Application of the Quadrature Method to Flexural Vibration Analysis of a Geometrically Nonlinear Beam

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Abstract. The quadrature method (QM) has been used in structural analysis only in recent years. In this study, QM is applied to flexural vibration analysis of a geometrically nonlinear beam. The numerical results by QM agree with the results by the finite element method. It is believed that this is the first attempt to solve a nonlinear dynamic problem by the quadrature method.

Key words: Beams, free vibration, nonlinear oscillation, quadrature method.

Nomenclature

A	area of beam cross section.
a	amplitude of first mode.
E	modulus of elasticity.
H	energy functional over the period T.
I	centroidal moment of inertia of beam cross section.
K	kinetic energy.
L	length of beam.
$L\{\ \}$	linear operator.
m	mass per unit length.
N	dynamic axial force.
r	radius of gyration of beam cross section.
T	period.
t	time.
U	strain energy.
$u(x)$	axial displacement of beam.
V_i	discrete nonlinear normal mode.
$v(x)$	nonlinear normal mode.
W_{ij}	weighting coefficient.
$w(x, t)$	instantaneous lateral deflection function of x and t.
x	beam axial position coordinate.
ξ	dimensionless beam axial position coordinate.
ϵ_x	strain in the x-direction.
κ_x	curvature.
ω	fundamental nonlinear frequency.
ω^*	fundamental dimensionless nonlinear frequency.
ω_L	fundamental linear frequency.

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Introduction

Beams oscillating with a large amplitude are nonlinear systems and the small deflection theory is no longer applicable in this case. Considerable research effort has been devoted to solving nonlinear oscillating beam problems. An extensive literature survey was given by Nayfeh and Mook [1] and a comparative evaluation was recently presented by Singh *et al.* [2]. The procedures used include analytical, perturbation, and finite element methods. In the present study, the quadrature method is used to solve the free vibration problem of a geometrically nonlinear beam. The method of differential quadrature was introduced by Bellman and Casti [3]. Mingle [4] also applied this idea to the nonlinear diffusion problem, while Civan and Sliepcevich solved transport process problems by this technique [5] and extended it to include integrals [6]. In structural engineering, Bert *et al.* (see [7] and seven of its references) solved a variety of problems involving linear and nonlinear static situations and several linear dynamic problems. It is believed that the present study is the first attempt to solve a nonlinear dynamic problem by the quadrature method.

Formulation

Consider a beam oscillating with a large amplitude on immovable supports. It is assumed that plane sections remain plane and normal to the deflected middle surface and that the stress-strain law is linear. The nonlinear strain-displacement and curvature-displacement relations of the beam can be described as:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \kappa_x = \frac{\partial^2 w}{\partial x^2}. \tag{1}$$

Upon neglect of the axial inertia force, the dynamic axial force N can be written in the form:

$$N(x, t) = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]. \tag{2}$$

Assuming that the ends are axially immovable, i.e., $u(o, t) = u(L, t) = 0$, then

$$N(x, t) = N(t) = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x. \tag{3}$$

It is evident that the axial force is independent of x and thus depends only on time.

With the above assumptions and considerations, the strain energy U and the kinetic energy K for a beam can be written in the form:

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^L \frac{N^2}{EA}\,\mathrm{d}x, \tag{4}$$

$$K = \frac{1}{2}\int_0^L m\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x. \tag{5}$$

Assume that:

$$w(x, t) = av(x)\cos \omega t. \tag{6}$$

The quantity $v(x)$ is the so-called nonlinear normal mode.

Define the energy functional H as the Lagrangian functional:

$$H = \frac{1}{T} \int_0^T (K - U)\, dt . \tag{7}$$

Hence:

$$H(v, a) = \frac{a^2\omega^2}{4} \int_0^L mv^2(x)\, dx - \frac{a^2}{4} \int_0^L EI\left[\frac{d^2v}{dx^2}\right]^2 dx - \frac{3a^4}{16} \frac{EA}{4L} \left[\int_0^L \left(\frac{dv}{dx}\right)^2 dx\right]^2 . \tag{8}$$

By Hamilton's principle, let $\delta_v H(v, a) = 0$ to obtain the equation of motion:

$$EI \frac{d^4v}{dx^4} - \frac{3}{4}\left[\frac{EAa^2}{2L} \int_0^L \left(\frac{dv}{dx}\right)^2 dx\right] \frac{d^2v}{dx^2} - \omega^2 mv = 0 . \tag{9}$$

Let:

$$\xi = \frac{x}{L} , \qquad (\omega^*)^2 = \omega^2 \frac{mL^4}{EI} , \qquad r^2 = \frac{I}{A} .$$

Then equation (9) can be rewritten in dimensionless form:

$$\frac{d^4v}{d\xi^4} - \frac{3}{4}\left[\frac{1}{2} \frac{a^2}{r^2} \int_0^1 \left(\frac{dv}{d\xi}\right)^2 d\xi\right] \frac{d^2v}{d\xi^2} - (\omega^*)^2 v = 0 . \tag{10}$$

The Quadrature Method

The quadrature method approximates the partial derivative of a function with respect to a space variable at a given discrete point as a weighted linear sum of the function values at all discrete points. The integral can also be approximated by the quadrature method [6].

A quadrature approximation at the ith discrete point is given by:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij} f(x_j) , \qquad i = 1, 2, \ldots, N . \tag{11}$$

Here $L\{\ \}$ is a linear operator applied to a function $f(x)$, where x is the independent variable and x_j (where $j = 1, 2, \ldots, N$) are the sample points obtained by dividing the x-variable into N discrete values; $f(x_j)$ are the function values at these points; and W_{ij} are the weights attached to these function values.

In order for one to determine the weighting coefficients W_{ij}, equation (11) must be exact for all polynomials of degree less than or equal to $(N - 1)$. A general term in such a polynomial or test function is then given by:

$$f_k(x) = x^{k-1} , \qquad k = 1, 2, \ldots, N . \tag{12}$$

Then equation (11) leads to:

$$L\{x^{k-1}\}_i = \sum_{j=1}^{N} W_{ij} x_j^{k-1} , \qquad i, k = 1, 2, \ldots, N . \tag{13}$$

If the linear operator represents an nth derivative, then the left hand side of equation (13) can be replaced as follows:

$$L\{x^{k-1}\}_i = (k-1)(k-2)\cdot \cdots \cdot (k-n)x_i^{(k-n-1)} , \qquad i, k = 1, 2, \ldots, N . \tag{14}$$

If the linear operator represents an mth integral, then the left hand side of equation (13) can be replaced as follows:

$$L\{x^{k-1}\}_i = x_i^{(k+m-1)}/[k(k+1)(k+2)\cdot \cdots \cdot (k+m-1)] , \qquad i, k = 1, 2, \ldots, N . \tag{15}$$

The equation (13) represents N sets of N linear algebraic equations for the determination of the weighting coefficients W_{ij}. It is noted that the sets have a unique solution for the weighting coefficients W_{ij}, since the matrix of element x_j^{k-1} represents a Vandermonde matrix which always has an inverse as described by Hamming [8]. The weighting coefficients are then used in equation (13) to express the derivatives or integrals of a function at a discrete point in terms of all the discrete function values. It is emphasized that the quadrature must be of higher order than the order of any partial derivatives and integrals, i.e., $N > n$ and $N > m$.

Application to a Nonlinear Vibrating Beam

There are four linear operators contained in equation (10), namely:

$$L_1 = \frac{d}{dx} , \quad L_2 = \frac{d^2}{dx^2} , \quad L_3 = \frac{d^4}{dx^4} , \quad L_4 = \int_0^{x_0} dx .$$

The weighting coefficients A_{ij}, B_{ij}, D_{ij}, and G_{ij} are associated with L_1, L_2, L_3, and L_4 respectively.

Applying the quadrature method to equation (10), one obtains:

$$\sum_{j=1}^{N} D_{ij} V_j + \frac{3}{8} \frac{a^2}{r^2} \left[\sum_{r,j=1}^{N} G_{rj} \Big(\sum_{s=1}^{N} A_{rs} V_s \Big) \Big(\sum_{k=1}^{N} A_{jk} V_k \Big) \right] \Big(\sum_{s=1}^{N} B_{is} V_s \Big) - (\omega^*)^2 V_i = 0$$

$$i = 3, 4, \ldots, N-2 . \tag{16}$$

For a simply supported beam, the deflection and moment are zero at the ends. The boundary conditions can be written as:

$$v(0) = v_{,\xi\xi}(0) = v(1) = v_{,\xi\xi}(1) = 0 , \tag{17}$$

where $(\ )_{,\xi\xi}$ denotes $d^2(\ )/d\xi^2$. In terms of quadrature:

$$V_1 = 0, \sum_{j=1}^{N} B_{2j} V_j = 0, \sum_{j=1}^{N} B_{(N-1)j} V_j = 0, V_N = 0 . \tag{18}$$

For a clamped beam, the deflection and slope are zero at each boundary. Thus:

$$v(0) = v_{,\xi}(0) = v(1) = v_{,\xi}(1) = 0 . \tag{19}$$

Applying differential quadrature yields:

$$V_1 = 0, \sum_{j=1}^{N} A_{2j} V_j = 0, \sum_{j=1}^{N} A_{(N-1)j} V_j = 0, V_N = 0 . \tag{20}$$

To solve the nonlinear eigenvalue problem, an iterative scheme is used [9]. First, set the amplitude a equal to zero and solve the resulting linear eigenvalue problem. The linear eigenvalues and eigenvectors are then used to obtain nonlinear coefficients. Solve the eigenvalue problem again to obtain nonlinear eigenvalues and eigenvectors and repeat the process. It does not take too many times to get convergence of eigenvalues and eigenvectors.

Results

The numerical results are obtained by taking $N = 7$. All the calculations are conducted on IBM 3081 K (main frame). The results of nonlinear frequencies for both simply supported and clamped beams along with Mei's finite element results [10] are listed in Table I. Amplitude-frequency curves are plotted in Figure 1.

Notice that the results for the clamped case agree better than that for the simply supported case. The reason is that the beam deforms less in the clamped case, so that the beam is closer to linear oscillation.

TABLE I.
Numerical results for the ratio of the nonlinear frequency to the linear frequency (ω/ω_L) as a function of the dimensionless amplitude

Simply Supported Beam			Clamped Beam	
a/r	QM	FEM	QM	FEM
0.1	1.0010	1.0009	1.0003	1.0003
0.2	1.0043	1.0037	1.0011	1.0012
0.4	1.0170	1.0148	1.0044	1.0048
0.6	1.0384	1.0339	1.0100	1.0107
0.8	1.0673	1.0578	1.0178	1.0190
1.0	1.1030	1.0889	1.0278	1.0295
1.5	1.2045	1.1902	1.0628	1.0650
2.0	1.3170	1.3022	1.1119	1.1127

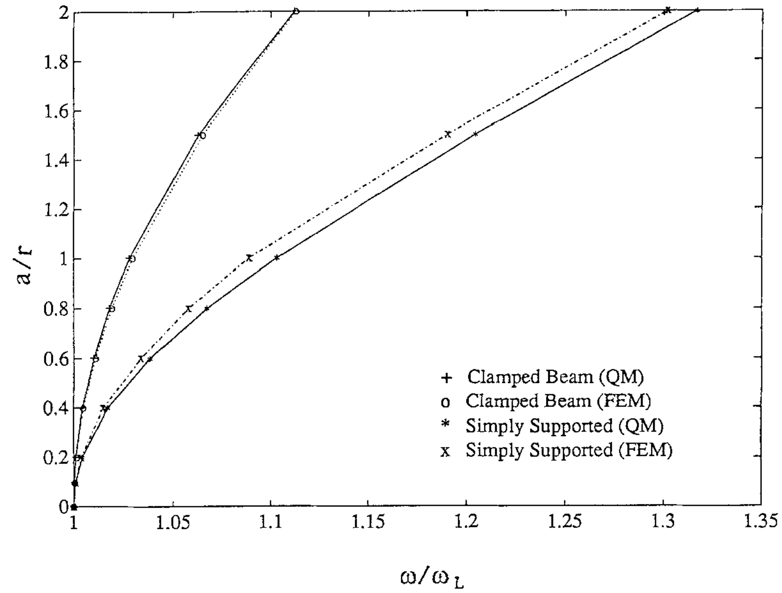

Fig. 1. Dimensionless amplitude-frequency curves of a geometrically nonlinear beam.

Conclusion

It is concluded that the quadrature method is very convenient to use. It requires less computational effort than the finite element method. For a beam problem, it only takes 2 seconds of CPU time, and the results obtained by the quadrature method agree with that by the finite element method especially in the clamped case.

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