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A bifurcation problem for a compressible nonlinearly elastic medium: growth of a micro-void

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Abstract

In this paper, we carry out an explicit analysis of a bifurcation problem for a solid circular cylinder composed of a particular *compressible* nonlinearly elastic material. This problem is concerned with the bifurcation of a solid body into a configuration involving an internal cavity. A discussion of its physical interpretation is then carried out. In particular, it is shown that this model may be used to describe the nucleation of a void from a *pre-existing* micro-void.

Introduction

In a recent paper [1], J.M. Ball has made an extensive study of a class of bifurcation problems for the equations of nonlinear elasticity. It is suggested in [1] that these problems are relevant to the phenomenon of internal rupture, in which a hole forms in the interior of a solid body which contains no hole in the undeformed state.

The purpose of the present paper is to study a bifurcation problem of the type considered in [1] for a particular elastic material under plane strain conditions, and to interpret this problem in terms of the growth of a *pre-existing* micro-void, rather than in terms of rupture.

It is shown that under remotely applied stretch $\lambda > 1$, an infinitesimally small pre-existing void (a "micro-void") does not grow until λ reaches a certain critical value λ_{cr}. Sudden growth takes place thereafter. (See Figure 1). It is found that the deformation field (for all values of λ), as well as the value λ_{cr} in this problem are in fact *identical* to the corresponding quantities arising in the bifurcation problem. Thus, the latter may be viewed as providing an idealized model for describing the sudden growth of a micro-void.

In Section 2, we consider a plane strain bifurcation problem for a *solid* circular cylinder of radius *a* composed of a particular homogeneous isotropic compressible elastic material, namely the "Blatz-Ko material." This well-known model, characterizing the constitutive behavior in plane strain of a certain foam rubberlike material, was proposed by Blatz and Ko [2] on the basis of experiments carried out by them. The

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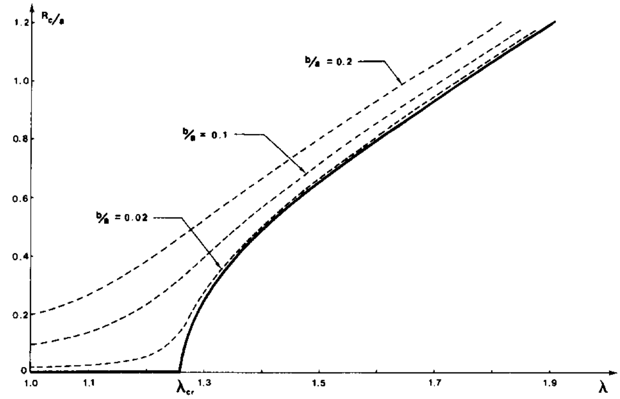

Figure 1. Variation of the deformed cavity radius R_c with prescribed stretch λ. The solid curve pertains to the bifurcation of a solid cylinder (Section 2) while the dashed curves describe the growth of a pre-existing void for different undeformed void radii b (Section 3). The solid curve also describes the limiting case of a micro-void ($b \to 0+$).

surface of the cylinder is subjected to a prescribed radial stretch $\lambda > 1$. One solution to this problem is that of a pure homogeneous stretching in which the cylinder expands radially. However, when λ exceeds a critical value λ_{cr}, ($\lambda_{cr} \simeq 1.25954$), we find in addition a *second* solution involving a cylindrical internal cavity.

It should be noted that the Blatz-Ko material does *not* satisfy many of the constitutive hypotheses imposed by Ball in his bifurcation studies (see Section 7 of [1]). Thus, the results obtained in [1] using the direct method of the calculus of variations, are not directly applicable to the bifurcation problem considered here.

In order to interpret physically the mathematical bifurcation problem studied in Section 2 we turn our attention in Section 3 to the uniform radial extension of a *hollow* circular cylinder with undeformed radii b, a ($b < a$). The inner surface is free of traction while the outer is subjected to prescribed surface displacement, with $\lambda > 1$ denoting the applied stretch. The body is composed, again, of the Blatz-Ko material. A closed form solution to this problem is obtained. Attention is then focused on features of this solution for the case $b \to 0+$ corresponding to a micro-void. In particular, it is shown that (in this limit) the radius R_c of the deformed cavity remains zero for all values of the applied stretch on the range $1 < \lambda < \lambda_{cr}$. For $\lambda > \lambda_{cr}$, this radius is positive and increases rapidly with increasing λ. See Figure 1. This behavior is analogous to the phenomenon of cavitation in fluids where infinitesimal pre-existing nuclei of undissolved gas grow into clearly visible cavities ([3], p. 482). [It is worth recalling that such behavior does not, of course, occur in the corresponding problem in classical linear elasticity. In the latter context the radius R_c of the inner cavity after deformation is

$$R_c = b + \frac{2(1-\nu)a^2 b(\lambda - 1)}{(1-2\nu)a^2 + b^2},$$

where ν is Poisson's ratio, $0 < \nu < 1/2$. Thus as $b \to 0+$, we see that $R_c \to 0$ no matter what the value of the applied stretch $\lambda > 1$.]

Finally in Section 4, we carry out a comparison between the solutions to the bifurcation and micro-void problems. It is shown that the deformation fields in both problems coincide identically for all values of λ. Consequently, the bifurcation problem may be viewed as providing an idealized model describing the growth of a pre-existing micro-void.

1. Preliminaries from finite elastostatics

Let the open region D_0 of the (x_1, x_2)-plane denote the cross-section of a right cylinder in its undeformed configuration. A plane deformation of the body, parallel to the (x_1, x_2)-plane, is described by a sufficiently smooth transformation

$$y = y(x) = x + u(x) \quad \text{on } D_0, \tag{1.1}$$

which maps D_0 onto a domain D. Here y is the position vector after deformation of the particle which in the undeformed configuration was located at x, while $u(x)$ denotes the displacement vector field. For the moment, $y(x)$ is assumed to be twice continuously differentiable and invertible on D_0. The deformation gradient tensor F is defined by

$$F = I + \nabla \mathbf{u}, \quad J = \det \mathbf{F} > 0 \quad \text{on } D_0. \tag{1.2}$$

Let $\tau(y)$ be the (in-plane) true stress tensor field accompanying the deformation at hand. Equilibrium, in the absence of body forces, then demands that

$$\operatorname{div} \tau(y) = 0, \quad \tau = \tau^T \quad \text{on } D. \tag{1.3}$$

Suppose now that the body under consideration is homogeneous, compressible and elastic and possesses an elastic potential $W(F)$ representing the strain-energy density per unit undeformed volume. Then the constitutive law for the in-plane true stress is

$$\tau = \frac{1}{J} W_F F^T. \tag{1.4}$$

In the bifurcation problem considered first, the undeformed cylinder is solid and thus D_0 is simple-connected. The boundary of D_0 is assumed to be subjected to the prescribed displacement

$$u(x) = (\lambda - 1)x \quad \text{on } \partial D_0, \tag{1.5}$$

where the parameter $\lambda > 1$ is prescribed and denotes the applied stretch. The analysis of this problem necessarily involves a deformation which is *not* one-to-one, and so in order to investigate this, the preceding regularity conditions must be relaxed. Thus we allow for the possibility that the mapping (1.1) is one-to-one everywhere on D_0 *except* at a single point x_0. *In this event*, x_0 is assumed to map onto a closed regular curve C, while the simply-connected domain D_0 then maps onto a doubly-connected domain D, with C denoting its inner boundary. Thus, in this situation, equation (1.2) is required to hold merely on the domain D_0 with x_0 deleted, while equation (1.3) holds on the doubly-connected domain D. The inner boundary C is now assumed to be traction-free and so

$$\tau n = 0 \quad \text{on } C, \tag{1.6}$$

where n denotes the unit outward normal vector on C, and τ is the limiting value of the true stress (presumed to exist) as a point on C is approached from within D.

The boundary-value problem of concern in Section 2 thus consists of the determination of displacement fields $\mathbf{u}(\mathbf{x})$ satisfying the field equations (1.2)-(1.4) subject to the boundary condition (1.5); when C exists, the boundary condition (1.6) must also be satisfied, and (1.2) is not required to hold at the point $\boldsymbol{x}_0$. Obviously, one solution to this problem is

$$\boldsymbol{u}(\boldsymbol{x}) = (\lambda - 1)\boldsymbol{x} \quad \text{on } D_0, \tag{1.7}$$

describing a pure homogeneous deformation of the body. Our purpose in the next Section is to seek a second solution, describing a bifurcated configuration of the cylinder involving a cylindrical cavity.

2. Bifurcation problem for a solid cylinder

2.1. Formulation

We suppose now that the cylinder is circular with radius a, so that its cross-section $D_0 = \{(r, \theta) \,|\, 0 \leqslant r < a, 0 < \theta \leqslant 2\pi\}$ and that the point $\boldsymbol{x}_0$ coincides with the origin. The cylinder is subjected to a prescribed radial displacement at its surface $r = a$. The resulting deformation is a mapping which takes the point $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ to the point $(y_1, y_2) = (R\cos\Theta, R\sin\Theta)$. We assume that the deformation is an axisymmetric plane strain one so that $\Theta = \theta$ and $R = R(r)$. In order to avoid interpenetration, it is required that

$$R = R(r) > 0 \quad \text{on } 0 < r < a, \quad R(0) \geqslant 0. \tag{2.1}$$

Observe that if $R(0) > 0$, the deformation (1.1) is not one-to-one at the origin.

The polar components of the deformation gradient tensor $\boldsymbol{F}$ associated with the deformation are given by

$$F_{rr} = \dot{R}(r), \quad F_{\theta\theta} = R(r)/r, \quad F_{r\theta} = F_{\theta r} = 0, \tag{2.2}$$

where the dot denotes differentiation with respect to the argument. The Jacobian determinant $J = \det \boldsymbol{F}$ is assumed to be positive on $0 < r < a$ and so $\dot{R}(r) > 0$ in this interval. The corresponding principal stretches λ_1, λ_2 are

$$\lambda_1 = \lambda_r \equiv \dot{R}(r), \quad \lambda_2 = \lambda_\theta \equiv R(r)/r, \quad 0 < r < a. \tag{2.3}$$

We turn now to the constitutive relation for an isotropic compressible elastic material characterized, in plane strain, by its elastic potential $W(\lambda_1, \lambda_2)$ representing the strain energy per unit undeformed volume. The principal components τ_1, τ_2 of the true stress tensor $\boldsymbol{\tau}$ are given by

$$\tau_1 = \frac{1}{\lambda_2}\frac{\partial W}{\partial \lambda_1}, \quad \tau_2 = \frac{1}{\lambda_1}\frac{\partial W}{\partial \lambda_2}. \tag{2.4}$$

In the particular case of the Blatz-Ko material * [2] W is given by

$$W(\lambda_1, \lambda_2) = \frac{\mu}{2}\left(\lambda_1^{-2} + \lambda_2^{-2} + 2\lambda_1\lambda_2 - 4\right), \tag{2.5}$$

where μ denotes the shear modulus of the material at infinitesimal deformations. Since

* An extensive discussion of the stress response of this material to various states of deformation is given by Knowles and Sternberg in [4].

in the present problem the radial and hoop stresses τ_{RR}, $\tau_{\Theta\Theta}$ are in fact the principal stresses, one finds from (2.3)-(2.5) that

$$\tau_{RR}(r) = \mu\left(1 - \frac{r}{R(r)\dot{R}^3(r)}\right), \quad \tau_{\Theta\Theta}(r) = \mu\left(1 - \frac{r^3}{R^3(r)\dot{R}(r)}\right). \tag{2.6}^*$$

The equilibrium equations in the absence of body forces in the present case reduce to the single equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\tau_{RR} + \frac{\dot{R}}{R}(\tau_{RR} - \tau_{\Theta\Theta}) = 0, \quad 0 < r < a. \tag{2.7}$$

This, together with (2.6), yields the following nonlinear singular second-order ordinary differential equation for $R(r)$:

$$3rR^3\ddot{R} - R^3\dot{R} + r^3\dot{R}^4 = 0, \quad 0 < r < a. \tag{2.8}$$

Since the cylinder is subjected to prescribed uniform radial displacement of its boundary $r = a$, we have

$$R(a) = \lambda a, \tag{2.9}$$

where $\lambda > 1$ is the prescribed radial stretch.

One solution (the "homogeneous solution") of the differential equation (2.8) satisfying the boundary condition (2.9) and condition (2.1) is

$$R(r) = \lambda r. \tag{2.10}$$

Note that the second of (2.1) holds with equality in this case. This corresponds to a homogeneous deformation in which the cylinder expands radially and the one-to-one mapping (1.1) maps the undeformed solid circular region D_0 onto the deformed solid circular region D.

Our purpose here is to exhibit explicitly a solution of (2.8), (2.9), (2.1) satisfying the second of (2.1) with inequality, i.e.

$$R(0) > 0, \tag{2.11}$$

corresponding to the situation in which an internal hole has appeared at the origin. The boundary C of this cavity, $y_1^2 + y_2^2 = R(0)^2$, will be assumed to be traction-free. It will be demonstrated that such a solution exists only when the prescribed stretch λ exceeds a critical value λ_{cr} ($\simeq 1.25954$).

The traction-free boundary condition on the cavity arises as a natural boundary condition in the variational treatment of Ball [1]. It should be noted that the bifurcation studies in [1] are carried out for displacement and traction boundary-value problems in n-dimensions for both incompressible and compressible materials. For incompressible materials the results are comprehensive and explicit while those in the more difficult compressible case are comparatively limited.

2.2. Solution

In a recent paper [5] the present authors have shown how the differential equation (2.8) may be solved explicitly. Thus on making the substitution

$$t(r) = r\frac{\dot{R}(r)}{R(r)} (= \lambda_r/\lambda_\theta) > 0, \quad 0 < r < a, \tag{2.12}$$

* It is convenient for our purposes to consider $\tau(r)$ rather than the more conventional $\tau(R)$.

equation (2.8) may be written as the first-order equation

$$3r\dot{t} - t(1-t)(t^2+t+4) = 0, \quad 0 < r < a, \tag{2.13}$$

where $\dot{t} = \mathrm{d}t/\mathrm{d}r$. There are two cases to consider. First of all, we observe that $t(r) \equiv 1$ on $0 < r < a$ is a solution of (2.13), and (2.12) then shows that $R(r) = cr$, where c is a constant. In this way, one recovers the homogeneous solution (2.10). Suppose then that $t \not\equiv 1$. Using arguments similar to those presented in [5] it can be shown that without loss of generality $t(r)$ may be assumed to be less than unity on $0 < r < a$ so that from (2.12), (2.13) it follows that

$$0 < t < 1, \quad \mathrm{d}t/\mathrm{d}r > 0 \quad \text{for } 0 < r < a. \tag{2.14}$$

Equation (2.13) may be readily integrated to yield

$$r^8 = \frac{Ct^6 h(t)}{(1-t)^4(t^2+t+4)}, \tag{2.15}$$

where $C > 0$ is a constant of integration and we have set

$$h(t) = \exp\left\{\frac{6}{\sqrt{15}} \tan^{-1}\left(\frac{2t+1}{\sqrt{15}}\right)\right\}(>0). \tag{2.16}$$

On the other hand (2.12) and (2.13) give

$$\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{3}{(1-t)(t^2+t+4)}(>0), \tag{2.17}$$

which in turn yields

$$R^4 = \frac{D(t^2+t+4)h(t)}{(1-t)^2}. \tag{2.18}$$

Again, $D > 0$ is a constant of integration. Observe from (2.17), (2.14) that the deformed and undeformed radial coordinates $(R,\ r)$ vary monotonically with t. Equations (2.15), (2.16), (2.18) provide a *parametric solution* to the differential equation (2.8). The range of the parameter t is

$$0 \leqslant t \leqslant t_a \tag{2.19}$$

where $t_a (0 < t_a < 1)$ is the value of t corresponding to $r = a$ and is to be determined from (2.15), that is,

$$\frac{Ct_a^6 h(t_a)}{(1-t_a)^4(t_a^2+t_a+4)} = a^8. \tag{2.20}$$

It is clear from (2.15) that $t \to 0$ as $r \to 0$.

Turning now to the boundary conditions, first of all we see that (2.9) together with (2.18) requires that

$$\lambda^4 a^4 = \frac{D(t_a^2+t_a+4)h(t_a)}{(1-t_a)^2}. \tag{2.21}$$

Since (2.18) yields

$$R(0) = (4h(0)D)^{1/4}, \tag{2.22}$$

we see that (2.11) will indeed by satisfied (and so an internal hole may develop) provided a *positive* constant D can be found. In this event, the radius of the resulting cavity is given by (2.22). Finally, to satisfy the traction-free boundary condition on the cavity surface ($t=0$), we require an expression for the normal stress component τ_{RR} which follows from (2.6), (2.15)-(2.18) as

$$\tau_{RR}=\mu\left\{1-\frac{\sqrt{C/D^2}}{\sqrt{h(t)}\,(t^2+t+4)^{3/2}}\right\},\quad 0\leqslant t\leqslant t_a. \tag{2.23}$$

Thus we obtain

$$C/D^2=64h(0). \tag{2.24}$$

Equations (2.20), (2.21) and (2.24) are three equations for the unknowns C, D and t_a corresponding to prescribed values of the stretch λ.

On substitution from (2.24) in (2.20) and (2.21) it follows that

$$D^2=\frac{a^8(1-t_a)^4(t_a^2+t_a+4)}{64h(0)t_a^6h(t_a)}, \tag{2.25}$$

$$\frac{h(t_a)(t_a^2+t_a+4)^3}{t_a^6}=64h(0)\lambda^8. \tag{2.26}$$

Thus if, for prescribed $\lambda>1$, (2.26) *can be solved for a number* t_a *such that* $0<t_a<1$, *then* (2.24), (2.25) *provide the values of the constants* C, D (>0) *and* (2.15), (2.16) (2.18) *with* $0\leqslant t\leqslant t_a$ *is the bifurcated solution involving an internal cavity.*

To verify that (2.26) can indeed be solved for an appropriate value of t_a, we simply observe that the auxiliary function $G(t)$ defined by

$$G(t)=h(t)(t^2+t+4)^3t^{-6}\quad\text{for } 0\leqslant t\leqslant 1, \tag{2.27}$$

(associated with the left hand side of (2.26)) can be easily shown to be monotone decreasing and $G(0)=\infty$, $G(1)=216h(1)$. It follows that (2.26) *can* be solved, in fact uniquely, for a root t_a, $0<t_a<1$ provided

$$\lambda^8>\frac{216h(1)}{64h(0)}, \tag{2.28}$$

that is,

$$\lambda>\lambda_{cr}\equiv\left(\frac{216h(1)}{64h(0)}\right)^{1/8}\simeq 1.25954. \tag{2.29}$$

Thus whenever the prescribed stretch λ *is greater than* λ_{cr}, *the existence of a bifurcated solution involving a cavity is guaranteed and this solution is given by* (2.15), (2.16), (2.18), (2.19), (2.24)-(2.26).

2.3. Discussion

When the prescribed stretch λ does not exceed the value λ_{cr} we have seen that a homogeneous solution (2.10) exists describing a uniform expansion of the cylinder. On the other hand, when $\lambda>\lambda_{cr}$ we have obtained, *in addition*, a bifurcated solution involving an internal cylindrical hole. The radius $R_c \equiv R(0)$ of the cavity in the deformed configuration is given by (2.22) and (2.25) i.e.

$$\left(\frac{R_c}{a}\right)^8 = \frac{(1-t_a)^4(t_a^2+t_a+4)h(0)}{4h(t_a)t_a^6}(>0) \tag{2.30}$$

where t_a is the root of (2.26). It is not difficult to confirm from (2.30) and (2.16) that R_c increases with decreasing t_a. Furthermore it follows from (2.26) and the monotonic decreasing character of $G(t)$ in (2.27) that t_a decreases with increasing λ. Thus the cavity radius R_c increases monotonically as the prescribed stretch λ is increased. Note that as $\lambda \to \infty$ (and so from (2.26), as $t_a \to 0$) it follows from (2.30) that $R_c \to \infty$. The variation of the cavity radius with prescribed stretch as described by (2.26), (2.30) is depicted by the solid curve in Figure 1 (on the range $\lambda > \lambda_{cr}$).

We note from (2.3), (2.12), (2.15)-(2.18) that $\lambda_r \to 0$, $\lambda_\theta \to \infty$ and $J = \lambda_r\lambda_\theta \to \infty$ as one approaches the cavity wall (i.e. as $t \to 0$). This is hardly surprising since the center of the undeformed cylinder maps onto every point of the cavity wall and the deformation is not one-to-one at the origin. It can also be easily verified that in the limit $\lambda \to \lambda_{cr}+$ (i.e. as $t_a \to 1$) the bifurcated solution $R(r)$ converges to the homogeneous solution λr at each value of r, $0 \leqslant r \leqslant a$.

It is of interest to examine the character of the system of governing partial differential equations, namely the displacement equations of equilibrium

$$c_{\alpha\beta\gamma\delta}(\boldsymbol{F})u_{\gamma,\delta\beta} = 0 \quad \text{where } c_{\alpha\beta\gamma\delta}(\boldsymbol{F}) = \frac{\partial^2 W(\boldsymbol{F})}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}}, \tag{2.31}$$

at the solutions obtained here. The usual cartesian tensor notation has been employed in (2.31). Necessary and sufficient conditions for the ellipticity of the system of equations (2.31) have been obtained by Knowles and Sternberg in [6]. In the special case of the Blatz-Ko material, W is given by (2.5) and these conditions are particularly simple. Thus from equation (2.57) of [6] it follows that the equations (2.31) are elliptic at the axisymmetric solutions here if and only if the principal stretches λ_r, λ_θ are such that ($t = \lambda_r/\lambda_\theta$)

$$2-\sqrt{3} < t < 2+\sqrt{3}. \tag{2.32}$$

Since in the present problem we have $t \leqslant 1$, it follows that the right hand inequality in (2.32) always holds; ellipticity will be lost whenever the left hand inequality is violated. For the homogeneous solution $R(r) = \lambda r$ one has $t = 1$ at all points in the body and at all values of the prescribed stretch. The governing equations are thus elliptic at this solution no matter how large the value of λ. On the other hand in the case of the bifurcated solution, t increases monotonically from the value $t = 0$ at the cavity to the value $t = t_a$ at the outer boundary. Thus ellipticity is *always* lost at the bifurcated solution in a zone adjacent to the cavity.

3. Growth of a micro-void

3.1. Formulation and solution

In order to better understand the physical implications of the preceding bifurcation problem, we turn our attention here to consider the growth of a small *pre-existing* void under remotely applied stretch (the "micro-void problem"). In this section we summarize the solution to the latter problem.

Consider now a circular cylinder which is hollow in its undeformed configuration and has inner and outer radii b and a respectively. The cylinder is subjected to a prescribed uniform radial displacement at its outer boundary so that $R(a)=\lambda a$, $\lambda > 1$. The inner surface is free of traction. The differential equation governing the deformation $R(r)$ continues to be given by (2.8) which now holds on $b<r<a$. The associated stress components are found from (2.6).

The solution procedure given in Section 2 can be readily adapted to the present problem. The resulting deformation field is, again, given parametrically by (2.15), (2.16), (2.18) with the parameter t now being in the range

$$t_b \leqslant t \leqslant t_a. \tag{3.1}$$

Here $t_a(<1)$ and $t_b(>0)$ are the values of t corresponding to $r=a$ and $r=b$ respectively and are to be determined, according to (2.15), by

$$\frac{Ct_a^6 h(t_a)}{(1-t_a)^4(t_a^2+t_a+4)} = a^8, \quad \frac{Ct_b^6 h(t_b)}{(1-t_b)^4(t_b^2+t_b+4)} = b^8. \tag{3.2}$$

On using (2.18), the boundary condition at the outer surface yields

$$\frac{D(t_a^2+t_a+4)h(t_a)}{(1-t_a)^2} = \lambda^4 a^4. \tag{3.3}$$

Finally, the vanishing of the radial stress τ_{RR} at $r=b$ results in

$$h(t_b)(t_b^2+t_b+4)^3 = C/D^2, \tag{3.4}$$

in view of $(2.6)_1$, (2.12), (2.15), (2.18).

Thus for prescribed $\lambda > 1$, the solution is given by (2.15), (2.16), (2.18) with t in the range (3.1) and the four constants t_a, t_b, $C>0$, $D>0$ are to be found from (3.2)-(3.4). Since we are chiefly concerned with small voids ($b \ll a$), we will not carry out a general discussion of the foregoing solution.

It is convenient for our purposes to eliminate C, D between (3.2)-(3.4) and so obtain

$$\frac{(t_a^2+t_a+4)^3 h(t_a)}{t_a^6(t_b^2+t_b+4)^3 h(t_b)} = \lambda^8. \tag{3.5}$$

3.2. *Discussion*

Observe first that the *radius* R_c of the deformed void is given by (2.18) evaluated at $t=t_b$, which, on using (3.2)-(3.4), can be written as

$$R_c = b t_b^{-3/4}, \tag{3.6}$$

with t_b given by (3.2)-(3.4). Graphs of R_c/a versus λ, obtained through numerical calculations, are shown by the dashed curves in Figure 1. Observe that for small values of the undeformed void radius b, (e.g. $b/a = 0.02$) the void first grows *slowly* until λ approaches a critical value. *Rapid* growth takes place beyond this point. As $b \to 0+$, the dashed curves approach the solid curve, corresponding to a void of zero radius – a "micro-void". In this case, the micro-void remains unchanged for $\lambda \leqslant \lambda_{cr}$ and suddenly begins to grow as λ exceeds λ_{cr}.

We now examine this limiting case of a *micro-void* in some detail. Thus, we consider the limit of the preceding solution as $b \to 0+$ with a, λ and r held fixed. Note from $(3.2)_2$ and (2.16) that there are two cases to be considered, namely that in which * $t_b \to 0$ and that in which $C \to 0$.

Case (1) $t_b \to 0$:

Equations (3.3)-(3.5) are readily seen to lead to

$$t_a^{-6}\left(t_a^2+t_a+4\right)^3 h(t_a) = 64h(0)\lambda^8, \tag{3.7}$$

$$D^2 = \frac{a^8(1-t_a)^4\left(t_a^2+t_a+4\right)}{64h(0)t_a^6 h(t_a)}, \tag{3.8}$$

$$C = 64h(0)D^2, \tag{3.9}$$

in this limit. Consequently, in this limit, the *solution* is formally given by (2.15), (2.16), (2.18) with the parameter t in the range

$$0 \leqslant t \leqslant t_a, \tag{3.10}$$

and with the positive constants $t_a(<1)$, C, D now given by (3.7)-(3.9).

The existence of this solution is ensured, provided (3.7) can be solved for a number t_a such that $0 < t_a < 1$, corresponding to the prescribed value of $\lambda > 1$. As shown in Section 2 in connection with (2.26), such a (unique) root t_a of (3.7) exists when $\lambda > \lambda_{cr}$ where λ_{cr} is given by (2.29). Consequently *case* (1), in fact, *applies in the range* $\lambda > \lambda_{cr}$, and on this interval, the deformation converges as $b \to 0+$ to the solution described in the preceding paragraph.

To further illustrate this convergence, it is instructive to derive an expression for the radius R_c of the deformed void in the present limit. A straightforward calculation using (3.6), (3.2), (3.4) leads to

$$\left(\frac{R_c}{a}\right)^8 = \frac{Ch(0)}{4a^8} + \frac{3}{2}\left(\frac{4-3t_a}{1-t_a}\right)\left(\frac{Ch(0)}{4a^8}\right)^{5/6}\left(\frac{b}{a}\right)^{4/3} + o\left(b^{4/3}\right), \tag{3.11}$$

as $b \to 0+$, where t_a and C are found from (3.7)-(3.9). To leading order this gives

$$\left(\frac{R_c}{a}\right)^8 = \frac{(1-t_a)^4\left(t_a^2+t_a+4\right)h(0)}{4h(t_a)t_a^6}(>0). \tag{3.12}$$

Note that in the present case, the deformed void radius R_c does *not* tend to zero as the undeformed radius $b \to 0+$. Equations (3.7), (3.12) describe the solid curve in Figure 1 (on the range $\lambda > \lambda_{cr}$).

Case (2) $C \to 0$:

Note from $(3.2)_1$ that in this case $t_a \to 1$ as $b \to 0+$ and so (3.5) yields

$$h(t_b)\left(t_b^2+t_b+4\right)^3 = 216h(1)\lambda^{-8} \tag{3.13}$$

* In what follows, it is to be understood that all limits involving the quantities t_a, t_b, C and D are the appropriate one-sided limits.

in this limit. Furthermore observe from (3.3) that necessarily $D \to 0$. However, the ratio C/D^2 remains bounded and is given by (3.4), (3.13) as

$$C/D^2 = 216h(1)\lambda^{-8}. \tag{3.14}$$

Monotonicity arguments similar to those used in Case (1) show that (3.13) can be solved for a value of t_b $(0 < t_b < 1)$ if and only if $\lambda < \lambda_{cr}$, and consequently *case* (2), in fact, *applies in the range* $1 < \lambda < \lambda_{cr}$.

Thus, when $\lambda < \lambda_{cr}$, the deformation converges as $b \to 0+$ to (2.15), (2.18) where t lies in the interval $t_b \leqslant t \leqslant 1$, and t_b, C/D^2 are given by (3.13), (3.14), with $C \to 0$, $D \to 0$. We now show that this deformation field, in fact, coincides with a *homogeneous deformation*. Observe from (2.15), (2.18) that

$$\left(\frac{R}{r}\right)^8 = \frac{D^2}{C}\frac{(t^2+t+4)^3 h(t)}{t^6}. \tag{3.15}$$

Furthermore it follows from (2.15) that, since $C \to 0$, we must have $t(r) \to 1$ for each fixed value of r in $0 < r \leqslant a$. Thus, in this limit we find from (3.15) and (3.14) that

$$R = \lambda r. \tag{3.16}$$

Moreover, it follows from (2.12) and (3.16) that $\dot{R}(r) \to \lambda$ for $0 < r \leqslant a$ as $b \to 0+$. However, since t_b tends to the root of (3.13) and not to unity, the derivative $\dot{R}(b)$ does *not* approach λ as $b \to 0+$.

Again, it is of interest to derive an expression for the radius R_c of the deformed void when $\lambda < \lambda_{cr}$. This is given by (2.18) evaluated at $t = t_b$ which can be written, on using $(3.2)_2$, (3.4) as

$$R_c = bt_b^{-3/4} + o(b), \quad \text{as } b \to 0+, \tag{3.17}$$

provided t_b is found from (3.13). Thus in the present case the deformed radius $R_c \to 0+$ as the undeformed radius $b \to 0+$, and so the dashed curves in Figure 1 (in the range $1 < \lambda < \lambda_{cr}$) approach the horizontal axis.

4. Concluding remarks

Finally, we proceed to a comparison between the solutions to the two problems discussed in Sections 2 and 3. Observe first that in the case $\lambda > \lambda_{cr}$, equations (2.24)-(2.26) pertaining to the bifurcated solution are identical to (3.7)-(3.9) obtained for the solution of the micro-void problem. Thus the values of the constants t_a, D and C are exactly the same in both solutions. Moreover, the ranges of the parameter t in both solutions, given by (2.19), (3.10), then coincide and so the deformation fields in both problems are *identical*.

Secondly, when $\lambda < \lambda_{cr}$, equation (2.10) pertaining to the homogeneous solution of the bifurcation problem is exactly the same as equation (3.16) for the micro-void problem and so the deformation fields in both problems are *identical* for this range of λ also.

Consequently, the bifurcation problem of Section 2 may be viewed as providing an idealized model describing the growth of a pre-existing micro-void. In particular, the solid curve in Figure 1, initially obtained for the bifurcation problem, is now seen to also describe micro-void growth.

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Note added in proof

The results obtained in this paper may be readily extended to the corresponding *three-dimensional* axisymmetric problem involving nucleation of a spherical void. (See [7] for other results on spherically symmetric deformations of the Blatz-Ko material). In this case, the corresponding value of the critical stretch is given by $\lambda_{cr} = 1.30874$.

Since this paper was written (October, 1984), other related papers have come to our attention. These are listed as References [8–10] above.