5 = \alpha_5 k_1^4, \quad \beta_6 = \alpha_6 k_1^4, \quad \beta_7 = \alpha_7 k_1^4, \quad \beta_3 = \alpha_3 k_1^2,$$

where

$$D = \frac{Eh_0}{1-\eta^2}, \quad B = \frac{Eh_0^3}{12(1-\eta^2)}, \tag{8.10}$$

E is Young's modulus, η is Poisson's ratio and h_0 is the initial thickness of the plate.
From (5.10), (8.3) and (8.7), we have

$$-\xi^2 f = \frac{N^{11}}{k_1^2} = k_1^2[\alpha_1 L_1 + \alpha_2(k_1^2-1) + \alpha_9 \delta_3], \quad \alpha_4 \delta_3 + \alpha_9 L_1 = 0, \tag{8.11}$$

so that

$$-\xi^2 f = \frac{N^{11}}{k_1^2} = \frac{N^{22}}{k_1^2} = \frac{k_1^2(k_1^2-1)Eh_0}{2(1-\eta)}, \quad d = \delta_3 + 1 = 1 - \frac{\eta(k_1^2-1)}{1-\eta}, \tag{8.12}$$

in view of (8.9). Also, (8.6) becomes

$$f\theta_1^2 = \frac{2d^2 k_1^4 B\pi^2/a^2}{1 + \dfrac{2\pi^2 k_1^2 h_0^2}{5(1-\eta)a^2}}. \tag{8.13}$$

If the strains are small then we may replace d and k_1 by unity, approximately, in (8.13). Then, in view of (8.5), the stability condition reduces to

$$-N^{11} < \frac{2B\pi^2/a^2}{1 + \dfrac{2\pi^2 h_0^2}{5(1-\eta)a^2}}. \tag{8.14}$$

The corresponding result obtained by classical stability theory * is found from (8.14) by letting $h_0/a \to 0$ and this results in

$$-N_c^{11} < 2B\pi^2/a^2. \tag{8.15}$$

The critical value given by (8.14) is always less than that in (8.15). As a typical example, suppose that

$$\eta = \tfrac{1}{3}, \quad \frac{h_0}{2a} = \tfrac{1}{10}. \tag{8.16}$$

Then

$$-N_c^{11} < 2B\pi^2/a^2, \quad -N^{11} < (0.806)2B\pi^2/a^2, \tag{8.17}$$

approximately, so that the present theory gives a critical compressive force which is considerably lower than that found from classical theory.

9. Note on vibrations of initially stressed plate

The transverse vibrations of an isotropic plate which has been subjected to prior large exten-

* By the classical theory we have reference to classical bending theory of plates in which the effect of 'transverse shear deformation' is absent.

sions in two perpendicular directions can be discussed using equations (6.17). In order to illustrate the character of the results obtained from such equations, we consider an isotropic square plate with its lateral faces free from assigned force and subjected to all around compression, so that the coefficients $h^{\alpha\beta\lambda\mu}$, $h^{\alpha\beta}$, $f^{\alpha\beta}$ are given by (8.2)–(8.3). We consider solutions of (6.17) subject to the boundary conditions (6.19) for simply supported edges after the large all around compression has been imposed.

Let p denote the angular frequency of vibrations. Then, by a straightforward calculation, it can be shown that p^2 satisfies the quadratic equation

$$j\rho_0^2 p^4 - \rho_0 p^2 \left[\beta_3 + \frac{\pi^2}{a^2}(m^2+n^2)\{\beta_5+\beta_6+\beta_7+j(\beta_3 d^2 - \xi^2 f)\}\right]$$

$$+\frac{\pi^2}{a^2}(m^2+n^2)\left[\frac{\pi^2 d^2}{a^2}\beta_3(\beta_5+\beta_6+\beta_7)(m^2+n^2)\right.$$

$$\left.-\xi^2 f\left\{\beta_3+\frac{\pi^2}{a^2}(\beta_5+\beta_6+\beta_7)(m^2+n^2)\right\}\right] = 0, \tag{9.1}$$

where m, n are positive integers. The inertia coefficient j can be determined from formulae in [6] based on a three-dimensonal analysis of plates and is found to be

$$j = \frac{h_0^2}{12}. \tag{9.2}$$

In order to obtain some specific results from (9.1), we limit the remaining discussion to the case when the strain energy function is given by (8.7) and the coefficients β_3, β_5, β_6, β_7 have the values indicated in (8.9) *. In addition, in (8.9) and (9.1) we put $k_1 = d = 1$ for plates in which the initial deformation is small. If the all round compressive force is less than the critical force computed in section 8, we set

$$\xi^2 f = \frac{\pi^2 B\phi}{a^2} \tag{9.3}$$

and the frequency equation (9.1) becomes

$$\frac{h_0^4\rho_0^2 p^4}{12} - 5(1-\eta)\rho_0 p^2 B\left[1+\frac{\pi^2 h_0^2(m^2+n^2)}{5(1-\eta)a^2}\left(\frac{17-5\eta}{12}-\frac{\pi^2 h_0^2\phi}{12a^2}\right)\right]$$

$$+\frac{5(1-\eta)\pi^4 B^2}{a^4}(m^2+n^2)\left(m^2+n^2-\phi-\frac{\pi^2 h_0^2(m^2+n^2)\phi}{5(1-\eta)a^2}\right) = 0. \tag{9.4}$$

The frequency equation obtained from the classical theory of thin plates may be obtained by letting $h_0/a \to 0$ in (9.4). Let p_c be the classical frequency, then

$$\rho_0 p_c^2 = \frac{B\pi^4}{a^4}(m^2+n^2)(m^2+n^2-\phi). \tag{9.5}$$

In contrast to (9.5), equation (9.4) gives, in general, two values for p^2 and these may readily be compared with p_c^2 derived from (9.5), for various values of m, n, ϕ and h_0/a. As a typical example, suppose that

$$\eta = \tfrac{1}{4},\quad \frac{h_0}{a} = \tfrac{1}{5},\quad m = n = 1,\quad \phi = 0. \tag{9.6}$$

* We retain the approximate value for α_3 given in (8.9). However, as noted in sec. 8 [see the footnote preceding (8.9)], a slightly different value for α_3 may be adopted for dynamical problems.

Then, the two roots p_1^2, p_2^2 of (9.4) are such that

$$p_1^2/p_c^2 = 0.79, \quad p_2^2/p_c^2 = 35, \tag{9.7}$$

approximately. Thus p_1 is less than p_c but near to it in value, whereas p_2 is very much greater than p_c and is not predicted by classical plate theory.

It is of interest to observe that when the compressive force reaches its critical value according to classical stability theory, then $\phi = 2$ and the corresponding frequency p_c for the case $m = n = 1$ vanishes. On the other hand, when the critical compressive force is given by (8.14), i.e., when ϕ has the value

$$\phi = \frac{2}{1 + \dfrac{2\pi^2 h_0^2}{5(1-\eta)a^2}}$$

and when $m = n = 1$, one frequency given by (9.4) vanishes; but, provided h_0/a is small, there is a second frequency which is large compared with the corresponding frequency of the unstressed plate.

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