A Control Problem for Burgers' Equation with Bounded Input/Output

JOHN A. BURNS and SUNGKWON KANG

Department of Mathematics and Interdisciplinary Center for Applied Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, U.S.A.

(Received: 30 May 1990; accepted: 8 April 1991)

Abstract. A stabilization problem for Burgers' equation is considered. Using linearization, various controllers are constructed which minimize certain weighted energy functionals. These controllers produce the desired degree of stability for the closed-loop nonlinear system. A numerical scheme for computing the feedback gain functional is developed and several numerical experiments are performed to illustrate the theoretical results.

Key words: Burgers' equation, stabilizability, optimal controls, approximations.

1. Introduction

During the past five years considerable attention has been given to the problem of active control of fluid flows (including feedback). This interest is motivated by a number of potential applications such as control of flow separation, combustion, fluid-structure interactions and supermaneuverable aircraft. The Navier–Stokes equations often play a central role in the modelling and in the development of computational algorithms for such applications. For example, the problem of using oscillatory motions of a circular cylinder to control the wake and flow separation requires that viscosity be included in the model. The development of practical computational algorithms for active control design for the Navier–Stokes equations is the long term goal of several current research projects.

We consider this paper as a first step in the development of rigorous and practical computational algorithms for control of those nonlinear partial differential equations that describe physically interesting problems of this nature. Burgers' equation is 'simple enough' to be used as a first case study, yet it also captures many of the important fluid flow characteristics (shocks, viscous dissipation, etc.). Therefore, in this paper we concentrate on the problem of computing feedback laws for Burgers' equation. We restrict our attention here to bounded input operators. The more complex boundary control problem requires different techniques and will be the subject of a forthcoming paper.

Burgers' equation

$$\frac{\partial}{\partial t} z(t, x) + z(t, x) \frac{\partial}{\partial x} z(t, x) = \epsilon \frac{\partial^2}{\partial x^2} z(t, x) \tag{1}$$

was introduced by Burgers [3, 4, 5] as a simple model for turbulence, where $\epsilon > 0$ is a viscosity coefficient. Since then, many researchers have considered the conservation law

Nonlinear Dynamics **2**: 235–262, 1991

$$\frac{\partial}{\partial t} z(t, x) + z(t, x) \frac{\partial}{\partial x} z(t, x) = 0 \tag{2}$$

and the 'viscosity solution'

$$z(t, x) = \lim_{\epsilon \downarrow 0} z^{\epsilon}(t, x) , \tag{3}$$

where $z^{\epsilon}(t, x)$ satisfies equation (1), see [7, 13, 15, 19, 21, 22, 23].

Oleĭnik [23] proved that for any L^{∞}-initial data, there is a unique viscosity solution for equation (2) and the solution satisfies the 'entropy condition'

$$\frac{z(t, x + a) - z(t, x)}{a} < \frac{E}{t} \tag{4}$$

for all $t > 0$, $a > 0$, $-\infty < x < \infty$ and for some constant $E > 0$. A complete discussion of these results may be found in [29].

Almost no results exist for the control associated with Burgers' equation. Chen, Wang and Weerakoon [6, 32] considered an optimal control problem for equation (2) with $-\infty < x < \infty$. The problem was to select an initial function to minimize a specific cost functional J. They obtained sufficient conditions for the differentiability of J with respect to the initial function and an explicit expression of the entropy solution of (2) in terms of initial data.

In this paper we consider a control problem for Burgers' equation (1) defined on a finite interval. Specifically, we will find several feedback laws stabilizing the nonlinear system (1) with a prescribed exponential decay rate. The feedback laws are obtained from the linearized equation. Curtain [9] has considered a stabilization problem for certain semilinear evolution equations. Using Kielhöfer's stability results for semilinear evolution equations [17], she showed that there exists a finite dimensional compensator which produces a stable closed-loop system. These finite dimensional compensators are also obtained from the linearized control system. Applying her results to Burgers' equation (1) with Dirichlet boundary condition, one can obtain stabilizability results for the closed-loop system which are similar to ours. However, in [9], there is a restriction on the action of the output operators. The domain of the output operator was required to be a certain subspace of L^2 which contains the Sobolev space H_0^1. In this paper we investigate optimal feedback laws.

Well-posedness and stability results for the open-loop system are obtained in Section 2. In Section 3, a 'shifted' linear control problem $(\mathrm{LQR})_{\hat{\alpha}}$ is introduced. Under appropriate selection of input and output operators, $(\mathrm{LQR})_{\hat{\alpha}}$ is stabilizable and detectable. The feedback control law obtained from $(\mathrm{LQR})_{\hat{\alpha}}$ produces the desired degree of stability for the closed-loop nonlinear system (Theorem 3.9). In Section 4, a numerical scheme for computing the 'feedback functional gains' is developed and several numerical experiments are performed.

We shall use standard notation. If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, then $\mathscr{L}(X, Y)$ will denote the space of all bounded linear operators from X to Y and for any $A \in \mathscr{L}(X, Y)$, $\|A\|$ or $\|A\|_{\mathscr{L}(X,Y)}$ will denote the operator norm on the space $\mathscr{L}(X, Y)$. In the event that $X = Y$ we denote $\mathscr{L}(X, Y)$ by $\mathscr{L}(X)$. From time to time we will use $\|\cdot\|$ without any subindex for vector or operator norm. In all such cases the appropriate index for $\|\cdot\|$ will be understood from the context. For a Hilbert space X, we denote the inner product on $X \times X$ by

$\langle \cdot , \cdot \rangle_X$. Given a linear operator A from X into itself, we denote its domain, spectrum, resolvent and adjoint by $\mathscr{D}(A)$, $\sigma(A)$, $\rho(A)$ and A^*, respectively. For real numbers a, b with $a < b$, $L^p(a, b; X)$, $1 < p < \infty$, will be the space of all Lebesgue measurable functions f from (a, b) to X such that $\|f\|_{L^p(a,b)} = (\int_a^b |f(x)|^p \, dx)^{1/p} < \infty$. The spaces $H^k(a, b)$ and $H_0^k(a, b)$ are the standard Sobolev spaces defined by $H^k(a, b) = \{f \in L^2(a, b) | f^{(j)} \in L^2(a, b),\ j = 0, 1, \ldots, k\}$ and $H_0^k(a, b) = \{f \in H^k(a, b) | f^{(j)}(a) = f^{(j)}(b) = 0,\ j = 0, 1, \ldots, k-1\}$, respectively. The dual space $H^{-k}(a, b)$ of $H_0^k(a, b)$ is the space of all continuous linear functionals on $H_0^k(a, b)$ represented by the inner product $\langle \cdot , \cdot \rangle_{L^2(a,b)}$.

Finally, we note that although Burgers' equation if often used to test computation methods developed for fluid flow problems, this equation also can be used as a reasonable mathematical model for several other physical problems. We present a physical example that may be found in most standard references to motivate the control problem for Burgers' equation. Other examples involving traffic flows, supersonic flow about airfoils, acoustic transmission and turbulence in hydrodynamic flows can be found in [10] and the references given there. The following example is taken almost directly from [10].

EXAMPLE (Shock Waves). An impulsively-started piston moving at a constant velocity into a tube containing a compressible fluid initially at rest creates compression waves. The compression waves will coalesce and form a single shock wave. The one-dimensional unsteady motion of the fluid is governed by the continuity equation

$$\frac{\partial}{\partial t} \rho(t, x) + \rho(t, x) \frac{\partial}{\partial x} v(t, x) + v(t, x) \frac{\partial}{\partial x} \rho(t, x) = 0 \tag{5}$$

and the x-momentum equation

$$\frac{\partial}{\partial t} v(t, x) + v(t, x) \frac{\partial}{\partial x} v(t, x) + \left(\frac{\partial}{\partial x} p(t, x) \right) / \rho(t, x) = \delta \frac{\partial^2}{\partial x^2} v(t, x) , \tag{6}$$

where ρ is the density, v is the velocity, p is the pressure and δ is the 'diffusivity of sound'. If one replaces the density by the sound speed, $a = a(t, x)$ by $a(t, x)/a_0 = (\rho(t, x)/\rho_0)^{\gamma - 1/2}$, where $\gamma > 1$ is the specific heat ratio and the subscript 0 refers to the undisturbed values [20], then equations (5) and (6) become

$$\frac{\partial}{\partial t} a(t, x) + v(t, x) \frac{\partial}{\partial x} a(t, x) + \frac{\gamma - 1}{2} a(t, x) \frac{\partial}{\partial x} v(t, x) = 0 \tag{7}$$

and

$$\frac{\partial}{\partial t} v(t, x) + v(t, x) \frac{\partial}{\partial x} v(t, x) + \frac{2}{\gamma - 1} a(t, x) \frac{\partial}{\partial x} a(t, x) = \delta \frac{\partial^2}{\partial x^2} v(t, x) . \tag{8}$$

Here δ is a function of the undisturbed viscosity, density, specific heat and thermal conductivity of the medium. Equations (7) and (8) are simplified by introducing the Riemann invariants,

$$r(t, x) = \frac{a(t, x)}{\gamma - 1} + \frac{v(t, x)}{2} , \quad s(t, x) = \frac{a(t, x)}{\gamma - 1} - \frac{v(t, x)}{2} \tag{9}$$

to give

$$\frac{\partial}{\partial t} r(t,x) + (a(t,x) + v(t,x)) \frac{\partial}{\partial x} r(t,x) = \frac{\delta}{2} \frac{\partial^2}{\partial x^2} (r(t,x) - s(t,x)) \tag{10}$$

and

$$\frac{\partial}{\partial t} s(t,x) - (a(t,x) - v(t,x)) \frac{\partial}{\partial x} s(t,x) = \frac{\delta}{2} \frac{\partial^2}{\partial x^2} (s(t,x) - r(t,x)) \tag{11}$$

Consider the propagation of a disturbance into an initially undisturbed region, $s = s_0$ where $s_0 = \frac{a_0}{\gamma - 1}$. This problem is governed by equation (10) and if one applies equation (9), then it follows that

$$a(t,x) + v(t,x) = \frac{\gamma+1}{2} r(t,x) + \frac{\gamma-3}{2} s_0 , \tag{12}$$

and equation (10) becomes

$$\frac{\partial}{\partial t} r(t,x) + \left(\frac{\gamma+1}{2} r(t,x) + \frac{\gamma-3}{2} s_0\right) \frac{\partial}{\partial x} r(t,x) = \frac{\delta}{2} \frac{\partial^2}{\partial x^2} r(t,x) . \tag{13}$$

The change of variables

$$z(t,x) = \frac{\gamma+1}{2} (r(t,x) - r_0) , \quad \xi = x - a_0 t \tag{14}$$

leads to Burgers' equation

$$\frac{\partial}{\partial t} z(t,\xi) + z(t,\xi) \frac{\partial}{\partial \xi} z(t,\xi) = \frac{\delta}{2} \frac{\partial^2}{\partial \xi^2} z(t,\xi) . \tag{15}$$

It follows that $z(t,\xi) = \{a(t,\xi) + v(t,\xi)\} - \{v_0 + a_0\}$ is the excess wavelet velocity (the difference between propagation speeds of disturbance in stagnation and nonstagnation conditions) and the coordinate ξ is measured relative to a frame of reference moving with the undisturbed speed of sound a_0.

2. Well-Posedness and Stability of Burgers' Equation

In this section we consider well-posedness and stability properties of the solution of Burgers' equation with Dirichlet boundary condition. These results will be needed in the analysis of our control problems in the next section. We first consider an abstract version of this problem and then specialize to Burgers' equation.

Consider an initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \mathcal{A} z(t) + f(t, z(t)) , \quad z(t_0) = z_0 , \quad (t > t_0) , \tag{16}$$

on a Hilbert space X, where $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $S(t)$ satisfying $\|S(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}$, $t \geq t_0$, for some constants $M = M(\omega) \geq 1$ and $\omega \geq 0$. Since $S(t)$ is analytic, the fractional powers of $\mathscr{A}_1 = -\mathscr{A} + aI$ are well-defined for any $a > \omega$ [24, Chapters 1, 2]. Also $0 \in \rho(\mathscr{A}_1)$ and hence $\mathscr{A}_1^{\mu}$ is invertible for all $0 \leq \mu \leq 1$. Therefore, the graph norm $\|z\| + \|\mathscr{A}_1^{\mu} z\|$ on the domain $\mathscr{D}(\mathscr{A}_1^{\mu})$ of $\mathscr{A}_1^{\mu}$ is equivalent to the norm $\|z\|_{\mu} = \|\mathscr{A}_1^{\mu} z\|$. We denote the Hilbert space $\mathscr{D}(\mathscr{A}_1^{\mu})$ with the norm $\|z\| + \|\mathscr{A}_1^{\mu} z\|$ or $\|\mathscr{A}_1^{\mu} z\|$ by $(X_{\mu}, \|\cdot\|_{\mu})$.

HYPOTHESIS (F): Let U be an open subset of $[t_0, \infty) \times X_{\mu}$. The function $f: U \to X$ satisfies the hypothesis (F) if for every $(t, z) \in U$ there is a neighborhood $V \subset U$ and constants $L \geq 0$, $0 < \theta \leq 1$ such that

$$\|f(t_1, z_1) - f(t_2, z_2)\|_X \leq L(|t_1 - t_2|^{\theta} + \|z_1 - z_2\|_{\mu}) \tag{17}$$

for all $(t_i, z_i) \in V$, $i = 1, 2$, i.e., f is locally Hölder continuous in t, locally Lipschitzian in z, on U.

The following local existence theorem appears as Theorem 3.3.3 in [14].

THEOREM 2.1. *Let $\mathscr{A}$ be as before. If f satisfies hypothesis (F), then for any $(t_0, z_0) \in U \subset \mathbf{R}^+ \times X_{\mu}$, there exists $T = T(t_0, z_0) > 0$ such that equation (16) has a unique (strong) solution $z(t)$ on $[t_0, t_0 + T)$ with initial value $z(t_0) = z_0$.*

Now consider Burgers' equation with Dirichlet boundary condition on a finite interval $[0, l]$ given by

$$\begin{aligned} &\frac{\partial}{\partial t} z(t, x) = \epsilon \frac{\partial^2}{\partial x^2} z(t, x) - z(t, x) \frac{\partial}{\partial x} z(t, x), \quad 0 < x < l, \quad t > 0, \\ &z(t, 0) = z(t, l) = 0, \quad z(0, x) = z_0(x), \end{aligned} \tag{18}$$

where $\epsilon = \frac{1}{\mathrm{Re}} > 0$ and Re is the Reynolds number. In order to place the system (18) into a semigroup framework let $z(t)(\cdot) = z(t, \cdot)$, $z_0(\cdot) = z(0, \cdot)$ and $H = L^2(0, l)$. Define the operator A_{ϵ} by

$$A_{\epsilon}\phi = \epsilon\phi'' \tag{19}$$

for all $\phi \in \mathscr{D}(A_{\epsilon}) = H^2(0, l) \cap H_0^1(0, l)$. The system (18) can now be written as the initial value problem

$$\frac{\partial}{\partial t} z(t) = A_{\epsilon} z(t) + f(t, z(t)), \quad z(0) = z_0, \quad (t > 0), \tag{20}$$

on the space H, where $f(t, z) = -zz'$ is defined on the space $H_0^1(0, l)$. It is well-known [14, 24, 13] that A_{ϵ} generates an analytic semigroup $S(t)$ on H. We summarize the basic properties of the infinitesimal generator A_{ϵ} and its semigroup $S(t)$, $t \geq 0$, in the following remark.

REMARK 2.2. (i) The spectrum $\sigma(A_{\epsilon})$ of A_{ϵ} consists of all eigenvalues $\lambda_n = -\epsilon n^2\pi^2/l^2$, $n = 1, 2, \ldots,$ *and for each eigenvalue* λ_n the corresponding eigenfunction ϕ_n is given by

$$\phi_n(x) = \sqrt{\frac{2}{l}} \sin \frac{n\pi}{l} x , \quad 0 < x < l . \tag{21}$$

(ii) The operator A_ϵ is self-adjoint, i.e., $A_\epsilon = A_\epsilon^*$, and the semigroup $S(t)$ can be represented by the formula

$$S(t)z + \sum_{n=1}^{\infty} e^{-(\epsilon n^2\pi^2/l^2)t} \langle z, \phi_n \rangle \phi_n \tag{22}$$

for all $z \in H$, where ϕ_n's are defined by equation (21). Moreover, from equation (22) it is easy to see that $S(t)$ has the stability property

$$\|S(t)\|_{\mathcal{L}(H)} \le e^{-(\epsilon\pi^2/l^2)t} , \quad t \ge 0 . \tag{23}$$

A simple application of Schwartz inequality yields the following first Poincaré inequality [33, p. 116].

LEMMA 2.3. *For any* $z \in H_0^1(0, l)$,

$$\|z\|_H \le l \|z'\|_H , \tag{24}$$

where $H = L^2(0, l)$.

REMARK 2.4. (i) The above lemma provides an equivalent norm $\|z\|_{H_0^1} \equiv \|z'\|_{L^2}$ on the space $H_0^1(0, l)$. (ii) It is well-known that $\mathcal{D}((-A_\epsilon)^{1/2}) = H_0^1(0, l)$ [14, p. 29], [18, p. 326].

LEMMA 2.5. *For any* $z \in H_0^1(0, l) = \mathcal{D}((-A_\epsilon)^{1/2})$, *the following inequalities hold:*

$$(i)\ \|S(t)z\|_{H_0^1} \le e^{-\gamma t} \|z\|_{H_0^1} , \quad (t \ge 0) \tag{25}$$

$$(ii)\ \|S(t)z\|_{H_0^1} \le \left(\frac{1+l}{\sqrt{2\epsilon}} \frac{1}{\sqrt{t}} + \frac{\pi(1+l)}{l} \right) e^{-\gamma t} \|z\|_{L^2(0,l)} , \quad (t > 0) , \tag{26}$$

where $\gamma = \epsilon\pi^2/l^2$.

Proof. For any $z \in H_0^1(0, l)$, we know that $(-A_\epsilon)^{1/2} S(t)z = S(t)(-A_\epsilon)^{1/2} z$. It follows from Remark 2.2 that

$$\|S(t)z\|_{H_0^1} = \|S(t)z\|_{L^2} + \|(-A_\epsilon)^{1/2} S(t)z\|_{L^2} \le \|S(t)\|(\|z\| + \|(-A_\epsilon)^{1/2} z\|) \le e^{-\gamma t} \|z\|_{H_0^1} .$$

Inequality (ii) follows from Remark 2.2, Lemma 2.3 and the estimates

$$\begin{aligned} \|S(t)z\|_{H_0^1} &\le (1+l)\|(-A_\epsilon)^{1/2} S(t)z\|_H = (1+l)\|(-A_\epsilon)^{1/2} \sum_{n=1}^{\infty} e^{\lambda_n t} \langle z, \phi_n \rangle \phi_n \| \\ &= (1+l) \left\| \sum_{n=1}^{\infty} \frac{n\pi}{l} e^{\lambda_n t} \langle z, \phi_n \rangle \sqrt{\frac{2}{l}} \cos \frac{n\pi}{l} x \right\| \\ &\le (1+l) \left(\sup\left\{ \frac{n\pi}{l} e^{(\lambda_n + \gamma)t} : n = 1, 2, \ldots \right\} \right) e^{-\gamma t} \|z\|_H \end{aligned}$$

and

$$\sup\left\{\frac{n\pi}{l}e^{(\lambda_n+\gamma)t}: n=1,2,\ldots\right\}\le\begin{cases}\frac{1}{\sqrt{2\epsilon}}\frac{1}{\sqrt{t}}, & 0<t\le\frac{l^2}{2\epsilon\pi^2}\\ \frac{\pi}{l}, & t\ge\frac{l^2}{2\epsilon\pi^2},\end{cases}$$

where $\lambda_n=-\epsilon n^2\pi^2/l^2$ and $\phi_n(x)=\sqrt{\frac{2}{l}}\sin\frac{n\pi}{l}x$, $n=1,2,\ldots$.

REMARK 2.6. The inequality (26) holds for every $z\in H=L^2(0,l)$, since the semigroup $S(t)$ is analytic.

We now have the well-posedness and stability properties of Burgers' equation (8) on the space $H_0^1(0,l)$. The following theorem follows from an application of Theorem 5.1.1 in [14].

THEOREM 2.7. *For any given β with $0<\beta<\gamma=\epsilon\pi^2/l^2$, there is a $\rho=\rho(l,\epsilon,\beta)>0$ such that for any initial data $z_0\in H_0^1(0,l)$, with $\|z_0\|_{H_0^1}\le\frac{\rho}{2}$, there is a unique solution $z(t)=z(t,0;z_0)\in H_0^1(0,l)$ of equation (18). Moreover, the solution satisfies the inequality*

$$\|z(t,0;z_0)\|_{H_0^1}\le 2e^{-\beta t}\|z_0\|_{H_0^1}\quad(t\ge 0) \tag{27}$$

and $\rho=\rho(l,\epsilon,\beta)>0$ can be chosen to satisfy

$$0<\rho<\frac{\sqrt{\epsilon}l(\gamma-\beta)}{\sqrt{2}\pi(1+l)(l\sqrt{\gamma-\beta}+\sqrt{2\pi\epsilon})}. \tag{28}$$

Proof. Note that 0 is an equilibrium point for the system (20) and $\mathcal{D}((-A_\epsilon)^{1/2})=H_0^1$. If the nonlinear term $f(z)=-zz'$ satisfies the hypothesis (F) with index $\mu=\frac{1}{2}$ then, by Theorem 2.1 we have a unique local solution $z(t,0;z_0)$ on the space H_0^1. It is easy to see that $\|f(z_1)-f(z_2)\|_{L^2}\le(\|z_1\|_{H_0^1}+\|z_2\|_{H_0^1})\|z_1-z_2\|_{H_0^1}$ for all z_1, $z_2\in H_0^1$, uniformly in $t>0$. Hence, f satisfies the hypothesis (F).

For the global existence and uniqueness of the solution $z(t,0;z_0)\in H_0^1$ let z_0 be any initial data in H_0^1 with $\|z_0\|_{H_0^1}\le\frac{\rho}{2}$, where $\rho=\rho(l,\epsilon,\beta)$ satisfies the condition (28). It follows that

$$\rho\left\{\frac{1+l}{\sqrt{2\epsilon}}\int_0^\infty\frac{1}{\sqrt{s}}e^{-(\gamma-\beta)s}\,ds+\frac{(1+l)\pi}{l(\gamma-\beta)}\right\}<\frac{1}{2}, \tag{29}$$

since $\int_0^\infty\frac{1}{\sqrt{s}}e^{-(\gamma-\beta)s}\,ds=\frac{1}{\sqrt{\gamma-\beta}}\frac{1}{\Gamma(\frac{1}{2})}\frac{\pi}{\sin\frac{\pi}{2}}=\frac{\sqrt{\pi}}{\sqrt{\gamma-\beta}}$, where Γ is the Gamma function and $\Gamma(\frac{1}{2})=\sqrt{\pi}$. If $\|z_0\|_{H_0^1}\le\frac{\rho}{2}$, then by the local existence property there is a unique solution $z(t,0;z_0)\in H_0^1$ satisfying the inequality $\|z(t,0;z_0)\|_{H_0^1}<\rho$ on an interval $[0,t_1)$ for some $t_1>0$, where t_1 is chosen as large as possible. If t_1 is finite, then we must have $\|z(t_1)\|_{H_0^1}\ge\rho$. Note that on the interval $[0,t_1)$,

$$\|f(z(t))\|_{L^2}=\|-z(t)z'(t)\|_{L^2}\le(\|z(t)\|_{H_0^1})^2\le\rho^2, \tag{30}$$

where $'=\frac{d}{dx}$. Lemma 2.5, Remark 2.6 together with inequalities (29), (30) yield

$$\begin{aligned}\|z(t_1)\|_{H_0^1} &= \|S(t_1)z_0 + \int_0^{t_1} S(t_1 - s)f(z(s))\,\mathrm{d}s\|_{H_0^1}\\ &\le \|S(t_1)z_0\|_{H_0^1} + \int_0^{t_1} \|S(t_1 - s)f(z(s))\|_{H_0^1}\,\mathrm{d}s\\ &\le e^{-\gamma t_1}\|z_0\|_{H_0^1} + \rho^2 \int_0^{t_1} \left\{\frac{1+l}{\sqrt{2\epsilon}}\frac{1}{\sqrt{t_1 - s}} + \frac{(1+l)\pi}{l}\right\} e^{-\gamma(t_1 - s)}\,\mathrm{d}s\\ &\le \frac{\rho}{2} + \rho^2\left\{\frac{1+l}{\sqrt{2\epsilon}}\int_0^{\infty}\frac{1}{\sqrt{s}}e^{-\gamma s}\,\mathrm{d}s + \frac{(1+l)\pi}{l}\int_0^{\infty} e^{-\gamma s}\,\mathrm{d}s\right\}\\ &= \frac{\rho}{2} + \rho\left\{\rho\left(\frac{1+l}{\sqrt{2\epsilon}}\sqrt{\frac{\pi}{\gamma}} + \frac{(1+l)\pi}{l\gamma}\right)\right\} < \frac{\rho}{2} + \frac{\rho}{2} = \rho\,.\end{aligned}$$

Therefore, the unique global solution $z(t, 0; z_0)$ exists and if $\|z_0\|_{H_0^1} \le \frac{\rho}{2}$, then $\|z(t, 0; z_0)\|_{H_0^1} < \rho$ for all $t \in [0, \infty)$.

Finally, we will derive the stability result (27). Let $w(t) = \sup\{\|z(s)\|_{H_0^1} e^{\beta s} : 0 \le s \le t\}$. It follows that

$$\begin{aligned}\|z(t)\|_{H_0^1} e^{\beta t} &\le e^{\beta t}\left(\|S(t)z_0\|_{H_0^1} + \int_0^t \|S(t-s)f(z(s))\|_{H_0^1}\,\mathrm{d}s\right)\\ &\le e^{\beta t}\left(e^{-\gamma t}\|z_0\|_{H_0^1} + \int_0^t \left\{\frac{1+l}{\sqrt{2\epsilon}}\frac{1}{\sqrt{t-s}} + \frac{(1+l)\pi}{l}\right\} e^{-\gamma(t-s)}\|z(s)\|_{H_0^1}^2\,\mathrm{d}s\right)\\ &\le e^{-(\gamma-\beta)t}\|z_0\|_{H_0^1} + \rho\int_0^t \left(\frac{1+l}{\sqrt{2\epsilon}}\frac{1}{\sqrt{t-s}} + \frac{(1+l)\pi}{l}\right) e^{-(\gamma-\beta)(t-s)}\|z(s)\|_{H_0^1} e^{\beta s}\,\mathrm{d}s\\ &\le \|z_0\|_{H_0^1} + \frac{1+l}{\sqrt{2\epsilon}}\,\rho\left(\int_0^t \frac{1}{\sqrt{t-s}} e^{-(\gamma-\beta)(t-s)}\,\mathrm{d}s\right) w(t) + \frac{(1+l)\pi}{l}\,\rho\left(\int_0^t e^{-(\gamma-\beta)s}\,\mathrm{d}s\right) w(t)\\ &\le \|z_0\|_{H_0^1} + \rho\left\{\frac{1+l}{\sqrt{2\epsilon}}\int_0^{\infty}\frac{1}{\sqrt{s}} e^{-(\gamma-\beta)s}\,\mathrm{d}s + \frac{(1+l(\pi}{l(\gamma+\beta)}\right\} w(t) \le \|z_0\|_{H_0^1} + \frac{1}{2}\,w(t)\,.\end{aligned}$$

Therefore, $w(t) \le 2\|z_0\|_{H_0^1}$ and $\|z(t)\|_{H_0^1} \le 2e^{-\beta t}\|z_0\|_{H_0^1}$. ■

REMARK 2.8. Rankin [26] considered well-posedness questions for a certain type of semi-linear evolution where the nonlinear terms are in divergence form. According to his results, we can see that equation (18) has a unique (strong) solution for initial data in $L^p(0, l)$, $p \ge 4$. To get this result, he used the analyticity of the semigroup $S(t)$ and the fact that the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ on $H_0^1(0, l)$ can be represented by

$$\sqrt{\epsilon}\,\frac{\mathrm{d}}{\mathrm{d}x} = (-A_\epsilon)^{1/2} B \tag{31}$$

for some bounded operator $B \in \mathscr{L}(H)$, where $H = L^2(0, l)$. In general, it is not true that $\sqrt{\epsilon}\,\frac{\mathrm{d}}{\mathrm{d}x} = (-A_\epsilon)^{1/2}$. This result could be used to analyze the stability property of the solution of Burgers' equation with initial data in $L^p(0, l)$, $p \ge 4$.

3. Linear Control Problem

As we noted in Section 2, the open-loop (no control) solution of Burgers' equation (18) decays exponentially in the topology of the energy space $H_0^1(0, l)$ (see Theorem 2.7). However, the decay rate depends on the viscosity $\epsilon > 0$. We now explore the possibility of obtaining an exponential decay rate independent of viscosity by feedback laws.

The basic model is governed by an abstract system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = Az(t) + Bu(t) , \quad z(0) = z_0 \in H , \qquad y(t) = Cz(t) , \quad t \geq 0 , \tag{32}$$

where H, U and Y are Hilbert spaces, $u(\cdot) \in L^2(0, \infty; U)$, $y(\cdot) \in L^2(0, \infty; Y)$, and A is the infinitesimal generator of an analytic semigroup $S(t)$ on a Hilbert space H. Assume that $B \in \mathscr{L}(U, H)$, $C \in \mathscr{L}(H, Y)$ and A is self-adjoint with compact resolvent. Solutions of the system (32) are taken to be in mild form, i.e.,

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s , \tag{33}$$

$$y(t) = CS(t)z_0 + C\int_0^t S(t-s)Bu(s)\,\mathrm{d}s , \quad t \geq 0 . \tag{34}$$

We first consider the performance index

$$J(u) = \int_0^\infty \{\|y(t)\|_Y^2 + R\|u(t)\|_U^2\}\,\mathrm{d}t , \tag{35}$$

where $y(t)$ is given by equation (34), and $R > 0$. The linear quadratic regulator problem is:

(LQR) *Find* $u(\cdot) \in L^2(0, \infty; U)$ *minimizing the cost functional* J *given by equation* (35) *subject to the system* (33)–(34).

The existence of an admissible control u such that $J(u) < \infty$ and the exponential stability of the closed-loop system depend on the following two hypotheses.

(H1). The system (32) is *stabilizable* in the sense that there is a feedback operator $K \in \mathscr{L}(H, U)$ such that the closed loop semigroup $S_K(t) \in \mathscr{L}(H)$ given by

$$S_K(t)z = S(t)z + \int_0^t S(t-s)BKS_K(s)z\,\mathrm{d}s \tag{36}$$

for all $t \geq 0$ and $z \in H$ decays exponentially.

(H2). The system (32) is *detectable* in the sense that there exists an operator $F \in \mathscr{L}(Y, H)$ such that the output injection semigroup $S_F(t) \in \mathscr{L}(H)$ given by

$$S_F(t)z = S(t)z + \int_0^t S_F(t-s)FCS(s)z\,\mathrm{d}s \tag{37}$$

for all $t \geq 0$ and $z \in H$ decays exponentially.

REMARK 3.1. [25, pp. 134–135] (i) If (H1) is satisfied, then for any $z_0 \in H$, there is an admissible control $u_{z0}(\cdot) \in L^2(0,\infty; U)$ such that $J(u_{z_0}) < \infty$. (ii) Let (H2) be satisfied. Then for any $z_0 \in H$ and $u(\cdot) \in L^2(0,\infty; U)$ with $J(u) < \infty$, $z(t)$ defined by equation (33) is in $L^2(0,\infty; H)$.

Now we state the following fundamental results for the (LQR) problem, see [2, 11].

THEOREM 3.2. *If hypotheses (H1) and (H2) are satisfied, then there is a unique optimal control $\bar{u}(\cdot) \in L^2(0,\infty; U)$ for the linear quadratic regulator (LQR) problem and $\bar{u}(\cdot)$ is given by the feedback law*

$$\bar{u}(t) = -R^{-1}B^*\Pi\bar{z}(t)\,, \quad t \geq 0\,, \tag{38}$$

where $\bar{z}(t)$ is the corresponding optimal trajectory. The operator $\Pi \in \mathscr{L}(H)$ is the unique nonnegative self-adjoint solution of the algebraic Riccati equation

$$A^*\Pi z + \Pi A z - \Pi B R^{-1} B^* \Pi z + C^*Cz = 0 \tag{39}$$

for every $z \in \mathscr{D}(A)$. Moreover, the closed-loop semigroup $S_\Pi(t) \in \mathscr{L}(H)$ generated by the operator $A - BR^{-1}B^*\Pi$ decays exponentially.

REMARK 3.3. The algebraic Riccati equation (39) can be defined for all $z \in H$, since the right hand side of the equation

$$\Pi A z = -A^*\Pi z + \Pi B R^{-1} B^*\Pi z - C^*Cz \tag{40}$$

is well-defined for all $z \in H$ and $\mathscr{D}(A)$ is densely embedded in H. Hence, we can extend the left hand side of equation (40) to $z \in H$ continuously.

We now consider the control problem for Burgers' equation. Define the operator A_ϵ, as in Section 2, by $A_\epsilon\phi = \epsilon\phi''$ for all $\phi \in \mathscr{D}(A_\epsilon) = H^2(0,l) \cap H_0^1(0,l)$. For the control input operator B and the observation output operator C we consider the Hilbert spaces $H = L^2(0,l)$, $U = \mathbf{R}$ and $Y = \mathbf{R}^k$. Assume that $B \in \mathscr{L}(U,H)$ and $C \in \mathscr{L}(W,Y)$ are defined by

$$Bu = b(\cdot)u \quad \text{and} \quad Cz = (\tilde{z}(\bar{x}_1), \ldots, \tilde{z}(\bar{x}_k))\,, \tag{41}$$

where $b(\cdot) \in H$, $u \in U$, and $\bar{x}_i \in (0,l)$, $1 \leq i \leq k$, $\tilde{z}(x_i)$ are defined by

$$\tilde{z}(\bar{x}_i) = \frac{1}{2\delta}\int_{\bar{x}_i-\delta}^{\bar{x}_i+\delta} z(x)\,\mathrm{d}x\,. \tag{42}$$

In equation (42), $\delta > 0$ is chosen so that $(\bar{x}_i - \delta, \bar{x}_i + \delta) \subset (0,l)$ for all $1 \leq i \leq k$. Consider the following linear control problem.

$(\text{LQR})_\alpha$ *Find $\bar{u}(\cdot) \in L^2(0,\infty; U)$ minimizing the weighted performance index*

$$J(u) = \int_0^\infty \{\|y(t)\|_Y^2 + R\|u(t)\|_U^2\}e^{2\alpha t}\,\mathrm{d}t\,, \quad (\alpha > 0) \tag{43}$$

subject to the governing equations

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = A_\epsilon z(t) + Bu(t), \quad z(0) = z_0 \tag{44}$$

$$y(t) = Cz(t), \quad t \geq 0, \tag{45}$$

where A_ϵ, B, C are as above and $R > 0$.

REMARK 3.4. (i) Equation (44) corresponds to the system

$$\begin{aligned} &\frac{\partial}{\partial t} z(t, x) = \epsilon \frac{\partial^2}{\partial x^2} z(t, x) + b(x)u(t), \quad t > 0, \quad 0 < x < l, \\ &v(t, 0) = v(t, l) = 0, \\ &v(0, x) = v_0(x). \end{aligned} \tag{46}$$

(ii) For each i, $1 \leq i \leq k$, $\tilde{z}(\bar{x}_i)$ given by equation (42) represents an average value of $z(x)$ over a small neighborhood of $\bar{x}_i$. We can regard each $\bar{x}_i$, $1 \leq i \leq k$, as the location of an output to be controlled.

(iii) The weight function $e^{2\alpha t}$ in the definition (43) of the cost functional J will play an important role in the exponential decay rate (see Theorems 3.8 and 3.9). However, it also gives rise to the question of existence of an admissible control $u(\cdot)$ such that $J(u) < \infty$.

For the control problem $(\mathrm{LQR})_\alpha$, we introduce an 'α-shifted' control system [12]. Let $\hat{z}(t) = z(t)e^{\alpha t}$, $\hat{u}(t) = u(t)e^{\alpha t}$ and $\hat{y}(t) = y(t)e^{\alpha t}$. We then have a modified linear control problem

$(\mathrm{LQR})_{\hat{\alpha}}$ *Find $\bar{\hat{u}} \in L^2(0, \infty; U)$ minimizing the cost functional*

$$\hat{J}(\hat{u}) = \int_0^\infty \{\|\hat{y}(t)\|_Y^2 + R\|\hat{u}(t)\|_U^2\}\, \mathrm{d}t \tag{47}$$

subject to

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{z}(t) = (A_\epsilon + \alpha I)\hat{z}(t) + B\hat{u}(t), \quad \hat{z}(0) = z_0, \quad \hat{y}(t) = C\hat{z}(t), \quad t \geq 0. \tag{48}$$

The solutions for the system (48) again are taken as mild solutions. If we solve the problem $(\mathrm{LQR})_{\hat{\alpha}}$ and apply

$$\bar{u}_\alpha(t) = e^{-\alpha t}\bar{\hat{u}}(t) \quad (t \geq 0) \tag{49}$$

to the original control system (44)–(45), then the resulting optimal trajectory $\bar{z}_\alpha(t)$ will satisfy the inequality

$$\|\bar{z}_\alpha(t)\|_H \leq Me^{-\alpha t}\|z_0\|_H, \tag{50}$$

where $M \geq 1$ is a constant and $\alpha > 0$ is the desired degree of stability.

REMARK 3.5. A discussion of the 'α-shifted' problem for finite dimensional systems appears in [1]. Anderson and Moore established that, for finite dimensional systems, the control problem $(\mathrm{LQR})_\alpha$ is 'equivalent' to $(\mathrm{LQR})_{\hat{\alpha}}$ in the following sense:

(i) The minimum value of J defined by equation (43) is the same as the minimum value of $\hat{J}$ given by equation (47). (ii) If $\tilde{u}(t) = g(\hat{z}(t))$ is the optimal control for $(\mathrm{LQR})_{\hat{\alpha}}$ for some function g, then $\bar{u}(t) = e^{-\alpha t} g(z(t)e^{\alpha t})$ is the optimal control for $(\mathrm{LQR})_\alpha$ and conversely.

REMARK 3.6. (i) From Remark 2.2, the spectrum $\sigma(A_\epsilon + \alpha I)$ of the infinitesimal generator $A_\epsilon + \alpha I$ consists of all eigenvalues $\lambda_{\alpha,n}$, $n = 1, 2, \ldots$, given by $\lambda_{\alpha,n} = \alpha - \epsilon n^2\pi^2/l^2$ and for each n, $n = 1, 2, \ldots$, the eigenfunction $\phi_{\alpha,n}$ corresponding to $\lambda_{\alpha,n}$ is given by $\phi_{\alpha,n}(x) = \sqrt{\frac{2}{l}} \sin \frac{n\pi}{l} x$. (ii) We are interested in the stabilization problem for the system (46) with small viscosity $\epsilon = \frac{1}{\mathrm{Re}} > 0$, i.e., high Reynolds number. Let $\alpha_0 = \frac{\epsilon\pi^2}{l^2}$. If $\alpha > \alpha_0$, then there is at least one positive eigenvalue $\lambda_{\alpha,1} = \alpha - \frac{\epsilon\pi^2}{l^2}$. Moreover, if $\alpha > \alpha_0$ and $0 < \gamma \leq \epsilon$, then the first eigenvalue $\alpha - \frac{\gamma\pi^2}{l^2}$ of $A_\gamma + \alpha I$ satisfies $\alpha - \frac{\gamma\pi^2}{l^2} > \alpha - \frac{\epsilon\pi^2}{l^2} > 0$ and hence $A_\gamma + \alpha I$ will have at least one positive eigenvalue for all $0 < \gamma \leq \epsilon$. This will become important when we apply feedback laws computed from low Reynolds number to investigate the closed-loop response of the nonlinear Burgers' equation for high Reynolds numbers (see Example 3 in Section 4).

Let $\alpha > \alpha_0 = \frac{\epsilon\pi^2}{l^2}$ be given and let

$$n_\alpha = \max\left\{ n \in \mathbf{N} : \lambda_{\alpha,n} = \alpha - \frac{\epsilon n^2\pi^2}{l^2} \geq 0 \right\}. \tag{51}$$

Since A_ϵ is self-adjoint (see Remark 2.2) and the set $\{\phi_{\alpha,n} : n = 1, 2, \ldots\}$ is a basis for $H = L^2(0, l)$, we can identify $z \in H$ with the sequence $\{\langle z, \phi_{\alpha,n}\rangle\}_{n\in\mathbf{N}}$. Assume that $b_n \in U$ and $c_n \in Y$ satisfy

$$Bu = \{\langle b_n, u\rangle\}_{n\in\mathbf{N}} \quad \text{and} \quad Cz = \sum_{n=1}^{\infty} c_n \langle z, \phi_{\alpha,n}\rangle \tag{52}$$

with $\sum_{n=1}^{\infty} \|b_n\|_U^2 < \infty$ and $\sum_{n=1}^{\infty} \|c_n\|_Y^2 < \infty$, see [25, pp. 137–143].

The following lemma is an application of stabilizability and detectability results given by Pritchard and Salamon [25, Section 4.2].

LEMMA 3.7. *For each $n = 1, 2, \ldots, n_\alpha$, let*

$$X_{\alpha,n} = \left\{ \frac{il}{n} : i = 1, 2, \ldots, n-1 \right\}. \tag{53}$$

Then the following statements hold.

(*a*) $b_n = \langle b(\cdot), \phi_{\alpha,n}\rangle \neq 0$ *for all $n = 1, 2, \ldots, n_\alpha$, if and only if the system* (48) *is stabilizable in H.*

(*b*) *If $\delta > 0$ satisfies $\delta < \frac{l}{2n_\alpha}$, then for each $n = 1, 2, \ldots, n_\alpha$, there exists at least one $\bar{x}_i$, $1 \leq i \leq k$, such that $\bar{x}_i \notin X_{\alpha,n}$ if and only if the system (48) is detectable through $C \in \mathscr{L}(H, Y)$.*

Proof. (a) From Remark 3.6, we know that the spectrum $\sigma(A_\epsilon + \alpha I)$ of $A_\epsilon + \alpha I$ consists of all eigenvalues $\lambda_{\alpha,n} = \alpha - \epsilon n^2\pi^2/l^2$. Thus, for all $n \geq n_\alpha + 1$, we know that $\lambda_{\alpha,n} < 0$. Let H_u be the linear span of eigenfunctions $\phi_{\alpha,1}, \ldots, \phi_{\alpha,n_\alpha}$. The dimension of H_u is n_α and hence the system

(48) is stabilizable if and only if the projection of (48) onto H_u is controllable, or equivalently, if and only if $b_n = \langle b(\cdot), \phi_{\alpha,n} \rangle \neq 0$ for all n, $n = 1, 2, \ldots, n_\alpha$.

To prove (b), let $c_n = (c_{n1}, \ldots, c_{nk})$, $n = 1, 2, \ldots, c_{nk})$, $n = 1, 2, \ldots$, be defined by equation (32). We have that

$$c_{ni} = \frac{\sqrt{2l}}{n\pi\delta} \sin \frac{n\pi \bar{x}_i}{l} \sin \frac{n\pi\delta}{l} \tag{54}$$

for $1 \le i \le k$. By duality, the system (48) is detectable through $C \in \mathscr{L}(H, Y)$ if and only if $c_n \neq 0$ for all $n = 1, 2, \ldots, n_\alpha$. Hence, (b) holds. ■

We now return to the original control problem $(\text{LQR})_\alpha$. The following theorem is the main result for our control problem $(\text{LQR})_\alpha$.

THEOREM 3.8. *Let $\alpha > \alpha_0$ be given. Suppose that $b(\cdot) \in H = L^2(0, l)$, $\delta > 0$, $\bar{x}_i$, $1 \le i \le k$, satisfy the conditions (a) and (b) of Lemma 3.7. Then there is a unique optimal control $\bar{u}_{\alpha,\epsilon} \in L^2(0, \infty; \mathbf{R})$ for the problem $(LQR)_\alpha$ such that*

$$\bar{u}_{\alpha,\epsilon}(t) = -R^{-1}B^*\Pi_{\alpha,\epsilon}\bar{z}_{\alpha,\epsilon}(t), \quad t \ge 0, \tag{55}$$

where $\bar{z}_{\alpha,\epsilon}(t)$ is the corresponding optimal trajectory and $\Pi_{\alpha,\epsilon} \in \mathscr{L}(H)$ is the unique nonnegative self-adjoint operator satisfying the algebraic Riccati equation

$$(A_\epsilon + \alpha I)^*\Pi_{\alpha,\epsilon}z + \Pi_{\alpha,\epsilon}(A_\epsilon + \alpha I)z - \Pi_{\alpha,\epsilon}BR^{-1}B^*\Pi_{\alpha,\epsilon}z + C^*Cz = 0 \tag{56}$$

for every $z \in \mathscr{D}(A_\epsilon) = H^2(0, l) \cap H_0^1(0, l)$. Moreover, the closed loop semigroup $S_{\Pi_{\alpha,\epsilon}}(t) \in \mathscr{L}(H)$ satisfies the following stability property

$$\|S_{\Pi_{\alpha,\epsilon}}(t)\|_{\mathscr{L}(H)} \le Me^{-(\alpha+\omega)t} \tag{57}$$

for some constants $M = M(\alpha, \epsilon) \ge 1$ and $\omega = \omega(\alpha, \epsilon) > 0$.

Proof. By Lemma 3.7, we know that the α-shifted control system (48) satisfies all hypotheses (H1) and (H2) (with $z(t)$, $y(t)$, $u(t)$, A, $S(t)$ and J replaced by $\hat{z}(t)$, $\hat{y}(t)$, $\hat{u}(t)$, $A_\epsilon + \alpha I$, $\hat{S}(t)$ and $\hat{J}$, respectively). Hence, by Theorem 3.2, there is a unique optimal control $\bar{\hat{u}}(t)$ for $(\text{LQR})_{\hat{\alpha}}$ and the corresponding closed-loop semigroup $\hat{S}(t)$ decays exponentially, i.e.,

$$\|\hat{S}(t)\|_{\mathscr{L}(H)} \le \hat{M}e^{-\omega t} \quad t \ge 0 \tag{58}$$

for some constants $\hat{M} = \hat{M}(\alpha, \epsilon) \ge 1$ and $\omega = \omega(\alpha, \epsilon) > 0$. Moreover, $\bar{\hat{u}}(t)$ is given by

$$\bar{\hat{u}}(t) = -R^{-1}B^*\Pi_{\alpha,\epsilon}\bar{\hat{z}}(t) \tag{59}$$

where $\bar{\hat{z}}(t)$ is the resulting optimal trajectory for the α-shifted system (48) and $\Pi_{\alpha,\epsilon}$ is the unique nonnegative self-adjoint solution of equation (56). Since the semigroup $\hat{S}(t)$ is generated by $A_\epsilon + \alpha I - BR^{-1}B^*\Pi_{\alpha,\epsilon}$, the infinitesimal generator of the closed-loop semigroup $S_{\Pi_{\alpha,\epsilon}}(t)$ for the original system (44) is $A_\epsilon - BR^{-1}B^*\Pi_{\alpha,\epsilon}$. Hence, $S_{\Pi_{\alpha,\epsilon}}(t) = \hat{S}(t)e^{-\alpha t}$ and, by the relation (58), $S_{\Pi_{\alpha,\epsilon}}(t)$ satisfies the inequality (57) with $M = \hat{M}$. Moreover, the optimal control $\bar{u}_{\alpha,\epsilon}(t)$ for $(\mathrm{LQR})_\alpha$ is given by the formula (55), since $\bar{u}_{\alpha,\epsilon}(t) = \bar{\hat{u}}(t)e^{-\alpha t} = -R^{-1}B^*\Pi_{\alpha,\epsilon}\bar{\hat{z}}(t)e^{-\alpha t} = -R^{-1}B^*\Pi_{\alpha,\epsilon}\bar{z}_{\alpha,\epsilon}(t)$, where $\bar{z}_{\alpha,\epsilon}(t) = \bar{\hat{z}}(t)e^{-\alpha t}$ is the corresponding optimal trajectory for the original system (44). This completes the proof. ■

The optimal control $\bar{u}_{\alpha,\epsilon}(\cdot) \in L^2(0,\infty;\mathbf{R})$ obtained in Theorem 3.8 is given by the feedback law (55). Note that $\bar{u}_{\alpha,\epsilon}(t)$ depends on α and ϵ. Define the feedback operator $K_{\alpha,\epsilon} \in \mathcal{L}(H, U)$ by

$$K_{\alpha,\epsilon} = -R^{-1}B^*\Pi_{\alpha,\epsilon}\,. \tag{60}$$

Then the optimal control $\bar{u}_{\alpha,\epsilon}(t)$, $t \geq 0$, is given by

$$\bar{u}_{\alpha,\epsilon}(t) = K_{\alpha,\epsilon}\bar{z}_{\alpha,\epsilon}(t) \tag{61}$$

and the infinitesimal generator for the closed-loop semigroup $S_{\Pi_{\alpha,\epsilon}}(t)$ is

$$A_\epsilon + BK_{\alpha,\epsilon} = A_\epsilon - BR^{-1}B^*\Pi_{\alpha,\epsilon}\,. \tag{62}$$

Recall that $H = L^2(0,l)$ and $U = \mathbf{R}$. Thus, the Riesz Representation Theorem (see, e.g., [8, p. 13]) implies that there is a unique *feedback gain function* $k_{\alpha,\epsilon}(\cdot) \in L^2(0,l)$ such that

$$K_{\alpha,\epsilon}z = \int_0^l k_{\alpha,\epsilon}(s)z(s)\,ds \tag{63}$$

for all $z \in L^2(0,l)$.

THEOREM 3.9. *Let $\alpha > \alpha_0$ be given. Suppose that $b(\cdot) \in H = L^2(0,l)$, $\delta > 0$ and $\bar{x}_i$, $1 \leq i \leq k$, satisfy the conditions (a) and (b) of Lemma 3.7. If $k_{\alpha,\epsilon}(\cdot) \in H$ is the linear feedback gain function defined by the formula (63), then there exist constants $\rho = \rho(\alpha,\epsilon) > 0$ and $M = M(\alpha,\epsilon) \geq 1$ such that for any initial data $z_0(\cdot) \in H_0^1(0,l)$, with $\|z_0\|_{H_0^1} \leq \frac{\rho}{2M}$, the controlled Burgers' equation*

$$\begin{aligned}
&\frac{\partial}{\partial t} z(t,x) = \epsilon \frac{\partial^2}{\partial x^2} z(t,x) - z(t,x)\frac{\partial}{\partial x} z(t,x) + b(x)\int_0^l k_{\alpha,\epsilon}(s)z(t,s)\,ds \\
&z(t,0) = z(t,l) = 0\,, \\
&z(0,x) = z_0(x) \in H_0^1(0,l)
\end{aligned} \tag{64}$$

has a unique (strong) solution and the solution $z(t)\cdot) = z(t,\cdot)$ satisfies the following stability property

$$\|z(t)\|_{H_0^1} \leq 2Me^{-\alpha t}\|z_0(\cdot)\|_{H_0^1}\,. \tag{65}$$

Proof. Let the operators A_ϵ, B, C and $K_{\alpha,\epsilon}$ be given by equations (19), (41) and (60). Define the nonlinear function $f: H_0^1(0,l) \to L^2(0,l)$ by

$$f(z) = BK_{\alpha,\epsilon}z - zz' \tag{66}$$

where $' = \frac{d}{dx}$. Then, the map f satisfies the hypothesis (F) in Section 2, since for any z_1, $z_2 \in H_0^1(0, l)$,

$$\|f(z_1) - f(z_2)\|_{L^2(0,l)} \le (\|BK_{\alpha,\epsilon}\|_{\mathcal{L}(H)} + \|z_1\|_{H_0^1} + \|z_2\|_{H_0^1})\|z_1 - z_2\|_{H_0^1} . \tag{67}$$

Note that the operator $BK_{\alpha,\epsilon}$ is bounded on the state space $H = L^2(0, l)$. Thus, by Theorem 2.1, we have a unique local (strong) solution of equation (64).

Let $S_{K_{\alpha,\epsilon}}(t)$, $t \ge 0$, be the analytic semigroup on H generated by the operator $A_\epsilon + BK_{\alpha,\epsilon}$. Theorem 3.8 implies that $S_{K_{\alpha,\epsilon}}(t)$ satisfies the inequality

$$\|S_{K_{\alpha,\epsilon}}(t)z\|_H \le M_\alpha e^{-(\alpha+\omega)t}\|z\|_H \tag{68}$$

for all $z \in H$ and for some constants $M_\alpha = M_\alpha(\alpha, \epsilon) \ge 1$ and $\omega = \omega(\alpha, \epsilon) > 0$. If $\alpha < \beta < \alpha + \omega$, then there is a constant $\tilde{M}_\alpha = \tilde{M}_\alpha(\alpha, \epsilon, \beta) \ge 1$ such that

$$\|S_{K_{\alpha,\epsilon}}(t)z\|_{H_0^1} \le M_\alpha e^{-\beta t}\|z\|_{H_0^1} \tag{69}$$

$$\|S_{K_{\alpha,\epsilon}}(t)z\|_{H_0^1} \le \tilde{M}_\alpha \frac{1}{\sqrt{t}} e^{-\beta t}\|z\|_H \tag{70}$$

for all $z \in H_0^1$. Let $M = \max\{M_\alpha, \tilde{M}_\alpha\}$ and choose $\rho > 0$ with $0 < \rho < \frac{\sqrt{\beta - \alpha}}{2\sqrt{\pi}M}$. It is easy to see that

$$\rho M \int_0^\infty \frac{1}{\sqrt{s}} e^{-(\beta-\alpha)s}\, ds < \frac{1}{2} . \tag{71}$$

Consequently, arguments similar to those in the proof of Theorem 2.7 together with inequalities (69)–(70) and the expression

$$z(t) = S_{K_{\alpha,\epsilon}}(t)z_0 + \int_0^t S_{K_{\alpha,\epsilon}}(t-s)g(z(s))\, ds , \tag{72}$$

imply that the unique global solution $z(t)(\cdot) = z(t, \cdot)$ for the controlled Burgers' equation (64) exists and satisfies the inequality (65), where $g(z(t)) = -z(t)z'(t)$. ■

4. Approximation and Numerical Results

In Section 3, we considered a linear quadratic regulator $(\mathrm{LQR})_\alpha$ to obtain a desired degree of stability for the solution of the closed-loop Burgers' equation. In this section we consider an approximation scheme for $(\mathrm{LQR})_\alpha$ and give some numerical results.

We first introduce an abstract approximation scheme for the problem (LQR) based on the results of Banks, Kunish [2] and Ito [16] and then apply the scheme to obtain the optimal control $\bar{u}_{\alpha,\epsilon}(\cdot) \in L^2(0, \infty; \mathbf{R})$. Throughout this section we assume that $R = I$, the identity operator on the control space U.

We formulate a sequence of approximate regulator problems and present a convergence result for the corresponding Riccati operators. Throughout this section, we use superscript N in the designation of subspaces, operators and matrices in the N-th approximating system and control problem, like H^N, A^N, B^N, etc. Hence the superscript N indicates the order of approximation.

Let H^N, $N = 1, 2, \ldots$, be a sequence of finite dimensional linear subspaces of H and $P^N : H \to H^N$ be the canonical orthogonal projections. Assume that $S^N(t)$ is a sequence of C_0 – semigroups on H^N with infinitesimal generators $A^N \in \mathscr{L}(H^N)$. Consider the family of regulator problems:

$(\mathrm{LQR})^N$ *Minimize* $J^N(z_0^N, u)$ *over* $u \in L^2(0, \infty; U)$ *subject to the control system*

$$z^N(t) = S^N(t)z_0^N + \int_0^t S^N(t-s)B^N u(s)\,\mathrm{d}s\,, \tag{73}$$

$$y^N(t) = C^N z^N(t)\,, \tag{74}$$

where $z^N(0) = z_0^N \equiv P^N z_0$ *and*

$$J^N(z_0^N, u) = \int_0^\infty \{\|y^N(t)\|_Y^2 + \|u(t)\|_U^2\}\,\mathrm{d}t\,. \tag{75}$$

REMARK 4.1. If for each N (A^N, B^N) is stabilizable and (A^N, C^N) is detectable, then Theorem 3.2 yields a unique optimal control $\bar{u}^N(t)$ for the finite dimensional problem $(\mathrm{LQR})^N$ given by

$$\bar{u}^N(t) = -(B^N)^*\Pi^N S_\Pi^N(t) z_0^N\,, \tag{76}$$

where $S_\Pi^N(t)$ is the C_0-semigroup on H^N generated by $A^N - B^N(B^N)^*\Pi^N$ and $\Pi^N \in \mathscr{L}(H^N)$ is the unique nonnegative self-adjoint solution of

$$(A^N)^*\Pi^N + \Pi^N A^N - \Pi^N B^N (B^N)^* \Pi^N + (C^N)^* C^N = 0\,. \tag{77}$$

In general, it is not clear that (A^N, B^N) is stabilizable even if the original system (A, B) is stabilizable. Similarly, it is not clear that the detectability property of (A, C) is preserved under the finite dimensional projections. Another question to consider is the convergence of approximates Π^N and $\bar{u}^N(t)$ to the infinite dimensional solutions Π and $\bar{u}(t)$, respectively. In order to address these issues we let $S^N(t) = e^{A^N t}$, $t \geq 0$ and make the following assumptions:

$(A1)$: For each $z \in H$, $S^N(t)P^N z \to S(t)z$ and $S^N(t)^* P^N z \to S(t)^* z$, where the convergences are uniform in t on bounded subsets of $[0, \infty)$.

$(A2)$: (i) For each $u \in U$, $B^N u \to Bu$ and for each $z \in H$, $(B^N)^* P^N z \to B^* z$.
(ii) For each $z \in H$, $C^N P^N z \to Cz$ and for each $y \in Y$, $(C^N)^* y \to C^* y$.

$(A3)$: (i) The family of systems (A^N, B^N) is *uniformly stabilizable*, i.e., there exists a sequence of operators $K^N \in \mathscr{L}(H^N, U)$ such that $\sup\|K^N\| < \infty$ and

$$\|e^{(A^N + B^N K^N)t} P^N\|_{\mathscr{L}(H)} \leq M_1 e^{-\omega_1 t}\,, \quad t \leq 0\,,$$

for some positive constants $M_1 \geq 1$ and $\omega_1 > 0$ which are independent of N.
(ii) The family of pairs (A^N, C^N) is *uniformly detectable*, i.e., there exists a sequence of operators $F^N \in \mathscr{L}(Y, H^N)$ such that $\sup\|F^N\| < \infty$ and

$$\|e^{(A^N+F^NC^N)t}P^N\|_{\mathscr{L}(H)} \leq M_2 e^{-\omega_2 t}, \quad t \leq 0,$$

for some constants $M_2 \geq 1$ and $\omega_2 > 0$ which are independent of N.

REMARK 4.2. (i) The condition $(A3)$(ii) is a relaxation of the coercivity assumption in [2] (see also [16, p. 3]). (ii) Suppose that $B^N = P^N B$ and $C^N = CP^N$. Then $(A2)$ holds, since it follows from $(A1)$ that $P^N z \to z$ for all $z \in H$.

By simple modification of results from [16, Theorem 2.1] and [2, Theorem 2.2], we have the following fundamental convergence results.

THEOREM 4.3. *Let (A, B) be stabilizable and (A, C) be detectable. If $(A1)$–$(A3)$ are satisfied, then for each N the finite dimensional algebraic Riccati Equation ((77) admits a unique nonnegative self-adjoint solution Π^N such that $\sup\{\|\Pi^N\|_{\mathscr{L}(H^N)} : N = 1, 2, \ldots\} < \infty$ and*

$$\Pi^N P^N z \to \Pi z \tag{78}$$

for every $z \in H$. Moreover, there exist positive constants $M_3 \geq 1$ and ω_3 (independent of N) such that

$$\|e^{(A^N - B^N(B^N)^*\Pi^N)t}P^N\|_{\mathscr{L}(H)} \leq M_3 e^{-\omega_3 t}, \quad t \geq 0. \tag{79}$$

It is helpful to introduce a sesquilinear form $a_\epsilon(\cdot,\cdot): V \times V \to \mathbf{C}$ defined by

$$a_\epsilon(z, w) = \int_0^1 \epsilon z'(x)\bar{w}'(x)\,dx, \quad z, w \in V, \tag{80}$$

where $V = H_0^1(0, 1)$. It is easy to see that the sesquilinear form $a_\epsilon(\cdot,\cdot)$ is V-coercive [33, p. 274], i.e.,

$$|a_\epsilon(z, w)| \leq \epsilon \|z\|_V \|w\|_V, \quad \text{(continuity)}, \tag{81}$$

$$\mathscr{R}e\, a_\epsilon(z, z) + \gamma \|z\|_H^2 \geq \epsilon \|z\|_V^2, \quad \text{(Gårding's inequality)}, \tag{82}$$

for all $z, w \in V$ and $\gamma \geq \epsilon > 0$. Furthermore, it follows from the bounds (81) and (82) that there exists a unique operator $A_\epsilon \in \mathscr{L}(V, V^*)$ such that

$$a_\epsilon(z, w) = \langle -A_\epsilon z, w\rangle \quad \text{and} \quad \overline{a_\epsilon(z, w)} = \langle -A_\epsilon{}^* w, z\rangle \tag{83}$$

for all $z, w \in V$ (see, e.g., [33, pp. 271–275]).

Turning next to specific approximations for $(LQR)_\alpha$ we divide the unit interval $[0, 1]$ into $N + 1$ equal subintervals to get $[x_i, x_{i+1}]$, $x_i = \frac{i}{N+1}$, $i = 0, 1, \ldots, N$. For each i, $1 \leq i \leq N$, let $h_i^N(x)$ denote the linear spline basis function defined by

$$h_i^N(x) = \begin{cases} (N+1)(x - x_{i-1}), & x_{i-1} \leq x \leq x_i \\ -(N+1)(x - x_{i+1}), & x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise}. \end{cases} \tag{84}$$

If H^N denotes the N-dimensional finite element space given by

$$H^N = \left\{ \sum_{i=1}^{N} z_i h_i^N(x) : z_i \in \mathbf{R}, \quad i = 1, 2, \ldots, N \right\}, \tag{85}$$

then each H^N is a finite dimensional (real) subspaces satisfying $H^N \subset V$, $N = 1, 2, \ldots$. Moreover, it is well-known [28], [16, p. 15] that the family of H^N satisfies the following approximation condition:

(APP) *For each $z \in V$, there exists an element $z^N \in H^N$ such that*

$\|z - z^N\|_V \le \varepsilon(N)$, *where* $\varepsilon(N) \to \infty$.

Let $P: H \to H^N$ be the canonical orthogonal projection onto H^N. The approximation property (APP) implies that $P^N z \to z$ as $N \to \infty$, for $z \in H$. For the finite dimensional regulator problem $(LQR)_\alpha^N$ we choose

$$B^N = P^N B \quad \text{and} \quad C^N = CP^N . \tag{86}$$

Conditions $(A2)$(i), (ii) will follow from Remark 4.2.

A representation A_ϵ^N of A_ϵ on H^N can be obtained by using the restriction of the sesquilinear form $a_\epsilon(\cdot, \cdot)$ to $H^N \times H^N$. The representation A_ϵ^N of A_ϵ satisfies

$$a_\epsilon(z, w) = \langle -A_\epsilon^N z, w \rangle \quad \text{and} \quad a_\epsilon(z, w) = \langle -(A_\epsilon^N)^* w, z \rangle \tag{87}$$

for all z, $2 \in H^N$. Equation (87) follows from the fact that H^N is a real Hilbert space. We know also that $A_\epsilon^N = (A_\epsilon^N)^*$, since $A_\epsilon = A_\epsilon^*$.

REMARK 4.4. Since $H^N \subset H$ it is easy to see that for any $\lambda \in \sigma(A_\epsilon^N)$, $\mathcal{R}e\lambda \le -\gamma \le -\epsilon$.

Let $S^N(t)$ be the C_0-semigroup generated by A_ϵ^N. Then the conditions $(A1)$(i), (ii) follow from the results of Banks and Kunish [2, Lemma 3.2]. Note that $S^N(t) = (S^N(t))^*$. For the condition $(A3)$(i) we need a certain *preservation of exponential stabilizability under approximation* ((POES) in [2]). The following result is taken from [2, Lemma 3.3].

THEOREM 4.5. *Let (A_ϵ, B) be (exponentially) stabilizable. If the approximation condition (APP) holds, then the approximations defined through equations (86)–(87) satisfy the condition (A3)(i), i.e., the family of pairs (A_ϵ^N, B^N) is uniformly stabilizable.*

By the dual of the arguments used in Theorem 4.5 we can see that the condition $(A3)$ (ii) holds under the assumption that (A_ϵ, C) is detectable. We summarize our discussion up to this point in the following theorem.

THEOREM 4.6. *Let $(A_\epsilon + \alpha I, B)$ be stabilizable and $(A_\epsilon + \alpha I, C)$ be detectable. If A_ϵ^N, B^N, C^N are defined as in equations (4.14) and (4.15), then*

$$\Pi_{\alpha,\epsilon}^N P^N z \to \Pi_{\alpha,\epsilon} z , \quad z \in H , \tag{88}$$

and

$$S^N(t)P^N z \to S(t)z\,, \quad z \in H\,, \tag{89}$$

where the convergence is uniform in t *on bounded subsets of* $[0,\infty)$ *and* P^N *is the orthogonal projection onto* H^N. *Here* $\Pi^N_{\alpha,\epsilon}$ *satisfies the Riccati equation*

$$(A^N_\epsilon + \alpha I^N)^*\Pi^N_{\alpha,\epsilon} + \Pi^N_{\alpha,\epsilon}(A^N_\epsilon + \alpha I^N) - \Pi^N_{\alpha,\epsilon}B^N(B^N)^*\Pi^N_{\alpha,\epsilon} + (C^N)^*C^N = 0\,. \tag{90}$$

REMARK 4.7. Note that $S(t) = \hat{S}(t)e^{-\alpha t}$ and $S^N(t) = \hat{S}^N(t)e^{-\alpha t}$ where $S(t)$, $S^N(t)$, $\hat{S}(t)$ and $\hat{S}^N(t)$ are semigroups generated by A_ϵ, A^N_ϵ, $A_\epsilon + \alpha I$ and $A^N_\epsilon + \alpha I$, respectively.

Next, consider the matrix representations of operators on the space H^N. Let the approximate solution $z^N(t,x)$ of $z(t,x)$ on H^N be given by

$$z^N(t,x) = \sum_{i=1}^{N} z^N_i(t)h^N_i(x) \tag{91}$$

for some $z^N_i(t) \in \mathbf{R}$, $i = 1, \ldots, N$. Equations (86) and (87) yield the finite dimensional ODE system

$$[G^N]\frac{\mathrm{d}}{\mathrm{d}t}\{z^N(t)\} = [\tilde{A}^N_\epsilon]\{z^N(t)\} + [\tilde{B}^N]u(t)\,, \tag{92}$$

where $\{z^N(t)\} = [z^N_1(t), \ldots, z^N_N(t)]^T$,

$$[G^N] = [\langle h^N_j, h^N_i\rangle]_{N\times N}$$

$$= \frac{1}{6(N+1)}\begin{bmatrix} 4 & 1 & 0 & 0 & & \cdots & 0 \\ 1 & 4 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & 4 & 1 & & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 1 & 4 & 1 \\ 0 & & \cdots & & 0 & 1 & 4 \end{bmatrix}_{N\times N}, \tag{93}$$

$$[\tilde{A}^N_\epsilon] = \epsilon(N+1)\begin{bmatrix} -2 & 1 & 0 & 0 & & \cdots & 0 \\ 1 & -2 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & -2 & 1 & & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 1 & -2 & 1 \\ 0 & & \cdots & & 0 & 1 & -2 \end{bmatrix}_{N\times N}, \tag{94}$$

and

$$[\tilde{B}^N] = [\langle b, h^N_1\rangle, \langle b, h^N_2\rangle, \ldots, \langle b, h^N_N\rangle]^T\,, \tag{95}$$

where $\langle b, h^N_j\rangle = \int_0^1 b(x)h^N_j(x)\,\mathrm{d}x$, $1 \le j \le N$. Since $[G^N]$ is invertible we may multiply both sides of (92) by $[G^N]^{-1}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\{z^N(t)\} = [A_\epsilon^N]\{z^N(t)\} + [B^N]u(t), \quad \{z^N(0)\} = \{z_0^N\}, \tag{96}$$

where

$$[A_\epsilon^N] = [G^N]^{-1}[\tilde{A}_\epsilon^N], \quad [B^N] = [G^N]^{-1}[\tilde{B}^N] \tag{97}$$

and $\{z_0^N\} = [G^N]^{-1}[\langle z_0, h_1^N\rangle, \ldots, \langle z_0, h_N^N\rangle]^T$.

Next, we consider the representation $[C^N]$ of the operator C on H^N. It is easy to see that $C^N : H^N \to \mathbf{R}^k$ has the matrix representation

$$[C^N] = [\tilde{h}_j^N(\bar{x}_i)]_{k\times N}, \tag{98}$$

where $\tilde{h}_j^N(\bar{x}_i) = \frac{1}{2\delta}\int_{\bar{x}_i-\delta}^{\bar{x}_i+\delta} h_j^N(x)\,\mathrm{d}x$, $1 \le i \le k$, $1 \le j \le N$.

Finally, the finite dimensional Riccati equation (90) yields the corresponding feedback gain operator $K_{\alpha,\epsilon}^N$ given by

$$[K_{\alpha,\epsilon}^N] = -[B^N]^*[\Pi_{\alpha,\epsilon}^N]. \tag{99}$$

Therefore, the closed loop system (96) can be represented by

$$\frac{\mathrm{d}}{\mathrm{d}t}\{z^N(t)\} = [A_\epsilon^N + B^N K_{\alpha,\epsilon}^N]\{z^N(t)\}, \quad \{z^N(0)\} = \{z_0^N\}. \tag{100}$$

Now, we discuss an algorithm for finding the unique nonnegative self-adjoint Riccati solution for equation (90). We employ the Potter's method [27] to obtain $\Pi_{\alpha,\epsilon}^N$. The first step in Potter's method is to form $2N \times 2N$ matrix

$$[M^N] = \begin{bmatrix} [A_\epsilon^N + \alpha I^N]^* & [C^N]^*[C^N] \\ [B^N][B^N]^* & -[A_\epsilon^N + \alpha I^N] \end{bmatrix}. \tag{101}$$

Next, find all eigenvalues and eigenvectors of M^N and form the matrix

$$[Z^N] = \begin{bmatrix} Q_1^N \\ Q_2^N \end{bmatrix}, \tag{102}$$

where the columns of $[Z^N]$ are the eigenvectors of $[M^N]$ corresponding to the eigenvalues with positive real part. When eigenvalues occur in complex conjugate pairs, so do the eigenvectors. In this case, the real and imaginary part of the eigenvector each forms a column of $[Z^N]$. Finally, the solution to the Riccati equation (90) is given by the formula $[\Pi_{\alpha,\epsilon}^N] = [Q_1^N][Q_2^N]^{-1}$.

REMARK 4.8. From the numerical results we found that the Riccati solution operators $\Pi_{\alpha,\epsilon}^N$ become unbounded when the viscosity $\epsilon > 0$ goes to 0 for fixed $\alpha > 0$. Also, when α goes to infinity with ϵ fixed, the same phenomenon has been observed.

Finally, the finite dimensional approximation for the controlled nonlinear Burgers' equation (64) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\{z^N(t)\} = ([A_\epsilon^N] + [B^N][K_{\alpha,\epsilon}^N])\{z^N(t)\} + f^N(\{z^N(t)\}), \quad \{z^N(0)\} = \{z_0^N\}, \tag{103}$$

where $[A_\epsilon^N]$, $[B^N]$ are defined as in equation (97), and

$$f^N(\{z^N(t)\}) = [G^N]^{-1}\tilde{f}^N(\{z^N(t)\}),$$

$$\tilde{f}^N(\{z^N(t)\}) = -\frac{1}{6}\begin{bmatrix} z_1^N(t)z_2^N(t) + (z_2^N(t))^2 \\ -(z_1^N(t))^2 - (z_1^N(t))(z_2^N(t)) + (z_2^N(t))(z_3^N(t)) + (z_3^N(t))^2 \\ \vdots \\ -(z_{N-2}^N(t))^2 - (z_{N-2}^N(t))(z_{N-1}^N(t)) + (z_{N-1}^N(t))(z_N^N(t)) + (z_N^N(t))^2 \\ -(z_{N-1}^N(t))^2 - (z_{N-1}^N(t))(z_N^N(t)) \end{bmatrix},$$

where $[G^N]$ is defined by equation (93). To solve the nonlinear ODE system (103) we use a 4-th order Runge–Kutta method [30].

For numerical examples, the length l for space domain, the Reynolds number, Re, the initial function $z_0(\cdot) \in H_0^1(0, 1)$ and the control input function $b(\cdot) \in L^2(0, 1)$ will be chosen as 1, 60, $\sin \pi x$ and e^x, respectively. Thus, the closed-loop equation is given by

$$\frac{\partial}{\partial t} z(t, x) = \frac{1}{60}\frac{\partial^2}{\partial x^2} z(t, x) - z(t, x)\frac{\partial}{\partial x} z(t, x) + e^x \int_0^1 k_{\alpha,\epsilon}(s) z(t, s)\,\mathrm{d}s$$

$$z(t, 0) = z(t, 1) = 0$$

$$z(0, x) = \sin \pi x, \tag{104}$$

where the feedback gain function $k_{\alpha,\epsilon}(\cdot) \in L^2(0, 1)$ will be determined by the desired degree $\alpha > 0$ of stability and the action of output operator C.

The 'robustness' of the feedback controller exhibited, for example, in Figure 2, will be discussed in Example 3. For this particular example, Reynolds numbers 60, 80, 100 and 120 are chosen.

REMARK 4.9. (i) From the numerical experiments, we found that if Reynolds number is less than 60, then the diffusion phenomena dominate convection phenomena. In this case, the formation of a steep gradient due to convection term $-z(t, x)\frac{\partial}{\partial x}z(t, x)$ of the open-loop solution, i.e., $k_{\alpha,\epsilon}(\cdot) \equiv 0$ in equation (104), is not clear. For Reynolds number greater than 60, the open-loop solution creates a 'sharp' gradient in finite time, (see Figures 1, 10, 12 and 14). The solution dies out eventually because of the diffusion term $\frac{1}{\mathrm{Re}}\frac{\partial^2}{\partial x^2} z(t, x)$. (ii) The control input function $b(x) = e^x$ is defined for all $x \in [0, 1]$. Thus, the feedback control acts on the whole domain $[0, 1]$. However, one can choose any L^2-function $b(\cdot) \in L^2(0, 1)$ satisfying the stabilizability condition in Lemma 3.7. In fact, $b(x) = e^x$ satisfies the stabilizability condition for any desired degree of stability $\alpha > 0$, since the coefficients b_n, $n = 1, 2, \ldots$, representing input function $b(\cdot)$ are not zero, i.e. $b_n = \langle b(\cdot), \sin \pi x\rangle_{L^2(0,1)} = \int_0^1 e^x \sin \pi x\,\mathrm{d}x \neq 0$ for all $n = 1, 2, \ldots$, (see Lemma 3.7). (iii) The initial function $z_0(x) = \sin \pi x$ is chosen for our numerical experiments. Other

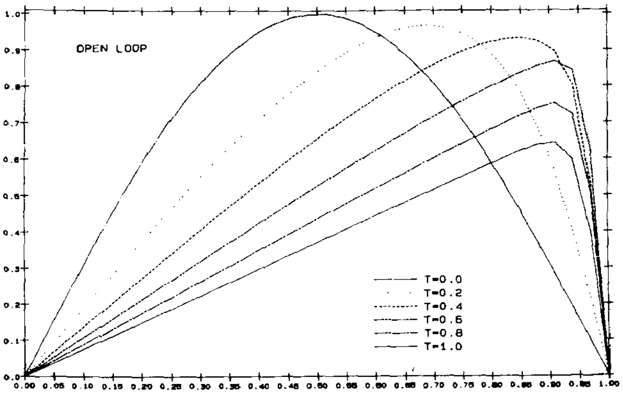

Fig. 1. Open loop ($N = 32$, $\mathrm{Re} = 60$, $z_0(x) = \sin \pi x$).

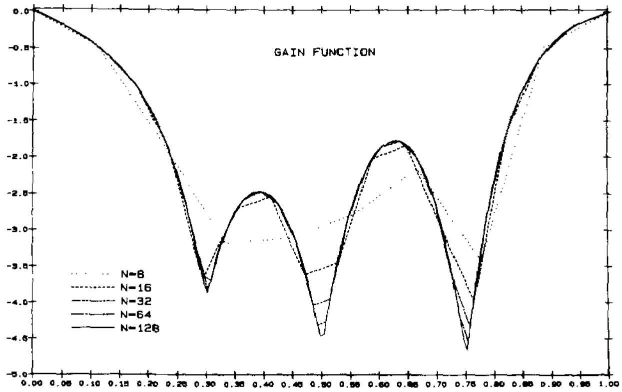

Fig. 2. Gain function $k^1_{\epsilon,\varphi}(\cdot)$ ($\alpha = 0.3$, $C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75))$).

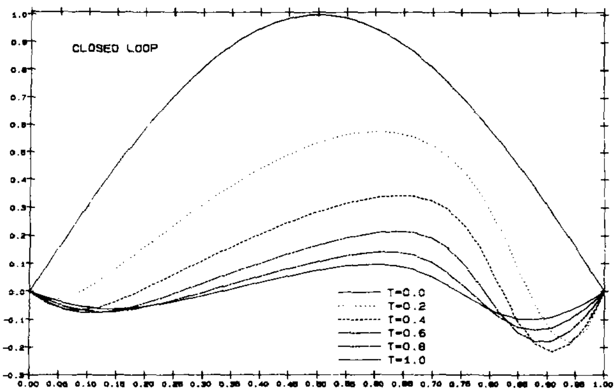

Fig. 3. Closed loop (feedback by $k^1_{\alpha,\epsilon}(\cdot)$).

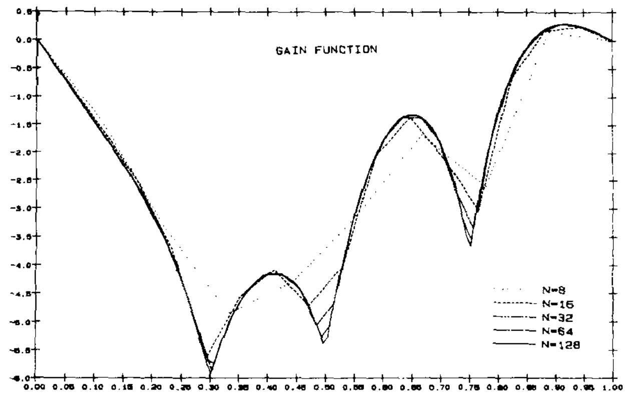

Fig. 4. Gain function $k^2_{\alpha,\epsilon}(\cdot)$ ($\alpha = 0.6$, $C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75))$).

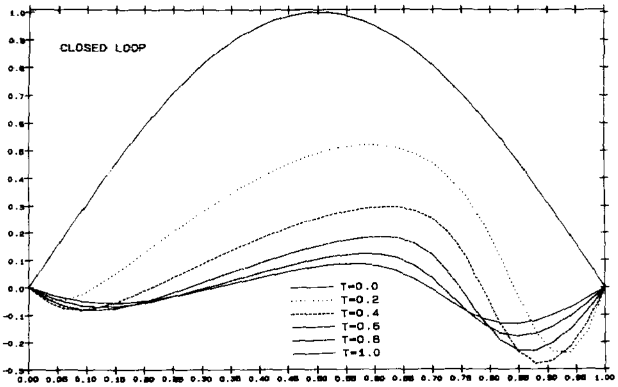

Fig. 5. Closed loop (feedback by $k^2_{\alpha,\epsilon}(\cdot)$).

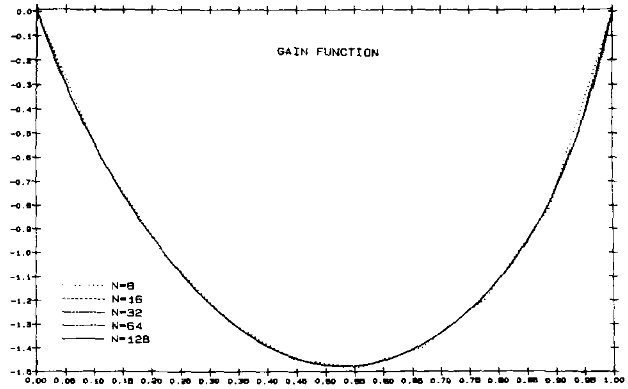

Fig. 6. Gain function $k^3_{\alpha,\epsilon}$ ($\alpha = 0.3$, $C = I$).

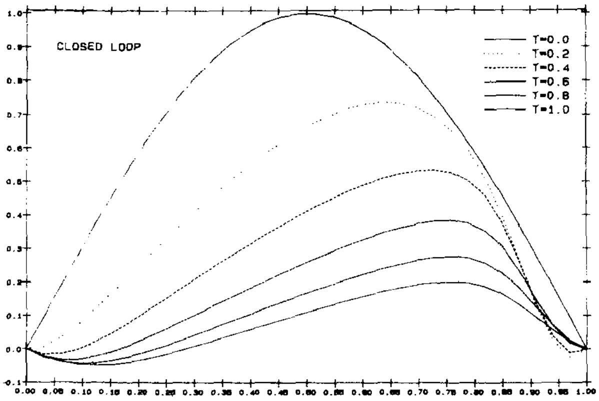

Fig. 7. Closed loop (feedback by $k^3_{\alpha,\epsilon}(\cdot)$).

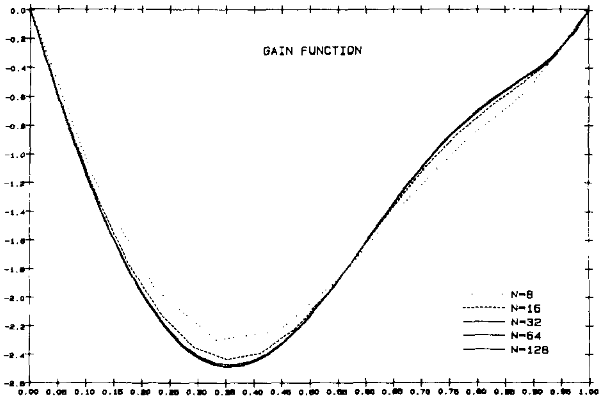

Fig. 8. Gain function $k^4_{\alpha,\epsilon}(\cdot)$ ($\alpha = 0.6$, $C = I$).

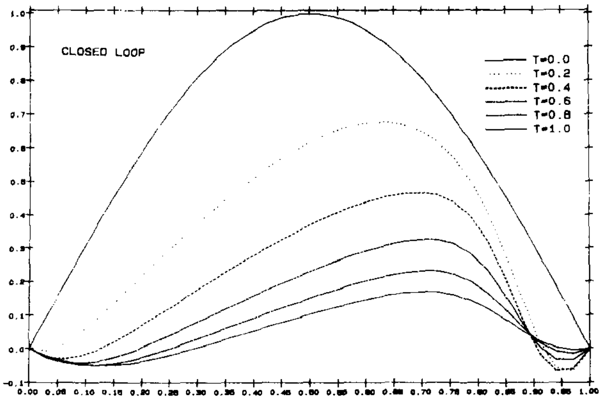

Fig. 9. Closed loop (feedback by $k^4_{\alpha,\epsilon}(\cdot)$).

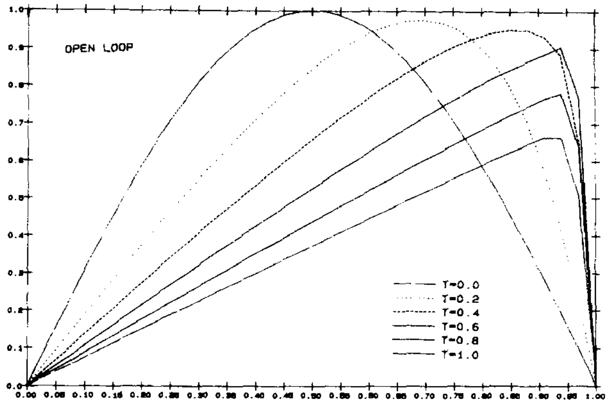

Fig. 10. Open loop ($N = 32$, Re $= 80$, $z_0(x) = \sin \pi x$).

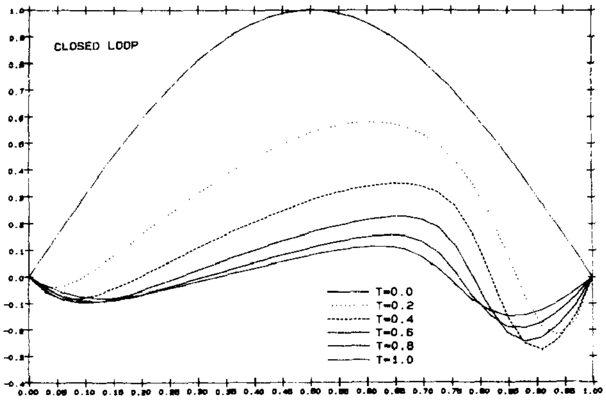

Fig. 11. Closed loop (Re $= 80$, feedback by $k^1_{\alpha,\epsilon}(\cdot)$).

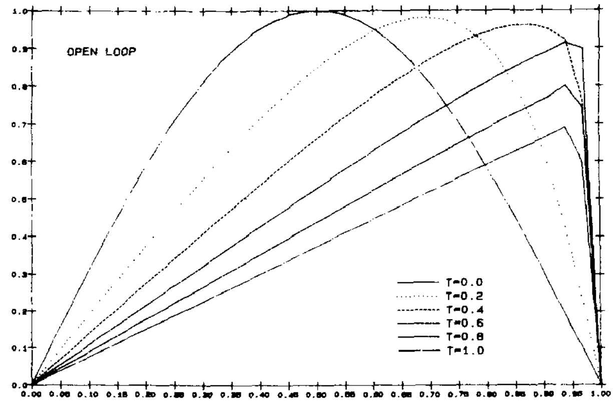

Fig. 12. Open loop ($N = 32$, Re $= 100$, $z_0(x) = \sin \pi x$).

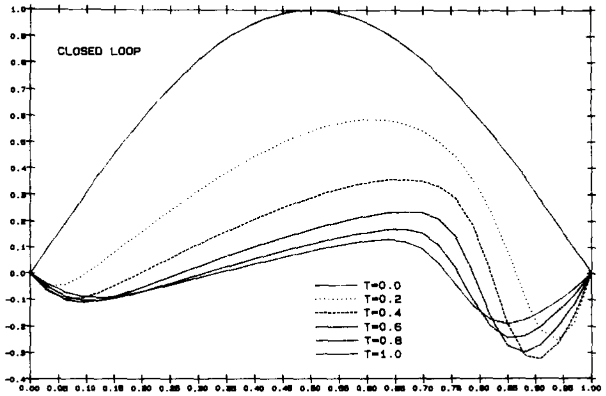

Fig. 13. Closed loop (Re = 100, feedback by $k^{1}_{\alpha,\epsilon}(\cdot)$).

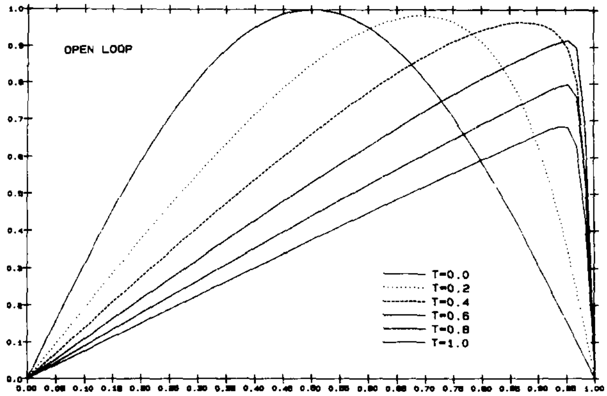

Fig. 14. Open loop ($N = 64$, Re = 120, $z_0(x) = \sin \pi x$).

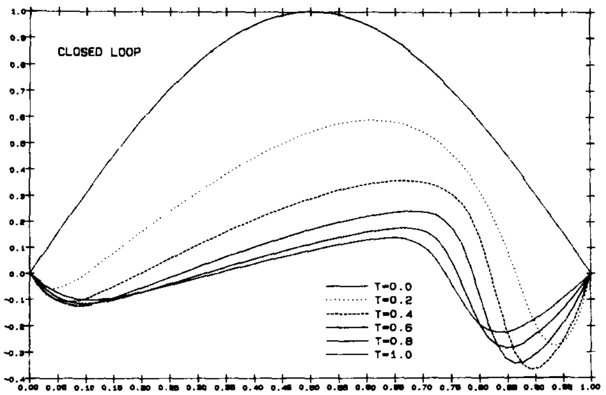

Fig. 15. Closed loop ($N = 64$, Re = 120, feedback by $k^{1}_{\alpha,\epsilon}(\cdot)$).

typical H_0^1-functions such as the 'hat function' defined by

$$z_0(x) = \begin{cases} 2x\,, & x \in [0, \frac{1}{2}] \\ -2x + 2\,, & x \in [\frac{1}{2}, 1] \end{cases} \tag{105}$$

were used for initial data. However, we found that the solution of Burgers' equation (104) with initial data $z_0(x)$ given by equation (105) is similar to the solution with initial data $z_0(x) = \sin \pi x$.

To illustrate the trajectories of open-loop and closed-loop solutions, the order N of approximation is chosen as $N = 32$ for both cases, and the corresponding trajectories from time $t = 0.0$ to $t = 1.0$ will be shown. The convergence of the feedback gain functions $k_{\alpha,\epsilon}(\cdot) \in L^2(0, 1)$ will be illustrated by selecting values of $N = 8$, 16, 32, 64 and 128.

EXAMPLE 1 (Bounded Input/Output). The observation operator $C \in \mathscr{L}(L^2(0, 1), \mathbf{R}^3)$ for this example is given by

$$C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75))\,, \tag{106}$$

where $\tilde{z}(\bar{x})$ is the average value of $z(\cdot) \in L^2(0, 1)$ in a small neighborhood of $\bar{x}$, $\bar{x} = 0.3, 0.5, 0.75$, and defined by equation (42), $\tilde{z}(\bar{x}) = \frac{1}{2\delta} \int_{\bar{x}-\delta}^{\bar{x}+\delta} z(s)\, ds$. Here, $\delta > 0$ is chosen so small that each open interval $(\bar{x} - \delta, \bar{x} + \delta)$ is contained in the whole domain $(0, 1)$. The desired degree α of stability is chosen to be 0.3 and 0.6 for Figures 3 and 5, respectively. For both cases, $n_\alpha = \max\{n \in \mathbf{N} : \alpha - \frac{1}{60} n^2 \pi^2 \geq 0\} = 1$ and hence the set $X_{\alpha,1}$ defined in Lemma 3.7 is empty. Thus, all assumption in Theorem 3.9 are satisfied.

The feedback gain functions $k_{\alpha,\epsilon}(\cdot)$ are shown in Figures 2 and 4. From these plots, it is easy to see that control action depends on the location of the points $\bar{x}_i$. This phenomenon is natural, since the optimal control is obtained to minimize the cost functional J defined by equation (47) whose first term $\|\hat{y}(t)\|_Y^2 = \|C\hat{z}(t)\|_{\mathbf{R}^3}^2 = \Sigma_{i=1}^3 |\tilde{\hat{z}}(\bar{x}_i)|^2$, where $\bar{x}_i = 0.3$, 0.5 and 0.75 for $i = 1, 2$ and 3, respectively. The corresponding closed-loop trajectories are shown in Figure 3 (for $\alpha = 0.3$) and Figure 5 (for $\alpha = 0.6$). From Figures 3 and 5, we can see how the controllers contribute to stabilization of the steep gradient as well as the solution itself.

EXAMPLE 2 (Identity Output Operator). For this example, we take the identity operator I on $L^2(0, 1)$ for the output operator C. In this case, the output space Y is $L^2(0, 1)$. The convergence of gain functions and corresponding closed-loop trajectories for $\alpha = 0.3$ and 0.6 is shown in Figures 6–7 and 8–9, respectively. Since the observation operator is the identity, this example gives information about the maximal control action. We note the following observation concerning the convergence rate of gain function. Theoretically, the rate is $O(\frac{1}{N})$ [16, p. 15]. But, in this example, the rate seems to be faster than $O(\frac{1}{N})$, (see Figures 2 and 4). Another observation is concerned with the location of maximal control action. The location moves to the left portion of domain as the degree of stability $\alpha > 0$ increases. In other words, we should put more action on the front part of domain to get a higher exponential decay rate $\alpha > 0$. (See also Figures 2 and 4).

EXAMPLE 3 (Robustness). In this example we show the robustness of the feedback controller given in Figure 2. The feedback controller is obtained from the control system with $\mathrm{Re} = 60$, $\alpha = 0.3$, $b(x) = e^x$ and the output operator C defined by equation (106).

Figures 1, 10, 12 and 14 show open-loop trajectories for Reynolds numbers 60, 80, 100 and 120, respectively. The corresponding closed-loop trajectories are shown in Figures 3, 11, 13 and 15. The order N of approximation is chosen as $N = 32$ for $\mathrm{Re} = 60$, 80, 100 and $N = 64$ for $\mathrm{Re} = 120$. From these examples, it is easy to see that the feedback controller obtained for $\mathrm{Re} = 60$ stabilizes the steep gradient of the solution for Burgers' equation with various Reynolds numbers. However, we see that the sharp gradient is relaxed slowly as the Reynolds number increases, (see Figures 3, 11, 13 and 15). Recall that Theorem 3.9 implies that the closed-loop solution $z(t)$ of the nonlinear system (64) satisfies the stability property

$$\|z(t)\|_{H_0^1} \leq 2Me^{-\alpha t}\|z_0(\cdot)\|_{H_0^1} . \tag{107}$$

Although the exponential decay rate α is independent of Reynolds number, the constant $M = M(\alpha, \mathrm{Re})$ depends strongly on the Reynolds number.

5. Conclusion

In this paper, we considered a feedback control problem for a nonlinear Burgers' equation. The method consists of linearization of the nonlinear equation. We used the linear quadratic regulator (LQR) problem to find optimal feedback gains. It was also proved that, under appropriate selection of the input and the output functions, the LQR problem for the linearized problem is detectable and stabilizable. We then analyzed a 'shifted quadratic cost' to construct gains which produce a fixed decay rate. In particular, we proved that the closed-loop system satisfies the inequality

$$\|z(t, 0; z_0)\|_{H_0^1} \leq M(\epsilon)e^{-\alpha t}\|z_0\|_{H_0^1} ,$$

where $\alpha > 0$ does not depend on the Reynolds number, but $M(\epsilon)$ does (see Theorem 3.9).

We also developed a numerical scheme for computing the feedback gain functions. Several numerical experiments were performed and the following observations were made:

(1) The gain functions depend strongly on output map. For example, if the output operator C is given by

$$C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75)) ,$$

then the gain function has sharp peaks at the locations 0.3, 0.5 and 0.75 (see Figures 2 and 4).

(2) The closed-loop nonlinear system is stabilized (as predicted) by linear feedback laws. Moreover, the steep gradients (for $\epsilon \approx 0$) are smoothed by feedback.

(3) To test the 'robustness' of the feedback control law, one experiment was performed. We obtained the functional gain $k^1_{\alpha,\epsilon}(\cdot)$ from the control system at the Reynolds number, $\mathrm{Re} = 60$, and applied it to the closed-loop system

$$\frac{\partial}{\partial t} z(t, x) = \frac{1}{\mathrm{Re}} \frac{\partial^2}{\partial x^2} z(t, x) - z(t, x) \frac{\partial}{\partial x} z(t, x) + e^x \int_0^1 k^1_{\alpha,\epsilon}(s) z(t, s)\, \mathrm{d}s \tag{108}$$

at $\mathrm{Re} = 80$, 100 and 120. The closed-loop responses are shown in Figures 11, 13 and 15. Although the performance was decreased, the system (108) was still stabilized and smoothed. These results provided some insight into the possibility of using linear feedback laws for nonlinear distributed parameter systems.

In closing, we note that considerable physical insight can be gained from the numerical results presented above. For example, the shape of the optimal functional gains (Figures 2, 4, 6, 8) provides information about the spatial location of sensors and actuators. Note that the maximum absolute value of the functional gain moves to the left as the desired damping rate $\alpha > 0$ is increased. One can conclude that 'state' sensors should be physically located near the place where this absolute value is maximized.

Acknowledgements

This work was supported in part by the Air Force Office of Scientific Research under grant AFOSR-89-0001 and the Defense Advanced Research Project Agency under contracts F49620-87-C-0116 and N00014-88-K-0721, and by the National Aeronautics and Space Administration under NASA contract No. NAS1-18605 while the first author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

References

1. Anderson, B. D. O. and Moore, J. B., 'Linear system optimisation with prescribed degree of stability', *Proc. IEEE* **116**, 1969, 2083–2087.
2. Banks, H. T. and Kunish, K., 'The linear regulator problem for parabolic systems', *SIAM J. Control and Optim.* **22**, 1984, 684–698.
3. Burgers, J. M., 'Mathematical examples illustrating relations occuring in the theory of turbulent fluid motion', *Trans. Roy. Neth. Acad. Sci.* **17**, Amsterdam, 1939, 1–53.
4. Burgers, J. M., 'A mathematical model illustrating the theory of turbulence', *Adv. in Appl. Mech.* **1**, 1948, 171–199.
5. Burgers, J. M., 'Statistical problems connected with asymptotic solution of one-dimensional nonlinear diffusion equation', in M. Rosenblatt and C. van Atta (eds.), *Statistical Models and Turbulence*, Springer, Berlin, 1972, 41–60.
6. Chen, G., Wang, H. K. and Weerakoon, S., 'An initial value control problem for Burgers' equation', in F. Kappel, K. Kunish and W. Schappacher (eds.), *Distributed Parameter Systems*, Lecture Notes in Control and Information Sciences **75**, 1985, 52–76.
7. Cole, J. D., 'On a quasi-linear parabolic equation occuring in aerodynamics', *Quart. Appl. Math.* **IX**, 1951, 225–236.
8. Conway, J. B., *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.
9. Curtain, R. F., 'Stability of semilinear evolution equations in Hilbert space', *J. Math. Pures et Appl.* **63**, 1984, 121–128.
10. Fletcher, C. A. J., 'Burgers' equation: a model for all reasons', in J. Noye (ed.), *Numerical Solution of Partial Differential Equations*, North-Holland, 1982, 139–225.
11. Gibson, J. S., 'The Riccati integral equations for optimal control problems on Hilbert spaces', *SIAM J. Control and Optim.* **17**, 1979, 537–565.
12. Gibson, J. S. and Rosen, I. G., 'Shifting the closed-loop spectrum in the optimal linear quadratic regulator problem for hereditary systems', Institute for Computer Applications for Science and Engineering, ICASE Report **86–16**, 1986, NASA Langley Research Center, Hampton, VA.
13. Glimm, J. and Lax, P., *Decay of Solutions of Systems of Nonlinear Hyperbolic Conservation Laws*, Amer. Math. Soc. Memoir **101**, A.M.S., Providence, 1970.
14. Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, New York, 1981.
15. Hopf, E., 'The partial differential equation $u_t + uu_x = \mu u_{xx}$', *Comm. Pure and Appl. Math* **3**, 1950, 201–230.
16. Ito, K., 'Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces', LCDS/CCS Report **87–15**, Brown University, 1987.

17. Kielhöfer, H., 'Stability and semilinear evolution equations in Hilbert space', *Arch. Rat. Mech. Anal.* **57**, 1974, 150–165.
18. Lasiecka, I., 'Unified theory for abstract parabolic boundary problems – a semigroup appoach', *Appl. Math. Optim.* **6**, 1980, 287–333.
19. Lax, P. D., *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMS-NSF Regional Conference Series in Applied Mathematics **11**, SIAM, 1973.
20. Lighthill, M. J., 'Viscosity effects in sound waves of finite amplitude', in G. K. Bachelor and R. M. Davies (eds.), *Surveys in Mechanics*, Cambridge University Press, Cambridge, 1956, 250–351.
21. Maslov, V. P., 'On a new principle of superposition for optimization problems', *Uspekhi Mat. Nauk* **42**, 1987, 39–48.
22. Maslov, V. P., 'A new approach to generalized solutions of nonlinear systems', *Soviet Math. Dokl* **35**, 1987, 29–33.
23. Oleĭnik, O. A., 'Discontinuous solutions of nonlinear differential equations', *Uspsekhi Math. Nauk* **12**, 1957, 3–73 (English Translation *Amer. Math. Soc. Trans.*, Series 2, **26**, 95–172).
24. Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
25. Pritchard, A. J. and Salamom, D., 'The linear quadratic control problem for infinite dimensional systems with unbounded input and output operators', *SIAM J. Control and Optim.* **25**, 1987, 121–144.
26. Rankin, S. M., 'Semilinear evolution equations in Banach spaces with application to parabolic partial differential equations', Preprint, 1989.
27. Russell, D. L., *Mathematics of Finite-Dimensional Control Systems: Theory and Design*, Marcel Dekker Inc., 1979.
28. Schultz, M. H., *Spline Analysis*, Prentice-Hall, N.J., 1973.
29. Smoller, J., *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, 1983.
30. Stoer, J. and Bulirsch, R., *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980.
31. Walker, J. A., *Dynamical Systems and Evolution Equations, Theory and Applications*, Plenum Press, New York, 1980.
32. Weerakoon, S., 'An initial value control problem for the Burgers equation', Ph.D. Thesis, Department of Mathematics, Pennsylvania State University, December, 1984.
33. Wloka, J., *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.