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Analytical Solution of Attitude Motion for Spacecraft with a Slewing Appendage

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Abstract. This paper investigates pitch motion and in orbital plane elastic vibration of a spacecraft with a flexible beam type appendage undergoing prescribed slew maneuver. The governing equations are transformed into a standard quasi-linear form, and then solved by Butenin's variation of parameters approach. Validity of the analytical solutions is assessed over a range of system parameters and initial conditions by comparing them with the results of numerical integration. The results show that they are very good approximations and provide extensive insight into the dynamical response of the system.

Key words: Spacecraft dynamics, nonlinear ordinary differential equations, approximate analytical solutions, variation of parameters.

1. Introduction

During various spacecraft missions, slew maneuvers are involved for reorienting directional antennas, solar panels for optimum production of power, telescopes aiming at distant galaxies, and other scientific instruments. Manipulators or robots mounted on the spacecraft have slew maneuvers, too. Slewing motion is likely to cause elastic vibrations and to disturb librational motion. Therefore the dynamics of libration and elastic vibration during slewing of appendages have received considerable attention. Turner and Junkins studied the large angle, single axis rotational maneuver of flexible spacecraft [1]. Mah and Modi [2] investigated the dynamical response during slewing and translational maneuvers of the Space Station based MRMS. Bainum and Li, studied the rapid in-plane maneuver of the flexible orbiting SCOLE [3]. Meirovitch and associates, have studied the retarget maneuver and control of flexible spacecraft [4–8].

As can be expected, the governing equations of such complex systems are nonautonomous, highly nonlinear and coupled and they do not admit of any known closed form solution. They are thus generally solved numerically. Approximate analytical solutions, if able to capture the essence of the problem, often provide better physical appreciation and insight into the system behavior. It can thus complement numerical analysis of complex problems in a useful way. To that end, Modi *et al.* have studied dynamics of spacecraft with two beam-type appendages [9, 10] using the K–B [11] and Butenin's method [12]. Modi and Misra [13] have presented an approximate analytical solution for a tethered satellite system during deployment and retrieval. Kalaycioglu and Misra [14] obtained the approximate analytical solution of flexible appendages during deployment too.

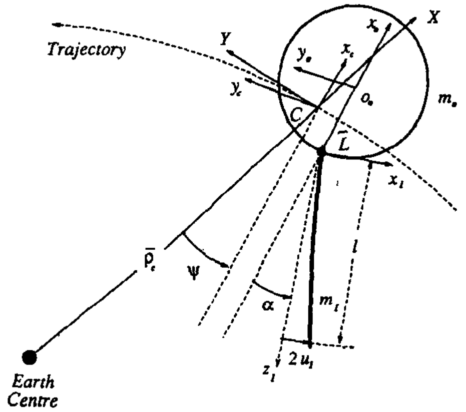

Fig. 1. A satellite with a beam-type slewing appendage undergoing planar motion.

This paper investigates dynamics of spacecraft with a beam-type appendage free to undergo librational and vibrational motions in the orbital plane during a prescribed slew maneuver. The governing equations are first transformed to conform to the standard nonlinear form and then solved by Butenin's variation of parameters method. Validity of the analytical solutions is assessed over a range of system parameters and initial conditions by comparing it with the results obtained by numerical integration of the exact equations of motion and the results show good correlation.

2. System Description

A spacecraft consisting of a tri-axial rigid central body and a beam-type appendage having prescribed slewing maneuver, as shown in Figure 1, is investigated. It has the following features:

- A circular orbit around the earth.
- The mass and inertia of the beam is much smaller than that of the central body.
- The beam is attached to one of the principal axes at the mass centre of the central body by a pin hinge permitting prescribed slew maneuver about it. The beam is much wider in the pin direction than in the direction normal to the pin, thus the elastic vibration is likely to occur in the same plane of the slew.
- The scope of this investigation includes pitch motion and elastic vibration of the beam in the orbital plane. Yaw and roll are assumed to be quiescent initially.

The inertial coordinate system $O - X_o, Y_o, Z_o$ has its origin at the earth centre. An orbiting frame $C - X, Y, Z$ is located at the instantaneous centre of mass of the system with the X-axis along the local vertical, the Y-axis along the local horizontal, and the Z-axis perpendicular to the orbital plane. There are also body fixed frames $o_i - x_i, y_i, z_i$ $(i = 0, 1)$ with their origins at the center of mass of the central body and at the pin position for the appendage. At the instantaneous mass centre of the spacecraft, 'system frame' $C - x_c, y_c, z_c$ is located with x_c, y_c, z_c parallel to x_o, y_o and z_o respectively.

The librational motion of the spacecraft is described by a set of three orientation angles λ, ϕ, ψ, which defines motion of the system frame $C - x_c, y_c, z_c$ w.r.t. the orbital frame $C - X$, Y, Z. Modified Euler angles (Bryant Angles), as shown in Figure 2, are used in this study.

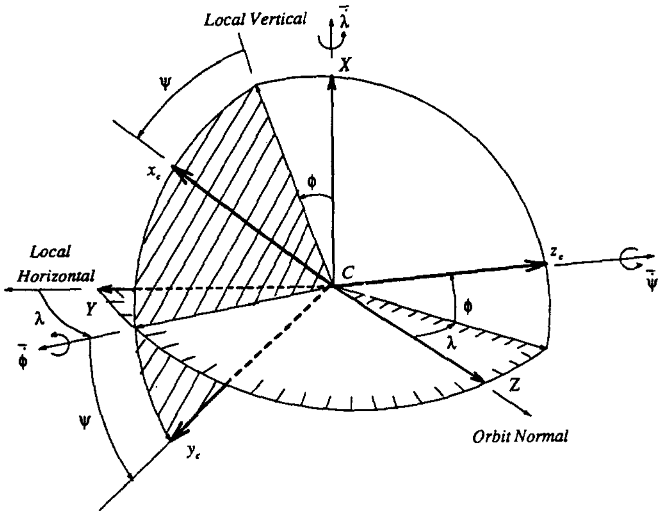

Fig. 2. Modified Euler angles (Bryant Angles).

Since only the planar motion, i.e. the motion of the spacecraft occurring in the orbital plane, is investigated, λ and ϕ as well as their derivatives are assumed to be zero; therefore ψ is about the orbit normal, and represents the pitch.

As for the elastic vibration of the beam, only the transverse vibration is considered, the longitudinal and torsional vibrations as well as foreshortening effects are purposely neglected. Further complication of the problem, it is felt, will mask appreciation of the influence of slew maneuver on pitch and elastic vibration which is the main objective of this investigation.

The vibratory displacement of a mass element on the beam is represented by admissible modal functions associated with time-varying generalized coordinates,

$$\bar{u} = \sum_{j=1}^{\infty} \bar{\Psi}_j(\bar{x}) u_j(t). \tag{1}$$

Since the mass of the central body is much larger than that of the appendage, it is reasonable to use the modes of a cantilever beam to represent the deformation of the appendage. The vibrational response in the second mode is at least two orders of magnitude smaller than that of the fundamental mode; for the same reason as mentioned above, only the first mode of a cantilever beam is taken:

$$\Psi_1(\xi) = \cosh(\lambda_1\xi) - \cos(\lambda_1\xi) - \sigma_1\{\sinh(\lambda_1\xi) - \sin(\lambda_1\xi)\} \quad (0 \le \xi \le 1) \tag{2}$$

with $\lambda_1 = 1.8751$ $\sigma_1 = 0.7341$.

3. The Equations of Motion

The governing equations are derived from the Lagrange equation:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i, \tag{3}$$

where T and V are the kinetic and potential energy respectively; q_i represents a generalized coordinate with $q_1 = \lambda$, $q_2 = \phi$, $q_3 = \psi$, and $q_4 = u(t)$, with $u(t)$ representing the transverse displacement of the beam. Q_i is the corresponding generalized force.

The kinetic energy of the system takes the following form:

$$T = \frac{1}{2}M\dot{\bar{\rho}}_c \bullet \dot{\bar{\rho}}_c + \frac{1}{2}(\Omega\bar{k} + \bar{\omega}_o)\bullet \bar{\bar{J}} \bullet(\Omega\bar{k} + \bar{\omega}_o)$$
$$+ (\Omega\bar{k} + \bar{\omega}_o) \bullet \bar{G} + \frac{1}{2}\left(\int \mathring{\bar{r}}_1 \bullet \mathring{\bar{r}}_1 \, dm_1 - M \mathring{\bar{R}}_o \bullet \mathring{\bar{R}}_o\right), \tag{4}$$

where M is the total mass of the spacecraft; $\bar{\bar{J}}$, its instantaneous inertia tensor and $\bar{G}$ its relative angular momentum w.r.t. the system frame $C - x_c, y_c, z_c$ due to slew and transverse vibration of the beam. $\Omega\bar{k}$ is the angular velocity of the orbital motion of the spacecraft; and $\bar{\omega}_o$, the angular velocity of libration of the central body. The notation $(\circ)$ implies time rate of change w.r.t. the system frame. Noting that $\bar{R}_o$ is the vector from the mass center of the spacecraft to that of the central body, it is understandable that $\mathring{\bar{R}}_o$ describes movement of the center of mass of the spacecraft due to its deformation.

In expression (4), the first term represents the kinetic energy contribution due to motion of the spacecraft as a point mass. The second term is the kinetic energy of the rotational motion of the spacecraft as a rigid body about its mass center C. The last term is that due to relative motions, including slew and vibration. The third term represents the kinetic energy due to coupling between the rotational motion and relative motions.

The potential energy of the system has two contributions: gravitational potential energy; and elastic strain energy. The gravitational potential energy is given by:

$$V_g = -\frac{KM}{\rho_c} - \frac{K}{2\rho_c^3}(tr.\ \bar{\bar{J}} - 3\bar{i} \bullet \bar{\bar{J}} \bullet \bar{i}), \tag{5}$$

where 'K' is the gravitational constant of the earth; $\bar{i}$, the unit vector along $\bar{\rho}_c$.

The strain energy of transverse vibration of a beam is:

$$V_s = \frac{E}{2}\int \left\{ I_{yy}\left(\frac{\partial^2 u_x}{\partial z^2}\right)^2 + I_{zz}\left(\frac{\partial^2 u_y}{\partial z^2}\right)^2 \right\} dz, \tag{6}$$

where EI_{xx}, EI_{yy} are the bending rigidity of the beam about the y and x axis, respectively. For the particular case under consideration, the strain energy takes the form

$$V_s = \frac{1}{2}ku^2, \tag{7}$$

with

$$k = EI_{yy}\lambda_1^4/l \int_0^1 (\Psi_1''(\xi))^2 \, d\xi, \tag{8}$$

and u is the ratio of half the tip deflection over the length of the beam.

Using Lagrange's equation and differentiating the kinetic energy and potential energy, and converting the independent variable from time to the true anomaly according to the following relations:

$$\frac{d}{dt} = \Omega\frac{d}{d\theta}; \qquad \frac{d^2}{dt^2} = \Omega^2\frac{d^2}{d\theta^2}, \tag{9}$$

the governing equations during slew are obtained as follows:

$$\lambda' = 0 \tag{10}$$

$$\phi' = 0 \tag{11}$$

$$\begin{aligned}
&\left\{\tilde{J}_{03} + \epsilon\epsilon_3\epsilon_l^2 + \frac{\epsilon}{1+\epsilon}(1 + \epsilon_l \cos\alpha - \epsilon_l\beta_1 u \sin\alpha)\right\}\psi'' + \frac{\epsilon\epsilon_l}{2}\left\{\epsilon_l\beta_2 + \frac{\beta_1(1-\epsilon\epsilon_l}{1+\epsilon}\right\}u'' \\
&+ \epsilon\epsilon_l\left\{\epsilon_3\epsilon_l - \frac{1}{2(1+\epsilon)}(\beta_1 u \sin\alpha - \cos\alpha)\right\}\alpha'' \\
&- \frac{\epsilon\epsilon_l}{1+\epsilon}\{\beta_1 \sin\alpha u' + (\beta_1\cos\alpha u + \sin\alpha)\alpha'\}\left(1 + \psi' + \frac{\alpha'}{2}\right) \\
&+ \frac{3}{2}\left\{\tilde{J}_{02} - \tilde{J}_{01} + \frac{\epsilon}{1+\epsilon}(1 + \epsilon_l\cos\alpha - \epsilon_l\beta_1 u \sin\alpha)\right. \\
&\left.+ \epsilon\epsilon_l^2(\epsilon_3\cos 2\alpha - \epsilon_4 \sin 2\alpha u)\right\}\sin 2\psi \\
&+ \frac{3}{2}\epsilon\epsilon_l\left\{\frac{1}{1+\epsilon}(\beta_1 u\cos\alpha + \sin\alpha) + \epsilon_l(\epsilon_3\sin 2\alpha + \epsilon_4\cos 2\alpha u)\right\}\cos 2\psi = 0
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\left\{\frac{\beta_1}{1+\epsilon} + \epsilon_l\left(\beta_2 - \frac{\epsilon\beta_1}{1+\epsilon}\right)\right\}\psi'' + \frac{\epsilon_l}{2}\left(1 - \frac{\epsilon\beta_1^2}{1+\epsilon}\right)u'' + \frac{\epsilon_l}{2}r_v^2 u \\
&+ \epsilon_l\epsilon_4\alpha'' + \frac{\beta_1}{1+\epsilon}\sin\alpha\{1.5 + \alpha' + \alpha'\psi' + 2\psi' + \psi'^2\} \\
&+ \frac{3}{2}\left\{\epsilon_l\epsilon_4\sin 2\alpha + \frac{\beta_1}{1+\epsilon}\sin\alpha\right\}\cos 2\psi \\
&+ \frac{3}{2}\left\{\epsilon_l\epsilon_4\cos 2\alpha + \frac{\beta_1}{1+\epsilon}\cos\alpha\right\}\sin 2\psi = 0,
\end{aligned} \tag{13}$$

where $\epsilon = m_1/m_0$; $\epsilon_l = l/L$; $r_v = \frac{1}{\Omega}\sqrt{k/m_1}$; $\tilde{J}_{0i} = J_{0i}/(m_0 L^2)$, $i = 1, 2, 3$; and

$$\epsilon_3 = \frac{1}{3} - \frac{\epsilon}{4(1+\epsilon)}; \qquad \epsilon_4 = \beta_2 - \frac{\epsilon\beta_1}{2(1+\epsilon)}$$

β_1 and β_2 are integrals of the shape function of equation (2):

$$\beta_1 = \int_0^1 \Psi_1(\xi)\, d\xi = 0.7830 \qquad \beta_2 = \int_0^1 \xi\Psi_1(\xi)\, d\xi = 0.5688.$$

Equations (10) and (11) show that yaw and roll are not excited by pitch and elastic vibration in the orbital plane if they are initially quiescent.

After slewing, since α is fixed at α_f, α' and α'' vanish, and the equations become simpler:

$$\begin{aligned}
&\left\{\tilde{J}_{03} + \epsilon\epsilon_3\epsilon_l^2 + \frac{\epsilon}{1+\epsilon}(1 + \epsilon_l\cos\alpha_f - \epsilon_l\beta_1 u\sin\alpha_f)\right\}\psi'' \\
&+ \frac{\epsilon\epsilon_l}{2}\left\{\epsilon_l\beta_2 + \frac{\beta_1(1-\epsilon\epsilon_l)}{1+\epsilon}\right\}u'' - \frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1\sin\alpha_f u'(1+\psi') \\
&+ \frac{3}{2}\left\{\tilde{J}_{02} - \tilde{J}_{01} + \frac{\epsilon}{1+\epsilon}(1 + \epsilon_l\cos\alpha_f - \epsilon_l\beta_1 u\sin\alpha_f)\right. \\
&\left.+ \epsilon\epsilon_l^2(\epsilon_3\cos 2\alpha_f - \epsilon_4\sin 2\alpha_f u)\right\}\sin 2\psi \\
&+ \frac{3}{2}\epsilon\epsilon_l\left\{\frac{1}{1+\epsilon}(\beta_1 u\cos\alpha_f + \sin\alpha_f) + \epsilon_l(\epsilon_3\sin 2\alpha_f + \epsilon_4\cos 2\alpha_f u)\right\}\cos 2\psi = 0
\end{aligned} \tag{14}$$

$$\left\{\frac{\beta_1}{1+\epsilon}+\epsilon_l\left(\beta_2-\frac{\epsilon\beta_1}{1+\epsilon}\right)\right\}\psi''+\frac{\epsilon_l}{2}\left(1-\frac{\epsilon\beta_1^2}{1+\epsilon}\right)u''$$
$$+\frac{\epsilon_l}{2}r_v^2u+\frac{\beta_1}{1+\epsilon}\sin\alpha_f\{1.5+2\psi'+\psi'^2\}$$
$$+\frac{3}{2}\left\{\epsilon_l\epsilon_4\sin 2\alpha_f+\frac{\beta_1}{1+\epsilon}\sin\alpha_f\right\}\cos 2\psi$$
$$+\frac{3}{2}\left\{\epsilon_l\epsilon_4\cos 2\alpha_f+\frac{\beta_1}{1+\epsilon}\cos\alpha_f\right\}\sin 2\psi=0. \tag{15}$$

4. Transformation of Equations

Equations (12–15) are non-autonomous, non-linear and coupled. The variation of parameter method is intended for differential equations with small nonlinearities and almost-constant coefficients, i.e.:

1. The dependent variables and their first derivatives are small;
2. The coefficients in the equations are either constants or vary slowly over a small range.

The source of the non-autonomous coefficients and terms in the equations of motion is the slew maneuver. The time history of the slew maneuver is taken to be sinusoidal, so that both the angular velocity and the angular acceleration vanish at the beginning and the end of the slew maneuver:

$$\alpha=\alpha_0+\frac{\Delta\alpha}{\Delta\theta}\left\{\theta-\frac{\Delta\theta}{2\pi}\sin\left(\frac{2\pi}{\Delta\theta}\theta\right)\right\}. \tag{16}$$

It can be seen by examining equations (12, 13) that, condition (2) is satisfied if the slewing range $\Delta\alpha$ and the slewing 'time' $\Delta\theta$ are not large. In this investigation, $\Delta\alpha$ is taken to be 30°, and $\Delta\theta$ is in the interval of 5° – 10°.

To satisfy the first condition, the dependent variables should vary around 'zero' equilibria. The equilibrium of ψ may not be zero because of the orientation change of the beam with respect to the central body. Hence it is assumed that: $\psi=\psi_0+\tilde{\psi}$, where ψ_0 is a constant angle representing the equilibrium and $\tilde{\psi}$ is the new variable replacing ψ. To determine ψ_0 during slewing, set the 'average gravitational moment' (with the small terms having 'u' ignored) to be zero. Then

$$\tan 2\psi_0=-\frac{\frac{\epsilon\epsilon_l}{1+\epsilon}\overline{\sin\alpha}+\epsilon\epsilon_l^2\epsilon_3\overline{\sin 2\alpha}}{\tilde{J}_{02}-\tilde{J}_{01}+\frac{\epsilon}{1+\epsilon}(1+\epsilon_l\overline{\cos\alpha})+\epsilon\epsilon_l^2\epsilon_3\overline{\cos 2\alpha}}, \tag{17}$$

where the overline represents average value of the function beneath it during slewing and are calculated by:

$$\overline{\sin\alpha} = \frac{1}{\Delta\alpha}\{\cos\alpha_0-\cos\alpha_f\} \tag{18}$$
$$\overline{\cos\alpha} = \frac{1}{\Delta\alpha}\{\sin\alpha_f-\sin\alpha_0\}. \tag{19}$$

To determine ψ_0 after slewing, formula (17) is still applicable, only that:

$$\overline{\sin\alpha}=\sin\alpha_f,\qquad \overline{\cos\alpha}=\cos\alpha_f. \tag{20}$$

Butenin's Method deals with equations of form:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \tilde{\psi}'' \\ u'' \end{bmatrix} + \begin{bmatrix} C_{13} & C_{14} \\ C_{23} & C_{24} \end{bmatrix} \begin{bmatrix} \tilde{\psi} \\ u \end{bmatrix}$$
$$= \Sigma_i \begin{bmatrix} D_{1i} \\ D_{2i} \end{bmatrix} \sin(\omega_i\theta + \gamma_i) + \begin{bmatrix} f(\theta, \tilde{\psi}, \tilde{\psi}', \tilde{\psi}'', u, u') \\ g(\theta, \tilde{\psi}, \tilde{\psi}', u, u') \end{bmatrix}. \tag{21}$$

In equation (21), all the entries in the inertia matrix and stiffness matrix are constants. The frequency ω_i, the initial phase γ_i and amplitude D_{1i}, D_{2i} of forcing terms are constants too. The functions f and g have small values. To solve equations (12–15), the first step is to transform them into the form of equation (21).

Equations (12) and (13) are transformed first.

The entries of the inertia matrix are easy to obtain except for C_{11}. To get a close value for C_{11}, the term involving 'u' in the coefficient of $\psi''(=\tilde{\psi}'')$ is put into the function f; and the average value of $\cos\alpha$ is taken. Therefore,

$$\begin{aligned}
C_{11} &= \tilde{J}_{03} + \epsilon\epsilon_3\epsilon_l^2 + \frac{\epsilon}{1+\epsilon}(1 + \epsilon_l\overline{\cos\alpha}) \\
C_{12} &= \frac{\epsilon\epsilon_l}{2}\left\{\epsilon_l\beta_2 + \frac{\beta_1(1-\epsilon\epsilon_l)}{1+\epsilon}\right\} \\
C_{21} &= \frac{\beta_1}{1+\epsilon} + \epsilon_l\left(\beta_2 - \frac{\epsilon}{l+\epsilon}\beta_1\right) \\
C_{22} &= \frac{1}{2}\epsilon_l\left(1 - \frac{\epsilon}{1+\epsilon}\beta_1^2\right).
\end{aligned} \tag{22}$$

Entries for the stiffness matrix are included in the terms involving $\sin 2\psi$ and $\cos 2\psi$, except C_{24}. Using the following formulas:

$$\begin{aligned}
\sin 2\psi &= \sin 2\psi_0 \cos 2\tilde{\psi} + \cos 2\psi_0 \sin 2\tilde{\psi} \\
\cos 2\psi &= \cos 2\psi_0 \cos 2\tilde{\psi} - \sin 2\psi_0 \sin 2\tilde{\psi}
\end{aligned} \tag{23}$$

and

$$\cos 2\tilde{\psi} \approx 1 - 2\tilde{\psi}^2 \qquad \sin 2\tilde{\psi} \approx 2\tilde{\psi} - \frac{4}{3}\tilde{\psi}^3 \tag{24}$$

terms involving $\sin 2\psi$ and $\cos 2\psi$ in equation (12) become:

$$\begin{aligned}
&\frac{3}{2}\left\{\left(\tilde{J}_{02} - \tilde{J}_{01} + \frac{\epsilon}{1+\epsilon}\right)\sin 2\psi_0 + \frac{\epsilon\epsilon_l}{1+\epsilon}\sin(2\psi_0 + \alpha)\right. \\
&\quad \left. + \epsilon\epsilon_l^2\epsilon_3 \sin 2(\psi_0 + \alpha)\right\}(1 - 2\tilde{\psi}^2) \\
&\quad + \frac{3}{2}\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1\cos(2\psi_0 + \alpha) + \epsilon\epsilon_l^2\epsilon_4 \cos 2(\psi_0 + \alpha)\right\}u(1 - 2\tilde{\psi}^2) \\
&\quad + 3\left\{\left(\tilde{J}_{01} - \tilde{J}_{03} + \frac{\epsilon}{1+\epsilon}\right)\cos 2\psi_0 + \frac{\epsilon\epsilon_l}{1+\epsilon}\cos(2\psi_0 + \alpha)\right. \\
&\quad \left. + \epsilon\epsilon_l^2\cos 2(\psi_0 + \alpha)\right\}\left(\tilde{\psi} - \frac{2}{3}\tilde{\psi}^3\right) \\
&\quad - 3\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1\sin(2\psi_0 + \alpha) + \epsilon\epsilon_l^2\epsilon_4 \sin 2(\psi_0 + \alpha)\right\}\tilde{\psi}u.
\end{aligned} \tag{25}$$

Terms involving $\sin 2\psi$ and $\cos 2\psi$ in equation (13) become:

$$\frac{3}{2}\left\{\frac{\beta_1}{1+\epsilon}\sin(2\psi_0+\alpha)+\epsilon_l\epsilon_4\sin 2(\psi_0+\alpha)\right\}(1-2\tilde{\psi}^2)$$
$$+3\left\{\frac{\beta_1}{1+\epsilon}\cos(2\psi_0+\alpha)+\epsilon_l\epsilon_4\cos 2(\psi_0+\alpha)\right\}\left(\tilde{\psi}-\frac{2}{3}\tilde{\psi}^3\right). \tag{26}$$

Thus, the stiffness constants C_{13}, C_{14}, and C_{23} are obtained as the coefficients of $\tilde{\psi}$ and u in equations (25) and (26). It is obtained that:

$$\begin{aligned}
C_{13} &= 3\left\{\left(\tilde{J}_{02}-\tilde{J}_{01}+\frac{\epsilon}{1+\epsilon}\right)\cos 2\psi_0+\frac{\epsilon\epsilon_l}{1+\epsilon}\overline{\cos(2\psi_0+\alpha)}+\epsilon\epsilon_l^2\epsilon_3\overline{\cos 2(\psi_0+\alpha)}\right\}\\
C_{14} &= \frac{3}{2}\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1\overline{\cos(2\psi_0+\alpha)}+\epsilon\epsilon_l^2\epsilon_4\overline{\cos 2(\psi_0+\alpha)}\right\}\\
C_{23} &= 3\left\{\frac{\beta_1}{1+\epsilon}\overline{\cos(2\psi_0+\alpha)}+\epsilon_l\epsilon_4\overline{\cos 2(\psi_0+\alpha)}\right\}\\
C_{24} &= 0.5\epsilon_l r_v^2.
\end{aligned} \tag{27}$$

For other parts in expressions (25), (26) and what is left in equations (12) and (13), terms without dependent variables are treated as forcing terms; nonlinear terms of $\tilde{\psi}$ and u as well as terms involving $\tilde{\psi}'$ and u' are included in the functions f and g.

Now determining forcing components. From equation (16),

$$\alpha'=\frac{\Delta\alpha}{\Delta\theta}\left\{1-\cos\left(\frac{2\pi}{\Delta\theta}\theta\right)\right\}\qquad \alpha''=\frac{2\pi\Delta\alpha}{\Delta\theta^2}\sin\left(\frac{2\pi}{\Delta\theta}\theta\right). \tag{28}$$

Since $\Delta\theta/2\pi \ll 1$, $\alpha \approx \alpha_0+(\Delta\alpha/\Delta\theta)\theta$; thus

$$\sin\alpha\approx\sin\left(\frac{\Delta\alpha}{\Delta\theta}\theta+\alpha_0\right)\qquad \cos\alpha\approx\cos\left(\frac{\Delta\alpha}{\Delta\theta}\theta+\alpha_0\right). \tag{29}$$

Collecting and arranging forcing terms in equations (12) and (13) and applying trigonometric formulas, seven forcing components are identified. Their frequencies, initial phases and amplitudes are:

$$\begin{aligned}
\omega_3 &= 0 & \gamma_3 &= 2\psi_0\\
\omega_4 &= \tfrac{\Delta\alpha}{\Delta\theta} & \gamma_4 &= \alpha_0\\
\omega_5 &= \tfrac{2\pi}{\Delta\theta} & \gamma_5 &= 0\\
\omega_6 &= \tfrac{\Delta\alpha}{\Delta\theta} & \gamma_6 &= 2\psi_0+\alpha_0\\
\omega_7 &= 2\tfrac{\Delta\alpha}{\Delta\theta} & \gamma_7 &= 2(\psi_0+\alpha_0)\\
\omega_8 &= \tfrac{2\pi+\Delta\alpha}{\Delta\theta} & \gamma_8 &= \alpha_0\\
\omega_9 &= \tfrac{2\pi-\Delta\alpha}{\Delta\theta} & \gamma_9 &= -\alpha_0
\end{aligned} \tag{30}$$

$$\begin{aligned}
D_{13} &= -\tfrac{3}{2}\left(\tilde{J}_{02}-\tilde{J}_{01}+\tfrac{\epsilon}{1+\epsilon}\right) & D_{23} &= 0\\
D_{14} &= \tfrac{\epsilon\epsilon_l}{1+\epsilon}\tfrac{\Delta\alpha}{\Delta\theta}\left(1+\tfrac{3}{4}\tfrac{\Delta\alpha}{\Delta\theta}\right) & D_{24} &= -\tfrac{\beta_1}{1+\epsilon}\left(\tfrac{3}{2}+\tfrac{\Delta\alpha}{\Delta\theta}\right)\\
D_{15} &= -\epsilon\epsilon_l^2\epsilon_3\tfrac{2\pi\Delta\alpha}{\Delta\theta^2} & D_{25} &= -\epsilon_l\epsilon_4\tfrac{2\pi\Delta\alpha}{\Delta\theta^2}\\
D_{16} &= -\tfrac{3}{2}\tfrac{\epsilon\epsilon_l}{1+\epsilon} & D_{26} &= -\tfrac{3}{2}\tfrac{\beta_1}{1+\epsilon}\\
D_{17} &= -\tfrac{3}{2}\epsilon\epsilon_l^2\epsilon_3 & D_{27} &= -\tfrac{3}{2}\epsilon_l\epsilon_4\\
D_{18} &= -\tfrac{1}{2}\tfrac{\epsilon\epsilon_l}{1+\epsilon}\tfrac{\Delta\alpha}{\Delta\theta}\left(1+\tfrac{\Delta\alpha}{\Delta\theta}+\tfrac{\pi}{\Delta\theta}\right) & D_{28} &= \tfrac{1}{2}\tfrac{\beta_1}{1+\epsilon}\tfrac{\Delta\alpha}{\Delta\theta}\\
D_{19} &= \tfrac{1}{2}\tfrac{\epsilon\epsilon_l}{1+\epsilon}\tfrac{\Delta\alpha}{\Delta\theta}\left(1+\tfrac{\Delta\alpha}{\Delta\theta}-\tfrac{\pi}{\Delta\theta}\right) & D_{29} &= -\tfrac{1}{2}\tfrac{\beta_1}{1+\epsilon}\tfrac{\Delta\alpha}{\Delta\theta}.
\end{aligned} \tag{31}$$

The frequencies $\Delta\alpha/\Delta\theta$ and $2\pi/\Delta\theta$ are attributed to the orientation change of the beam w.r.t. the central body and the inertia force respectively, caused by the slew maneuver. Other frequencies are combinations of them.

Finally, $f(\theta, \tilde{\psi}, \tilde{\psi}', \tilde{\psi}'', u, u')$ and $g(\theta, \tilde{\psi}, \tilde{\psi}', u, u')$ are:

$$\begin{aligned}
f &= 3\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1 \sin(2\psi_0+\alpha) + \epsilon\epsilon_l^2\epsilon_4 \sin 2(\psi_0+\alpha)\right\}\tilde{\psi}u \\
&+ 3\left\{\left(\tilde{J}_{01} - \tilde{J}_{03} + \frac{\epsilon}{1+\epsilon}\right)\sin 2\psi_0 + \frac{\epsilon\epsilon_l}{1+\epsilon}\sin(2\psi_0+\alpha) + \epsilon\epsilon_l^2\epsilon_3 \sin 2(\psi_0+\alpha)\right\}\tilde{\psi}^2 \\
&+ 3\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1 \cos(2\psi_0+\alpha) + \epsilon\epsilon_l^2\epsilon_4 \cos 2(\psi_0+\alpha)\right\}\tilde{\psi}^2 u \\
&+ 2\left\{\left(\tilde{J}_{02} - \tilde{J}_{01} + \frac{\epsilon}{1+\epsilon}\right)\cos 2\psi_0 + \frac{\epsilon\epsilon_l}{1+\epsilon}\cos(2\psi_0+\alpha) + \epsilon\epsilon_l^2\epsilon_3 \cos 2(\psi_0+\alpha)\right\}\tilde{\psi}^3 \\
&+ \frac{\epsilon}{1+\epsilon}\epsilon_l^2 \sin\alpha\alpha'\tilde{\psi}' + \frac{\epsilon}{1+\epsilon}\epsilon_l\beta_1\left\{\left(1+\frac{\alpha'}{2}\right)\cos\alpha\alpha' + \frac{1}{2}\sin\alpha\alpha'' + \sin\alpha\tilde{\psi}''\right\}u \\
&+ \frac{\epsilon}{1+\epsilon}\epsilon_l\beta_1\left(1+\frac{\alpha'}{2}+\tilde{\psi}'\right)\sin\alpha u' + \frac{\epsilon}{1+\epsilon}\epsilon_l\beta_1\cos\alpha\alpha'\tilde{\psi}'u
\end{aligned} \tag{32}$$

$$\begin{aligned}
g &= 3\left\{\frac{\beta_1}{1+\epsilon}\sin(2\psi_0+\alpha) + \epsilon_l\epsilon_4 \sin 2(\psi_0+\alpha)\right\}\tilde{\psi}^2 \\
&+ 2\left\{\frac{\beta_1}{1+\epsilon}\cos(2\psi_0+\alpha) + \epsilon_l\epsilon_4 \cos 2(\psi_0+\alpha)\right\}\tilde{\psi}^3 \\
&- \frac{\beta_1}{1+\epsilon}\sin\alpha(2\tilde{\psi}' + \alpha'\tilde{\psi}' + \tilde{\psi}'^2).
\end{aligned} \tag{33}$$

By the same method, equations (14) and (15) for motion after slewing are transformed into:

$$\begin{bmatrix} C_{11}^* & C_{12}^* \\ C_{21}^* & C_{22}^* \end{bmatrix}\begin{bmatrix} \tilde{\psi}'' \\ u'' \end{bmatrix} + \begin{bmatrix} C_{13}^* & C_{14}^* \\ C_{23}^* & C_{24}^* \end{bmatrix}\begin{bmatrix} \tilde{\psi} \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ D_{23}^* \end{bmatrix} + \begin{bmatrix} f^*(\theta, \tilde{\psi}, \tilde{\psi}', \tilde{\psi}'', u, u') \\ g^*(\theta, \tilde{\psi}, \tilde{\psi}', u, u') \end{bmatrix}, \tag{34}$$

where

$$\begin{aligned}
C_{11}^* &= \tilde{J}_{03} + \epsilon\epsilon_3\epsilon_l^2 + \frac{\epsilon}{1+\epsilon}\{1+\epsilon_l\cos\alpha_f\} \\
C_{12}^* &= C_{12} \quad C_{21}^* = C_{21} \quad C_{22}^* = C_{22}
\end{aligned} \tag{35}$$

$$\begin{aligned}
C_{13}^* &= 3\left\{\left(\tilde{J}_{02} - \tilde{J}_{01} + \frac{\epsilon}{1+\epsilon}\right)\cos 2\psi_0 + \frac{\epsilon\epsilon_l}{1+\epsilon}\cos(2\psi_0+\alpha_f) + \epsilon\epsilon_l^2\epsilon_3\cos 2(\psi_0+a_f)\right\} \\
C_{14}^* &= \frac{3}{2}\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1\cos(2\psi_0+\alpha_f) + \epsilon\epsilon_l^2\epsilon_4\cos 2(\psi_0+\alpha_f)\right\}
\end{aligned} \tag{36}$$

$$\begin{aligned}
C_{23}^* &= 3\left\{\frac{\beta_1}{1+\epsilon}\cos(2\psi_0+\alpha_f) + \epsilon_l\epsilon_4\cos 2(\psi_0+\alpha_f)\right\} \\
C_{24}^* &= 0.5\epsilon_l r_v^2 \\
D_{23}^* &= -\frac{3}{2}\left\{\frac{\beta_1}{1+\epsilon}\sin(2\psi_0+\alpha_f) + \epsilon_l\epsilon_4\sin 2(\psi_0+\alpha_f)\right\} - \frac{3\beta_1}{2(1+\epsilon)}\sin\alpha_f
\end{aligned} \tag{37}$$

$$
\begin{aligned}
f^* &= +3\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1\sin(2\psi_0+\alpha_f)+\epsilon\epsilon_l^2\epsilon_4\sin 2(\psi_0+\alpha_f)\right\}\tilde{\psi}u \\
&\quad+3\left\{\frac{\epsilon\epsilon_l}{1+\epsilon}\beta_1\cos(2\psi_0+\alpha_f)+\epsilon\epsilon_l^2\epsilon_4\cos 2(\psi_0+\alpha_f)\right\}\tilde{\psi}^2u \\
&\quad+2\left\{\left(\tilde{J}_{02}-\tilde{J}_{01}+\frac{\epsilon}{1+\epsilon}\right)\cos 2\psi_0\right. \\
&\quad\left.+\frac{\epsilon\epsilon_l}{1+\epsilon}\cos(2\psi_0+\alpha_f)+\epsilon\epsilon_l^2\epsilon_3\cos 2(\psi_0+\alpha_f)\right\}\tilde{\psi}^3 \\
&\quad+\frac{\epsilon}{1+\epsilon}\epsilon_l\beta_1\sin\alpha_f(u'+\tilde{\psi}'u'+\tilde{\psi}''u)
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
g^* &= 3\left\{\frac{\beta_1}{1+\epsilon}\sin(2\psi_0+\alpha_f)+\epsilon_l\epsilon_4\sin 2(\psi_0+\alpha_f)\right\}\tilde{\psi}^2 \\
&\quad+2\left\{\frac{\beta_1}{1+\epsilon}\cos(2\psi_0+\alpha_f)+\epsilon_l\epsilon_4\cos 2(\psi_0+\alpha_f)\right\}\tilde{\psi}^3 \\
&\quad-\frac{\beta_1}{1+\epsilon}\sin\alpha_f(2\tilde{\psi}'+\tilde{\psi}'^2).
\end{aligned}
\tag{39}
$$

5. Solution of Equations of Motion

The equations of motion during slewing are solved first. To do so, first solve the linearized equation with f and g ignored. The solution is:

$$
\begin{aligned}
\begin{bmatrix}\tilde{\psi}\\ u\end{bmatrix} &= A_p\begin{bmatrix}1\\ \alpha_v\end{bmatrix}\sin(\omega_p\theta+\gamma_p)+A_v\begin{bmatrix}\alpha_p\\ 1\end{bmatrix}\sin(\omega_v\theta+\gamma_v) \\
&\quad+\sum_{i=3}^{9}\begin{bmatrix}A_{1i}\\ A_{2i}\end{bmatrix}\sin(\omega_i\theta+\gamma_i),
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
\omega_p &= \left\{\frac{1}{2(C_{11}C_{22}-C_{12}C_{21})}\{C_{11}C_{24}+C_{13}C_{22}-C_{12}C_{23}-C_{14}C_{21}\right. \\
&\quad-\left\{(C_{12}C_{23}+C_{14}C_{21}-C_{11}C_{24}-C_{13}C_{22})^2\right. \\
&\quad\left.\left.-4(C_{11}C_{22}-C_{12}C_{21})(C_{13}C_{24}-C_{14}C_{23})\}^{0.5}\right\}\right\}^{1/2}
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\omega_v &= \left\{\frac{1}{2(C_{11}C_{22}-C_{12}C_{21})}\{C_{11}C_{24}+C_{13}C_{22}-C_{12}C_{23}-C_{14}C_{21}\right. \\
&\quad+\left\{(C_{12}C_{23}+C_{14}C_{21}-C_{11}C_{24}-C_{13}C_{22})^2\right. \\
&\quad\left.\left.-4(C_{11}C_{22}-C_{12}C_{21})(C_{13}C_{24}-C_{14}C_{23})\}^{0.5}\right\}\right\}^{1/2}
\end{aligned}
\tag{42}
$$

$$
\alpha_p = -\frac{C_{14}-\omega_v^2C_{12}}{C_{13}-\omega_v^2C_{11}}\qquad \alpha_v = -\frac{C_{23}-\omega_p^2C_{21}}{C_{24}-\omega_p^2C_{22}}
\tag{43}
$$

$$
A_{1i} = \frac{1}{\Delta_i}\det\begin{bmatrix}D_{1i} & C_{14}-\omega_i^2C_{12}\\ D_{2i} & C_{24}-\omega_i^2C_{22}\end{bmatrix}
\tag{44}
$$

$$
A_{2i} = \frac{1}{\Delta_i}\det\begin{bmatrix}C_{13}-\omega_i^2C_{11} & D_{1i}\\ C_{23}-\omega_i^2C_{21} & D_{2i}\end{bmatrix}
\tag{45}
$$

with

$$\Delta_i = \det\begin{bmatrix} C_{13} - \omega_i^2 C_{11} & C_{14} - \omega_i^2 C_{12} \\ C_{23} - \omega_i^2 C_{21} & C_{24} - \omega_i^2 C_{22} \end{bmatrix}.$$

As to the nonlinear system (21), it is assumed that the forced vibration remains the same, while the amplitudes and phases of free vibrations vary slowly with θ.

It is also assumed that $\tilde{\psi}'$ and u' take the form of the amplitudes and phases being constants. This assumption produces two equations for $A_p'(\theta)$, $A_v'(\theta)$ and $\gamma_p'(\theta)$, $\gamma_v'(\theta)$. Substituting expression (40) into equation (21) produces another two equations. These four algebraic linear equations are written as:

$$\begin{bmatrix} \sin\tau_1 & \cos\tau_1 & \alpha_p \sin\tau_2 & \alpha_p\cos\tau_2 \\ \alpha_v \sin\tau_1 & \alpha_v\cos\tau_1 & \sin\tau_2 & \cos\tau_2 \\ \omega_p\cos\tau_1 & -\omega_p\sin\tau_1 & \omega_v\alpha_p\cos\tau_2 & -\omega_v\alpha_p\sin\tau_2 \\ \omega_p\alpha_v\cos\tau_1 & -\omega_p\alpha_v\sin\tau_1 & \omega_v\cos\tau_2 & -\omega_v\sin\tau_2 \end{bmatrix} \begin{bmatrix} A_p'(\theta) \\ A_p(\theta)\gamma_p'(\theta) \\ A_v'(\theta) \\ A_v(\theta)\gamma_v'(\theta) \end{bmatrix} = \frac{1}{C_{11}C_{22} - C_{12}C_{21}} \begin{bmatrix} 0 \\ 0 \\ C_{22}f - C_{12}g \\ C_{11}g - C_{21}f \end{bmatrix} \tag{46}$$

with $\tau_1 = \omega_p\theta + \gamma_p(\theta)$; and $\tau_2 = \omega_v\theta + \gamma_v(\theta)$.

Solving equation (46);

$$\begin{aligned} A_p'(\theta) &= \frac{1}{\Delta\omega_p}\{(C_{22} + \alpha_p C_{21})f - (C_{12} + \alpha_p C_{11})g\}\cos\tau_1 \\ A_p(\theta)\gamma_p'(\theta) &= -\frac{1}{\Delta\omega_p}\{(C_{22} + \alpha_p C_{21})f - (C_{12} + \alpha_p C_{11})g\}\sin\tau_1 \\ A_v'(\theta) &= -\frac{1}{\Delta\omega_v}\{(C_{21} + \alpha_v C_{22})f - (C_{11} + \alpha_v C_{12})g\}\cos\tau_2 \\ A_v(\theta)\gamma_v'(\theta) &= \frac{1}{\Delta\omega_v}\{(C_{21} + \alpha_v C_{22})f - (C_{11} + \alpha_v C_{12})g\}\sin\tau_2 \end{aligned} \tag{47}$$

with

$$\Delta = (C_{11}C_{22} - C_{12}C_{21})(1 - \alpha_p\alpha_v). \tag{48}$$

Since the rates of change of $A_p(\theta)$, $A_v(\theta)$, $\gamma_p(\theta)$ and $\gamma_v(\theta)$ are small, the average values of the right sides of expressions (47) are taken. Calculating the following average values by formulas (18), (19) and trigonometric identities:

$$\begin{aligned} F_1 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} f\sin\tau_1\,\mathrm{d}\theta & F_2 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} f\cos\tau_1\,\mathrm{d}\theta \\ F_3 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} f\sin\tau_2\,\mathrm{d}\theta & F_4 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} f\cos\tau_2\,\mathrm{d}\theta \\ G_1 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} g\sin\tau_1\,\mathrm{d}\theta & G_2 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} g\cos\tau_1\,\mathrm{d}\theta \\ G_3 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} g\sin\tau_2\,\mathrm{d}\theta & G_4 &= \frac{1}{\Delta\theta}\int_0^{\Delta\theta} g\cos\tau_2\,\mathrm{d}\theta \end{aligned} \tag{49}$$

then

$$A_p' \approx \frac{1}{\Delta\omega_p}\{(C_{22} + \alpha_p C_{21})F_2 - (C_{12} + \alpha_p C_{11})G_2\}$$

$$\begin{aligned}\gamma_p' &\approx -\frac{1}{\Delta\omega_p A_p(0)}\{(C_{22}+\alpha_p C_{21})F_1-(C_{12}+\alpha_p C_{11})G_1\}\\ A_v' &\approx -\frac{1}{\Delta\omega_v}\{(C_{21}+\alpha_v C_{22})F_4-(C_{11}+\alpha_v C_{12})G_4\}\\ \gamma_v' &\approx \frac{1}{\Delta\omega_v A_v(0)}\{(C_{21}+\alpha_v C_{22})+\alpha_v C_{22})F_3-(C_{11}+\alpha_v C_{12})G_3\}. \end{aligned} \tag{50}$$

Eventually, the approximate closed form solution for the system during slewing is found to be:

$$\begin{aligned}\begin{bmatrix}\tilde{\psi}\\ u\end{bmatrix} &= (A_p'\theta+A_p(0))\begin{bmatrix}1\\ \alpha_v\end{bmatrix}\sin\{(\omega_p+\gamma_p')\theta+\gamma_p(0)\}\\ &\quad+(A_v'\theta+A_v(0))\begin{bmatrix}\alpha_p\\ 1\end{bmatrix}\sin\{(\omega_v+\gamma_v')\theta+\gamma_v(0)\}\\ &\quad+\sum_{i=3}^{9}\begin{bmatrix}A_{1i}\\ A_{2i}\end{bmatrix}\sin(\omega_i\theta+\gamma_i),\end{aligned} \tag{51}$$

where $A_p(0)$, $\gamma_p(0)$, $A_v(0)$ and $\gamma_v(0)$ are the initial conditions.

Equation (34) for motion after slewing can be solved by the same method. The solution is:

$$\begin{aligned}\begin{bmatrix}\tilde{\psi}\\ u\end{bmatrix} &= A_p^*\begin{bmatrix}1\\ \alpha_v^*\end{bmatrix}\sin\{(\omega_p^*+\gamma_p'^*)\theta+\gamma_p^*\}\\ &\quad+A_v^*\begin{bmatrix}\alpha_p^*\\ 1\end{bmatrix}\sin\{(\omega_v^*+\gamma_v'^*)\theta+\gamma_v^*\}+\begin{bmatrix}A_{13}^*\\ A_{23}^*\end{bmatrix}\end{aligned} \tag{52}$$

with

$$\begin{aligned}A_{13}^* &= -D_{23}^*C_{14}^*/(C_{13}^*C_{24}^*-C_{23}^*C_{14}^*)\\ A_{23}^* &= C_{13}^*D_{23}^*/(C_{13}^*C_{24}^*-C_{23}^*C_{14}^*).\end{aligned} \tag{53}$$

Other parameters, ω_p^*, ω_v^* and α_p^*, α_v^* are calculated by formulas like (41), (42) and (43); $\gamma_p'^*$ and $\gamma_v'^*$ and $A_p'^*$, $A_v'^*$ are evaluated by formulas like (50), but $A_p'^*$ and $A_v'^*$ vanish in this case; A_p^*, γ_p^* and A_v^*, γ_v^* are determined by the values of state variables right after slewing.

6. Results and Discussion

The validity of the closed form solutions (51) and (52) is assessed over a range of system parameters and initial conditions, through its comparison with the numerical solutions of the exact equations of motion as given by (12–15). As can be expected, the amount of information is literally enormous. For conciseness, only the results of eight test cases are presented here to help establish trends.

For all these cases, components of the moment of inertia of the central body are: yaw – $\tilde{J}_{01}=0.1$; roll – $\tilde{J}_{02}=0.5$; pitch – $\tilde{J}_{03}=0.6$. The ratio of the appendage mass to that of the central body is taken as 1 : 50. The length ration ($\epsilon_l = 1/L$) is varied as $2:1$, $3:1$, and $5:1$. In general, the duration of integration is 1/12 of an orbit. However, for the first four cases, the pitch response over the entire orbit was obtained. The analytical solutions are represented by lines, and the numerical results with dots. Except for Case 8, the two sets of results virtually coincided and hence the dots are not visible. The detailed analytical solutions for the cases 1–4 are also presented and the dominant vibration components during slewing are indicated by an underline.

Case 1 may be taken as the reference. It represents slew of the beam-type appendage, with a stiffness of 60 cycles/orbit, through 30° as the spacecraft traverses 10° of the circular orbit. Initial conditions are all zero. The parameter affected by a change in a given case is indicated by an underline. Case 2 studies the effect of initial conditions, with all the other parameters held fixed at the values used in Case 1. Similarly, Case 3 assesses the influence of fast slew (6° per orbital degree), higher stiffness, and shorter appendage length; while Case 4 adds to this a pitch initial disturbance. Case 5 is similar to Case 2 except that the length ratio ϵ_l is larger. In Case 6, the initial orientation of the beam-type appendage with respect to the central body is 60° instead of 0°. Case 7 shows that even if the slewing angle changes to 45°, i.e. the slewing rate increases to 4.5° per orbital degree, the correlation between the response results continues to be quite good. Perhaps a demanding situation would be represented by faster slew of a beam with lower stiffness (compared to Case 4). This is shown in Case 8. A small discrepancy in the pitch response appears which increases with time. However, the vibrational response of the appendage continues to be predicted accurately. Even in this extreme situation, the response results would be considered sufficiently accurate during the initial design stage.

The plots and the analytical solutions provide much information and insight into the behavior of the system; and the latter is a useful complement and explanation for the former. The following characteristics are indicated:

- Slew maneuver excites both pitch motion and elastic vibration and the level of the motion excited depends on slewing 'time' and slewing angle as well as the mass and inertia ratio of the beam over the central body.
- In the pitch response during slew, besides the free vibration at the pitch frequency, another dominant component is the one with frequence $\Delta\alpha/\Delta\theta$, showing that the influence of slew on pitch is mainly due to the orientation change of the beam with respect to the central body. Furthermore, from equation (31) it can be seen that the amplitude of this component is proportional to $\Delta\alpha/\Delta\theta$, which represents the angular velocity of the slew maneuver. If the slew maneuver is violent, the inertia force caused by slew maneuver is also an important factor affecting the pitch response.
- It can be observed from the analytical solution that besides the free vibration, the term of the inertial force caused by slew dominates the vibration of the beam; implying that influence of slew on elastic vibration of the beam is mainly attributed to the inertial force. Moreover, the amplitude depends on $\Delta\alpha/\Delta\theta^2$, i.e. the angular acceleration of the slew maneuver of the beam (equations (28) and (31)). The effect of the slew on elastic vibration also depends on the rigidity of the beam. A soft beam is easily affected by slew maneuver. This explains why the elastic vibration of the beam reaches the greatest amplitude in Case 8.
- The response after the slew maneuver is a pure free vibration for both the pitch and the beam vibration. At the termination of the slew maneuver, the deformed state acts as an initial condition for the subsequent periodic motion of the system.
- It can be observed from the plots that the librational response is modulated by the high frequency contribution from the beam vibration. Strength of the modulation depends on the mass and the inertia ratio of the beam over the central body as well as the

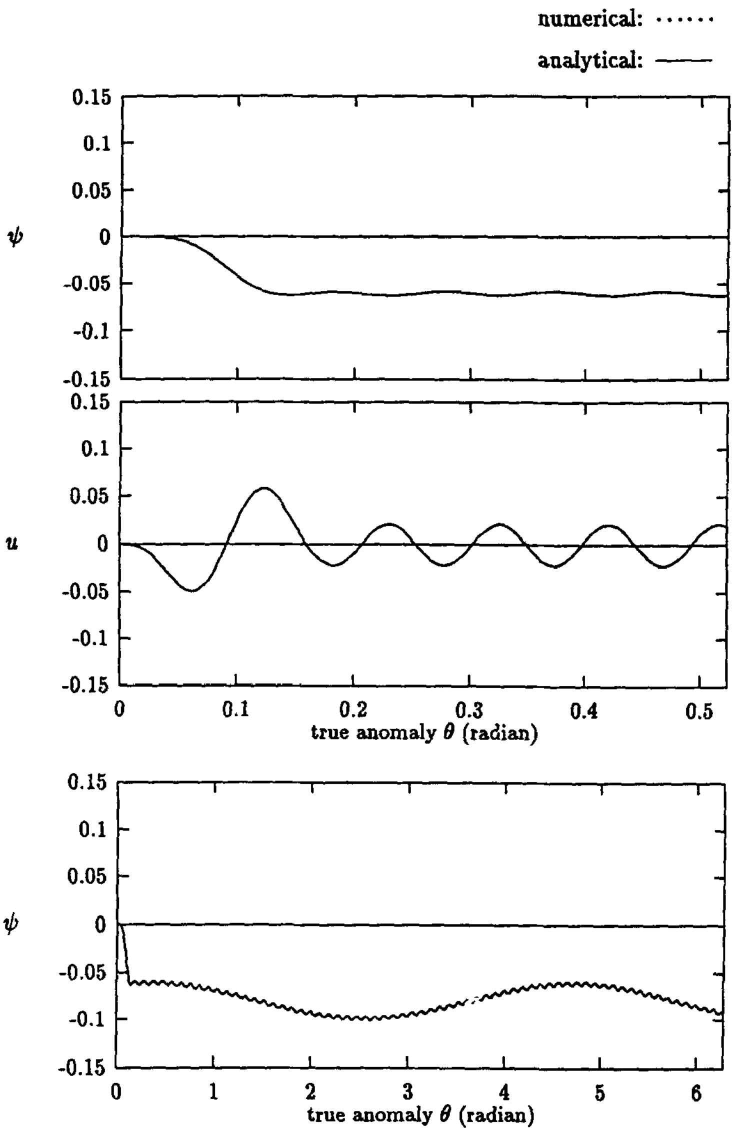

Fig. 3. The pitch and vibrational response during the slew maneuver under the nominal conditions.

amplitude of the beam vibration. While the pitch motion hardly affects beam vibration. This observation is substantiated by the solution after slew shown in cases 1–4.

- Slew maneuver changes the ellipsoid of inertia of the spacecraft; and the pitch motion after the maneuver is around the new equilibrium.

Case 1.

$$\epsilon = 0.02 \;\; \Delta\theta = 10^\circ \;\; \psi(0) = 0^\circ \quad u(0) = 0.0 \;\; r_v = 60$$
$$\epsilon_1 = 3.0 \;\; \Delta\alpha + 30^\circ \;\; \psi'(0) = 0.0 \;\; u'(0) = 0.0 \;\; \alpha_0 = 0^\circ$$

During slewing:

$$\psi = \underline{(0.0297\theta + 0.0639)\sin(1.4669\theta + 2.9826)} - 0.0102\sin(66.02128\theta - 0.0002)$$

$$
\begin{aligned}
& -0.1138\sin(3.0006\theta) + 0.0175\sin(3.0006\theta - 0.0823) \\
& +0.0109\sin(36.0068\theta) + 0.0036\sin(6.0011\theta - 0.0823) \\
& +0.0015\sin(33.0062\theta) + 0.0015\sin(39.0074\theta) - 0.0084 \\
u \;=\; & -0.0003(0.0297\theta + 0.0639)\sin(1.4669\theta + 2.9826) \\
& +0.0216\sin(66.0212\theta - 0.0002) \\
& -0.0010\sin(3.0006\theta) - 0.0002\sin(3.0006\theta - 0.0823) \\
& -0.0424\sin(36.0068\theta) - 0.0004\sin(6.0011\theta - 0.0923) \\
& +0.0007\sin(33.0062\theta) + 0.0022\sin(39.0074\theta).
\end{aligned}
$$

After slewing:

$$
\begin{aligned}
\psi &= -0.0798 + 0.0190\sin(1.4431\theta + 1.0589) - 0.0021\sin(66.0603\theta - 7.3535) \\
u &= -0.0005 + 0.0213\sin(66.0603\theta - 7.3535).
\end{aligned}
$$

Case 2.

$$
\begin{array}{lllll}
\epsilon + 0.02 & \Delta\theta = 10^\circ & \psi(0) = 5.7^\circ & u(0) = 0.1 & r_v = 60 \\
\epsilon_1 = 3.0 & \Delta\alpha = 30^\circ & \psi'(0) = 0.0 & u'(0) = 0.0 & \alpha_0 = 0^\circ
\end{array}
$$

During slewing:

$$
\begin{aligned}
\psi \;=\; & -0.0084 + (0.0037\theta + 0.1375)\sin(1.5397\theta + 2.0548) \\
& -0.0995(0.0007\theta + 0.1023)\sin(66.0254\theta + 1.3585) - 0.1138\sin(3.0006\theta) \\
& +0.0109\sin(36.0068\theta) + 0.0015\sin(33.0062\theta) + 0.0015\sin(39.0074\theta) \\
& +0.0175\sin(3.0006\theta - 0.0823) + 0.0036\sin(6.0011\theta - 0.0823) \\
u \;=\; & -0.0003(0.0037\theta + 0.1375)\sin(1.5397\theta + 2.0548) \\
& +(0.0007\theta + 0.1023)\sin(66.0254\theta + 1.3585) + 0.0010\sin(3.0006\theta) \\
& -0.0424\sin(36.0068\theta) + 0.0007\sin(33.0062\theta) + 0.0022\sin(39.0074\theta) \\
& -0.0002\sin(3.0006\theta - 0.0823) - 0.0004\sin(6.0011\theta - 0.0823).
\end{aligned}
$$

After slewing:

$$
\begin{aligned}
\psi &= -0.0798 + 0.1257\sin(1.4376\theta + 1.4557) - 0.0082\sin(66.0604\theta - 11.1283) \\
u &= -0.0005 + 0.0821\sin(66.0604\theta - 11.1283).
\end{aligned}
$$

Case 3.

$$
\begin{array}{lllll}
\epsilon = 0.02 & \Delta\theta = 5^\circ & \psi(0) = 0^\circ & u(0) = 0.0 & r_v = 120 \\
\epsilon_1 = 2.0 & \Delta\alpha = 30^\circ & \psi'(0) = 0.0 & u'(0) = 0.0 & \alpha_0 = 0^\circ
\end{array}
$$

During slewing:

$$
\begin{aligned}
\psi \;=\; & -0.0029 + (0.0606\theta + 0.0595)\sin(1.420\theta + 3.0898) \\
& -0.0551(-0.0001\theta + 0.0246)\sin(127.7\theta) - 0.0558\sin(5.998\theta)
\end{aligned}
$$

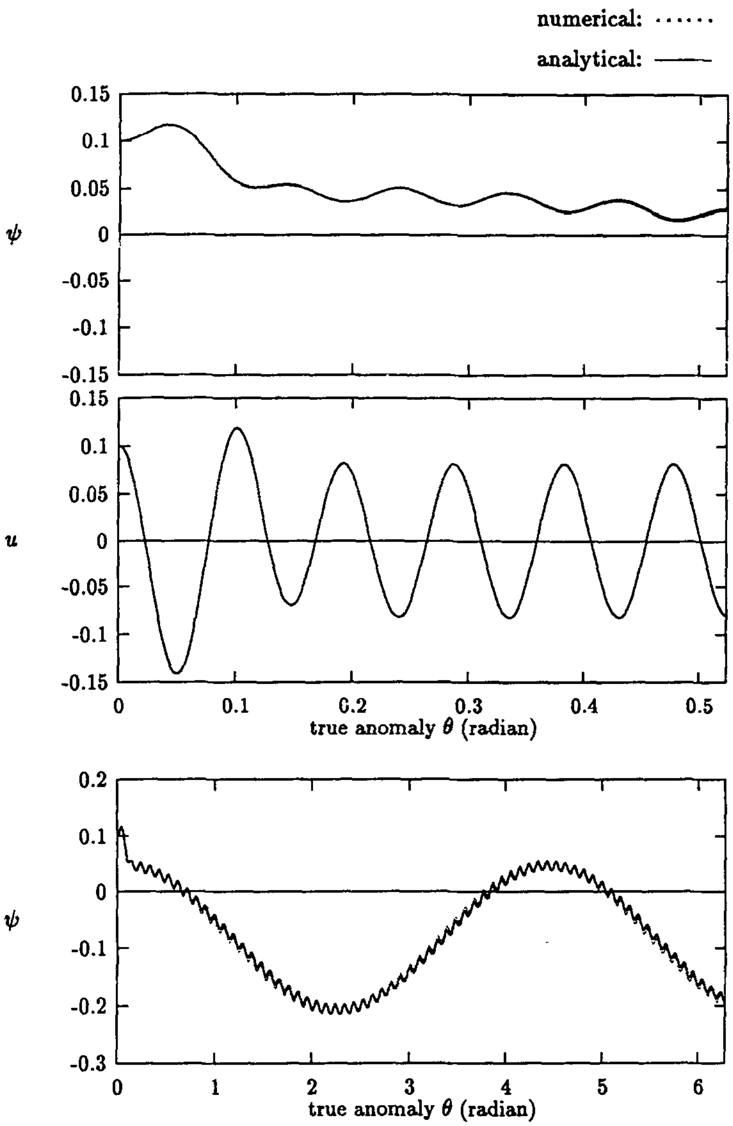

Fig. 4. Effect of the initial pitch and appendage disturbances on the system response as compared to Case

$$
\begin{aligned}
&+0.0057\sin(71.97\theta)+0.0011\sin(65.97\theta)+0.0011\sin(77.97\theta)\\
&+0.0025\sin(5.998\theta-0.0471)+0.0004\sin(12.00\theta-0.0471)\\
u &= -0.0001(0.0606\theta+0.0595)\sin(1.420\theta+3.0898)\\
&+(-0.0001\theta+0.0246)\sin(127.7\theta)-0.0006\sin(5.998\theta)\\
&-0.0461\sin(71.97\theta)+0.0007\sin(65.97\theta)+0.0018\sin(77.97\theta)\\
&-0.0001\sin(5.998\theta-0.0471)-0.0001\sin(12.00\theta-0.0471).
\end{aligned}
$$

After slewing:

$$
\begin{aligned}
\psi &= -0.0453+0.013\sin(1.438\theta+0.8995)-0.0018\sin(127.7\theta-7.1555)\\
u &= -0.0002+0.0320\sin(127.7\theta-7.1555).
\end{aligned}
$$

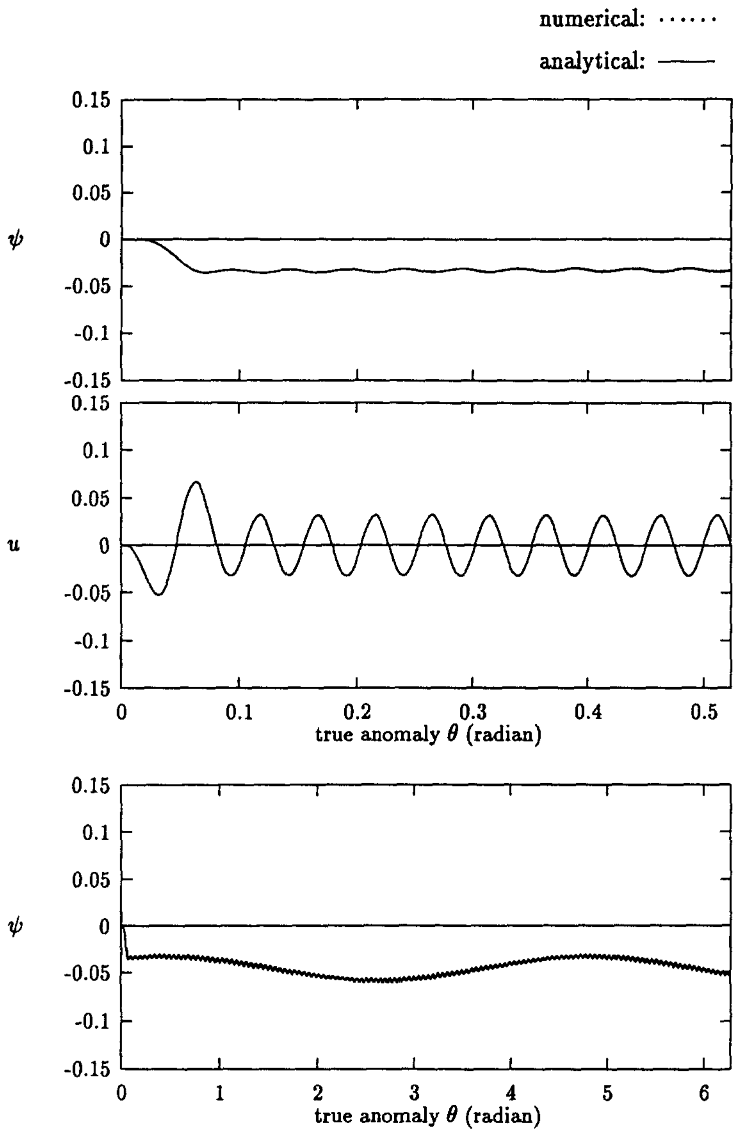

Fig. 5. Influence of faster slew of the appendage, with higher stiffness and shorter length, on the system response.

Case 4.

$$\epsilon = 0.02 \quad \Delta\theta = 5^\circ \quad \psi(0) = 5.7^\circ \quad u(0) = 0.1 \quad r_v = 120$$
$$\epsilon_1 = 2.0 \quad \Delta\alpha = 30^\circ \quad \psi'(0) = 0.0 \quad u'(0) = 0.0 \quad \alpha_0 = 0^\circ$$

During slewing:

$$\begin{aligned} \psi &= -0.0029 + (0.0151\theta + 0.1238)\sin(1.687\theta + 2.0715) \\ &\quad 0.00551(0.0004\theta + 0.1030)\sin(127.7\theta + 1.3295) - 0.0558\sin(5.998\theta) \\ &\quad +0.0057\sin(71.97\theta) + 0.0011\sin(65.97\theta) + 0.0011\sin(77.97\theta) \\ &\quad +0.0025\sin(5.998\theta - 0.00471) + 0.0004\sin(12.00\theta - 0.0471) \\ u &= -0.0001(0.0151\theta + 0.1238)\sin(1.687\theta + 2.0715) \end{aligned}$$

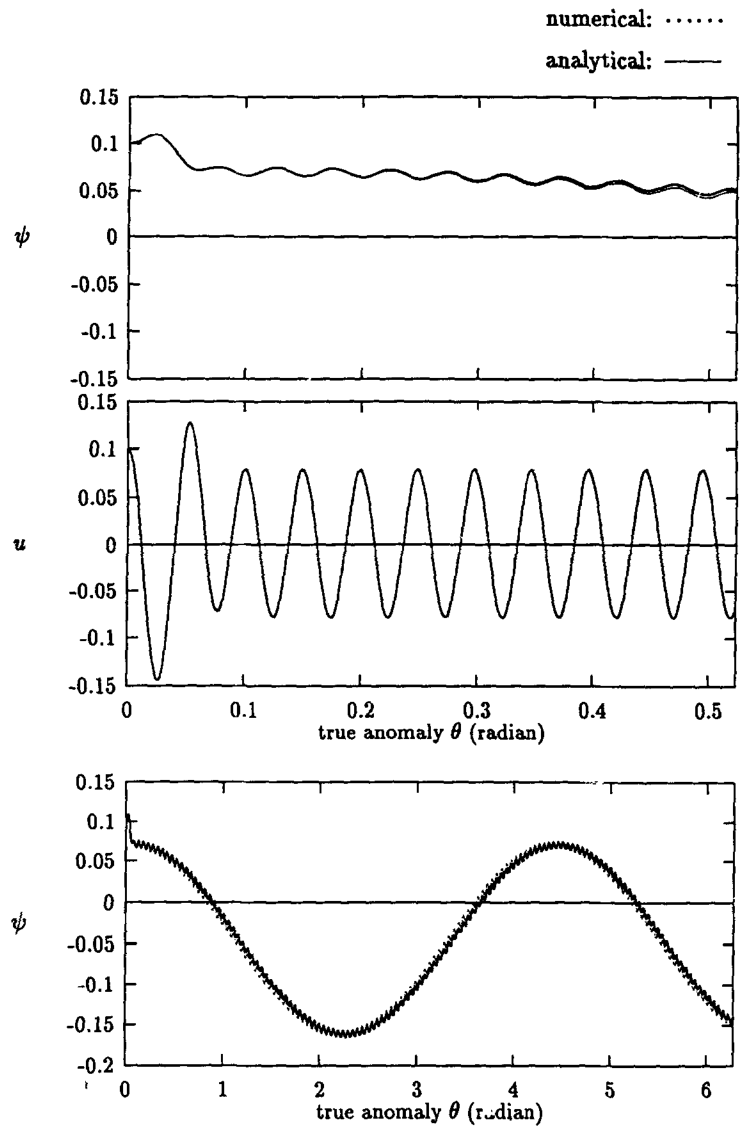

Fig. 6. Effect of the initial pitch and appendage disturbances on the system response as compared to Case 3

$$+(0.0004\theta + 0.1030)\sin(127.7\theta + 1.3295) - 0.0006\sin(5.998\theta)$$
$$-0.0461\sin(71.97\theta) + 0.0007\sin(65.97\theta) + 0.0018\sin(77.97\theta)$$
$$-0.0001\sin(5.998\theta - 0.0471) - 0.0001\sin(12.00\theta - 0.0471).$$

After slewing:

$$\psi = -0.0453 + 0.1157\sin(1.4332\theta + 1.4599) - 0.0043\sin(127.7\theta - 11.2645)$$
$$u = -0.0002 + 0.0783\sin(127.7\theta - 11.2645).$$

Case 5.

$\epsilon + 0.02$ $\Delta\theta = 10°$ $\psi(0) = 5.7°$ $u(0) = 0.1$ $r_v = 60$
$\epsilon_1 = 5.0$ $\Delta\alpha = 30°$ $\psi'(0) = 0.0$ $u'(0) = 0.0$ $\alpha_0 = 0°$

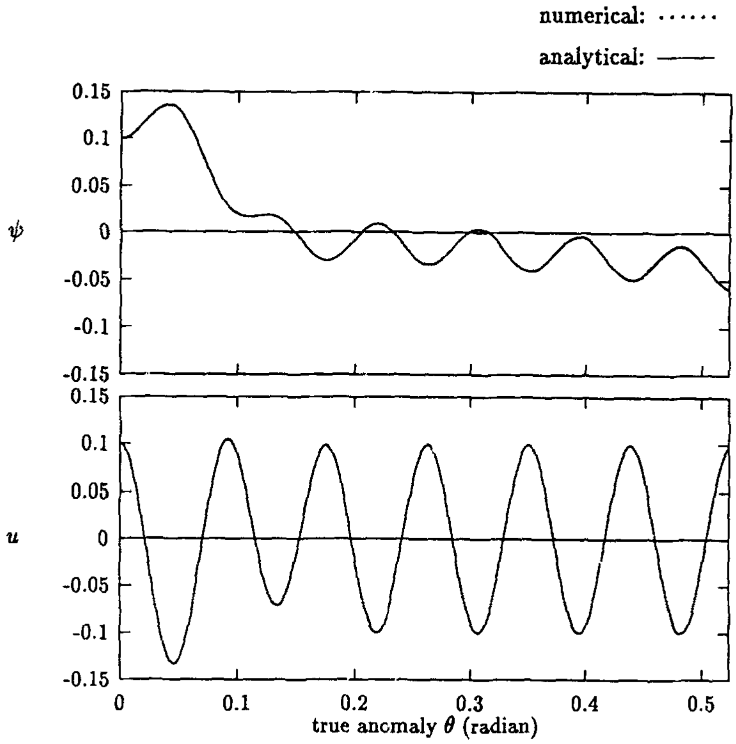

Fig. 7. System response during the nominal slew of the beam with its length much longer then that of the central body.

Case 6.

$$\begin{array}{lllll} \epsilon = 0.02 & \Delta\theta = 10^\circ & \psi(0) = 0^\circ & \underline{u(0) = 0.1} & r_v = 60 \\ \epsilon_1 = 3.0 & \Delta\alpha = 30^\circ & \psi'(0) = 0.0 & u'(0) = 0.0 & \underline{\alpha_0 = 60^\circ} \end{array}$$

Case 7.

$$\begin{array}{lllll} \epsilon = 0.02 & \Delta\theta = 10^\circ & \underline{\psi(0) = 5.7^\circ} & \underline{u(0) = 0.1} & r_v = 60 \\ \epsilon_1 = 3.0 & \underline{\Delta\alpha = 45^\circ} & \psi'(0) = 0.0 & u'(0) = 0.0 & \alpha_0 = 0^\circ \end{array}$$

Case 8.

$$\begin{array}{lllll} \epsilon = 0.02 & \underline{\Delta\theta = 5^\circ} & \underline{\psi(0) = 5.7^\circ} & \underline{u(0) = 0.1} & \underline{r_v = 60} \\ \epsilon_1 = 3.0 & \Delta\alpha = 30^\circ & \psi'(0) = 0.0 & u'(0) = 0.0 & \alpha_0 = 0^\circ \end{array}$$

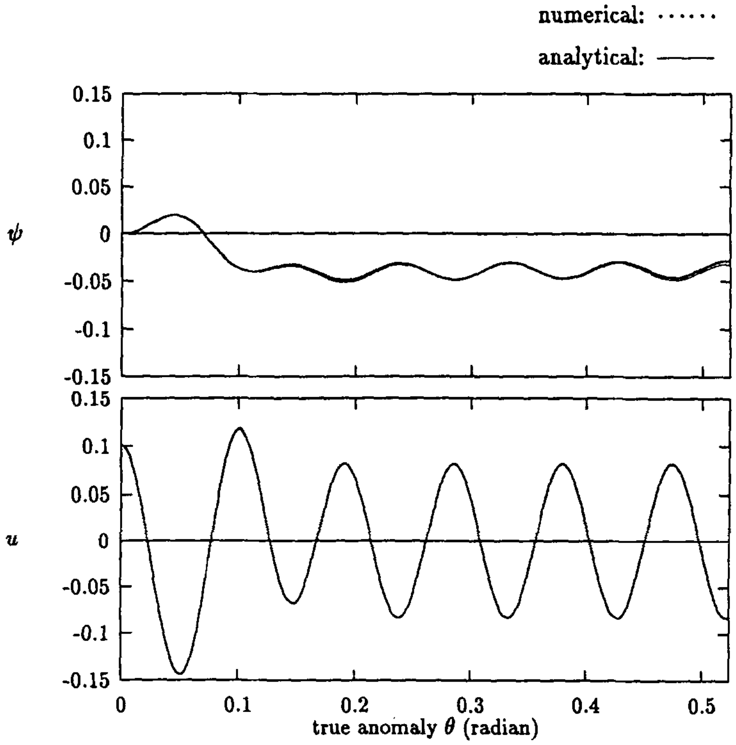

Fig. 8. System response during the slew maneuver of the beam which initially has an 60° offset with respect to the central body.

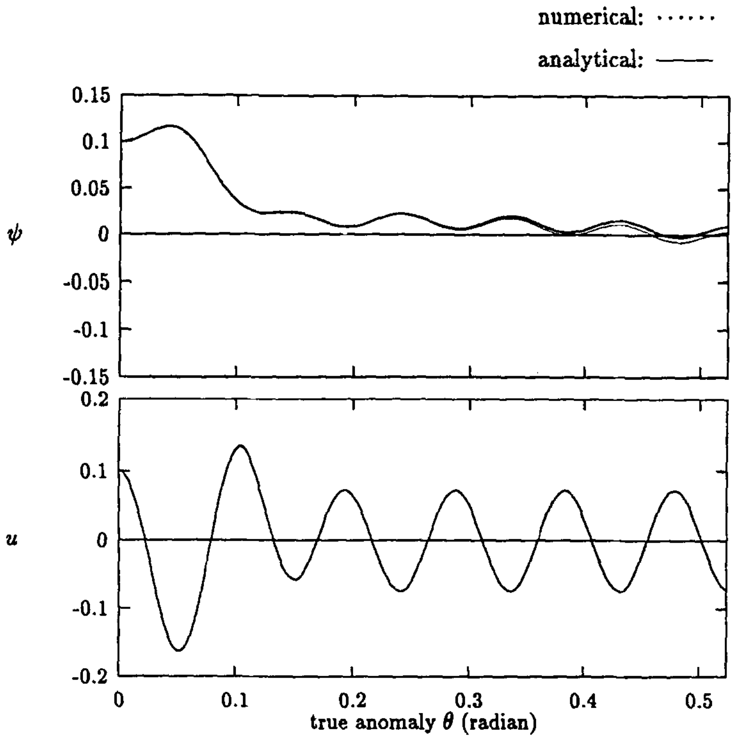

Fig. 9. Effect of a larger slewing magnitude and higher slewing rate on the system response.

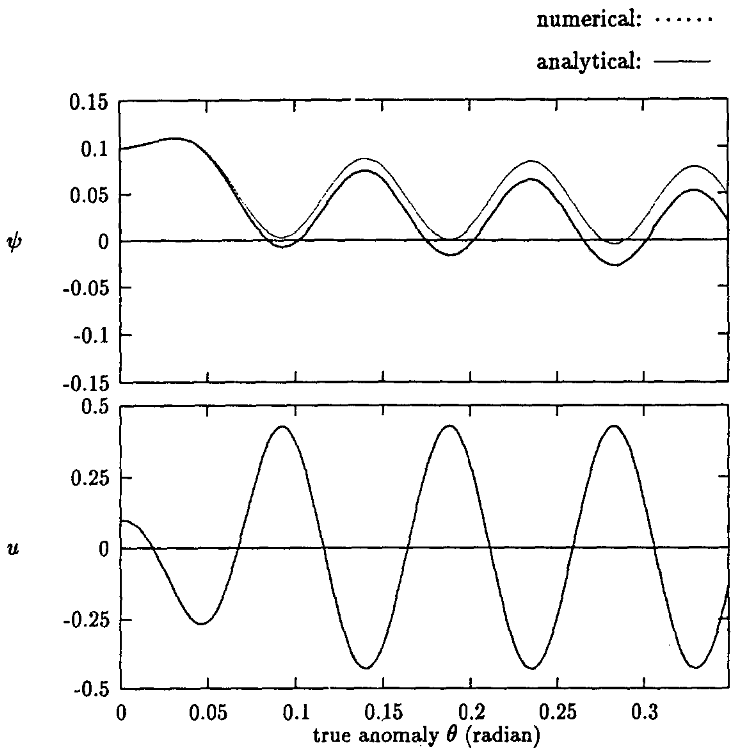

Fig. 10. System response during the faster slew maneuver of the beam with lower stiffness.

7. Conclusions

This paper investigates the planar dynamics of a spacecraft with a flexible beam-type appendage undergoing slew maneuver. The nonlinear, nonautonomous and coupled equations for pitch motion of the whole system and elastic vibration of the beam in the orbital plane are solved by the variation of parameters method suggested by Butenin. The objective is to study the effects of slew maneuvers on the system response through approximate analytical solutions which provide comparisons to the results obtained by numerical integration. To do so, the original governing equations of motion are first transformed into a standard quasi-linear form by careful expansion of the nonlinear terms and truncating the series at an appropriate order depending on their relative magnitude. Up to eight vibration components were identified and hence the analytical results are of relatively high accuracy. The analytical solution, substantiated by the numerical results, provides better understanding into the system performance and allows for predictions, particularly those concerning the influence of the slew maneuvers on the system response.

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