Log-Likelihood Maximization and Response Surface in Reliability Assessment

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Abstract. Nonlinear dynamics problems can generally be solved only in a numerical way. This prevents from a direct application of standard reliability methods. A technique which makes use of iterated response-surface analytical approximations of the system performance function was therefore proposed in view of reliability assessment. The limitation of this technique was of working in a standard normalized space, so that appropriate space transformations are preliminarly required.

This paper shows how this response-surface iterative scheme can also be used in the original space of the random variables, provided a maximum log-likelihood constrained optimization problem is solved. Moreover, asymptotic theory also provides a better estimate of the probability of failure of the dynamical system against any assigned limit state.

Key words: Hysteretic oscillator, maximum likelihood, reliability, response surface, stochastic dynamics.

1. Introduction

A classical problem in nonlinear stochastic mechanics is the calculation of the reliability of the system. Standard methods of structural reliability require an analytical form for the limit state function which indicates the state of the structure. Any nonlinear dynamics problem is generally characterized by the impossibility of obtaining such analytical solutions since numerical methods are required in practical applications.

A new procedure of reliability assessment, capable of working even if the analysis of the systems can only be conducted in a numerical way, was presented in previous papers [1, 2]. This procedure makes use of a response surface scheme in combination with classical reliability methods. Indeed the results obtained by numerical calculations are used to compute an analytical form of the relationship between the structural response and the input parameters: generally, it is modelled by a second order polynomial as described in Section 2.

A number of analytical approximation methods were developed for the assessment of the probability of failure of a structural system. The basic idea of these methods, which are outlined at the beginning of Section 3, was to transform all random variables into standard normal variables.

In order to transform arbitrary non-normal random vectors with dependent components into standard normal random vectors with independent components, Hohenbichler and Rackwitz [3] proposed the use of the Rosenblatt transformation. However this method clearly shows two drawbacks. First, in the case of an n-dimensional random vector, it is necessary to use n conditional distribution functions, which must either be known beforehand or be computed by

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numerical differentiation. Second, by such a transformation the structure of the problem is completely changed, as it can be seen in the example with dependent exponential random variables in [3]. Here a linear boundary is transformed into a non-linear boundary of complicate shape. The practical use of this method was therefore limited, and restricted, to examples where the conditional distributions were known analytically. An asymptotic method which avoids the problem of this transformation is described in Section 3.

A combined reliability approach, making use of the response surface methodology in conjunction with the asymptotic method of Section 3, is eventually presented and some numerical examples are illustrated.

2. Governing Relations of the Response Surface Scheme

Let the vector $\{\mathbf{X}\}$ denote the random variables in the mechanical system under investigation (here and in the following random quantities are distinguished by symbols printed in bold). The vector $\{\mathbf{X}\}$ can be partitioned into:

$$\{\mathbf{X}\}^T = \{\{\mathbf{X}_v\}^T, \{\mathbf{X}_s\}^T\}$$

where $\{\mathbf{X}_v\}$ is a vector of random variables and $\{\mathbf{X}_s\}$ groups those random variables whose spatial or temporal distribution cannot be neglected. Any component $\mathbf{X}_{si}$ of $\{\mathbf{X}_s\}$ can be written in one of the two alternative forms:

$$\mathbf{X}_{si}(t, x, y, z) = \mathbf{X}_{svi} + \mathbf{X}_{ssi}(t, x, y, z)$$
$$\mathbf{X}_{si}(t, x, y, z) = \mathbf{X}_{svi}\mathbf{X}_{ssi}(t, x, y, z)$$

where $\mathbf{X}_{svi}$ is any random central value of $\mathbf{X}_{si}$ and $\mathbf{X}_{ssi}$ denotes the deviations of $\mathbf{X}_{si}$ from the central value $\mathbf{X}_{svi}$. Discretization could lead one to regard $\mathbf{X}_{ssi}$ as a random vector rather than a random field.

The extended response surface model formulated by the second author [4, 5, 6] is based on the relationship

$$\mathbf{g}(\{X\} \mid \{X_{ss}\}) = F_R(\{X_v\}, \{X_{sv}\}, \{\theta\}) + \mathbf{e}(\{X_{ss}\}) \tag{1}$$

where, as usual, $F_R(\cdot)$ is a polynomial with coefficients $\{\theta\}$ and the random term $\mathbf{e}$ takes into account the error due to the lack of fit and the randomness of the variables $\{X_{ss}\}$ which do not appear explicitly in equation (1).

The coefficients of the response surface method and the error must be evaluated by a regression analysis on the results of numerical structural analyses whose input parameters are selected in accordance with experiment design theory [6].

The variables $\{X_v\}$ and $\{X_{sv}\}$ are mapped in the standardized space Z where all the variables are uncorrelated, and have zero mean and unit variance just in order to have uniformity in the experiment design. This can be done by any transformation of classical reliability theory. It follows that the performance function in the space Z can be given through the parametrized form:

$$\mathbf{G}(\{Z\} \mid \{X_{ss}\}) = F_R(\{Z\}, \{\Theta\}) + \boldsymbol{\epsilon}(\{X_{ss}\}) . \tag{2}$$

The calculations are strongly simplified when $\boldsymbol{\epsilon}$ is assumed nearly constant with $\{X_{ss}\}$.

The reader is referred to [2] for the more important aspects met in modelling equation (2).

3. Reliability Assessment

The problem of calculating failure probabilities involves in general the computation of multi-dimensional integrals with implicitly defined boundaries. Several approximation methods were developed to obtain more and more efficient solution methods for this problem.

The usual formulation of the problem is the following. The given items are a random vector $\{\mathbf{X}\} = \{\mathbf{X}_1, \ldots, \mathbf{X}_n\}^T$, which describes the random influences on a structure and a limit state function $g(X_1, \ldots, X_n) = g(\{X\})$, which indicates the state of the structure when the random vector $\{\mathbf{X}\}$ has the realization $\{X\} = \{X_1, \ldots, X_n\}^T$. If $g(\{X\}) > 0$, the structure is safe, if $g(\{X\}) \leq 0$, it is unsafe. If $f(\{X\})$ is the joint probability density function (JPDF) of the random vector $\{\mathbf{X}\}$, the probability of failure $P(F)$, with the failure $F = \{\{X\}; g(\{X\}) \leq 0\}$, is then

$$P(F) = \int_{g(\{X\}) \leq 0} f(\{X\})\{\mathrm{d}X\} .$$

In general, the dimension n is large and the function $g(\{X\})$ has no simple analytical form.

The first approximation methods, which were used to calculate this probability $P(F)$, are the so called FORM (First Order Reliability Methods) procedures. As shown in [3] every random vector $\{\mathbf{X}\}$ with independent components $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and continuous PDF can be transformed into a standard Gaussian vector $\{\mathbf{U}\} = (\{\mathbf{U}_1, \ldots, \mathbf{U}_n\})^T$ with independent components. This random vector $\{\mathbf{U}\}$ has then the joint PDF:

$$f(\{U\}) = (2\pi)^{-n/2} \exp\left(-1/2 \sum_{i=1}^{n} (U_i^2)\right) = (2\pi)^{-n/2} \exp(-|\{U\}|^2/2) .$$

Here $|\{U\}| = \sqrt{(\Sigma_{i=1}^{n} (U_i^2))}$ is the Euclidean norm of the vector $\{\mathbf{U}\}$. If a failure domain F is given in a standard normal space, an approximation for its probability content $P(F)$ was derived by simple geometric arguments. First, the point $\{X^0\} \in F$ with minimal distance to the origin was determined, i.e., the point $\{X^0\} \in F$ with $|\{X^0\}| = \min_{\{X\} \in F} |\{X\}|$. Since $g(\{X\})$ is continuous and F is defined by $F\{\{X\}; g(\{X\}) \leq 0\}$, this point must lie on the limit state surface $B = \{\{X\}; g(\{X\}) = 0\}$. Second, at this point a first order Taylor expansion of $g(\{X\})$ was made and then $g(\{X\})$ is replaced by the linear function $g_L(\{X\})$ obtained by this expansion

$$g_L(\{X\}) = g(\{X^0\}) + \{\nabla g(\{X^0\})\}^T(\{X\} - \{X^0\}) = \{\nabla g(\{X^0\})\}^T(\{X\} - \{X^0\}) = 0 .$$

Instead of $P(F)$ then the probability content of the domain $F_L = \{\{X\}; g_L(\{X\}) \leq 0\}$ is calculated. This can be done easily, since F_L is a half-space and, in the standard normal space, the probability content of such a domain is just $\Phi(-\beta)$ with β the distance of F_L to the origin, if F_L does not contain the origin in its interior. It can be shown that in this case $\beta = |\{X^0\}|$.

What can be done in the general case of dependent random variables? The Rosenblatt transformation for transforming random vectors with dependent components into standard normal vectors is applicable only in special cases, as already mentioned in the introduction.

Three questions in connection with these approximation methods remain open.

1. How to calculate approximations, if there is no unique minimal distance point?
2. What can be said about the quality of these approximations?
3. Are the transformations into a standard normal space necessary?

The first two problems were solved by the first author [7] using concepts of asymptotic analysis. It was shown that for FORM approximations no precise mathematical justification can be given, but this can be done for the so-called SORM (Second Order Reliability Methods) procedures. These methods are analogous to the FORM's, but, instead of the first order Taylor expansion $g_L(\{X\})$, a second order expansion $g_Q(\{X\})$ is made. For the probability content $P(F_Q)$ of a region $F_Q = \{\{X\}; g_Q(\{X\}) \leq 0\}$, bounded by such a quadratic function, a simple asymptotic approximation can be found. Further it can be proved that $P(F_Q) \approx P(F)$, if $P(F) \to 0$. The asymptotic approximation is

$$P(F) \sim \Phi(-\beta) \prod_{i=1}^{n-1} (1 - \beta\kappa_i)^{-1/2} .$$

Here the κ_i's are the main curvatures of the surface $B = \{\{X\}; g(\{X\}) = 0\}$ at the minimal distance point $\{X^0\}$. Further it was outlined that, in the case of several minimal distance points, an approximation is obtained by calculating for each point, separately, the approximation in the equation above. All these contributions must then be added.

The last more important question was again answered by the first author in [8]. Here it was proved that it is not necessary to transform random vectors into standard normal random vectors to be able to calculate asymptotic approximations. The basic idea is to understand the meaning of the minimization of the distance to the origin in FORM and SORM. If we look at the Gaussian PDF $f(\{U\})$, we get for its logarithm

$$\ln(f(\{U\})) = -n/2 \ln(2\pi) - |\{U\}|^2/2 .$$

The minimization of the distance $|\{U\}|$ corresponds to the maximization of the logarithm of the PDF. In mathematical statistics this function is called the log-likelihood. Therefore the probabilistic meaning of this is the maximization of the log-likelihood. When this is clear, asymptotic approximations can be derived also for non-normal random vectors with a PDF $f(\{X\})$ and log-likelihood $l(\{X\}) = \ln(f(\{X\}))$. The point $\{X^*\}$ is calculated for which:

$$l(\{X^*\}) = \max_{\{X\} \in F} l(\{X\}) .$$

Then at this point a suitable expansion of the log-likelihood is made. We get as approximation (see [8]):

$$P(F) \sim (2\pi)^{(n-1)/2} \frac{f(\{X^*\})}{|\nabla l(\{X^*\})| \sqrt{(|\det[H^*]|)}} . \tag{3}$$

Here the matrix $[H^*(\{X^*\})]$ has the form

$$[H^*] = [P][H][P]^T - \{\nu\}\{\nu\}^T$$

with:

(1) $\{\nu\} = |\{\nabla l(\{X^*\})\}|^{-1}\{\nabla l(\{X^*\})\}$ the normal to the surface.

(2) $[P] = [I_n] - \{\nu\}\{\nu\}^T([I_n]$ being the n-dimensional unity matrix)

(3) $[H] = (l_{,ij}(\{X^*\}) - \dfrac{|\{\nabla l(\{X^*\})\}|}{|\{\nabla g(\{X^*\})\}|} g_{,ij}(\{X^*\}))$ $\quad i, j = 1, \ldots, n$

the Hessian of $l(\{X\})$ in local coordinates.

An important advantage of the method of log-likelihood approximation is the fact that it can be used also for dependent random variables, if only the joint PDF $f(\{X\})$ and its log-likelihood $l(\{X\}) = \ln(f(\{X\}))$ are known. In this case we calculate the point of maximum likelihood, then the first and second derivatives of the likelihood function at this point are computed and inserted into the approximation equation. Here no conditional distribution function, as needed in the Rosenblatt transformation, must be computed. Further we can calculate directly from the log-likelihood function in the original space importance and sensitivity factors (see [9]).

4. Available Extensions for the Log-Likelihood Maximization Approach

4.1. Importance of the Random Variables

At the maximum log-likelihood point $\{X^*\}$ the gradient of this function gives information about the importance of the components, in a way similar to the α-factors in the case of standard normal random variables. The probability of failure $P(F)$ satisfies in first approximation the relation (see Breitung [8]):

$$\partial P(F)/\partial X_i \approx (\partial l(\{X^*\})/\partial X_i)P(F)\,.$$

This relation gives the relative importance of the various components, if standardized. This gives for the relative importance

$$(\partial l(\{X^*\})/\partial X_i)\Big/\left[\sum_{i=1}^{n} \partial l(\{X^*\})/\partial X_i\right]^2. \tag{4}$$

In the case of independent X_i's, one has in particular

$$\partial l(\{X^*\})/\partial X_i = \partial l_i(X_i^*)/\partial X_i$$

l_i being the log-likelihood function of the marginal PDF of $\mathbf{X}_i$.

4.2. Several Maximum Likelihood Points

If there are k points $\{X\}_1, \ldots, \{X\}_k$ on the boundary of the failure domain, where the log-likelihood function has a maximum with respect to F, we have to distinguish between local and global maxima. The asymptotic theory says that only global maxima are of importance asymptotically, but, since we are never in the infinite local maxima case, their contributions should be retained for two reasons. First they give an idea of the structure of the limit state function and second, due to uncertainties in the parameter estimation process, it may happen that local maxima become global and vice versa for small changes in the parameter values. Therefore, when the

log-likelihood has local maxima in k points $\{X\}_1, \ldots, \{X\}_k$, an approximation for $P(F)$ is given by

$$P(F) \approx (2\pi)^{(n-1)/2} \sum_{l=1}^{k} (f(\{X\}_l)/|\{\nabla l(\{X\}_l)\}|\sqrt{(|\det([H(\{X\}_l)])|)}$$

with

$$H(\{X\}_l) = [P]_l[H(\{X\}_l)[P]_l]^T - \{v\}_l\{v\}_l^T$$

$$[P] = [I_n] - \{v\}_l\{v\}_l^T$$

$$\{v\}_l = |\{\nabla l(\{X\}_l)\}|^{-1}\{\nabla l(\{X\}_l)\}$$

$$[H(\{X\}_l)] = (l_{,ij}(\{X\}_l) - (|\{\nabla l(\{X\}_l)\}|/|\nabla g(\{X\}_l)|)g_{,ij}(\{X\}_l)) \qquad i, j = 1, \ldots, n .$$

4.3. Asymptotic Approximation Methods for Crossing Rates of Random Processes

The asymptotic method which has been described can also be applied to calculate surface integrals. Such results can be used to achieve approximations for the crossing rates of stochastic vector processes. Let $\{\mathbf{X}(t)\} = \{\mathbf{X}_1(t), \ldots, \mathbf{X}_n(t)\}^T$ be a given stationary vector process with continuously differentiable sample paths. If the surface B in $\mathbf{R}^n$ is given, the expected number of crossings of this process through B during one time unit is given by:

$$E(B) = \int_B E(|\{v(\{y\})\}^T\{\mathbf{X}'(t)\}|; \{\mathbf{X}(t)\} = \{y\})f_{\{\mathbf{X}(t)\}}(\{y\})\, ds(\{y\})\,.$$

Here: $\{v(\{y\})\}$ is the normal to the surface B at $\{y\}$, $\{\mathbf{X}'(t)\} = \{\mathbf{X}_1'(t), \ldots, \mathbf{X}_n'(t)\}^T$ is the derivative process, $E(\mathbf{z}; \mathbf{y} = y)$ denotes the conditional expectation of the random variable $\mathbf{z}$ under the condition $\mathbf{y} = y$, $f_{\{\mathbf{X}(t)\}}(\{y\})$ is the PDF of the random vector $\{\mathbf{X}(t)\}$ at $\{y\}$, $ds(\{y\})$ denotes the integration surface differential.

For this integral, asymptotic approximations are derived in [10]. Such approximations can be calculated also for nonstationary processes. The univariate case is treated in [11] and the multivariate in [12].

5. The Method

Consider equation (2) written in the central region of all the variables in order to identify the actual location of $G(\cdot)$ in the $\{Z\}$ space. In this case standardization is a shift of the origin and a change of scale, but existing correlations are not removed. A linear polynomial form is preliminarly selected in order to model the response surface. An appropriate fractional replicate of a factorial design centered at the origin of the space Z defines the experiment plan [1]. This first response surface application (where the contribution of the quantities $\mathbf{X}_{ss}$ is neglected) leads to the following results:

– an estimation of the design point $\{Z_{(1)}^0\}$ and of the maximum likelihood point $\{Z_{(1)}^*\}$ for the linear model;

– the coefficients of the linear approximation $G_L(z)$.

The sensitivity factors of each variable Z_j can be estimated from the gradient of the log-likelihood function at the point $\{\mathbf{Z}_{(1)}^*\}$ (see equation (4)). No substantial error is generally introduced by considering the random variables with low sensitivity factors as fixed and equal to their means in

the further calculations. A better estimation is then pursued by a sequence of repeated response surface applications. Each of them consists of several deterministic structural analyses for different input data. They form an experiment design conceived both to fit the second-order polynomial response surface and to incorporate the terms $\mathbf{X}_{ss}$. At the i-th step, the resulting approximation for the response function is

$$\mathbf{G}_V^{(i)}(\{V_{(i)}\}) = F_R(\{V_{(i)}\}, \{\Theta_{(i)}\}) + \epsilon_{(i)}(\{X_{ss}\}) \tag{5}$$

where

$$\{V_{(i)}\} = \{Z\} - \{Z^{\circ}_{(i)}\} . \tag{6}$$

By equation (6), the origin of the V space is shifted along the vector from the origin of the Z space to the relevant design point $\{Z^{0}_{(i-1)}\}$. The convergence to the actual value is pursued by appropriately selecting the new origin on this line: this defines $\{Z^{\circ}_{(i)}\}$.

Practical reasons suggest that one should work, rather that in the space $\{V\}$, in a particular space $\{W\}$, obtained from it by rotation. The new first axis W_1 coincides with the normal to the function $G(\cdot) = \text{const}$ at the selected point. This iterative procedure stops when the square of the difference between the current and previous distances of the design point from the origin is lower than a fixed tolerance. At each step one also computes the current estimate of the maximum-likelihood $\{Z^{*}_{(i)}\}$ by the algorithm of constrained nonlinear optimization in [3].

Iterations toward the maximum likelihood point could also be pursued. In this case, however, the only convergence on the distance from the origin could be no longer satisfactory. Moreover, the dependency of the maximum log-likelihood point on the derivatives of the function G may result in a very slow convergence.

A final refinement consists of a response surface with its center at the maximum likelihood point. A distortion of the experiment plan along the first axis (so that all the experiments contribute to the definition of $G(\cdot) = 0$ [1]) was not tested in this case, since it may alter the derivatives which play a basic role in the evaluation of the maximum likelihood point.

Then, using equation (3), an asymptotic approximation for the probability content of the failure domain is derived.

6. Examples

The maximum-likelihood response-surface algorithm proposed in this paper for the reliability assessment of strongly non-linear mechanical systems is applied in this section to: (1) the classical reinforced-concrete cross section static problem in seven random variables, already discussed in [1]; and (2) a hysteretic oscillator under white-noise excitation, with uncertain system parameters.

6.1. Static Application

In [1] a vector $\{\mathbf{X}\}$ of mutually uncorrelated Gaussian random variables is introduced. Its seven components are: $\mathbf{X}_1$ = sectional bending moment; $\mathbf{X}_2$ = distance of the reinforcement from the compressed end; $\mathbf{X}_3$ = steel yielding stress; $\mathbf{X}_4$ = reinforcement area; $\mathbf{X}_5$ = factor of the stress

strain concrete relation; $\mathbf{X}_6$ = cross section width and $\mathbf{X}_7$ = maximum compressive strength of the concrete. The safety domain for a pure bending case can be written:

$$g(\{\mathbf{X}\}) = \mathbf{X}_2\mathbf{X}_3\mathbf{X}_4 - \frac{\mathbf{X}_5\mathbf{X}_3^2\mathbf{X}_4^2}{\mathbf{X}_6\mathbf{X}_7} - \mathbf{X}_1 > 0\,. \tag{7}$$

This problem is studied in the present example for the probabilistic definition of the components of $\{\mathbf{X}\}$ given in Table I.

The chi-square and Weibull distributions are introduced in order to point out the capability of the proposed procedure to manage these distributions in an analytical way without any numerical transformation procedure. Each variable is replaced by its standardized form:

$$\mathbf{Z}_i = (\mathbf{X}_i - E[\mathbf{X}_i])/(\mathrm{Var}[\mathbf{X}_i])^{1/2}$$

where, as usual, $E[\ ]$ denotes the mean value and Var[ ] the variance.

For this static example the relative performance function is known analytically and therefore the maximum likelihood point can be found exactly by the algorithm of [13]. Its coordinates in the Z-space are given in the first row of Table II.

The application of the approach proposed in this paper starts with a linear response surface approximation of equation (7).

TABLE I.
Probabilistic definition of the random variables of the example concerning the static application. The variable $\mathbf{X}_2$ is described by its transformation $\mathbf{X}_2' = (0.3158 - \mathbf{X}_2)0.100(\mathrm{m})$

Symbol	Distribution type	Mean value	Standard deviation
$\mathbf{X}_1$	Gaussian	0.01 MNm	0.003 MNm
$\mathbf{X}_2'$	Chi-square with 5 degrees of freedom	0 m	1 m
$\mathbf{X}_3$	Weibull	363.78 MPa	35.39 MPa
$\mathbf{X}_4$	Gaussian	$2.26\,10^{-4}$ m^2	$1.13\,10^{-5}$ m^2
$\mathbf{X}_5$	Gaussian	0.5	0.05
$\mathbf{X}_6$	Gaussian	0.12 m	0.006 m
$\mathbf{X}_7$	Weibull	39.24 MPa	5.82 MPa

TABLE II.
Maximum likelihood point in the Z-space for the static example. The first row gives the exact solution; the following rows provide the iteration process of the iterative approach proposed in the paper. l_{ott} is the optimal value of the likelihood function; P_F is the failure probability estimate obtained by equation (3)

Iteration	l_{ott}	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	ϵ	$P_F\,10^4$
Exact solution	−11.171	2.133	−0.341	−3.548	−0.591	0.020	−0.010	0.273	–	2.847 (2.828)
1 (linear)	−11.058	2.080	−0.094	−3.368	−0.845	0.055	−0.027	0.226	–	–
2	−11.193	2.130	−0.325	−3.561	−0.588	–	–	–	−0.027	2.831
3	−11.185	2.128	−0.328	−3.558	−0.591	–	–	–	−0.024	2.844
4	−11.184	2.129	−0.328	−3.557	−0.591	–	–	–	−0.022	2.845
5	−11.184	2.130	−0.329	−3.556	−0.591	–	–	–	−0.021	2.842
Final	−11.180	2.132	−0.341	−3.554	−0.590	–	–	–	−0.015	2.825

It provides a first approximation $\{Z^*_{(1)}\}$ for the maximum likelihood point $\{Z^*\}$ (Table II, second row) and the following importance factors (from Equation (4) written in $\{Z^*_{(1)}\}$): 2.080; 0.573; −1.656; −0.845; 0.055; −0.027; −0.079. Since the ratio between the highest factor and the absolute value of the last three ones is greater than 20, the variables $\mathbf{Z}_5$, $\mathbf{Z}_6$ and $\mathbf{Z}_7$ will be incorporated in the error term from the second step on. These steps work with a quadratic response surface in the space of the variables Z_1, Z_2, Z_3 and Z_4.

The iteration process moves the center of the experiment plan for the response surface analysis, from the previous point $Q_{(j)}$ to the point $Q_{(j+1)}$ along the direction of the design point $\{Z^0_{(j)}\}$. Its distance from the origin is increased with respect to the one of $Q_{(j)}$, of just 1/2 of the difference of the distances of $\{Z^0_{(j)}\}$ and $Q_{(j)}$. This corresponds to global distances from the origin which are 1/2, 1/1.34, 1/1.14 times $\{Z^0_{(j)}\}$, respectively, for $j = 1$, 2 and 3. The last (5th) iteration is centered exactly at $\{Z^0_{(4)}\}$, and the experiment plan is not "squashed". The coordinates of $\{Z^0_{(4)}\}$ are {2.909, 0.600, −2.197, −0.931} with error at −0.03 and coincides with the ones of $\{Z^0_{(5)}\}$. A final step makes use of the experiment plan centered at $\{Z^*_5\}$, i.e., the achieved maximum likelihood point.

The results of the iteration process are given in Table II. Comparison with the values in the first row shows that the optimal value of the likelihood function and the coordinates of the maximum log-likelihood point tend toward the exact ones as the iteration proceeds. The values in the last but one row corresponds to the convergence in the design point. The final step makes use of an experiment design centered at $\{Z^*_{(5)}\}$. The probabilities estimates depend on the point $\{Z^*_{(j)}\}$ but also on the current approximation for the performance function $G(\{Z\})$. For this reason any comparison must be done with both the actual value of the probability of failure $P(F) \simeq P_F$ ($= 2.847\,10^{-4}$) and its better estimate (2.828). Both values were computed in the actual point $\{Z^*\}$, but the first estimate makes use of the actual shape of the performance function while the latter value adopts a response surface approximation with experiment design centered at $\{Z^*\}$. Table II shows a very satisfactory accuracy of the iterative procedure in assessing the actual probability of failure together with a good convergence to its better estimate. In this sense this first example fully validates the method proposed in this paper.

6.2. *Dynamic Application*

Consider the hysteretic oscillator governed by the equation [6]:

$$\dot{\mathbf{u}} = \mathbf{y}$$

$$\dot{\mathbf{y}} = -\eta\omega_0^2\mathbf{u} - 2\zeta\omega_0\mathbf{y} - (1-\eta)\omega_0^2\mathbf{z} + \mathbf{w}(t)$$

$$\dot{\mathbf{z}} = \dot{\mathbf{u}} - c_1\dot{\mathbf{u}}|\mathbf{z}|^r - c_2\mathbf{z}|\dot{\mathbf{u}}|\,|\mathbf{z}|^{(r-1)}$$

where ω_0 is the circular frequency, ζ is the damping ratio, η is the ratio between the post and the pre-yielding stiffness and c_1 and c_2 are parameters of the hysteretic equation. r is a further parameter that is assumed equal to 1 in the following. The excitation $\mathbf{w}(t)$ is a white noise of power spectral density G_0. Let the performance function be assigned in the form

$$K - (1/2)[(\mathrm{Var}[u]/0.2)^{1/2} + \mathrm{Cov}[z\dot{u}]/0.5] \geq 0$$

where $\mathrm{Cov}[z\dot{u}]$ provides a measure of the dissipated energy rate.

Let the parameters $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$, $\mathbf{c}_1$ and $\mathbf{c}_2$ be of a random nature. The oscillator becomes a

TABLE III.
Probabilistic definition of the random variables of the example concerning the dynamic application. The variables ζ and η are described by their transformations $\zeta = 0.01\zeta' + 0.05$ and $\eta = 0.015\eta' + 0.05$, respectively

Symbol	Distribution	Mean value	Standard dev.	Correl.
ζ'	Chi-square with 5 d.o.f.	0.0	1.0	–
η'	Chi-square with 5 d.o.f.	0.0	1.0	–
$\mathbf{c}_1$ $\mathbf{c}_2$	Joint extreme value distribution	1.25	0.25	0.75
$\mathbf{K}$	Gaussian	0.1	0.05	–
$\mathbf{G}_0$	Gaussian	$5.11\frac{\text{sq.in}}{\text{rad.s}^3}$	$0.205\frac{\text{sq.in}}{\text{rad.s}^3}$	–

"disordered" system, i.e., a system with uncertain parameters. The properties of such parameters are stated in Table III.

The correlated extreme distribution [14] was selected to describe the joint probability density function (JPDF) of $\mathbf{c}_1$ and $\mathbf{c}_2$. The proposed procedure, in fact, just requires the analytical or numerical knowledge of the first and second derivatives of the joint log-likelihood function. Therefore, there is no need for numerical transformations and/or for numerical calculations of conditional densities which may offer numerical difficulties. In a more general case the JPDF of non-normal dependent variables will be obtained by inversion of the characteristic function but the procedure will be still applicable.

The standardized variables are, as usual,

$$\mathbf{Z}_i = \frac{\mathbf{X}_i - E[\mathbf{X}_i]}{(\text{Var}[\mathbf{X}_i])^{1/2}}.$$

Note that $\zeta \geq 0.0345$ and $\eta \geq 0.0268$.

Within the response surface scheme, experiments are planned according to the experiment design theory [6]. For each set of variable values the nonlinear dynamic analysis is repeated and the response surface is built to fit the results in terms of the performance function $G(\{Z\})$. For the first linear interpolation a single white noise realization is considered. In the subsequent nonlinear approximations, some realizations are used according to the blocking theory [6]. The governing equation becomes then a standard deterministic differential equation which is integrated numerically. The response variance Var$[u]$, as well as Cov$[z\dot{u}]$, are estimated over a period of 100 s after a time interval of $10/(2\zeta\omega)$ necessary to make negligible the effect of the initial conditions.

The results of the reliability analysis procedure proposed in this paper are summarized in Table IV. The error term ϵ is due to the stochastic variability of the excitation.

7. Conclusions

This paper presents a reliability analysis procedure which can be used for linear and nonlinear systems under static and dynamic actions. The procedure is obtained by the conjunction of an asymptotic approach, to the reliability calculation, and a response surface scheme, able to cover any stochastic finite element problem.

TABLE IV.
Maximum likelihood point estimates $\{Z^*_{(i)}\}$ and probability approximations for the dynamic example. The first step consider a linear response surface. The experiment design for the 4 step is centered at $\{Z^0\}(=\{-1.248, -0.286, 0.476, -0.090, -1.137, -1.860\}$. The last step adopts an experiment design centered at $\{Z^*_{(4)}\}$

Iteration	l_{out}	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	ϵ	$P_F \cdot 10^4$
1									
(linear)	−9.230	−0.997	−1.026	1.302	1.262	−1.226	1.534	–	–
2	−10.056	−1.169	−0.745	0.136	0.075	−1.162	2.179	−1.394	6.254
3	−9.793	−1.035	−0.782	−0.011	−0.148	−1.181	1.845	−1.842	6.314
4	−9.992	−1.031	−0.779	−0.001	−0.167	−1.237	1.875	−1.880	5.473
Final	−10.023	−1.070	−0.837	−0.204	−0.323	−1.190	2.022	−1.808	5.171

The advantage of the proposed procedure, when compared with existing methods in reliability analysis, is that it just requires that the first and second derivatives of the joint log-likelihood function are known in an analytical or numerical way. Neither numerical transformations or numerical calculations of conditional densities are required and so most of the numerical difficulties are avoided.

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Appendix: Approximation Formula for the Probability of Failure

The approximation formula for the probability of failure $P(F)$ derived in [8] by the first author has the form

$$P(F) \approx (2\pi)^{(n-1)/2} f(\{X^*\}) / \sqrt{|J_1|}$$

with

$$J_1 = \{\nabla l(\{X^*\})\}^T [C(\{X^*\})] \{\nabla l(\{X^*\})\} .$$

The matrix

$$[C(\{X^*\})] = \left(\operatorname{cof} \left[\frac{\partial^2 l(\{X^*\})}{\partial X_i \, \partial X_j} - \lambda_1 \frac{\partial^2 g(\{X^*\})}{\partial X_i \, \partial X_j} \right] \right) \qquad i, j = 1, \ldots, n$$

is the matrix of the cofactors of the matrix

$$[H(\{X^*\})] = \left[\frac{\partial^2 l(\{X^*\})}{\partial X_i \, \partial X_j} - \lambda_1 \frac{\partial^2 g(\{X^*\})}{\partial X_i \, \partial X_j} \right] \qquad i, j = 1, \ldots, n$$

and

$$\lambda_1 = |\{\nabla l(\{X^*\})\}| / |\{\nabla g(\{X^*\})\}| .$$

For a non-singular matrix $[H(\{X^*\})]$ there is the relation

$$[C(\{X^*\})] = \det([H(\{X^*\})])[H(\{X^*\})]^{-1}$$

i.e., the co-factor matrix is the inverse multiplied by the determinant.

This gives the factor J_1 another form

$$J_1 = \det([H(\{X^*\})])\{\nabla l(\{X^*\})\}^T[H(\{X^*\})]^{-1}\{\nabla l(\{X^*\})\}\,.$$

If the dimension of the space is large, it is useful to have a method for calculating the factor J_1 directly from the first and second derivatives without matrix inversion. Such a method is described in the following.

If we make a rotation of the coordinates in such a way that the unit vector $|\{\nabla l(\{X^*\})\}|^{-1}\{\nabla l(\{X^*\})\}$ becomes the direction of the n-th coordinate axis, in this new coordinate system the gradient $\{\nabla l(\{X^*\})\}$ has the form $\{\nabla l(\{X^*\})\} = (\{0, \ldots, 0, -|\{\nabla l(\{X^*\})\}|\})^T$, i.e., only the n-th component is non-zero. Therefore in the quadratic form J_1 all terms vanish except one term. We get in this coordinate system

$$J_1 = |\{\nabla l(\{X^*\})\}|^2 \cdot C_{nn}$$

with C_{nn} the cofactor of the element $(\partial^2 l(\{X^*\})/\partial X_n^2 - \lambda_1(\partial^2 g(\{X^*\}))/(\partial X_n^2))$ in the matrix $[H(\{X^*\})]$. But this cofactor is by definition the determinant of the matrix $[\bar{H}(\{X^*\})]$ which is an $(n-1)\times(n-1)$ matrix obtained from $[H(\{X^*\})]$ by deleting the last row and column, i.e.,

$$[\bar{H}(\{X^*\})] = \left(\frac{\partial^2 l(\{X^*\})}{\partial X_i\,\partial X_j} - \lambda_1 \frac{\partial^2 g(\{X^*\})}{\partial X_i\,\partial X_j}\right) \qquad i, j = 1, \ldots, n-1\,.$$

This determinant is non-positive, i.e., in the regular case negative, since it is the Hessian of the log-likelihood in local coordinates at the maximum likelihood point on the surface $g(\{X\}) = 0$.

If we now add to this matrix an n-th row and column with zeroes everywhere and -1 at the n-th component, we get an $n \times n$ matrix:

$$[H'(\{X^*\})] = \begin{bmatrix} [\bar{H}(\{X^*\})] & 0 \\ 0 & -1 \end{bmatrix}$$

which has the same determinant as the matrix $-[H(\{X^*\})]$. But this matrix can be written as the sum of two $n \times n$ matrices:

$$[H'\{X^*\})] = \begin{bmatrix} [\bar{H}(\{X^*\})] & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} [\bar{H}(\{X^*\})] & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} [0]_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$$

where $[0]_{n-1}$ denotes the $(n-1)$-dimensional zero matrix. It can be shown then, using the projection matrix $[P] = [I_n] - \{e_n\}\{e_n\}^T$, with $\{e_n\} = \{0, \ldots, 0, 1\}^T$, which projects a vector into the subspace spanned by the first $n-1$ unit axes, that

$$\begin{bmatrix} [\bar{H}(\{X^*\})] & 0 \\ 0 & 0 \end{bmatrix} = [P][H(\{X^*\})][P]^T$$

$$\begin{bmatrix} [0]_{n-1} & 0 \\ 0 & -1 \end{bmatrix} = -\{e_n\}\{e_n\}^T\,.$$

We have therefore

$$
\begin{aligned}
|C_{nn}| &= |\det([\bar{H}(\{X^*\})])| \\
&= |\det([H'(\{X^*\})])| \\
&= |\det([P][H(\{X^*\})][P]^T - \{e_n\}\{e_n\}^T)| .
\end{aligned}
$$

But this result is coordinate invariant; this means that in another coordinate system we replace the vector $\{e_n\}$ by the unit vector $|\{\nabla l(\{X^*\})\}|^{-1}\{\nabla l(\{X^*\})\}$ and

$$
[P] = [I_n] - |\{\nabla l(\{X^*\})\}|^{-2}\{\nabla l(\{X^*\})\}\{\nabla l(\{X^*\})\}^T .
$$

This gives finally the following form for the approximation:

$$
P(F) \simeq (2\pi)^{(n-1)/2} \frac{f(\{X^*\})}{|\{\nabla l(\{X^*\})\}||\det([P][H(\{X^*\}][P]^T - \{\nu\}\{\nu\}^T)|^{1/2}}
$$

with $\{\nu\} = |\nabla l(\{X^*\})|^{-1}\{\nabla l(\{X^*\})\}$. The advantage of this representation of the approximation formula is that here no matrix inversion is required.

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