Journal of Elasticity 37: 1–38, 1995.

Conewise linear elastic materials

ALAIN CURNIER, QI-CHANG HE and PHILIPPE ZYSSET
Laboratoire de Mécanique Appliquée, Département de Génie Mécanique, Ecole Polytechnique Fédérale de Lausanne, CH-1015, Lausanne, Switzerland

Received 27 May 1993; in revised form 21 June 1994

Abstract. *Conewise linear elastic* (CLE) materials are proposed as the proper generalization to two and three dimensions of one-dimensional *bimodular* models. The basic elements of classical smooth elasticity are extended to *nonsmooth* (or piecewise smooth) elasticity. Firstly, a necessary and sufficient condition for a stress–strain law to be *continuous* across the interface of the tension and compression subdomains is established. Secondly, a sufficient condition for the strain energy function to be strictly *convex* is derived. Thirdly, the representations of the energy function, stress–strain law and elasticity tensor are obtained for *orthotropic, transverse isotropic* and *isotropic* CLE materials. Finally, the previous results are specialized to a *piecewise linear* stress–strain law and it is found out that the pieces must be polyhedral convex cones, thus the CLE name.

1. Introduction

This work is concerned with the formulation of constitutive laws for *elastic materials* presenting a *different behavior in tension and compression* under small deformations, as sketched in Fig. 1. Many fiber-reinforced and granular composites such as boron/epoxy, kevlar/rubber and graphite/epoxy can be cited as examples of such materials [Jones (1977); Vijayakumar and Rao (1987)]. Moreover, a damaged brittle material such as concrete or ceramics, which contains microcracks and/or microvoids, generally exhibits different stiffnesses or compliances under tensile and compressive loading, owing to the unilateral nature of the damage [Horii and Nemat-Nasser (1983); Chaboche (1990); Mazars, Berthaud and Ramtani (1990); Costa, Fremond and Mamiya (1992)].

In the literature [see, e.g. Tabaddor (1979)], the word "bimodulus" is often used for referring to a material whose one-dimensional stress–strain law is *piecewise linear* as represented in Fig. 1, because it is characterized by different Young's moduli in tension and compression. Concentrating the non-linearity at the origin is probably the simplest way for describing a dissymmetry between tensile and compressive behavior. Extension of such models to two- and three-dimensional cases began with the works of Ambartsumyan and his collaborators [Ambartsumyan (1965); Ambartsumyan and Khachatryan

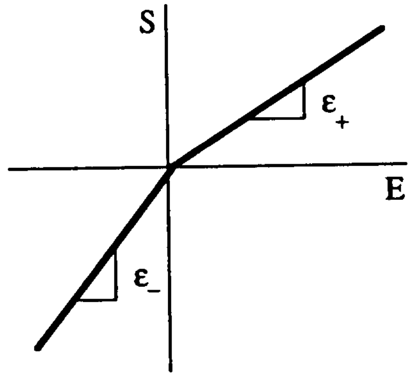

Fig. 1. One-dimensional stress–strain law for a "bimodulus" material.

(1966) and Ambartsumyan (1969)] and was then pursued by a number of other authors [Shapiro (1966); Novak and Bert (1968); Green and Mkrtichian (1977); Spence and Mkrtichian (1977); Bert (1977); Jones (1977) Tabaddor (1979); Kamiya (1979); Vijayakumar and Rao (1987) and Sacco and Reddy (1992)]. In several studies [e.g. Green and Mkrtichian (1977); Rigbi (1980) and Medri (1982)], the tension-compression non-linearity is mistaken with material anisotropy. Confusing the tension-compression non-linearity with material anisotropy is perhaps due to the fact that most previously proposed strain–stress relations were not written in invariant forms. For this same reason, it is sometimes difficult to verify if a given stress–strain relation respects the invariance conditions imposed by the material symmetries. Finally, a number of bimodulus constitutive laws [Green and Mkrtichian (1977); Rigbi (1980) and Sacco and Reddy (1992)] based on principal stresses or strains are not piecewise linear due to the nonlinear dependence of the eigenvalues of a tensor on this tensor. Moreover such stress–strain relations imply high computational costs since they require the resolution of the eigenvalue problem.

In this paper, we propose a definition of *conewise linear elastic* (CLE) materials as the proper generalization to two and three dimensions of one-dimensional bimodular models. To this end, some basic properties of classical smooth elasticity, namely existence, differentiability, convexity, (an-) isotropy and quadratic form of the elastic energy function [e.g., Gurtin (1981); Marsden and Hughes (1983) and Ciarlet (1988)], are extended to *nonsmooth* or *piecewise smooth* elasticity. Firstly, a necessary and sufficient condition for a stress–strain law to be *continuous* ($\mathscr{C}^0$) across the interface of the tension and compression subdomains, i.e. for the energy function to be differentiable ($\mathscr{C}^1$) across the same interface, is established. The effect of this condition on the interface shape is determined. Secondly, a sufficient condition for the strain energy function to be (wholewise) strictly *convex* is derived. The strain–stress relation is then obtained by inverting the stress–strain one and the complementary energy by using a Legendre transformation. Thirdly, the *invariant* forms of the energy function, stress–strain law and elasticity tensor are obtained for isotropic, transverse isotropic and orthotropic CLE materials, by using the invariants of the strain tensor [e.g. Boehler (1978) and Spencer (1982)]. Finally, the previous results are specialized to a piecewise quadratic energy function with a *piecewise linear* stress–strain law, relevant for small deformation analyses, and it is found out that the pieces must be pyramids, i.e. polyhedral convex cones, thus the CLE name attributed to such materials.

The paper is composed of five sections. In Section 2, the classical *smooth elasticity* background which is relevant for this work is summarized. In Section 3, the generalized framework outlined above for *nonsmooth elasticity* is proposed. In Section 4, the developed theory is *specialized* to orthotropic, transversely isotropic and isotropic CLE materials undergoing small deformations. A few conclusions are drawn in Section 5.

2. Smooth elasticity background

In this section, some elements of smooth nonlinear elasticity are summarized for later reference [e.g., Gurtin (1981); Marsden and Hughes (1983) and Ciarlet (1988)]. A few developments regarding the representation of the fourth-order elasticity tensor are novel.

The main *notations* used in the article are the following. The Green–Lagrange material strain $\mathbf{E}$ and the second Piola–Kirchhoff material stress $\mathbf{S}$ are used throughout. Both measures are symmetric $\mathbf{E} = \mathbf{E}^T$, $\mathbf{S} = \mathbf{S}^T$ and objective, i.e. invariant in a change of frame of reference $\mathbf{E}^* = \mathbf{E}$, $\mathbf{S}^* = \mathbf{S}$. Both $\mathbf{E}$ and $\mathbf{S}$ belong to "vector" spaces of symmetric second-order tensors, herein denoted $\mathscr{E}$ and $\mathscr{S}$ respectively and equipped with a norm and inner product, e.g., $\|\mathbf{E}\| = (\operatorname{tr} \mathbf{E}^2)^{1/2}$ and $\mathbf{E}:\mathbf{H} = \operatorname{tr}(\mathbf{E}^T\mathbf{H})$. The internal energy density product is again denoted $\mathbf{S}:\mathbf{E}$. Fourth order elasticity and compliance tensors are respectively denoted $\mathbb{S}$ and $\mathbb{E}$ and their vector spaces $\boldsymbol{\mathscr{S}}$ and $\boldsymbol{\mathscr{E}}$. The tensor product of two second-order tensors is defined by $[\mathbf{A} \otimes \mathbf{B}]\mathbf{X} = (\mathbf{B}:\mathbf{X})\mathbf{A}$, $\forall \mathbf{X} \in \mathscr{E}$. Another three useful similar products are defined by $[\mathbf{A} \underline{\otimes} \mathbf{B}]\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{B}^T$, $\forall \mathbf{X} \in \mathscr{E}$ [Halmos (1958), Del Piero (1979) and Lucchesi et al. (1990)], $[\mathbf{A} \bar{\otimes} \mathbf{B}]\mathbf{X} = \mathbf{A}\mathbf{X}^T\mathbf{B}^T$, $\forall \mathbf{X} \in \mathscr{E}$ and $[\mathbf{A} \underline{\bar{\otimes}} \mathbf{B}]\mathbf{X} = (\mathbf{A}\mathbf{X}\mathbf{B}^T + \mathbf{B}\mathbf{X}^T\mathbf{A}^T)/2$, $\forall \mathbf{X} \in \mathscr{E}$, i.e. $\mathbf{A} \underline{\bar{\otimes}} \mathbf{B} = [\mathbf{A} \underline{\otimes} \mathbf{B} + \mathbf{B} \bar{\otimes} \mathbf{A}]/2$ [Curnier (1993)]. The intrinsic (direct) notation is used throughout.

Although $\mathbf{E}$ and $\mathbf{S}$ are valid measures for large displacements and large strain formulations, the present study is limited to the large displacement but *small strain* case. Thus $\mathbf{E}$ belongs in fact to the convex subset $\mathscr{E}' = \{\mathbf{E} \in \mathscr{E} \mid \|\mathbf{E}\| \leqslant M \ll 1\}$.

2.1. Smooth hyperelasticity

Classically, a hyperelastic material is characterized by the existence of a smooth (twice continuously differentiable) potential energy density $W(\mathbf{E})$ from which the stress–strain law $\mathbf{S}(\mathbf{E})$ and the symmetric elasticity tensor $\mathbb{S}(\mathbf{E})$ are derived:

$$W = W(\mathbf{E}), \qquad W(0) = 0, \qquad W \in \mathscr{C}^2(\mathscr{E}, \mathscr{R}); \tag{2.1}$$

$$\mathbf{S} = \mathbf{S}(\mathbf{E}) = \nabla_{\mathbf{E}} W(\mathbf{E}), \qquad \mathbf{S}(\mathbf{0}) = \mathbf{0}, \qquad \mathbf{S} \in \mathscr{C}^1(\mathscr{E}, \mathscr{S}); \tag{2.2}$$

$$\mathbb{S} = \mathbb{S}(\mathbf{E}) = \nabla^2_{\mathbf{E}} W(\mathbf{E}), \qquad \mathbb{S} \in \mathscr{C}^0(\mathscr{E}, \mathbb{\mathscr{S}}). \tag{2.3}$$

In (2.1) and (2.2), the energy W and the stress $\mathbf{S}$ are assumed to be zero in the reference configuration, without loss of generality and for simplicity respectively (whereas $\mathbb{S}(\mathbf{0})$ must be positive definite as specified in (2.8)). The fourth-order Hessian tensor $\mathbb{S}$ possesses two minor symmetries resulting from those of $\mathbf{E}$ and $\mathbf{S}$ and a major one due to the existence of W:

$$\mathbb{S}(\mathbf{E})\mathbf{A} = \mathbf{0}, \ \forall \mathbf{E} \in \mathscr{E}, \ \forall \mathbf{A} \in \mathscr{A}; \quad \mathbb{S}(\mathbf{E})\mathbf{H} = [\mathbb{S}(\mathbf{E})\mathbf{H}]^T, \ \forall \mathbf{E}, \mathbf{H} \in \mathscr{E}; \tag{2.4}$$

$$\mathbf{H} : \mathbb{S}(\mathbf{E})\mathbf{G} = \mathbf{G} : \mathbb{S}(\mathbf{E})\mathbf{H}, \ \forall \mathbf{E}, \mathbf{G}, \mathbf{H} \in \mathscr{E} \Leftrightarrow \mathbb{S}^T(\mathbf{E}) = \mathbb{S}(\mathbf{E}); \tag{2.5}$$

where $\mathscr{A}$ is the space of skew second-order tensors. The major symmetry (2.5) is a necessary and sufficient condition for the existence of W [Marsden and Hughes (1983)].

2.2. Convexity. Monotony. Positiveness

For stability reasons in small strain situations, it is usually assumed that the material elastic potential W is strictly convex or equivalently that the stress–strain law $\mathbf{S}$ is strictly monotone or finally that the elasticity tensor $\mathbb{S}$ is positive definite, i.e. that $\forall \mathbf{E}, \mathbf{H} \in \mathscr{E}$, $\mathbf{E} \neq \mathbf{H}$,

$$W[\lambda \mathbf{E} + (1 - \lambda)\mathbf{H}] < \lambda W(\mathbf{E}) + (1 - \lambda) W(\mathbf{H}), \ \forall \lambda \in [0, 1], \tag{2.6a}$$

$$W(\mathbf{H}) > W(\mathbf{E}) + \mathbf{S}(\mathbf{E}) : (\mathbf{H} - \mathbf{E}), \tag{2.6b}$$

$$[\mathbf{S}(\mathbf{H}) - \mathbf{S}(\mathbf{E})] : (\mathbf{H} - \mathbf{E}) > 0, \tag{2.7}$$

$$(\mathbf{H} - \mathbf{E}) : \mathbb{S}(\mathbf{E})(\mathbf{H} - \mathbf{E}) > 0. \tag{2.8}$$

REMARK. Although convexity of the nominal energy function (in terms of the deformation gradient) is not acceptable at finite strains for multiple reasons (objectivity, non-uniqueness, basic inequalities...), convexity (2.6) of the material energy function is acceptable at least at small strains, i.e.

replacing $\mathscr{E}$ by $\mathscr{E}'$ in (2.6)–(2.8). Note that since $W \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$, (2.6a) $\Leftrightarrow$ (2.6b) $\Leftrightarrow$ (2.7) $\Leftarrow$ (2.8) whereas (2.7) $\Rightarrow$ (2.8) only holds in the non-strict case. In words, positive definiteness is sufficient but not necessary for strict convexity.

Due to strict convexity, the inverse strain–stress law exists and derives from an elastic complementary energy density given by a Legendre transformation

$$W^* = W^*(\mathbf{S}) = \mathbf{S}:\mathbf{E} - W(\mathbf{E}), \qquad W^*(\mathbf{0}) = 0, \tag{2.9}$$

$$\mathbf{E} = \mathbf{E}(\mathbf{S}) = \mathbf{S}^{-1}(\mathbf{S}) = \nabla_{\mathbf{S}} W^*(\mathbf{S}), \qquad \mathbf{E}(\mathbf{0}) = \mathbf{0}, \tag{2.10}$$

$$\mathbb{E} = \mathbb{E}(\mathbf{S}) = \mathbb{S}^{-1}(\mathbf{S}) = \nabla^2_{\mathbf{S}} W^*(\mathbf{S}). \tag{2.11}$$

The complementary energy and the strain are equal to zero at the stress origin, just as their inverses. The inverse elasticity tensor $\mathbb{E}$ is called the compliance tensor. It has the same symmetry and positiveness properties as $\mathbb{S}$.

2.3. Material symmetries

For a constitutive law to comply with the material symmetries, W and (thus) $\mathbf{S}$ and $\mathbb{S}$ must be invariant under the corresponding symmetry group $\mathscr{G}$, i.e. $\forall \mathbf{E} \in \mathscr{E}$, $\forall \mathbf{Q} \in \mathscr{G} \subseteq \mathscr{O}$,

$$W(\mathbf{QEQ}^T) = W(\mathbf{E}), \tag{2.12}$$

$$\mathbf{S}(\mathbf{QEQ}^T) = \mathbf{QS}(\mathbf{E})\mathbf{Q}^T, \tag{2.13}$$

$$\mathbb{S}(\mathbf{QEQ}^T) = [\mathbf{Q} \underline{\otimes} \mathbf{Q}]\mathbb{S}(\mathbf{E})[\mathbf{Q} \underline{\otimes} \mathbf{Q}]^T, \tag{2.14}$$

where $\mathscr{O}$ is the full proper orthogonal group. For an isotropic material $\mathscr{G} = \mathscr{O}$ whereas $\mathscr{G} \subset \mathscr{O}$ for an anisotropic one.

REMARK. The invariance conditions (2.12)–(2.14) involve congruences which can be systematically written by means of the tensor product $\underline{\otimes}$ as, $\forall \mathbf{E} \in \mathscr{E}$, $\forall \mathbf{Q} \in \mathscr{G} \subseteq \mathscr{O}$,

$$W([\mathbf{Q} \underline{\otimes} \mathbf{Q}]\mathbf{E}) = W(\mathbf{E}),$$

$$\mathbf{S}([\mathbf{Q} \underline{\otimes} \mathbf{Q}]\mathbf{E}) = [\mathbf{Q} \underline{\otimes} \mathbf{Q}]\mathbf{S}(\mathbf{E}),$$

$$\mathbb{S}([\mathbf{Q} \underline{\otimes} \mathbf{Q}]\mathbf{E}) = [(\mathbf{Q} \underline{\otimes} \mathbf{Q}) \underline{\otimes} (\mathbf{Q} \underline{\otimes} \mathbf{Q})]\mathbb{S}(\mathbf{E}).$$

It can be shown that (2.12) $\Rightarrow$ (2.13) $\Rightarrow$ (2.14).

Representation theorems give the most general form that W can take in terms of basic invariants of $\mathbf{E}$ under the group $\mathscr{G}$ and, by differentiation, the corresponding forms for $\mathbf{S}$ and $\mathbb{S}$:

$$W(\mathbf{E}) = \omega(I_{\mathbf{E}}), \quad I_{\mathbf{E}} := \{I_i(\mathbf{E}),\ i = 1, p\}, \tag{2.15}$$

$$\mathbf{S}(\mathbf{E}) = \omega_i \mathbf{G}_i, \quad \omega_i := \frac{\partial \omega}{\partial I_i}(I_{\mathbf{E}}), \quad \mathbf{G}_i := \nabla_{\mathbf{E}} I_i(\mathbf{E}), \quad (i = 1, p), \tag{2.16}$$

$$\begin{aligned} &\mathbb{S}(\mathbf{E}) = \omega_{ij} \mathbf{G}_i \otimes \mathbf{G}_j + \omega_i \nabla_{\mathbf{E}} \mathbf{G}_i, \quad (i, j = 1, p), \\ &\omega_{ij} := \frac{\partial^2 \omega}{\partial I_i \partial I_j}(I_{\mathbf{E}}) = \omega_{ji}, \quad \nabla_{\mathbf{E}} \mathbf{G}_i = \nabla^2_{\mathbf{E}} I_i(\mathbf{E}). \end{aligned} \tag{2.17}$$

In (2.15)–(2.17), I_i are invariants of $\mathbf{E}$ under $\mathscr{G}$ forming an irreducible "functional" basis for the energy function ω (with first derivatives $\omega_i = \partial\omega/\partial I_i$ and symmetric second derivatives $\omega_{ij} = \partial^2\omega/\partial I_i \partial I_j$), and $\mathbf{G}_i = \nabla_{\mathbf{E}} I_i$ are the corresponding generators, invariant under $\mathscr{G}$, for second-order symmetric tensors as well as for fourth-order symmetric tensors by means of their gradients $\nabla_{\mathbf{E}} \mathbf{G}_i$ and the tensor products $\mathbf{G}_i \otimes \mathbf{G}_j$. The summation/enumeration convention on dummy/frank repeated indices is used throughout.

The symmetry condition $\omega_{ij} = \omega_{ji}$ stated in (2.17) is necessary and sufficient for a law in the form $\mathbf{S}(\mathbf{E}) = \omega_i(I_{\mathbf{E}})\mathbf{G}_i(\mathbf{E})$ to derive from a potential $\omega(I_{\mathbf{E}})$. Equation (2.17) suggests a decomposition of the elasticity tensor into a "bulk" part and a "shear" part along

$$\mathbb{S}(\mathbf{E}) = \bar{\mathbb{S}}(\mathbf{E}) + \mathbb{S}'(\mathbf{E}), \quad \bar{\mathbb{S}}(\mathbf{E}) = \omega_{ij} \mathbf{G}_i \otimes \mathbf{G}_j, \quad \mathbb{S}'(\mathbf{E}) = \omega_i \nabla_{\mathbf{E}} \mathbf{G}_i.$$

For anisotropic materials, the group $\mathscr{G}$ is most conveniently characterized by means of texture tensors $\mathbf{A}_a$ as follows:

$$\mathscr{G} = \{\mathbf{Q} \in \mathscr{O} \mid \mathbf{Q}\mathbf{A}_a\mathbf{Q}^T = \mathbf{A}_a,\ a = 1, d\}. \tag{2.18}$$

Finding a representation of the $\mathscr{G}$-anisotropic function $W(\mathbf{E})$ is then equivalent to constructing an $\mathscr{O}$-isotropic function $\hat{W}(\mathbf{E}, \mathbf{A}_a)$ such that $W(\mathbf{E}) = \hat{W}(\mathbf{E}, \mathbf{A}_a)$ [e.g., Boehler (1978) and Liu (1982)]. The texture tensors $\mathbf{A}_a$, symmetry groups $\mathscr{G}$, corresponding invariants I_i, second-order generators $\mathbf{G}_i$ and fourth-order generators $\nabla_{\mathbf{E}} \mathbf{G}_i$ are summarized in Table 1 for isotropic ($\mathbf{A}_1 = \mathbf{I}$), transverse isotropic ($\mathbf{A}_1 = \mathbf{I}, \mathbf{A}_2 = \mathbf{A}$) and orthotropic ($\mathbf{A}_1 = \mathbf{I}, \mathbf{A}_2 = \mathbf{A}, \mathbf{A}_3 = \mathbf{B}$) materials. In the orthotropic case, the third direction vector and associated texture tensor are uniquely determined by orthogonality as $\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$ and $\mathbf{C} = \mathbf{I} - \mathbf{A} - \mathbf{B}$. Consequently, $\mathbf{I}$ can be replaced by $\mathbf{C}$, $\operatorname{tr} \mathbf{E}$ by $\operatorname{tr} \mathbf{CE}$ and $\operatorname{tr} \mathbf{E}^2/2$ by $\operatorname{tr} \mathbf{CE}^2$ ($\operatorname{tr} \mathbf{E}^3/3$ stays) in the isotropy column, in order to retrieve a cyclic symmetry over the three directions $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ ($\mathbf{A}_1 = \mathbf{A}, \mathbf{A}_2 = \mathbf{B}, \mathbf{A}_3 = \mathbf{C}$).

REMARKS. (i) Without the hyperelasticity hypothesis $\mathbf{S} = \nabla_{\mathbf{E}} W$ adopted here, the generator $\mathbf{G}_{p+1} = (\mathbf{AE}^2 + \mathbf{E}^2\mathbf{A})$ and its gradient

$$\nabla_{\mathbf{E}} \mathbf{G}_{p+1} = \mathbf{A} \,\bar{\underline{\otimes}}\, \mathbf{E} + \mathbf{E} \,\bar{\underline{\otimes}}\, \mathbf{A}$$

Table 1. Elements of representation for the main types of anisotropy in hierarchical order

Symmetry	Isotropy ($d = 1$, $p = 3$)	Transverse Isotropy (2, 5)	Orthotropy (3, 7)
$\mathbf{A}_a (a=1, d)$	$\mathbf{I}$	$\mathbf{A}=\mathbf{a}\otimes\mathbf{a}$, $\lVert\mathbf{a}\rVert=1$	$\mathbf{B}=\mathbf{b}\otimes\mathbf{b}$, $\lVert\mathbf{b}\rVert=1, \mathbf{a}\cdot\mathbf{b}=0$
$\mathscr{G}$	$\mathscr{O}=\{\mathbf{Q}\in\mathscr{E}\mid\mathbf{QIQ}^T=\mathbf{I}\}$	$\mathscr{T}=\{\mathbf{Q}\in\mathscr{O}\mid\mathbf{QAQ}^T=\mathbf{A}\}$	$\mathscr{P}=\{\mathbf{Q}\in\mathscr{T}\mid\mathbf{QBQ}^T=\mathbf{B}\}$
$I_i\ (i=1, p)$	$\operatorname{tr}\mathbf{E}$, $\operatorname{tr}\mathbf{E}^2/2$, $\operatorname{tr}\mathbf{E}^3/3$	$\operatorname{tr}\mathbf{AE}$, $\operatorname{tr}\mathbf{AE}^2$	$\operatorname{tr}\mathbf{BE}$, $\operatorname{tr}\mathbf{BE}^2$
$\mathbf{G}_i=\nabla_{\mathbf{E}} I_i$	$\mathbf{I}$, $\mathbf{E}$, $\mathbf{E}^2$	$\mathbf{A}$, $\mathbf{AE}+\mathbf{EA}$	$\mathbf{B}$, $\mathbf{BE}+\mathbf{EB}$
$\nabla_{\mathbf{E}}\mathbf{G}_i$	$\mathbb{O}$, $\mathbf{I}\,\underline{\bar{\otimes}}\,\mathbf{I}$, $\mathbf{E}\,\underline{\bar{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\bar{\otimes}}\,\mathbf{E}$	$\mathbb{O}$, $\mathbf{A}\,\underline{\bar{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\bar{\otimes}}\,\mathbf{A}$	$\mathbb{O}$, $\mathbf{B}\,\underline{\bar{\otimes}}\,\mathbf{I}+\mathbf{I}\,\underline{\bar{\otimes}}\,\mathbf{B}$

This table is *hierarchical* in the sense that, for transverse isotropy and orthotropy, only the *additional* invariants and generators with respect to isotropy and transverse isotropy respectively, are given. The zero tensors $\mathbb{O}$ are included in the last line in order to keep the number of fourth order generators equal to the numbers of invariants and second-order generators, which is necessary for an effectual application of the summation convention in (2.17).

must be added in the transversely isotropic column of Table 1, together with an $\omega_{p+1}(I_{\mathbf{E}})$ function in (2.16)–(2.17), and, with polynomial functions ω_i (instead of general ones), a generator $\mathbf{G}_{p+2} = (\mathbf{BE}^2 + \mathbf{E}^2\mathbf{B})$ and $\nabla_{\mathbf{E}}\mathbf{G}_{p+2}$ must be added in the orthotropic column with an $\omega_{p+2}(I_{\mathbf{E}})$ polynomial.

(ii) In Table 1, the first and second gradients of the invariants are calculated with the help of the directional derivative and by insisting on the symmetry of $\mathbf{E}$ as shown in Appendix A.

2.4. Linear elasticity

In the hypothesis of small strains, a linear stress–strain law (of the Kirchhoff–St. Venant's type), deriving from a quadratic potential involving a constant elasticity tensor (Fig. 2) often proves satisfactory:

$$W(\mathbf{E}) = \tfrac{1}{2}\mathbf{E}:\mathbb{S}\mathbf{E}, \tag{2.19}$$

$$\mathbf{S}(\mathbf{E}) = \mathbb{S}\mathbf{E}, \tag{2.20}$$

$$\mathbb{S} = \text{constant}, \tag{2.21}$$

Due to the symmetries (2.4) and (2.5), the tensor $\mathbb{S}$ is fully determined by 21 independent constants.

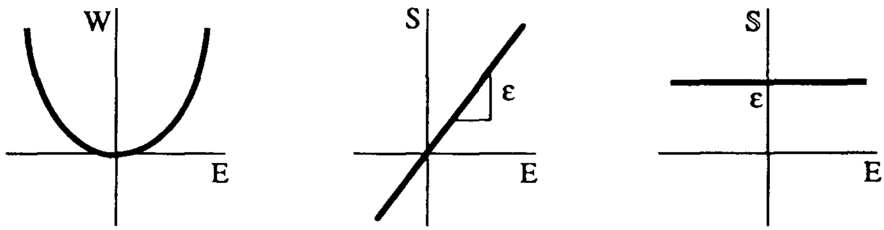

Fig. 2. (a) Quadratic potential, (b) linear stress–strain law, (c) constant elasticity modulus for a smooth elastic material in one dimension.

The complementary energy, inverse law and compliance tensor are given by

$$W^*(\mathbf{S}) = \tfrac{1}{2}\mathbf{S} : \mathbb{E}\mathbf{S}, \tag{2.22}$$

$$\mathbf{E}(\mathbf{S}) = \mathbb{E}\mathbf{S}, \tag{2.23}$$

$$\mathbb{E} = \mathbb{S}^{-1} = \text{constant}, \tag{2.24}$$

2.5. Linear orthotropic elasticity

Combining linearity with symmetries consists in retaining the complete quadratic combination of invariants in (2.15), linear combination of their gradients in (2.16) and constant combination of their Hesssians in (2.17), all three with constant "bulk" and "shear" Lamé-like coefficients λ_{ab} and μ_a:

$$W(\mathbf{E}) = \frac{\lambda_{ab}}{2}\,\mathrm{tr}(\mathbf{A}_a\mathbf{E})\,\mathrm{tr}(\mathbf{A}_b\mathbf{E}) + \mu_a\,\mathrm{tr}(\mathbf{A}_a\mathbf{E}^2), \quad (a, b = 1, d), \tag{2.25}$$

$$\mathbf{S}(\mathbf{E}) = \lambda_{ab}\,\mathrm{tr}(\mathbf{A}_a\mathbf{E})\mathbf{A}_b + \mu_a(\mathbf{A}_a\mathbf{E} + \mathbf{E}\mathbf{A}_a), \tag{2.26}$$

$$\mathbb{S} = \lambda_{ab}\mathbf{A}_a \otimes \mathbf{A}_b + \mu_a[\mathbf{A}_a \,\bar{\underline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\bar{\underline{\otimes}}\, \mathbf{A}_a], \quad \lambda_{ab} = \lambda_{ba}, \tag{2.27}$$

where $d = 1$ for *isotropy* (2 constants: the two usual Lamé constants λ and μ), $d = 2$ for *transverse isotropy* (5 constants) and $d = 3$ for *orthotropy* (9 constants). Hence, such combinations involve $(d^2 + d)$ terms and only $[d(d+1)/2 + d]$ constants. Explicit expressions are given in Appendix B.

The inverse relationships can be written as

$$W^*(\mathbf{S}) = -\frac{\nu_{ab}}{2\varepsilon_a}\,\mathrm{tr}(\mathbf{A}_a\mathbf{S})\,\mathrm{tr}(\mathbf{A}_b\mathbf{S}) + \frac{1 + \nu_{aa}}{2\varepsilon_a}\,\mathrm{tr}(\mathbf{A}_a\mathbf{S}^2), \quad (a, b = 1, d), \tag{2.28}$$

$$\mathbf{E}(\mathbf{S}) = -\frac{\nu_{ab}}{\varepsilon_a}\,\mathrm{tr}(\mathbf{A}_a\mathbf{S})\mathbf{A}_b + \frac{1 + \nu_{aa}}{2\varepsilon_a}(\mathbf{A}_a\mathbf{S} + \mathbf{S}\mathbf{A}_a), \tag{2.29}$$

$$\mathbb{E} = -\frac{\nu_{ab}}{\varepsilon_a}\mathbf{A}_a \otimes \mathbf{A}_b + \frac{1 + \nu_{aa}}{2\varepsilon_a}[\mathbf{A}_a \,\bar{\underline{\otimes}}\, \mathbf{I} + \mathbf{I} \,\bar{\underline{\otimes}}\, \mathbf{A}_a], \tag{2.30}$$

$$\frac{\nu_{ab}}{\varepsilon_a} = \frac{\nu_{ba}}{\varepsilon_b}, \quad (a, b = 1, d); \tag{2.31}$$

where ε_a are Young's like elastic moduli, ν_{ab} Poisson's like ratios with diagonal elements ν_{aa} related to the more usual shear moduli G_{ab} by

$$\frac{1 + \nu_{aa}}{\varepsilon_a} + \frac{1 + \nu_{bb}}{\varepsilon_b} = \frac{1}{G_{ab}} = \frac{2}{\mu_a + \mu_b}, \quad (a, b = 1, d). \tag{2.32}$$

Here W (respectively W^*) are strictly convex, **S** strictly monotone and $\mathbb{S}$ positive definite if and only if the bulk coefficient matrix $[\lambda_{ab}]$ ($[-\nu_{ab}/\varepsilon_a]$) is positive definite and the shear coefficients μ_a (G_{ab}) are positive. In the isotropic case, for instance, the conditions are

$$3\lambda + 2\mu > 0 \text{ and } \mu > 0 \quad \text{or} \quad \varepsilon > 0 \text{ and } -1 < \nu < 1/2,$$

where ε is Young's modulus and ν Poisson's ratio.

3. Nonsmooth elasticity framework

Among the four basic properties discussed in the previous summary, smoothness and linearity restrict the nonlinearity of the material law whereas convexity and symmetry ensure its invertibility and anisotropy respectively. In this section, the smoothness or linearity assumption is renounced in order to accommodate bimodular materials, whereas convexity and symmetry are maintained and exploited to restrict the possible forms of the resulting laws.

3.1. Piecewise smooth hyperelasticity

For modeling a hyperelastic material which behaves differently in tension and compression, it is natural to consider an elastic potential energy density which is (wholewise) continuously differentiable but only *piecewise* twice continuously differentiable. Indeed, the stress–strain law deriving from such a potential is (wholewise) continuous and piecewise continuously differentiable with a piecewise continuous elasticity tensor (discontinuous across the origin) and thus able to produce in a simple traction experiment a different response in tension and compression, with a *kink* at the origin as shown in Fig. 1.

3.1.1. Strain space division into a compression subdomain and a tension subdomain

To begin, let the strain space $\mathscr{E}$ be divided into a compression subdomain $\mathscr{E}_-$ and a tension subdomain $\mathscr{E}_+$ by means of a hypersurface I characterized by a *continuously differentiable* scalar valued function $g(\mathbf{E})$ as follows (Fig. 3):

$$\mathscr{I} := \{\mathbf{E} \in \mathscr{E} \mid g(\mathbf{E}) = 0\}, \quad g \in \mathscr{C}^1(\mathscr{E}, \mathscr{R}); \tag{3.1a}$$

$$\mathscr{E}_- := \{\mathbf{E} \in \mathscr{E} \mid g(\mathbf{E}) < 0\}, \quad \bar{\mathscr{E}}_- := \{\mathbf{E} \in \mathscr{E} \mid g(\mathbf{E}) \leqslant 0\}; \tag{3.1b}$$

$$\mathscr{E}_+ := \{\mathbf{E} \in \mathscr{E} \mid g(\mathbf{E}) > 0\}, \quad \bar{\mathscr{E}}_+ := \{\mathbf{E} \in \mathscr{E} \mid g(\mathbf{E}) \geqslant 0\}; \tag{3.1c}$$

$$\mathscr{E} = \mathscr{E}_- \cup \mathscr{E}_+ \cup \mathscr{I}, \quad \mathscr{E}_- \cap \mathscr{E}_+ = \varnothing, \quad \bar{\mathscr{E}}_- \cap \bar{\mathscr{E}}_+ = \mathscr{I}. \tag{3.1d}$$

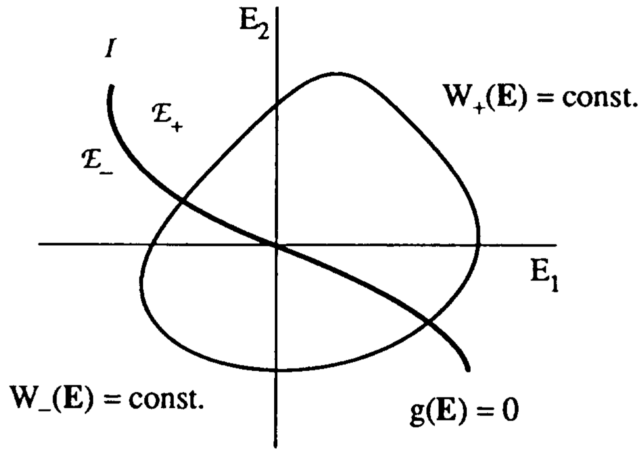

Fig. 3. Schematic partition of strain space $\mathscr{E}$ into a compression subdomain $\mathscr{E}_-$ and a tension subdomain $\mathscr{E}_+$ and typical equipotential lines of the two restrictions W_- and W_+ in $2D$ principal strain plane.

This subdivision of $\mathscr{E}$ being motivated by a different behavior in tension and compression, the interface must contain the strain space origin

$$\mathbf{0} \in \mathscr{I} \Leftrightarrow g(\mathbf{0}) = 0. \tag{3.2}$$

For instance, the simplest subdivision of the strain space into two *half-spaces* is provided by a *hyperplane* characterized by its unit normal $\mathbf{N} \in \mathscr{E}(\|\mathbf{N}\| = 1)$:

$$\mathscr{I} := \{\mathbf{E} \in \mathscr{E} \mid g(\mathbf{E}) = \mathbf{N}:\mathbf{E} = 0\}, \tag{3.3a}$$

$$\mathscr{E}_- := \{\mathbf{E} \in \mathscr{E} \mid \mathbf{N}:\mathbf{E} < 0\}, \quad \mathscr{E}_+ := \{\mathbf{E} \in \mathscr{E} \mid \mathbf{N}:\mathbf{E} > 0\}. \tag{3.3b}$$

More complicated subdivisions of $\mathscr{E}$ into several nonsmooth subdomains can be obtained by using several interfaces. For example, subdivisions into polyhedral convex cones (i.e. pyramids) will be discussed in Section 4.5 and an explicit subdivision into eight octants will be studied in Section 4.6.

REMARK. For the subdivision of $\mathscr{E}$ to be valid, the interface $\mathscr{I}$ must fulfill other properties than (3.2). For instance, $\mathscr{I}$ must be such that $\mathscr{E}_-$ and $\mathscr{E}_+$ are simply connected. In the sequel, $\mathscr{I}$ is assumed to possess *all* the properties necessary for the purpose of the study.

3.1.2. Piecewise smooth law

Next, assume that the material response can be separately described in compression and in tension by two elastic energy functions $W_-(\mathbf{E})$ and $W_+(\mathbf{E})$ with first and second derivatives. By using the above subdivision, these two functions can be restricted to their relevant subdomains $\mathscr{E}_-$ and $\mathscr{E}_+$ and then juxtaposed to give a single energy function $W(\mathbf{E})$ defined over $\mathscr{E}$. Provided proper continuity conditions are enforced across the interface, the first derivative $\mathbf{S}(\mathbf{E}) = \nabla_{\mathbf{E}} W(\mathbf{E})$ will exist everywhere and even be continuous across the interface. In this respect, it is important to understand that, if the elasticity tensor can be discontinuous across the interface, the stress–strain law must be continuous across it, for the material model to make sense.

More specifically, let $W_-(\mathbf{E})$ and $W_+(\mathbf{E})$ be two twice continuously differentiable energy functions defined over $\mathscr{E}$. A (piecewise twice, wholewise once) *continuously differentiable* energy function $W(\mathbf{E})$ is defined over $\mathscr{E}$, together with its first derivative $\mathbf{S}(\mathbf{E})$ and its piecewise second derivative $\mathbb{S}(\mathbf{E})$ where it exists, by the two *restrictions* $W_-(\mathbf{E})$ and $W_+(\mathbf{E})$ to the two *subdomains* $\mathscr{E}_-$ and $\mathscr{E}_+$ respectively, as follows:

$$W(\mathbf{E}) := \begin{cases} W_-(\mathbf{E}) & \text{if } g(\mathbf{E}) \leqslant 0, \quad W_- \in \mathscr{C}^2(\mathscr{E}, \mathscr{R}), \\ W_+(\mathbf{E}) & \text{if } g(\mathbf{E}) \geqslant 0, \quad W_+ \in \mathscr{C}^2(\mathscr{E}, \mathscr{R}), \end{cases} \qquad W \in \mathscr{C}^1(\mathscr{E}, \mathscr{R}); \tag{3.5}$$

$$\mathbf{S}(\mathbf{E}) = \nabla_{\mathbf{E}} W(\mathbf{E}) = \begin{cases} \mathbf{S}_-(\mathbf{E}) = \nabla_{\mathbf{E}} W_-(\mathbf{E}) & \text{if } g(\mathbf{E}) \leqslant 0, \\ \mathbf{S}_+(\mathbf{E}) = \nabla_{\mathbf{E}} W_+(\mathbf{E}) & \text{if } g(\mathbf{E}) \geqslant 0; \end{cases} \tag{3.6}$$

$$\mathbb{S}(\mathbf{E}) = \begin{cases} \mathbb{S}_-(\mathbf{E}) = \nabla^2_{\mathbf{E}} W_-(\mathbf{E}) & \text{if } g(\mathbf{E}) < 0, \\ \mathbb{S}_+(\mathbf{E}) = \nabla^2_{\mathbf{E}} W_+(\mathbf{E}) & \text{if } g(\mathbf{E}) > 0. \end{cases} \tag{3.7}$$

As in the smooth case, it is assumed that the energy and the stress are zero in the reference configuration, which in view of the continuity assumption imply

$$W(\mathbf{0}) = W_-(\mathbf{0}) = W_+(\mathbf{0}) = 0, \tag{3.8}$$

$$\mathbf{S}(\mathbf{0}) = \mathbf{S}_-(\mathbf{0}) = \mathbf{S}_+(\mathbf{0}) = \mathbf{0}. \tag{3.9}$$

On the interface $\mathscr{I}$, the Hessian elasticity tensor $\mathbb{S} = \nabla^2_{\mathbf{E}} W$ does not exist in a classical sense but, provided $\mathbf{S}$ is Lipschitz continuous (hereby excluding inextensible and incompressible materials), $\mathbb{S}$ can be replaced by a *set* $\partial_{\mathbf{E}}\mathbf{S}$ of elasticity tensors called the generalized Hessian [Clarke (1983)] and defined as the convex hull of the compression and tension Hessians $\mathbb{S}_-$ and $\mathbb{S}_+$, i.e.

$$\text{if } \exists k > 0 \mid \|\mathbf{S}(\mathbf{H}) - \mathbf{S}(\mathbf{E})\| \leqslant k\|\mathbf{H} - \mathbf{E}\|, \forall \mathbf{E}, \mathbf{H} \in \mathscr{E} \quad \text{then} \quad \forall \mathbf{E} \in \mathscr{I},$$
$$\partial_{\mathbf{E}}\mathbf{S}(\mathbf{E}) = \partial^2_{\mathbf{E}} W(\mathbf{E}) := \{\mathbb{S}(\mathbf{E}) \mid \mathbb{S}(\mathbf{E}) = (1 - \lambda)\mathbb{S}_-(\mathbf{E}) + \lambda\mathbb{S}_+(\mathbf{E}), \forall \lambda \in [0, 1]\}. \tag{3.10}$$

3.1.3. Continuity conditions

Continuity of the elastic energy function and its gradient, the stress–strain law, across the interface, implies

$$W(\mathbf{E}) = W_+(\mathbf{E}) = W_-(\mathbf{E}), \quad \forall \mathbf{E} \,|\, g(\mathbf{E}) = 0, \tag{3.11}$$

$$\mathbf{S}(\mathbf{E}) = \mathbf{S}_+(\mathbf{E}) = \mathbf{S}_-(\mathbf{E}), \quad \forall \mathbf{E} \,|\, g(\mathbf{E}) = 0, \tag{3.12}$$

$$[\![\mathbb{S}(\mathbf{E})]\!] := \mathbb{S}_+(\mathbf{E}) - \mathbb{S}_-(\mathbf{E}) = s(\mathbf{E})\nabla g(\mathbf{E}) \otimes \nabla g(\mathbf{E}), \quad \forall \mathbf{E} \,|\, g(\mathbf{E}) = 0; \tag{3.13}$$

where $s(\mathbf{E})$ is a continuous scalar valued function. The first two continuity conditions are obvious but the third discontinuity condition deserves an explanation. It expresses that the jump in the elasticity tensor across the interface is *normal* to the interface. There is no tangential discontinuity.

In fact, the normal jump condition (3.13) is a characterization of the stress–strain law continuity (3.12). This assertion is the purpose of the next proposition.

PROPOSITION 3.1. *The energy function W defined in* (3.5) *is continuously differentiable on $\mathscr{E}$, or* (*equivalently*) *the stress–strain law* $\mathbf{S} = \nabla_{\mathbf{E}} W$ *given by* (3.6) *is continuous on $\mathscr{E}$, if and only if* $\mathbf{S} = \nabla_{\mathbf{E}} W$ *is continuous at the strain origin as specified by* (3.9) *and the elasticity tensor normal jump condition* (3.13) *holds on $\mathscr{I}$.*

Proof. The demonstration is divided in two lemmas, the former being a preliminary.

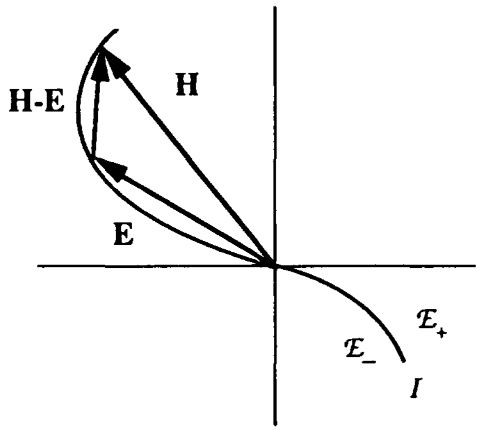

Fig. 4. Differentiation and integration on a hypersurface $\mathscr{I}$ in $\mathscr{E}$.

LEMMA 3.1. *The energy function W defined by* (3.5) *is continuous across $\mathscr{I}$ if and only if it satisfies* (3.8) *and*

$$[\![\nabla W(\mathbf{E})]\!] := \nabla W_+(\mathbf{E}) - \nabla W_-(\mathbf{E}) = w(\mathbf{E})\nabla g(\mathbf{E}), \quad \forall \mathbf{E} \in \mathscr{I}; \tag{3.14}$$

where $w(\mathbf{E})$ is a continuous scalar valued function.

Proof. Let $\mathbf{E}$ and $\mathbf{H}$ be two neighboring points on $\mathscr{I}$ (Fig. 4). Suppose W is continuous, then (3.8) is trivially verified and there simultaneously hold

$$\begin{aligned} &W_+(\mathbf{E}) - W_-(\mathbf{E}) = 0, \qquad &&W_+(\mathbf{H}) - W_-(\mathbf{H}) = 0,\\ &g(\mathbf{E}) = 0, &&g(\mathbf{H}) = 0. \end{aligned}$$

Since W_-, $W_+ \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$ and $g \in \mathscr{C}^1(\mathscr{E}, \mathscr{R})$, the following expansions are valid:

$$W_-(\mathbf{H}) = W_-(\mathbf{E}) + \nabla W_-(\mathbf{E}):(\mathbf{H} - \mathbf{E}) + \mathscr{O}(\|\mathbf{H} - \mathbf{E}\|^2),$$

$$W_+(\mathbf{H}) = W_+(\mathbf{E}) + \nabla W_+(\mathbf{E}):(\mathbf{H} - \mathbf{E}) + \mathscr{O}(\|\mathbf{H} - \mathbf{E}\|^2),$$

$$g(\mathbf{H}) = g(\mathbf{E}) + \nabla g(\mathbf{E}):(\mathbf{H} - \mathbf{E}) + \mathscr{O}(\|\mathbf{H} - \mathbf{E}\|^2).$$

Subtracting the former from the second and using the above continuity conditions yields

$$\nabla W_+(\mathbf{E}):(\mathbf{H} - \mathbf{E}) - \nabla W_-(\mathbf{E}):(\mathbf{H} - \mathbf{E}) = \mathscr{O}(\|\mathbf{H} - \mathbf{E}\|^2),$$

$$\nabla g(\mathbf{E}):(\mathbf{H} - \mathbf{E}) = \mathscr{O}(\|\mathbf{H} - \mathbf{E}\|^2).$$

Dividing throughout by $\|\mathbf{H} - \mathbf{E}\|$ and letting $\mathbf{H} \to \mathbf{E}$, the unit vector

$$(\mathbf{H} - \mathbf{E})/\|\mathbf{H} - \mathbf{E}\| \to \mathbf{T}(\mathbf{E}),$$

and it follows that

$$\left.\begin{aligned} &[\nabla W_+(\mathbf{E}) - \nabla W_-(\mathbf{E})]:\mathbf{T}(\mathbf{E}) = 0\\ &\nabla g(\mathbf{E}):\mathbf{T}(\mathbf{E}) = 0 \end{aligned}\right\}, \quad \forall \mathbf{E} \in \mathscr{I}, \forall \mathbf{T}(\mathbf{E}), \ \|\mathbf{T}(\mathbf{E})\| = 1.$$

This implies (3.14) with $w \in \mathscr{C}^0(\mathscr{E}, \mathscr{R})$.

Conversely, assume (3.8) and (3.14) hold. Then, (3.14) can be integrated from the strain origin $\mathbf{0}$ up to any strain $\mathbf{E}$ on $\mathscr{I}$, along it:

$$\int_{\mathbf{0}}^{\mathbf{E}} [\nabla W_+(\mathbf{H}) - \nabla W_-(\mathbf{H})]:d\mathbf{H} = \int_{\mathbf{0}}^{\mathbf{E}} w(\mathbf{H})\nabla g(\mathbf{H}):d\mathbf{H}, \quad \forall \mathbf{E}, \mathbf{H} \in \mathscr{I}.$$

Since $\nabla g(\mathbf{H}):d\mathbf{H} = 0$ on $\mathscr{I}$, W_-, $W_+ \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$ and $w \in \mathscr{C}^0(\mathscr{E}, \mathscr{R})$, the result is, in view of (3.8),

$$W_+(\mathbf{E}) - W_+(\mathbf{0}) - W_-(\mathbf{E}) + W_-(\mathbf{0}) = W_+(\mathbf{E}) - W_-(\mathbf{E}) = 0, \quad \forall \mathbf{E} \in \mathscr{I}. \qquad \square$$

REMARK. Geometrically, condition (3.14) expresses that the gradient of a scalar valued function, continuous across a smooth interface, can only suffer a jump in the normal direction to that interface. By introducing the unit normal $\mathbf{N}(\mathbf{E}) = \nabla g(\mathbf{E})/\|\nabla g(\mathbf{E})\|$ to the surface $\mathscr{I}$ at $\mathbf{E}$, Eq. (3.14) can be equivalently written as

$$[\mathbf{I} \,\bar{\underline{\otimes}}\, \mathbf{I} - \mathbf{N}(\mathbf{E}) \otimes \mathbf{N}(\mathbf{E})] [\![\nabla_{\mathbf{E}} W(\mathbf{E})]\!] = \mathbf{0}, \quad \forall \mathbf{E} \in \mathscr{I},$$

which expresses that the tangential jump is zero.

LEMMA 3.2. *The stress–strain law* $\mathbf{S} = \nabla_{\mathbf{E}} W$ *defined by* (3.6) *is continuous across* $\mathscr{I}$ *if and only if it satisfies* (3.9) *and*

$$[\![\nabla \mathbf{S}(\mathbf{E})]\!] := \nabla \mathbf{S}_+(\mathbf{E}) - \nabla \mathbf{S}_-(\mathbf{E}) = s(\mathbf{E}) \nabla g(\mathbf{E}) \otimes \nabla g(\mathbf{E}), \quad \forall \mathbf{E} \in \mathscr{I}, \tag{3.15}$$

where $s(\mathbf{E})$ *is a continuous scalar valued function.*

Proof. Suppose $\mathbf{S}$ is continuous, then (3.9) is trivially verified and applying the reasoning used in Lemma 3.1 to $\mathbf{S}(\mathbf{E})$ instead of $W(\mathbf{E})$ leads at once to

$$\left.\begin{aligned} [\nabla \mathbf{S}_+(\mathbf{E}) - \nabla \mathbf{S}_-(\mathbf{E})]\mathbf{T}(\mathbf{E}) &= \mathbf{0} \\ \nabla g(\mathbf{E}) : \mathbf{T}(\mathbf{E}) &= 0 \end{aligned}\right\}, \quad \forall \mathbf{E} \in \mathscr{I}, \forall \mathbf{T}(\mathbf{E}), \ \|\mathbf{T}(\mathbf{E})\| = 1.$$

On introducing the (five-dimensional) hyperplane $\mathscr{H} := \{\mathbf{T} \in \mathscr{E} \mid \nabla g(\mathbf{E}) : \mathbf{T} = 0\}$ tangent to $\mathscr{I}$ at $\mathbf{E}$, there are, according to linear algebra, only two possibilities for the above system to hold at a given $\mathbf{E}$:

$$\text{either } \nabla \mathbf{S}_+ = \nabla \mathbf{S}_- \ \text{ with } \operatorname{Ker}[\![\nabla \mathbf{S}]\!] = \mathscr{E} \text{ and } \operatorname{Im}[\![\nabla \mathbf{S}]\!] = \{\mathbf{0}\},$$

$$\text{or } \quad \nabla \mathbf{S}_+ \neq \nabla \mathbf{S}_- \ \text{ with } \operatorname{Ker}[\![\nabla \mathbf{S}]\!] = \mathscr{H} \text{ and } \operatorname{Im}[\![\nabla \mathbf{S}]\!] = \{\mathbf{S} \in \mathscr{S} \mid \mathbf{S} = s\mathbf{M}, s \in \mathscr{R}\};$$

where $\operatorname{Ker}(\mathbb{T})$ and $\operatorname{Im}(\mathbb{T})$ denote the kernel and image of $\mathbb{T}$ respectively, and $\mathbf{M} \in \mathscr{S}$ is a nonzero second-order tensor. The first possibility corresponds to the continuous case $[\![\nabla \mathbf{S}(\mathbf{E})]\!] = \mathbb{O}$ (choice of smooth elasticity). Therefore, the only nontrivial solution is the second one which expresses that for any fixed **E** on $\mathscr{I}$, the linear operator $[\![\nabla\mathbf{S}(\mathbf{E})]\!]$ must be singular and in fact of *rank one* in the form

$$\nabla\mathbf{S}_+(\mathbf{E}) - \nabla\mathbf{S}_-(\mathbf{E}) = s(\mathbf{E})\mathbf{M}(\mathbf{E}) \otimes \nabla g(\mathbf{E}).$$

The result (3.15) follows from the major symmetries of $\nabla\mathbf{S}_-(\mathbf{E})$ and $\nabla\mathbf{S}_+(\mathbf{E})$ with $s \in \mathscr{C}^0(\mathscr{E}, \mathscr{R})$.

Conversely, assume (3.9) and (3.15) hold. Then, (3.15) can be integrated from the strain origin **0** up to any strain **E** on $\mathscr{I}$, along it:

$$\int_{\mathbf{0}}^{\mathbf{E}} [\nabla\mathbf{S}_+(\mathbf{H}) - \nabla\mathbf{S}_-(\mathbf{H})] : d\mathbf{H} = \int_{\mathbf{0}}^{\mathbf{E}} s(\mathbf{H})[\nabla g(\mathbf{H}) \otimes \nabla g(\mathbf{H})]\, d\mathbf{H}, \quad \forall \mathbf{E}, \mathbf{H} \in \mathscr{I}.$$

Since $\nabla g(\mathbf{H}) : d\mathbf{H} = 0$ on $\mathscr{I}$, $\mathbf{S}_-$, $\mathbf{S}_+ \in \mathscr{C}^1(\mathscr{E}, \mathscr{S})$ and $s \in \mathscr{C}^0(\mathscr{E}, \mathscr{R})$, it follows, in view of (3.9) that,

$$\begin{aligned}\mathbf{S}_+(\mathbf{E}) - \mathbf{S}_+(\mathbf{0}) - \mathbf{S}_-(\mathbf{E}) + \mathbf{S}_-(\mathbf{0}) &= \mathbf{S}_+(\mathbf{E}) - \mathbf{S}_-(\mathbf{E}) \\ &= \int_{\mathbf{0}}^{\mathbf{E}} s(\mathbf{H})(\nabla g(\mathbf{H}) : d\mathbf{H})\nabla g(\mathbf{H}) = \mathbf{0}, \quad \forall \mathbf{E} \in \mathscr{I}.\end{aligned}$$ □

REMARK. Note that the major symmetry of $\mathbb{S}_-$ and $\mathbb{S}_+$ (or equivalently the existence of W_- and $W_+ \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$) plays a crucial role in arriving at a symmetric jump.

3.1.4. Hyperelasticity conditions

A stress–strain law (3.6) being the (continuous) gradient of an energy function (3.5) is hyperelastic by definition. The work it produces around any closed strain cycle is zero or, equivalently, the work it produces between any two given strains (say **0** and **E**) is independent of the path

$$\int_{\mathbf{0}}^{\mathbf{E}} \mathbf{S}(\mathbf{H}) : d\mathbf{H} = \int_{\mathbf{0}}^{\mathbf{E}} \nabla_{\mathbf{E}} W(\mathbf{H}) : d\mathbf{H} = W(\mathbf{E}) - W(\mathbf{0}) = W(\mathbf{E}), \quad \forall \mathbf{E} \in \mathscr{E}.$$

Conversely, if the work produced by a continuous elastic stress–strain law such as (3.6) (without $\mathbf{S} = \nabla_{\mathbf{E}} W$) between any two given strains is independent of the path, then this law must be the gradient of a continuously differentiable energy function such as (3.5) (thus $\mathbf{S} = \nabla_{\mathbf{E}} W$), just as in the smooth case. Indeed, taking a step orthogonal to $\mathbf{e}_1 \otimes \mathbf{e}_1$ followed by another parallel to it yields

$$W(\mathbf{E}) = \int_{\mathbf{0}}^{\mathbf{E}} \mathbf{S}(\mathbf{H}) : d\mathbf{H} = \int_{\mathbf{0}}^{\mathbf{P}} \mathbf{S}(\mathbf{H}) : d\mathbf{H} + \int_{\mathbf{P}}^{\mathbf{E}} \mathbf{S}(\mathbf{H}) : d\mathbf{H}$$

$$= W(\mathbf{P}) + \int_{0}^{E_{11}} S_{11}(H_{11}) dH_{11},$$

where $\mathbf{P} = \mathbf{E} - E_{11}\mathbf{e}_1 \otimes \mathbf{e}_1$ denotes the projection of $\mathbf{E}$ on the basis hyperplane orthogonal to $\mathbf{e}_1 \otimes \mathbf{e}_1$. Taking the derivative of W with respect to E_{11}, it is found out that $S_{11} = \partial W / \partial E_{11}$ since $\mathbf{P}$ and thus $W(\mathbf{P})$ are independent of E_{11}. The conclusion is reached by scanning the five other components.

Now, if a stress–strain law (3.6) is given without its elastic potential (3.5), the following hyperelasticity criteria will guarantee the latter existence.

PROPOSITION 3.2. *A continuous piecewise differentiable stress–strain law* $\mathbf{S}$, *as defined in* (3.6) (*without* $\mathbf{S} = \nabla_{\mathbf{E}} W$), *derives from an energy function* W (*thus* $\mathbf{S} = \nabla_{\mathbf{E}} W$) *if and only if each elasticity tensor has the major symmetry*

$$\mathbb{S}_-^T(\mathbf{E}) = \mathbb{S}_-(\mathbf{E}), \quad \forall \mathbf{E} \in \mathscr{E}, \tag{3.16}$$

$$\mathbb{S}_+^T(\mathbf{E}) = \mathbb{S}_+(\mathbf{E}), \quad \forall \mathbf{E} \in \mathscr{E}. \tag{3.17}$$

Moreover W *is in the form* (3.5), *provided it satisfies* (3.8).

Proof. If the law (3.6) derives from an energy function (3.5), then $\mathbb{S}_-(\mathbf{E})$ and $\mathbb{S}_+(\mathbf{E})$ satisfy the major symmetry conditions (3.16) and (3.17) by application of (2.5) to W_- and W_+.

Conversely, if (3.16) and (3.17) hold, then the potentials W_- and W_+ exist. Moreover, since the law (3.6) is continuous, by the argument preceding Proposition 3.2, there exists then a (wholewise) continuously differentiable energy function W. This function can only be the juxtaposition of W_- and W_+ as defined in (3.5), in view of (3.8). □

In other words, a *continuous* piecewise hyperelastic stress–strain law is hyperelastic. The key condition is the continuity of the stress–strain law across the interface, as expressed by (3.12) and ensured by the jump condition (3.13) on the elasticity tensors.

3.2. Convexity. Monotony. Positiveness

As in smooth elasticity, strict convexity (2.6) of the elastic potential W or, equivalently, strict monotony (2.7) of the stress–strain law $\mathbf{S}$ are needed at small strains and in particular at the strain origin $\mathbf{E} = \mathbf{0}$. Positive definiteness (2.8) of the elasticity tensor $\mathbb{S}$ breaks down however, since $\mathbb{S}$ is not defined on the interface $\mathscr{I}$ (including at $\mathbf{E} = \mathbf{0}$). It is thus natural to look for sufficient conditions on the restrictions which will ensure convexity and monotony on the whole and for an alternative characterization of positive definiteness at the elasticity tensor level.

3.2.1. Half-spacewise convexity

One way to arrive at a function W which is convex over $\mathscr{E}$, when starting from its definition (3.5), is to require that W_- be convex over $\mathscr{E}_-$ and W_+ be convex over $\mathscr{E}_+$. Bearing in mind that a convex function is (at least in the classical sense) necessarily defined on a convex domain (because such a domain is closed under convex combinations), it can be further assumed that $\mathscr{E}_-$ and $\mathscr{E}_+$ are *convex* subsets of $\mathscr{E}$ (although this assumption is over restrictive). But, in view of their definitions (3.1), $\mathscr{E}_-$ and $\mathscr{E}_+$ will both be convex if and only if they are *half-spaces* and their interface $\mathscr{I}$ is a *hyperplane* passing through the origin as defined in (3.3):

$$\mathscr{I} := \{\mathbf{E} \in \mathscr{E} \mid g(\mathbf{E}) = \mathbf{N}:\mathbf{E} = 0\}, \quad (\|\mathbf{N}\| = 1), \tag{3.3a$'$}$$

$$\mathscr{E}_- := \{\mathbf{E} \in \mathscr{E} \mid \mathbf{N}:\mathbf{E} < 0\}, \quad \mathscr{E}_+ := \{\mathbf{E} \in \mathscr{E} \mid \mathbf{N}:\mathbf{E} > 0\}. \tag{3.3b$'$}$$

This is a direct consequence of the separation theorem of convex sets [Rockafellar (1970)]. It follows that g is linear with a constant gradient $\nabla g(\mathbf{E}) = \mathbf{N}$. Accordingly the jump condition (3.13) simplifies into

$$[\![\mathbb{S}(\mathbf{E})]\!] = \mathbb{S}_+(\mathbf{E}) - \mathbb{S}_-(\mathbf{E}) = s(\mathbf{E})\mathbf{N} \otimes \mathbf{N}, \quad \forall \mathbf{E} \mid \mathbf{N}:\mathbf{E} = 0. \tag{3.18}$$

Now, even if the restrictions $W_- \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$ and $W_+ \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$ are convex over convex half-spaces $\mathscr{E}_-$ and $\mathscr{E}_+$, the function W resulting from (3.5) can be nonconvex over $\mathscr{E}$ if it is only continuous across $\mathscr{I}$, as illustrated in Fig. 5. The smoothness condition (3.12) across $\mathbf{N}:\mathbf{E} = 0$ or its corollary (3.18) turns out to be sufficient for global convexity.

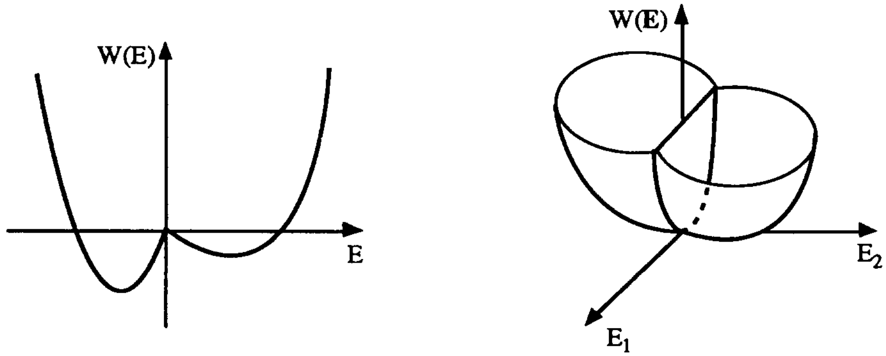

Fig. 5. Nonconvex piecewise convex functions.

PROPOSITION 3.3. *The continuously differentiable energy function W defined in* (3.5), *with an hyperplane $\mathscr{I} = \{\mathbf{E}\in\mathscr{E} \mid \mathbf{N}:\mathbf{E} = 0\}$ for interface, is strictly convex over $\mathscr{E}$ if the two restrictions W_- and W_+ are strictly convex.*

Proof. To show that W is strictly convex, it is enough to show that it satisfies (2.6b) (since W is differentiable), i.e.

$$W(\mathbf{H}) > W(\mathbf{E}) + \nabla_{\mathbf{E}}W(\mathbf{E}):(\mathbf{H} - \mathbf{E}),\ \forall \mathbf{E}, \mathbf{H}\in\mathscr{E},\ \mathbf{E} \neq \mathbf{H}. \tag{2.6b$'$}$$

Only the situation where $\mathbf{E}\in\mathscr{E}_-$, $\mathbf{H}\in\bar{\mathscr{E}}_+$ and the segment $[\mathbf{E},\mathbf{H}]$ intersects the hyperplane $\mathscr{I}$ at a single point $\mathbf{J}$ needs to be considered, since the two situations where $\mathbf{E},\mathbf{H}\in\bar{\mathscr{E}}_-$ (thus $W = W_-$) or $\mathbf{E}$, $\mathbf{H}\in\bar{\mathscr{E}}_+$ (thus $W = W_+$) are trivial and the situation where $\mathbf{H}\in\bar{\mathscr{E}}_-$, $\mathbf{E}\in\mathscr{E}_+$ merely is a permutation of the former. Thus, let

$$\mathbf{J} = (1 - \gamma)\mathbf{E} + \gamma\mathbf{H} \mid \mathbf{N}:\mathbf{J} = 0 \Rightarrow \gamma = -(\mathbf{N}:\mathbf{E})/[\mathbf{N}:(\mathbf{H} - \mathbf{E})].$$

Then, given the definition (3.5) of W and the hypothesis that W_- and W_+ are strictly convex,

$$W(\mathbf{J}) = W_-(\mathbf{J}) > W_-(\mathbf{E}) + \nabla_{\mathbf{E}}W_-(\mathbf{E}):(\mathbf{J} - \mathbf{E}) = W(\mathbf{E}) + \nabla_{\mathbf{E}}W(\mathbf{E}):(\mathbf{J} - \mathbf{E}),$$
$$W(\mathbf{H}) = W_+(\mathbf{H}) \geqslant W_+(\mathbf{J}) + \nabla_{\mathbf{E}}W_+(\mathbf{J}):(\mathbf{H} - \mathbf{J}) = W(\mathbf{J}) + \nabla_{\mathbf{E}}W(\mathbf{J}):(\mathbf{H} - \mathbf{J}).$$

Substituting the former in the latter yields

$$W(\mathbf{H}) > W(\mathbf{E}) + \nabla_{\mathbf{E}}W(\mathbf{E}):(\mathbf{J} - \mathbf{E}) + \nabla_{\mathbf{E}}W(\mathbf{J}):(\mathbf{H} - \mathbf{J}).$$

A direct calculation gives

$$\mathbf{J} - \mathbf{E} = \mathbf{H} - \mathbf{E} + \mathbf{J} - \mathbf{H}; \qquad \mathbf{J} - \mathbf{H} = (1 - \gamma)(\mathbf{E} - \mathbf{H}) = \frac{1-\gamma}{\gamma}(\mathbf{E} - \mathbf{J}) = -\frac{\mathbf{N}:\mathbf{H}}{\mathbf{N}:\mathbf{E}}(\mathbf{E} - \mathbf{J}).$$

Substituting these expressions in the above inequality gives

$$W(\mathbf{H}) > W(\mathbf{E}) + \nabla_{\mathbf{E}}W(\mathbf{E}):(\mathbf{H} - \mathbf{E}) - \frac{\mathbf{N}:\mathbf{H}}{\mathbf{N}:\mathbf{E}}[\nabla_{\mathbf{E}}W(\mathbf{E}) - \nabla_{\mathbf{E}}W(\mathbf{J})]:(\mathbf{E} - \mathbf{J}).$$

But $-(\mathbf{N}:\mathbf{H})/(\mathbf{N}:\mathbf{E}) \geqslant 0$ by assumption and

$$[\nabla_{\mathbf{E}}W(\mathbf{E}) - \nabla_{\mathbf{E}}W(\mathbf{J})]:(\mathbf{E} - \mathbf{J}) > 0$$

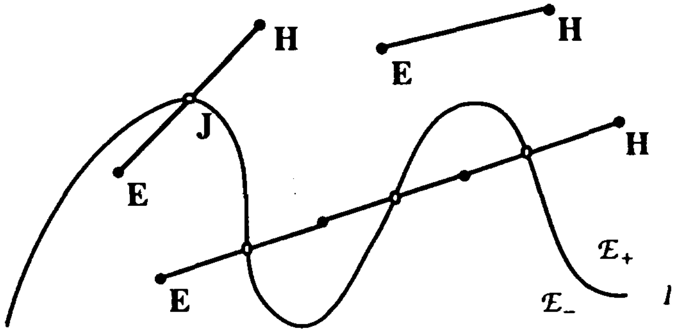

Fig. 6. Generic locations of **E**, **H** and **J**.

since $\nabla_{\mathbf{E}} W = \nabla_{\mathbf{E}} W_-$ is monotone on $\mathscr{E}_-$ (because it derives from $W = W_-$ convex). Consequently,

$$W(\mathbf{H}) > W(\mathbf{E}) + \nabla_{\mathbf{E}} W(\mathbf{E}) : (\mathbf{H} - \mathbf{E}). \qquad \square$$

REMARKS. (i) The preceding proof seems to remain valid if the subdomains $\mathscr{E}_-, \mathscr{E}_+$ are not convex, upon replacing **N** by $\nabla g(\mathbf{J})$ wherever it occurs. Indeed, even if the subdomains are separated by a curved or wavy $\mathscr{C}^1$ interface (Fig. 6), all possible situations for **E** and **H** are combinations of the ones considered in the proposition.

(ii) More complicated subdivisions of $\mathscr{E}$ into several convex subdomains are possible. The correct generalization of the two above half-spaces appears to be *polyhedral convex cones*, i.e. *pyramids* centered at the origin. This assertion is further discussed in Sections 4.5 and 4.6, where its relevance is more important.

It follows from Proposition 3.3 and Assumptions (3.8) and (3.9) that

$$W(\mathbf{E}) > W(\mathbf{J}) + \nabla_{\mathbf{E}} W(\mathbf{J}) : (\mathbf{E} - \mathbf{J}), \quad \forall \mathbf{E} \in \mathscr{E}, \forall \mathbf{J} \in \mathscr{J}, \mathbf{E} \neq \mathbf{J},$$
$$W(\mathbf{E}) > 0, \forall \mathbf{E} \in \mathscr{E}, \mathbf{E} \neq \mathbf{0}, \quad (W(\mathbf{0}) = 0). \tag{3.19}$$

3.2.2. Half-spacewise invertibility

If the potential W exists but is unknown or even if it does not exist as with an *unsymmetric* elasticity tensor, it is still possible to give an invertibility condition for the stress–strain law (3.6), provided it is Lipschitz continuous.

PROPOSITION 3.4. *A Lipschitz continuous half-spacewise differentiable stress–strain law* **S**(**E**) *is locally invertible at* $\mathbf{E} \in \mathscr{J}$ *if both* $\det[\mathbb{S}_-(\mathbf{E})] > 0$ *and* $\det[\mathbb{S}_+(\mathbf{E})] > 0$.

Proof. Consider the generalized Hessian $\partial\mathbf{S}(\mathbf{E})$ defined in (3.10). A theorem due to Kojima and Saigal (1980) and extended by Alart (1992) asserts, in essence, that if any element $\mathbb{S}(\mathbf{E})$ of the generalized Hessian $\partial\mathbf{S}(\mathbf{E})$ is invertible, then the operator S(E) is locally invertible at **E**. In the half-spacewise case at hand, this condition simply requires

$$\det[(1-\lambda)\mathbb{S}_-(\mathbf{E}) + \lambda\mathbb{S}_+(\mathbf{E})] > 0,\ \forall\lambda\in[0,1],\ \forall\mathbf{E}\in\mathscr{I} \Rightarrow \exists\mathbf{S}^{-1}. \tag{3.20}$$

Now, due to continuity, $\mathbb{S}_-$ and $\mathbb{S}_+$ differ by a rank-one tensor, in general *non-symmetric* in this case,

$$\mathbb{S}_+(\mathbf{E}) - \mathbb{S}_-(\mathbf{E}) = s(\mathbf{E})\mathbf{M}\otimes\mathbf{N}. \tag{3.21}$$

But

$$\det[(1-\lambda)\mathbb{S}_- + \lambda\mathbb{S}_+] = \det[\mathbb{S}_- + \lambda s\mathbf{M}\otimes\mathbf{N}] = \det\mathbb{S}_-[1+\lambda s\mathbf{N}:(\mathbb{S}_-^{-1}\mathbf{M})]$$
$$\geqslant \min\{\det\mathbb{S}_-, \det\mathbb{S}_+\}. \qquad \square$$

REMARK. Proposition 3.4 is applicable to the polyhedral conewise linear case (cf. §4.5–4.6), provided a tensor $\mathbb{T}$ can be exhibited such that any convex combination of this tensor with each elasticity tensor has a positive determinant.

In summary, a *continuous* piecewise monotone stress–strain law is monotone.

Once the stress–strain law is invertible, the complementary energy density is still given by the Legendre transformation (2.9) and its gradient by (2.10).

In stress space, the interface $\mathscr{I}^*$ is given by the composite function

$$\mathscr{I}^* := \{\mathbf{S}\in\mathscr{S}\,|\,h(\mathbf{S}) = 0\},\quad h(\mathbf{S}) = g[\mathbf{E}_-(\mathbf{S})] \text{ or } g[\mathbf{E}_+(\mathbf{S})],\quad h\in\mathscr{C}^1(\mathscr{S},\mathscr{R}). \tag{3.22}$$

The inverse relationships of (3.5), (3.6) and (3.7) take the forms

$$W^*(\mathbf{S}) := \begin{cases} W_-^*(\mathbf{S}) & \text{if } h(\mathbf{S}) \leqslant 0, & W_-^*\in\mathscr{C}^2(\mathscr{S},\mathscr{R}) \\ W_+^*(\mathbf{S}) & \text{if } h(\mathbf{S}) \geqslant 0, & W_+^*\in\mathscr{C}^2(\mathscr{S},\mathscr{R}) \end{cases},\quad W^*\in\mathscr{C}^1(\mathscr{S},\mathscr{R}); \tag{3.23}$$

$$\mathbf{E}(\mathbf{S}) = \nabla_\mathbf{S} W^*(\mathbf{S}) = \begin{cases} \mathbf{E}_-(\mathbf{S}) = \nabla_\mathbf{S} W_-^*(\mathbf{S}) & \text{if } h(\mathbf{S}) \leqslant 0 \\ \mathbf{E}_+(\mathbf{S}) = \nabla_\mathbf{S} W_+^*(\mathbf{S}) & \text{if } h(\mathbf{S}) \geqslant 0 \end{cases}; \tag{3.24}$$

$$\mathbb{E}(\mathbf{S}) = \mathbb{S}^{-1}(\mathbf{S}) = \begin{cases} \mathbb{E}_-(\mathbf{S}) = \nabla_\mathbf{S}^2 W_-^*(\mathbf{S}) & \text{if } h(\mathbf{S}) < 0 \\ \mathbb{E}_+(\mathbf{S}) = \nabla_\mathbf{S}^2 W_+^*(\mathbf{S}) & \text{if } h(\mathbf{S}) > 0 \end{cases}. \tag{3.25}$$

The continuity conditions for the inverse relationships are similar to (3.11)–(3.13), with $\nabla_\mathbf{E} g(\mathbf{E})$ replaced by $\nabla_\mathbf{S} h(\mathbf{S})$ and $s(\mathbf{E})$ by

$$e(\mathbf{S}) = -s(\mathbf{E})/[1 + s(\mathbf{E})\nabla_\mathbf{S} h(\mathbf{S}):\nabla_\mathbf{E} g(\mathbf{E})].$$

3.2.3. Half-spacewise positiveness

Finally a more convenient test for convexity and monotony is available in terms of the elasticity tensors.

COROLLARY 3.3. *A set of sufficient conditions for the convexity of W or monotony of* $\mathbf{S}$ *is*

$$(\mathbf{H}-\mathbf{E}):\mathbb{S}_-(\mathbf{E})(\mathbf{H}-\mathbf{E}) > 0, \forall \mathbf{E}, \mathbf{H} \in \mathscr{E}, \mathbf{H} \neq \mathbf{E}, \tag{3.26}$$

$$(\mathbf{H}-\mathbf{E}):\mathbb{S}_+(\mathbf{E})(\mathbf{H}-\mathbf{E}) > 0, \forall \mathbf{E}, \mathbf{H} \in \mathscr{E}, \mathbf{H} \neq \mathbf{E}, \tag{3.27}$$

$$\mathbb{S}_+(\mathbf{E}) - \mathbb{S}_-(\mathbf{E}) = s(\mathbf{E})\mathbf{N} \otimes \mathbf{N}, \forall \mathbf{E} \mid \mathbf{N}:\mathbf{E} = 0. \tag{3.18$'$}$$

Proof. Obvious from Proposition 3.3 and the equivalences (2.6a)$\Leftrightarrow$(2.6b)$\Leftrightarrow$(2.7)$\Leftarrow$(2.8) for $W_+, W_- \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$. □

If $\mathbb{S}_-$ is taken for reference and $\mathbb{S}_+$ regarded as a byproduct, then (3.27) can be replaced by

$$s(\mathbf{E}) > -\frac{(\mathbf{H}-\mathbf{E}):\mathbb{S}_-(\mathbf{E})(\mathbf{H}-\mathbf{E})}{((\mathbf{H}-\mathbf{E}):\mathbf{N})^2}, \quad \forall \mathbf{E}, \mathbf{H} \in \mathscr{E}, \mathbf{H} \neq \mathbf{E}. \tag{3.28}$$

3.3. Material symmetries

In the piecewise smooth case, it is important to distinguish between piecewise and wholewise symmetries. For instance, a compression-isotropic–tension-orthotropic piecewise smooth material is conceivable and should not be mistaken for an isotropic or orthotropic piecewise smooth material. In this article, only *wholewise symmetries* are considered.

3.3.1. Representations

With this provision in mind, for W-$\mathbf{S}$-$\mathbb{S}$ defined in (3.5), (3.6) and (3.7) to be compatible with the material symmetries, conditions (2.12), (2.13) and (2.14) must be satisfied. This will in particular be the case if, in addition to the restrictions W_- and W_+ defining W, the function g defining the interface is invariant under $\mathscr{G}$,

$$g(\mathbf{QEQ}^T) = g(\mathbf{E}), \forall \mathbf{E} \in \mathscr{E}, \forall \mathbf{Q} \in \mathscr{G}. \tag{3.29}$$

Therefore, W (and thus $\mathbf{S}$-$\mathbb{S}$) will remain invariant if g, W_- and W_+ (and thus $\mathbf{S}_-, \mathbf{S}_+, \mathbb{S}_-, \mathbb{S}_+$) have representations analogous to the smooth ones (2.15), (2.16) and (2.17), i.e.

$$g(\mathbf{E}) = \gamma(I_{\mathbf{E}}); \tag{3.30}$$

$$W(\mathbf{E}) = \omega(I_{\mathbf{E}}) := \begin{cases} \omega_-(I_{\mathbf{E}}) & \text{if } \gamma(I_{\mathbf{E}}) \leqslant 0 \\ \omega_+(I_{\mathbf{E}}) & \text{if } \gamma(I_{\mathbf{E}}) \geqslant 0 \end{cases}, \quad \omega \in \mathscr{C}^1(\mathscr{I}_{\mathscr{E}}, \mathscr{R}); \tag{3.31}$$

$$\mathbf{S}(\mathbf{E}) = \omega_i \mathbf{G}_i, \quad \mathbf{G}_i := \nabla_{\mathbf{E}} I_i(\mathbf{E}), \quad (i = 1, p), \tag{3.32}$$

$$\omega_i := \frac{\partial \omega}{\partial I_i}(I_{\mathbf{E}}) = \begin{cases} \omega_{-i}(I_{\mathbf{E}}) = \partial \omega_- / \partial I_i & \text{if } \gamma(I_{\mathbf{E}}) \leqslant 0 \\ \omega_{+i}(I_{\mathbf{E}}) = \partial \omega_+ / \partial I_i & \text{if } \gamma(I_{\mathbf{E}}) \geqslant 0 \end{cases}; \tag{3.33}$$

$$\mathbb{S}(\mathbf{E}) = \omega_{ij} \mathbf{G}_i \otimes \mathbf{G}_j + \omega_i \nabla_{\mathbf{E}} \mathbf{G}_i, \quad \nabla_{\mathbf{E}} \mathbf{G}_i = \nabla^2_{\mathbf{E}} I_i(\mathbf{E}), \quad (i, j = 1, p), \tag{3.34}$$

$$\omega_{ij} = \omega_{ji} := \begin{cases} \omega_{-ij}(I_{\mathbf{E}}) = \partial^2 \omega_- / \partial I_i \partial I_j & \text{if } \gamma(I_{\mathbf{E}}) < 0 \\ \omega_{+ij}(I_{\mathbf{E}}) = \partial^2 \omega_+ / \partial I_i \partial I_j & \text{if } \gamma(I_{\mathbf{E}}) > 0 \end{cases}; \tag{3.35}$$

where $I_{\mathbf{E}}$, $\mathbf{G}_i = \nabla_{\mathbf{E}} I_i$, $\mathbf{G}_i \otimes \mathbf{G}_j$ and $\nabla_{\mathbf{E}} \mathbf{G}_i$ are specified in Table 1 for isotropic, transversely isotropic and orthotropic materials. The generators are the *same* in the tension and compression subdomains because the material is assumed to have the same symmetries in tension and compression.

3.3.2. Continuity conditions

Since the generators $\mathbf{G}_i$ are independent, the continuity conditions (3.11) for the elastic energy function, (3.12) for the stress–strain law and the jump condition (3.13) for the elasticity tensor take the special forms

$$\omega(I_{\mathbf{E}}) = \omega_-(I_{\mathbf{E}}) = \omega_+(I_{\mathbf{E}}), \quad \forall I_{\mathbf{E}} \,|\, \gamma(I_{\mathbf{E}}) = 0, \tag{3.36}$$

$$\omega_i(I_{\mathbf{E}}) = \omega_{-i}(I_{\mathbf{E}}) = \omega_{+i}(I_{\mathbf{E}}), \quad (i = 1, p), \quad \forall I_{\mathbf{E}} \,|\, \gamma(I_{\mathbf{E}}) = 0, \tag{3.37}$$

$$[\![\omega_{ij}(I_{\mathbf{E}})]\!] = \omega_{+ij}(I_{\mathbf{E}}) - \omega_{-ij}(I_{\mathbf{E}}) = \sigma(I_{\mathbf{E}}) \gamma_i(I_{\mathbf{E}}) \gamma_j(I_{\mathbf{E}}), \quad (i, j = 1, p), \tag{3.38}$$

$$\sigma(I_{\mathbf{E}}) := s(\mathbf{E}); \qquad \gamma_i := \frac{\partial \gamma}{\partial I_i}(I_{\mathbf{E}}), \quad \forall I_{\mathbf{E}} \,|\, \gamma(I_{\mathbf{E}}) = 0.$$

Again, the first two continuity conditions are easy to interpret. The third discontinuity condition expresses that only the bulk part $\bar{\mathbb{S}}(\mathbf{E}) = \omega_{ij} \mathbf{G}_i \otimes \mathbf{G}_j$ of the elasticity tensor can suffer a jump across the interface; the shear part $\mathbb{S}'(\mathbf{E}) = \omega_i \nabla_{\mathbf{E}} \mathbf{G}_i$ remains continuous. The link between the three conditions can be specified as follows.

PROPOSITION 3.5. *The energy function* ω *defined in* (3.31) *is continuously differentiable on* $\mathscr{E}$, *or* (*equivalently*), *the stress–strain law* $\mathbf{S} = \omega_i \mathbf{G}_i$ *given by* (3.32) *and* (3.33) *is continuous on* $\mathscr{E}$ *if and only if the first derivatives* $\omega_i = \partial \omega / \partial I_i$ *are continuous at the strain invariant origin* $I_0 = I_{\mathbf{E}}(\mathbf{0})$ *and the jump condition*

(3.38) *on the second derivatives* ω_{+ij} *and* ω_{-ij} *holds on* $\mathscr{I}$.

Proof. The first part of the proposition (the equivalence) is a direct corollary of Proposition 3.1. Its proof is straightforward, upon observing that the generators $\mathbf{G}_i$ in (3.32) are linearly *independent* and *continuous* in $\mathbf{E}$. Thus, only (3.38) is proved. On the one hand, observing that the interface gradient is given by

$$\nabla g(\mathbf{E}) = \frac{\partial \gamma}{\partial I_i}(I_{\mathbf{E}})\nabla_{\mathbf{E}} I_i(\mathbf{E}) = \gamma_i(I_{\mathbf{E}})\mathbf{G}_i(\mathbf{E}), \tag{3.39}$$

the jump (3.13) in the elasticity tensor simplifies into

$$[\![\mathbb{S}(\mathbf{E})]\!] = s(\mathbf{E})[\gamma_i(I_{\mathbf{E}})\gamma_j(I_{\mathbf{E}})]\mathbf{G}_i(\mathbf{E}) \otimes \mathbf{G}_j(\mathbf{E}), \ (i, j = 1, p), \ \forall \mathbf{E} \,|\, g(\mathbf{E}) = 0. \tag{3.40}$$

On the other hand, a direct calculation based on (3.34) and the continuity condition (3.37) yields

$$[\![\mathbb{S}(\mathbf{E})]\!] = [\omega_{+ij}(I_{\mathbf{E}}) - \omega_{-ij}(I_{\mathbf{E}})]\mathbf{G}_i(\mathbf{E}) \otimes \mathbf{G}_j(\mathbf{E}), \ (i, j = 1, p), \ \forall \mathbf{E} \,|\, g(\mathbf{E}) = 0. \tag{3.41}$$

Comparison of (3.40) and (3.41) establishes the jump condition (3.38), since the $\mathbf{G}_i$ and thus the $\mathbf{G}_i \otimes \mathbf{G}_j$ are linearly independent. □

4. Conewise linear elasticity

In the hypothesis of small strains, it makes sense to specialize the previous piecewise smooth formulation into a piecewise linear one.

4.1. Half-spacewise linear elasticity

A piecewise *linear* stress–strain law $\mathbf{S}(\mathbf{E})$ deriving from a piecewise *quadratic* potential $W(\mathbf{E})$ and involving a piecewise *constant* elasticity tensor $\mathbb{S}(\mathbf{E})$ can be defined on two subdomains $\mathscr{E}_-$ and $\mathscr{E}_+$ by two quadratic restrictions

$$W_- = \tfrac{1}{2}\mathbf{E} : \mathbb{S}_- \mathbf{E} \quad \text{and} \quad W_+ = \tfrac{1}{2}\mathbf{E} : \mathbb{S}_+ \mathbf{E},$$

together with their linear gradients $\mathbf{S}_-(\mathbf{E}) = \mathbb{S}_-\mathbf{E}$ and $\mathbf{S}_+(\mathbf{E}) = \mathbb{S}_+\mathbf{E}$ and their constant Hessians $\mathbb{S}_-$ and $\mathbb{S}_+$. Note that (3.8) and (3.9) are automatically satisfied.

At this stage, the pieces $\mathscr{E}_-$ and $\mathscr{E}_+$ defined in (3.1) can again be taken of arbitrary shape. However, this freedom ceases as soon as continuous differentiability of $W \in \mathscr{C}^1(\mathscr{E}, \mathscr{R})$ is enforced. Indeed, if the continuity condition (3.12)

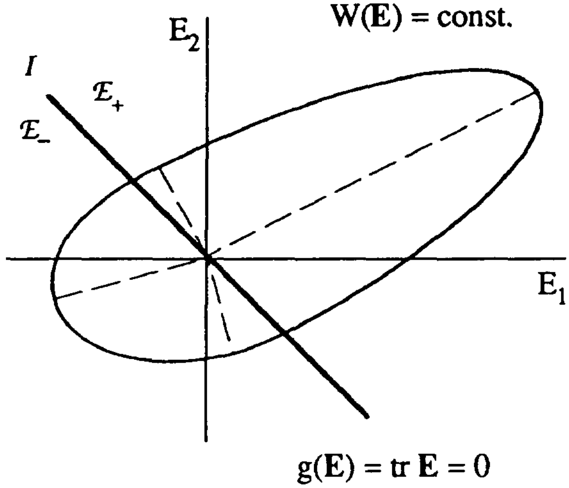

Fig. 7. Unique partition of strain space $\mathscr{E}$ into a compression half-space $\mathscr{E}_-$ and a tension half-space $\mathscr{E}_+$ and typical equipotential ellipses for a piecewise linear law in 2*D* principal strain plane.

is requested, its Corollary (3.13) imposes

$$[\![\mathbb{S}]\!] = \mathbb{S}_+ - \mathbb{S}_- = s(\mathbf{E})\nabla g(\mathbf{E}) \otimes \nabla g(\mathbf{E}), \quad \forall \mathbf{E} \,|\, g(\mathbf{E}) = 0. \tag{4.1}$$

Since $\mathbb{S}_-$ and $\mathbb{S}_+$ are constant, the interface normal must also be constant as well as the jump coefficient,

$$\nabla g(\mathbf{E}) = \mathbf{N}, \quad s(\mathbf{E}) = s. \tag{4.2}$$

It follows that g must be an affine function and in fact a *linear* one since $g(\mathbf{0}) = 0$. Therefore, the interface $\mathscr{I}$ is a *hyperplane* characterized by its unit normal N and the two pieces $\mathscr{E}_-$ and $\mathscr{E}_+$ are two *half-spaces* as defined in (3.3) and illustrated in Fig. 7. Hence, continuity of a piecewise linear law across the interface imposes the same planarity restriction on the interface shape than the convexity of the potential restrictions W_- and W_+.

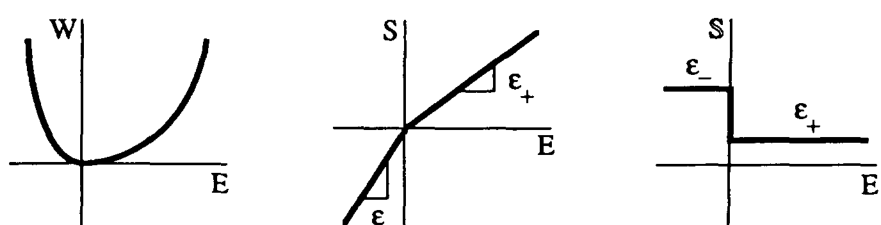

Fig. 8. (a) Piecewise quadratic potential, (b) piecewise linear stress–strain law, (c) piecewise constant elasticity modulus, for a non-smooth elastic material in one dimension.

In summary, a *piecewise linear continuous elastic law*, limited to two pieces (one for tension and one for compression) is *necessarily* a *half-spacewise linear law* deriving from a half-spacewise quadratic potential and involing a half-spacewise constant elasticity tensor (Fig. 8):

$$W(\mathbf{E}) = \tfrac{1}{2}\mathbf{E}:\mathbb{S}(\mathbf{E})\mathbf{E} = \begin{cases} W_-(\mathbf{E}) = \tfrac{1}{2}\mathbf{E}:\mathbb{S}_-\mathbf{E} & \text{if } \mathbf{N}:\mathbf{E} \leqslant 0 \\ W_+(\mathbf{E}) = \tfrac{1}{2}\mathbf{E}:\mathbb{S}_+\mathbf{E} & \text{if } \mathbf{N}:\mathbf{E} \geqslant 0 \end{cases}; \tag{4.3}$$

$$\mathbf{S}(\mathbf{E}) = \mathbb{S}(\mathbf{E})\mathbf{E} = \nabla_{\mathbf{E}} W(\mathbf{E}) = \begin{cases} \mathbf{S}_-(\mathbf{E}) = \mathbb{S}_-\mathbf{E} & \text{if } \mathbf{N}:\mathbf{E} \leqslant 0 \\ \mathbf{S}_+(\mathbf{E}) = \mathbb{S}_+\mathbf{E} & \text{if } \mathbf{N}:\mathbf{E} \geqslant 0 \end{cases}; \tag{4.4}$$

$$\mathbb{S}(\mathbf{E}) := \begin{cases} \mathbb{S}_- & \text{if } \mathbf{N}:\mathbf{E} < 0 \\ \mathbb{S}_+ & \text{if } \mathbf{N}:\mathbf{E} > 0 \end{cases}, \quad \mathbb{S}_+ - \mathbb{S}_- = s\mathbf{N} \otimes \mathbf{N}. \tag{4.5}$$

REMARKS. (i) Equivalent expressions of the law can be obtained in terms of $\mathbb{S}_-$ *or* $\mathbb{S}_+$ *and* s (instead of $\mathbb{S}_-$ *and* $\mathbb{S}_+$) by using the jump condition and the point to set distance function

$$d_{\mathscr{C}}(\mathbf{x}) \equiv \operatorname{dist}(\mathbf{x}, \mathscr{C}) := \min_{\mathbf{y} \in \mathscr{C}} (\|\mathbf{y} - \mathbf{x}\|); \; [d_{\mathscr{R}_+}(x) = \max(-x, 0) \text{ in } 1D];$$

$$W(\mathbf{E}) = \tfrac{1}{2}\mathbf{E}:\mathbb{S}_-\mathbf{E} + \frac{s}{2}\, d^2_{\mathscr{R}_+}(\mathbf{N}:\mathbf{E}) = \tfrac{1}{2}\mathbf{E}:\mathbb{S}_+\mathbf{E} - \frac{s}{2}\, d^2_{\mathscr{R}_-}(\mathbf{N}:\mathbf{E}),$$

$$\mathbf{S}(\mathbf{E}) = \mathbb{S}_-\mathbf{E} + s d_{\mathscr{R}_+}(\mathbf{N}:\mathbf{E})\mathbf{N} = \mathbb{S}_+\mathbf{E} - s d_{\mathscr{R}_-}(\mathbf{N}:\mathbf{E})\mathbf{N}.$$

(ii) The generalized Hessian reduces on the hyperplane interface, i.e. $\forall \mathbf{E} \in \mathscr{I}$, to

$$\begin{aligned} \partial_{\mathbf{E}}\mathbf{S}(\mathbf{E}) = \{\mathbb{S} = \lambda\mathbb{S}_- + (1-\lambda)\mathbb{S}_+ &= \mathbb{S}_- + (1-\lambda)s\mathbf{N} \otimes \mathbf{N} \\ &= \mathbb{S}_+ - \lambda s\mathbf{N} \otimes \mathbf{N},\ \forall \lambda \in [0,1]\}. \end{aligned}$$

The hyperelasticity condition (3.16) or (3.17) holds with constant $\mathbb{S}_+$ or $\mathbb{S}_-$.

4.2. Convexity-monotony criterion and inverse law

It follows from Corollary 3.1 that the convexity of W and monotony of $\mathbf{S}$ are ensured by the positive definiteness of both elasticity tensors, i.e. $\forall \mathbf{H} \in \mathscr{E}$, $\mathbf{H} \neq \mathbf{0}$, $\mathbf{H}:\mathbf{N} \neq 0$

$$\mathbf{H}:\mathbb{S}_-\mathbf{H} > 0 \quad \text{and} \quad \mathbf{H}:\mathbb{S}_+\mathbf{H} > 0 \quad \text{or} \quad s > -\frac{\mathbf{H}:\mathbb{S}_-\mathbf{H}}{(\mathbf{H}:\mathbf{N})^2}. \tag{4.6}$$

In stress space, the interface $\mathscr{I}^*$ is given by

$$h(\mathbf{S}) = \mathbf{M}:\mathbf{S} = 0, \quad \mathbf{M} := \mathbb{E}_-\mathbf{N} \quad \text{or} \quad \mathbb{E}_+\mathbf{N}. \tag{4.7}$$

The complementary energy, inverse law and compliance tensor are given by

$$W^*(\mathbf{S}) = \tfrac{1}{2}\mathbf{S}:\mathbb{E}(\mathbf{S})\mathbf{S}, \tag{4.8}$$

$$\mathbf{E}(\mathbf{S}) = \mathbb{E}(\mathbf{S})\mathbf{S}, \tag{4.9}$$

$$\mathbb{E}(\mathbf{S}) = \mathbb{S}^{-1}(\mathbf{S}) = \begin{cases} \mathbb{E}_- & \text{if } \mathbf{M}:\mathbf{S} < 0 \\ \mathbb{E}_+ & \text{if } \mathbf{M}:\mathbf{S} > 0 \end{cases}, \quad \mathbb{E}_+ - \mathbb{E}_- = e\mathbf{M} \otimes \mathbf{M}; \tag{4.10}$$

where

$$e = -s/(1 + s\mathbf{M}:\mathbf{N}). \tag{4.11}$$

4.3. Orthotropic half-spacewise linear elasticity

The interface function g being linear, its invariant form γ must be a linear combination of the linear invariants $I_a = \operatorname{tr} \mathbf{A}_a\mathbf{E}$ and its constant gradient $\nabla_{\mathbf{E}} g$, the hyperplane normal, must be a linear combination of the corresponding texture tensors $\mathbf{A}_a$:

$$g(\mathbf{E}) = \gamma(I_{\mathbf{E}}) = \gamma_a \operatorname{tr} \mathbf{A}_a\mathbf{E} = \operatorname{tr}[(\gamma_a\mathbf{A}_a)\mathbf{E}] = \mathbf{N}:\mathbf{E} = 0, \quad (a = 1, d), \tag{4.12}$$

$$\mathbf{N} = \nabla_{\mathbf{E}} g = \frac{\partial \gamma}{\partial I_a} \nabla_{\mathbf{E}} I_a = \gamma_a \mathbf{A}_a, \quad (\|\mathbf{N}\| = 1 \Leftrightarrow \sqrt{\gamma_a\gamma_a} = 1). \tag{4.13}$$

Similarly, the half-spacewise quadratic energy function (4.3), linear stress–strain law (4.4) and constant elasticity tensor (4.5) take, in analogy with (2.25), (2.26) and (2.27), the following forms:

$$W(\mathbf{E}) = \frac{\lambda_{ab}[\mathbf{N}:\mathbf{E}]}{2} \operatorname{tr}(\mathbf{A}_a\mathbf{E}) \operatorname{tr}(\mathbf{A}_b\mathbf{E}) + \mu_a \operatorname{tr}(\mathbf{A}_a\mathbf{E}^2), \quad (a, b = 1, d), \tag{4.14}$$

$$\mathbf{S}(\mathbf{E}) = \lambda_{ab}[\mathbf{N}:\mathbf{E}] \operatorname{tr}(\mathbf{A}_a\mathbf{E})\mathbf{A}_b + \mu_a(\mathbf{A}_a\mathbf{E} + \mathbf{E}\mathbf{A}_a), \tag{4.15}$$

$$\mathbb{S}(\mathbf{E}) = \lambda_{ab}[\mathbf{N}:\mathbf{E}]\mathbf{A}_a \otimes \mathbf{A}_b + \mu_a[\mathbf{A}_a \underline{\bar{\otimes}} \mathbf{I} + \mathbf{I} \underline{\bar{\otimes}} \mathbf{A}_a], \tag{4.16}$$

$$\lambda_{ab}[\mathbf{N}:\mathbf{E}] = \lambda_{ba} := \begin{cases} \lambda_{-ab} & \text{if } \mathbf{N}:\mathbf{E} < 0 \\ \lambda_{+ab} & \text{if } \mathbf{N}:\mathbf{E} > 0 \end{cases}, \qquad \lambda_{+ab} - \lambda_{-ab} = \sigma\gamma_a\gamma_b, \ \mathbf{N} = \gamma_a\mathbf{A}_a. \tag{4.17}$$

Therefore the anisotropic half-spacewise smooth relationships (4.14), (4.15) and (4.16) differ from the smooth ones (2.25), (2.26) and (2.27) only by *half-spacewise constant "bulk" functions* $\lambda_{ab}[\mathbf{N}:\mathbf{E}]$ defined in (4.17), instead of the usual constants λ_{ab}. The shear coefficients μ_a are the same in tension and compression.

Solving Eq. (4.17b) for the unit normal direction cosines γ_a (with $\sqrt{\gamma_a\gamma_a} = 1$) leads to the compatibility conditions

$$\gamma_a = \sqrt{(\lambda_{+aa} - \lambda_{-aa})/\sigma}, \quad \sigma = \lambda_{+bb} - \lambda_{-bb}, \quad (a, b = 1, d;\ \text{sum on } b), \tag{4.18}$$

$$\lambda_{+ab} - \lambda_{-ab} = \sqrt{(\lambda_{+aa} - \lambda_{-aa})(\lambda_{+bb} - \lambda_{-bb})}, \quad (a, b = 1, d;\ \text{no sum}). \tag{4.19}$$

This proves the following fact.

PROPOSITION 4.1. *For an orthotropic half-spacewise linear stress–strain law, the orientation of the hyperplane interface is determined by the jumps* $[\![\lambda_{aa}]\!]$ *in the diagonal "bulk" constants as defined in* (4.18), *which must be consistent with the off-diagonal ones* $[\![\lambda_{ab}]\!]$ ($a \neq b$) *as specified in* (4.19).

It follows that the half-spacewise relationships (4.14), (4.15) and (4.16) involve $[(d + 5)d/2]$ constants: $[(d + 1)d/2]$ "λ_-" + (d) "$[\![\lambda]\!]$" + (d) "μ".

The inverse relationships are given by (2.28)–(2.31), provided the constant elastic moduli ε_a and contraction ratios ν_{ab} are replaced by half-spacewise constant functions

$$\varepsilon_a[\mathbf{M}:\mathbf{S}] := \begin{cases} \varepsilon_{-a} & \text{if } \mathbf{M}:\mathbf{S} < 0 \\ \varepsilon_{+a} & \text{if } \mathbf{M}:\mathbf{S} > 0 \end{cases}, \quad \nu_{ab}[\mathbf{M}:\mathbf{S}] := \begin{cases} \nu_{-ab} & \text{if } \mathbf{M}:\mathbf{S} < 0 \\ \nu_{+ab} & \text{if } \mathbf{M}:\mathbf{S} > 0 \end{cases}; \tag{4.20}$$

where $\mathbf{M} := \mathbb{E}_-\mathbf{N}$ or $\mathbb{E}_+\mathbf{N}$. Symmetries of the bulk Lamé constants λ_{ab} require

$$\frac{\nu_{-ab}}{\varepsilon_{-a}} = \frac{\nu_{-ba}}{\varepsilon_{-b}}, \quad \frac{\nu_{+ab}}{\varepsilon_{+a}} = \frac{\nu_{+ba}}{\varepsilon_{+b}}. \tag{4.21}$$

The shear Lamé constants μ_a being the same in tension and compression due to continuity, the following restrictions must hold

$$\frac{1}{G_{ab}} = \frac{2}{\mu_a + \mu_b} = \frac{1 + \nu_{-aa}}{\varepsilon_{-a}} + \frac{1 + \nu_{-bb}}{\varepsilon_{-b}} = \frac{1 + \nu_{+aa}}{\varepsilon_{+a}} + \frac{1 + \nu_{+bb}}{\varepsilon_{+b}}, \quad (a, b = 1, d). \tag{4.22}$$

These constraints keep the number of independent constants down to $[(d+5)d/2]$.

The isotropic case is studied in more details in the next paragraph.

4.4. Isotropic half-spacewise linear elasticity

In the isotropic case, Eqs. (4.14)–(4.17) explicitly reduce to (Fig. 9)

$$W(\mathbf{E}) := \tfrac{1}{2}\lambda[\operatorname{tr}\mathbf{E}]\operatorname{tr}^2(\mathbf{E}) + \mu\operatorname{tr}(\mathbf{E}^2), \tag{4.23}$$

$$\mathbf{S}(\mathbf{E}) = \lambda[\operatorname{tr}\mathbf{E}](\operatorname{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \tag{4.24}$$

$$\mathbb{S}(\mathbf{E}) = \lambda[\operatorname{tr}\mathbf{E}]\mathbf{I}\otimes\mathbf{I} + 2\mu\mathbf{I}\,\underline{\bar{\otimes}}\,\mathbf{I}, \tag{4.25}$$

$$\lambda[\operatorname{tr}\mathbf{E}] := \begin{cases}\lambda_- & \text{if } \operatorname{tr}\mathbf{E} < 0\\ \lambda_+ & \text{if } \operatorname{tr}\mathbf{E} > 0\end{cases}, \tag{4.26}$$

which involve 3 constants: λ_-, λ_+ and μ. Equation (4.24) shows that the bulk (or volumic) part of the stress–strain relationship becomes half-spacewise linear whereas the shear (or deviatoric) part remains linear. Equation (4.26) shows that the interface $\mathscr{I}$ is the hyperplane defined by the trace normal $\mathbf{N} = \mathbf{I}$ or by the trace invariant $I_1 = \operatorname{tr}\mathbf{E} = 0$. The hyperplane normal is a common eigentensor of $\mathbb{S}_+$ and $\mathbb{S}_-$ since they commute (see Appendix C)

$$\mathbb{S}_+\mathbb{S}_- = \mathbb{S}_-\mathbb{S}_+ \Rightarrow \begin{cases}\mathbb{S}_+\mathbf{I} = (3\lambda_+ + 2\mu)\mathbf{I}\\ \mathbb{S}_-\mathbf{I} = (3\lambda_- + 2\mu)\mathbf{I}\end{cases}. \tag{4.27}$$

Characterizing the tension-compression dissymmetry by means of the sign of I_1 was already proposed in Shapiro (1966), but his argument was quite different.

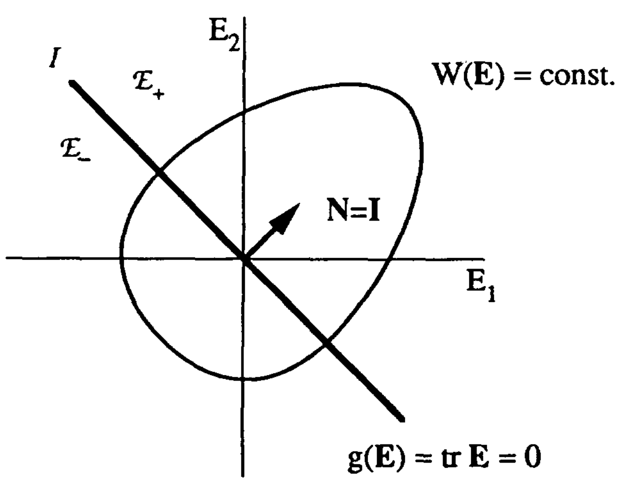

Fig. 9. Unique partition of strain space $\mathscr{E}$ into a compression half-space $\mathscr{E}_-$ and a tension half-space $\mathscr{E}_+$ and typical equipotential ellipses for a piecewise linear isotropic law in 2*D* principal strain plane.

The inverse relationships are in accordance with (2.28), (2.29) and (2.30):

$$W^*(\mathbf{S}) = -\frac{\nu[\operatorname{tr}\mathbf{S}]}{2\varepsilon[\operatorname{tr}\mathbf{S}]}\operatorname{tr}^2(\mathbf{S}) + \frac{1+\nu[\operatorname{tr}\mathbf{S}]}{2\varepsilon[\operatorname{tr}\mathbf{S}]}\operatorname{tr}(\mathbf{S}^2), \tag{4.28}$$

$$\mathbf{E}(\mathbf{S}) = -\frac{\nu[\operatorname{tr}\mathbf{S}]}{\varepsilon[\operatorname{tr}\mathbf{S}]}(\operatorname{tr}\mathbf{S})\mathbf{I} + \frac{1+\nu[\operatorname{tr}\mathbf{S}]}{\varepsilon[\operatorname{tr}\mathbf{S}]}\mathbf{S}, \tag{4.29}$$

$$\mathbb{E}(\mathbf{S}) = -\frac{\nu[\operatorname{tr}\mathbf{S}]}{\varepsilon[\operatorname{tr}\mathbf{S}]}\mathbf{I}\otimes\mathbf{I} + \frac{1+\nu[\operatorname{tr}\mathbf{S}]}{\varepsilon[\operatorname{tr}\mathbf{S}]}\mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{I}, \tag{4.30}$$

$$\varepsilon[\operatorname{tr}\mathbf{S}] := \begin{cases}\varepsilon_- & \text{if } \operatorname{tr}\mathbf{S} < 0\\ \varepsilon_+ & \text{if } \operatorname{tr}\mathbf{S} > 0\end{cases}, \qquad \nu[\operatorname{tr}\mathbf{S}] := \begin{cases}\nu_- & \text{if } \operatorname{tr}\mathbf{S} < 0\\ \nu_+ & \text{if } \operatorname{tr}\mathbf{S} > 0\end{cases}. \tag{4.31}$$

Continuity of the shear Lamé constant requires

$$2\mu = \frac{\varepsilon_-}{1+\nu_-} = \frac{\varepsilon_+}{1+\nu_+}. \tag{4.32}$$

The constraint (4.32) maintains the number of independent constants down to 3. The bulk Lamé constants are deduced from the familiar formulas

$$\lambda_- = 2\mu\frac{\nu_-}{1-2\nu_-}, \qquad \lambda_+ = 2\mu\frac{\nu_+}{1-2\nu_+}. \tag{4.33}$$

Substituting (4.32) in (4.28), (4.29) and (4.30) shows that the shear part of the strain–stress law remains continuous. For instance (4.29) can be rewritten as

$$\mathbf{E}(\mathbf{S}) = \frac{-\nu[\operatorname{tr}\mathbf{S}]}{\varepsilon[\operatorname{tr}\mathbf{S}]}(\operatorname{tr}\mathbf{S})\mathbf{I} + \frac{1}{2\mu}\mathbf{S}.$$

Usually, the constants which are measured in experiments are ε_-, ε_+, ν_- and ν_+ rather than λ_-, λ_+ and μ. A violation of the continuity condition (4.32) by measurements would mean that the present model is inappropriate. In particular, the hypothesis that the material is isotropic in compression *and* in tension should be reexamined.

4.5. Conewise linear elasticity

More complicated subdivisions of $\mathscr{E}$ can be obtained by using several hyperplanes. The correct generalization of the two above half-spaces are then *polyhedral convex cones,* i.e. *pyramids* centered at the origin (Fig. 10a). This assertion can be supported from two different perspectives: closedness of the pieces under positive linear combinations and continuity across the interfaces as in Section 3.1.3. A polyhedral convex cone is defined by a system of linear inequalities,

$$K := \{\mathbf{E} \in \mathscr{E} \mid \mathbb{A}\mathbf{E} \leqslant \mathbf{0}\}. \tag{4.34}$$

Its *negative* and *polar* (Fig. 10b) are also polyhedral convex cones defined by

$$-K := \{\mathbf{E} \in \mathscr{E} \mid -\mathbf{E} \in K\},$$

$$K^{\circ} := \{\mathbf{H} \in \mathscr{E} \mid \mathbf{H}:\mathbf{E} \leqslant 0,\ \forall \mathbf{E} \in K\} = \{\mathbf{H} = \mathbb{A}^T \mathbf{G} \mid \mathbf{G} \geqslant \mathbf{0}\}.$$

They are sometimes useful for simplifying the subdivision definition.

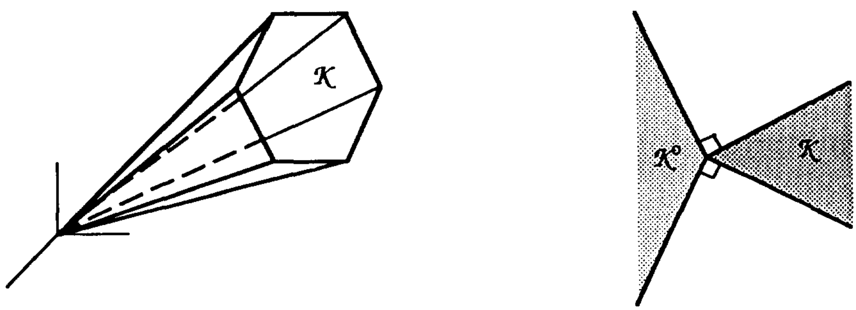

Fig. 10. (a) Polyhedral convex cone, (b) convex cone and its polar in $\mathscr{R}^2$.

For instance a simple possibility is studied in Section 4.6. Note however that the interface function $g(\mathbf{E}) = \mathbb{A}\mathbf{E}$ must respect the material symmetries which rules out certain combinations. An isotropic material, for example, is necessarily limited to two pieces: the two half-spaces already studied.

4.6. Orthotropic octantwise linear elasticity

For an orthotropic material, a subdivision of the strain space into *eight octants* delimited by the three orthogonal hyperplanes normal to the three texture tensors $\mathbf{A}_a = \mathbf{a}_a \otimes \mathbf{a}_a$ $(a = 1, 3)$ is natural:

$$\begin{aligned} \mathscr{I} &= \{\mathbf{E} \in \mathscr{E} \mid g_1(\mathbf{E}) = \mathbf{A}_1:\mathbf{E} = 0 \text{ or } g_2(\mathbf{E}) = \mathbf{A}_2:\mathbf{E} = 0 \text{ or } g_3(\mathbf{E}) = \mathbf{A}_3:\mathbf{E} = 0\},\\ &= \{\mathbf{E} \in \mathscr{E} \mid (\mathbf{A}_1:\mathbf{E})(\mathbf{A}_2:\mathbf{E})(\mathbf{A}_3:\mathbf{E}) = 0\} \end{aligned} \tag{4.36a}$$

$$\mathscr{E}^0_- = \{\mathbf{E} \in \mathscr{E} \mid \mathbf{A}_1:\mathbf{E} \leqslant 0,\ \mathbf{A}_2:\mathbf{E} \leqslant 0,\ \mathbf{A}_3:\mathbf{E} \leqslant 0\}, \quad \mathscr{E}^0_+ = -\mathscr{E}^0_- = \mathscr{E}^{0^{\circ}}_-, \tag{4.36b}$$

$$\mathscr{E}^1_- = \{\mathbf{E} \in \mathscr{E} \mid \mathbf{A}_1:\mathbf{E} \geqslant 0,\ \mathbf{A}_2:\mathbf{E} \leqslant 0,\ \mathbf{A}_3:\mathbf{E} \leqslant 0\}, \quad \mathscr{E}^1_+ = -\mathscr{E}^1_- = \mathscr{E}^{1^{\circ}}_-, \tag{4.36c}$$

$$\mathscr{E}^2_- = \{\mathbf{E} \in \mathscr{E} \mid \mathbf{A}_1:\mathbf{E} \leqslant 0,\ \mathbf{A}_2:\mathbf{E} \geqslant 0,\ \mathbf{A}_3:\mathbf{E} \leqslant 0\}, \quad \mathscr{E}^2_+ = -\mathscr{E}^2_- = \mathscr{E}^{2^{\circ}}_-, \tag{4.36d}$$

$$\mathscr{E}^3_- = \{\mathbf{E} \in \mathscr{E} \mid \mathbf{A}_1:\mathbf{E} \leqslant 0,\ \mathbf{A}_2:\mathbf{E} \leqslant 0,\ \mathbf{A}_3:\mathbf{E} \geqslant 0\}, \quad \mathscr{E}^3_+ = -\mathscr{E}^3_- = \mathscr{E}^{3^{\circ}}_-. \tag{4.36e}$$

The strain origin is a singular point in this case.

The energy function, stress–strain law and elasticity tensor are defined by means of *eight constant elasticity tensors* as

$$W(\mathbf{E}) = \tfrac{1}{2}\mathbf{E} : \mathbb{S}(\mathbf{E})\mathbf{E}, \tag{4.37}$$

$$\mathbf{S}(\mathbf{E}) = \mathbb{S}(\mathbf{E})\mathbf{E}, \tag{4.38}$$

$$\mathbb{S}(\mathbf{E}) := \mathbb{S}^i_\Sigma \text{ if } \mathbf{E} \in \mathscr{E}^i_\Sigma, \quad (i = 0, 3;\ \Sigma = \{-, +\}). \tag{4.39}$$

A set of seven necessary and sufficient conditions for the stress–strain law to be *continuous* across the interface (4.36a) is

$$\mathbb{S}^a_- = \mathbb{S}^0_- + s_a \mathbf{A}_a \otimes \mathbf{A}_a, \quad (a = 1, 3;\ \text{no sum}), \tag{4.40a}$$

$$\mathbb{S}^a_+ = \mathbb{S}^0_+ - s_a \mathbf{A}_a \otimes \mathbf{A}_a, \quad (a = 1, 3;\ \text{no sum}), \tag{4.40b}$$

$$\mathbb{S}^0_+ = \mathbb{S}^0_- + s_a \mathbf{A}_a \otimes \mathbf{A}_a, \quad (a = 1, 3;\ \text{sum}). \tag{4.40c}$$

These conditions are direct extensions of the single interface condition (4.5b). They show that an orthotropic octantwise linear stress–strain law involves *twelve* independent constants: 9 for $\mathbb{S}^0_-$ and 3 s_a. There are only three jump magnitudes s_a $(a = 1, 3)$ because there are just three hyperplanes $\mathbf{A}_a : \mathbf{E} = 0$.

Enforcing orthotropy, the octantwise quadratic energy function (4.37), linear stress–strain law (4.38) and constant elasticity tensor (4.39) take the forms (4.14) to (4.16), respectively, with $d = 3$ and

$$\lambda_{ab}[\mathbf{N} : \mathbf{E}] = \lambda_{ab}[\mathbf{E}] := \lambda^i_{\Sigma ab} \text{ if } \mathbf{E} \in \mathscr{E}^i_\Sigma, \ (i = 0, 3;\ \Sigma = \{-, +\}). \tag{4.41}$$

Therefore the model is defined by six *octantwise constant "bulk" functions* $\lambda_{ab}[\mathbf{E}]$ and three usual "shear" constants μ_a. The *continuity* conditions (4.46) become, in analogy with (4.17b),

$$\lambda^c_{-ab} = \lambda^0_{-ab} + \sigma_c \delta_{ac} \delta_{bc}, \quad (a, b, c = 1, 3;\ \text{no sum}), \tag{4.42a}$$

$$\lambda^c_{+ab} = \lambda^0_{+ab} - \sigma_c \delta_{ac} \delta_{bc}, \quad (a, b, c = 1, 3;\ \text{no sum}), \tag{4.42b}$$

$$\lambda^0_{+ab} = \lambda^0_{-ab} + \sigma_a \delta_{ab}, \quad (a, b = 1, 3;\ \text{no sum}). \tag{4.42c}$$

These conditions show that *only the diagonal bulk constants suffer a jump* across the interface. The *off-diagonal bulk constants* (and the shear constants) *are the* *same over the eight octants.* Hence, an octantwise linear orthotropic model can be effectively expressed as follows. For (a, $b = 1,3$; $b \neq a$).

$$W(\mathbf{E}) = \frac{\lambda_{aa}[\mathbf{A}_a:\mathbf{E}]}{2} \operatorname{tr}^2(\mathbf{A}_a\mathbf{E}) + \frac{\lambda_{ab}}{2} \operatorname{tr}(\mathbf{A}_b\mathbf{E}) \operatorname{tr}(\mathbf{A}_b\mathbf{E}) + \mu_a \operatorname{tr}(\mathbf{A}_a\mathbf{E}^2), \tag{4.43}$$

$$\mathbf{S}(\mathbf{E}) = \lambda_{aa}[\mathbf{A}_a:\mathbf{E}] \operatorname{tr}(\mathbf{A}_a\mathbf{E})\mathbf{A}_a + \lambda_{ab} \operatorname{tr}(\mathbf{A}_a\mathbf{E})\mathbf{A}_b + \mu_a(\mathbf{A}_a\mathbf{E} + \mathbf{E}\mathbf{A}_a), \tag{4.44}$$

$$\mathbb{S}(\mathbf{E}) = \lambda_{aa}[\mathbf{A}_a:\mathbf{E}]\mathbf{A}_a \otimes \mathbf{A}_a + \lambda_{ab}\mathbf{A}_a \otimes \mathbf{A}_b + \mu_a[\mathbf{A}_a \underline{\bar{\otimes}} \mathbf{I} + \mathbf{I} \underline{\bar{\otimes}} \mathbf{A}_a], \tag{4.45}$$

$$\lambda_{aa}[\mathbf{A}_a:\mathbf{E}] := \lambda_{\Sigma(a)aa}, \quad \Sigma(a) := \operatorname{sign}(\mathbf{A}_a:\mathbf{E}). \tag{4.46}$$

It involves only 12 constants: 6 "$\lambda\_$" + 3 "σ" + 3 "μ".

The inverse relationships are still given by (2.28)–(2.31), provided the constant elastic moduli ε_a and contraction ratios ν_{ab} are replaced by the octantwise constant functions

$$\varepsilon_a[\mathbf{A}:\mathbf{S}] := \varepsilon_{\Sigma(a)a}, \quad \nu_{ab}[\mathbf{A}_a:\mathbf{S}] := \nu_{\Sigma(a)ab}, \quad \Sigma(a) := \operatorname{sign}(\mathbf{A}_a:\mathbf{S}), \quad (a, b = 1,3). \tag{4.47}$$

Symmetry of the bulk Lamé constants requires

$$\frac{\nu_{\Sigma(a)ab}}{\varepsilon_{\Sigma(a)a}} = \frac{\nu_{\Sigma(a)ba}}{\varepsilon_{\Sigma(a)b}}, \quad (a, b = 1,3). \tag{4.48}$$

Continuity of the shear Lamé constants requires

$$\frac{1}{G_{ab}} = \frac{2}{\mu_a + \mu_b} = \frac{1 + \nu_{\Sigma(a)aa}}{\varepsilon_{\Sigma(a)a}} + \frac{1 + \nu_{\Sigma(a)bb}}{\varepsilon_{\Sigma(a)b}}, \quad (a, b = 1,3). \tag{4.49}$$

Since Ambartsumyan's initiating work in 1969, several *orthotropic* models have been developed for predicting the elastic behavior of fibrous composite materials with different elastic moduli in tension and compression [Tabaddor (1969); Bert (1977); Jones (1977) and Vijayakumar and Rao (1987)]. However these models were not formulated in invariant form like here.

4.7. Principle of similarity for small displacement conewise linear elasticity

To close this section, a fundamental property of conewise linear materials is stated together with its consequence on the solutions of boundary value problems in conewise linear elasticity.

4.7.1. Positive homogeneity

The stress–strain law (4.4) is neither additive nor homogeneous. It is only half-spacewise so. However it remains wholewise *positively homogeneous* of degree one. Accordingly, the energy function and the elasticity tensor are positively homogeneous functions of degrees two and zero, respectively

$$W(\lambda\mathbf{E}) = \lambda^2 W(\mathbf{E}), \quad \forall\lambda \geqslant 0; \tag{4.50}$$

$$\mathbf{S}(\lambda\mathbf{E}) = \lambda\mathbf{S}(\mathbf{E}), \quad \forall\lambda \geqslant 0; \tag{4.51}$$

$$\mathbb{S}(\lambda\mathbf{E}) = \mathbb{S}(\mathbf{E}), \quad \forall\lambda \geqslant 0. \tag{4.52}$$

This comes from the fact that a half-space in particular and a convex cone in general are closed under positive multiplication. Positive homogeneity is a basic characteristic of conewise linear elastic stress–strain laws and has an important consequence on the solution of the corresponding boundary value problems.

4.7.2. Boundary value problem

Using a standard notation and assuming small displacements $\|\mathbf{u}\| \ll 1$ (and thus small strains $\|\mathbf{E}\| \ll 1$), the boundary value problem can be stated as follows:

Data $\Omega \in \mathbf{R}^3$, $\partial\Omega = \Gamma_u \cup \Gamma_p$, $\mathbf{f}(\mathbf{x})$, $\bar{\mathbf{u}}(\mathbf{x})$, $\bar{\mathbf{s}}(\mathbf{x})$.

Find $\mathbf{u}(\mathbf{x})$ such that:

$$\text{Div}\,\mathbf{S}[\nabla\mathbf{u}(\mathbf{x})] + \mathbf{f}(\mathbf{x}) = \mathbf{0}, \quad \forall\mathbf{x} \in \Omega; \tag{4.53a}$$

$$\mathbf{S}[\nabla\mathbf{u}] = \mathbf{S}[\mathbf{E}], \quad \mathbf{E}(\nabla\mathbf{u}) = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T); \tag{4.53b}$$

$$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}), \ \forall\mathbf{u} \in \Gamma_u; \ \mathbf{S}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \bar{\mathbf{s}}(\mathbf{x}), \ \forall\,\mathbf{x} \in \Gamma_p. \tag{4.53c}$$

4.7.3. Principle of similarity

In the small displacement case, positive homogeneity is the root of a weakened form of the classical principle of superposition in linear elasticity, which can be called and stated as follows.

PRINCIPLE OF SIMILARITY. *If* $\mathbf{u}(\mathbf{x})$ *is the (unique) small displacement solution to a conewise linear elastic problem with external body force* $\mathbf{f}(\mathbf{x})$, *boundary conditions* $\bar{\mathbf{u}}(\mathbf{x})$ *on* $\partial\Omega_u$ *and* $\bar{\mathbf{s}}(\mathbf{x})$ *on* $\partial\Omega_p$, *then* $\lambda\mathbf{u}(\mathbf{x})$ *is the homothetic (similar) solution to the homothetic problem with body force* $\lambda\mathbf{f}(\mathbf{x})$, *boundary conditions* $\lambda\bar{\mathbf{u}}(\mathbf{x})$ *and* $\lambda\bar{\mathbf{s}}(\mathbf{x})$, *for any positive scalar* λ.

Proof. It is straightforward upon substituting property (4.51) into the governing equilibrium equation (4.53a) and invoking uniqueness of the solution in the small displacement case. □

5. Conclusions

In this study, *conewise linear* materials have been found out to be the proper generalization to two and three dimensions of one-dimensional bimodular materials. *Continuity* of the stress–strain law has turned out to be the key property for globalizing a piecewise property. Schematically,

piecewise property + *continuity* = *(wholewise) property.*

Indeed, it has been successively shown that

(1) a continuous piecewise hyperelastic stress–strain law is hyperelastic;
(2) a continuous piecewise monotone stress–strain law is monotone.

Another important conclusion is that a twicewise linear law (i.e. a law linear over *two* pieces) is a *half-spacewise* linear law with a hyperplane interface. If moreover the material has the same symmetry in tension and in compression, then the hyperplane orientation is uniquely determined by the tension and compression *bulk* constants. In particular, if the material is wholewise isotropic, then the tension-compression interface must be the traceless hyperplane with the identity as unit normal.

Finally, *positive homogeneity* of the stress–strain law has been identified as the most fundamental property of conewise linear materials since it leads to the *principle of similarity* of small displacement solutions to boundary value problems of conewise linear elasticity.

To close with a perspective, note that classical *unimodular* elastic materials which behave similarly in tension and compression are characterized by a very smooth elastic energy density $W \in \mathscr{C}^2(\mathscr{E}, \mathscr{R})$ and thus belong to the realm of *smooth* elasticity. *Bimodular* elastic materials which behave differently in tension and compression are characterized by a less smooth density $W \in \mathscr{C}^1(\mathscr{E}, \mathscr{R})$ and can be said to belong to *piecewise smooth* elasticity. Similarly, *amodular* elastic materials which cavitate in tension and are incompressible in compression or are inextensible in tension and oppose no resistance in compression are characterized by a continuous density $W \in \mathscr{C}^0(\mathscr{E}, \mathscr{R})$ and could be said to belong to *nonsmooth* elasticity. Their study could proceed along the lines sketched in this article.

Acknowledgement

The authors wish to thank Pierre Alart for his helpful comments regarding Paragraph 3.2.2.

References

1. P. Alart, Critères d'injectivité et de surjectivité pour certaines applications de R^N dans lui-même; Application à la mécanique des contacts. *RAIRO* (1992).
2. S.A. Ambartsumyan, The axisymmetric problem of circular cylindrical shell made of materials with different stiffnesses in tension and compression. *Izv. Akad. Nauk. SSSR Mekh.* 4 (1965) 77–85.
3. S.A. Ambartsumyan, Basic equations and relations of the different modulus theory of elasticity of an anisotropic body. *Mechanics of Solids* 4 (1969) 48–56.
4. S.A. Ambartsumyan and A.A. Khachatryan, The basic equations of the theory of elasticity for materials with different stiffnesses in tension and compression. *Mechanics of Solids* 1 (1966) 29–34.
5. C.W. Bert, Model for fibrous composites with different properties in tension and compression. *J. of Eng. Mat. & Tech.* 99 (1977) 344–349.
6. J.P. Boehler, Lois de comportement des milieux continus. *J. de Mécanique* 17 (1978) 153–190.
7. J.L. Chaboche, On the description of damage induced anisotropy and active/passive damage effect. In J.W. Ju, D. Krajcinovic and H.L. Schreyer (eds), *Damage Mechanics in Engineering Materials.* AMD/ASME (1990) pp. 153–166.
8. P.G. Ciarlet, *Mathematical Elasticity.* Elsevier (1988).
9. F.H. Clarke, *Optimization and Nonsmooth Analysis.* John Wiley & Sons (1983).
10. H. Costa-Mattos, M. Fremond and E.N. Mamiya, A simple model of the mechanical behaviour of ceramic-like materials. *Int. J. of Solids and Structures* 29 (1992) 3185–3200.
11. A. Curnier, *Méthodes numériques en mécanique des solides.* PPUR (1993).
12. G.P. Del Piero, Some properties of the set of fourth-order tensors, with application to elasticity. *J. of Elasticity* 9 (1979) 245–261.
13. A.E. Green and J.Z. Mkrtichian, Elastic solids with different moduli in tension and compression. *J. of Elasticity* 7 (1977) 369–386.
14. M.E. Gurtin, *An Introduction to Continuum Mechanics.* Academic Press (1981).
15. P.R. Halmos, *Finite-Dimensional Vector Spaces.* Springer-Verlag (1974).
16. H. Horii and S. Nemat-Nasser, Overall moduli of Solids with microcracks: load induced anisotropy. *J. of the Mechanics and Physics of Solids* 32 (1983) 155–171.
17. R.M. Jones, Stress–strain relations for materials with different moduli in tension and compression. *AIAA Journal* 15 (1977) 16–23.
18. N. Kamiya, Bimodulus thermoelasticity considering temperature dependent material properties. In C.W. Bert (ed.), *Mechanics of Bimodulus Materials.* AMD/ASME (1979) pp. 29–37.
19. M. Kojima and R. Saigal, On the relationship between conditions that ensure a PL mapping is a homeomorphism. *Mathematics of Operations Research* 5 (1980).
20. I.-S. Liu, On representations of anisotropic invariants. *Int. J. Engng. Sci.* 20 (1982) 1099–1109.
21. M. Lucchesi, D.R. Owen and P. Podio-Guidugli, Materials with elastic range: A theory with a view toward applications. Part III, Approximate constitutive relations. *Archives of Rational Mechanics and Analysis* 117 (1992) 53–96.
22. J.E. Marsden and T.J.R. Hughes, *Mathematical Foundations of Elasticity.* Prentice-Hall (1983).
23. J. Mazars, Y. Berthaud and S. Ramtani, The unilateral behaviour of damaged concrete. *Eng. Fract. Mech.* 35 (1990) 629–635.
24. G. Medri, A non linear elastic model for isotropic material with different behaviour in tension and compression. *J. of Eng. Mat. & Tech.* 104 (1982) 22–27.

25. R.C. Novak and C.W. Bert, Theoretical and experimental basis for more precise elastic properties of epoxy. *J. of Composite Materials* 2 (1968) 506–508.
26. Z. Rigbi, Some thoughts concerning the existence or otherwise of an isotropic bimodulus material. *J. of Eng. Mat. & Tech.* 99 (1980) 383–384.
27. R.T. Rockafellar, *Convex Analysis.* Princeton University Press (1970).
28. E. Sacco and J.N. Reddy, A constitutive model for bimodular materials with an application to plate bending. *J. of Applied Mechanics* 59 (1992) 220–221.
29. G.S. Shapiro, Deformation of bodies with different tension and compression stiffnesses. *Mechanics of Solids* 1 (1966) 85–86.
30. D.A. Spence and J.Z. Mkrtichian, The Boussinesq problem for a material with different moduli in tension and compresssion. *J. of Mechanics and Applied Mathematics* 30 (1977) 449–466.
31. A.J.M. Spencer, The formulation of constitutive equations for anisotropic solids. In J.P. Boehler (ed.), *Mechanical Behaviour of Anisotropic Solids.* Martinus Nijhoff Publishers, The Hague (1982) pp. 2–26.
32. F. Tabaddor, A survey of constitutive equations of bimodulus elastic materials. In C.W. Bert (ed.), *Mechanics of Bimodulus Materials.* AMD/ASME (1979) pp. 1–15.
33. K. Vijayakumar and K.P. Rao, Stress–strain relations for composites with different stiffnesses in tension and compression. *Comp. Mech.* 2 (1987) 167–175.

Appendix A: First and second gradients of symmetric strain tensor invariants

In Table 1, the invariant (e.g. I_5) and generator (e.g. $\mathbf{G}_5$) gradients are calculated with the help of the directional derivative and by insisting on the symmetry of $\mathbf{E}$ as follows.

$$\delta[I_i](\mathbf{E}, \delta\mathbf{E}) = \nabla_{\mathbf{E}} I_i(\mathbf{E}) : \delta\mathbf{E} = \lim_{\varepsilon \to 0} [D_i(\varepsilon)) := I_i(\mathbf{E} + \varepsilon\delta\mathbf{E}) - I_i(\mathbf{E})]/\varepsilon.$$

$$\begin{aligned} D_5(\varepsilon) &= \mathrm{tr}[\mathbf{A}(\mathbf{E}+\varepsilon\delta\mathbf{E})^2] - \mathrm{tr}[\mathbf{A}\mathbf{E}^2] = \mathrm{tr}\{\mathbf{A}[\mathbf{E}^2 + \varepsilon(\mathbf{E}\delta\mathbf{E} + \delta\mathbf{E}\mathbf{E}) + \mathcal{O}(\varepsilon^2)] - \mathbf{A}\mathbf{E}^2\} \\ &= \varepsilon\,\mathrm{tr}[\delta\mathbf{E}(\mathbf{A}\mathbf{E}+\mathbf{E}\mathbf{A})] + \mathcal{O}(\varepsilon^2) = \varepsilon\,\mathrm{tr}[(\mathbf{A}\mathbf{E}+\mathbf{E}\mathbf{A})^T\delta\mathbf{E}^T] + \mathcal{O}(\varepsilon^2) \\ &= \frac{\varepsilon}{2}[\mathbf{A}(\mathbf{E}+\mathbf{E}^T) + (\mathbf{E}+\mathbf{E}^T)\mathbf{A}]^T : \delta\mathbf{E} + \mathcal{O}(\varepsilon^2) = \varepsilon(\mathbf{A}\mathbf{E}+\mathbf{E}\mathbf{A}) : \delta\mathbf{E} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

$$\nabla_{\mathbf{E}}\,\mathrm{tr}(\mathbf{A}\mathbf{E}^2)(\mathbf{E}) = \mathbf{A}\mathbf{E} + \mathbf{E}\mathbf{A}. \tag{A.1}$$

$$\delta[\mathbf{G}_i](\mathbf{E}, \delta\mathbf{E}) = \nabla_{\mathbf{E}}\mathbf{G}_i(\mathbf{E})\delta\mathbf{E} = \lim_{\varepsilon \to 0} [\mathbf{D}_i(\varepsilon) := \mathbf{G}_i(\mathbf{E}+\varepsilon\delta\mathbf{E}) - \mathbf{G}_i(\mathbf{E})]/\varepsilon.$$

$$\begin{aligned} \mathbf{D}_5(e) &= [\mathbf{A}(\mathbf{E}+\varepsilon\delta\mathbf{E}) + (\mathbf{E}+\varepsilon\delta\mathbf{E})\mathbf{A}] - [\mathbf{A}\mathbf{E}+\mathbf{E}\mathbf{A}] = \varepsilon(\mathbf{A}\delta\mathbf{E} + \delta\mathbf{E}\mathbf{A}) + \mathcal{O}(\varepsilon^2) \\ &= \frac{\varepsilon}{2}[\mathbf{A}(\delta\mathbf{E}+\delta\mathbf{E}^T) + (\delta\mathbf{E}+\delta\mathbf{E}^T)\mathbf{A}] + \mathcal{O}(\varepsilon^2) \\ &= \frac{\varepsilon}{2}[\mathbf{A}\delta\mathbf{E}\mathbf{I} + \mathbf{A}\delta\mathbf{E}^T\mathbf{I} + \mathbf{I}\delta\mathbf{E}\mathbf{A} + \mathbf{I}\delta\mathbf{E}^T\mathbf{A}] + \mathcal{O}(\varepsilon^2) \\ &= \frac{\varepsilon}{2}[\mathbf{A} \underline{\otimes} \mathbf{I} + \mathbf{A} \bar{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{A} + \mathbf{I} \bar{\otimes} \mathbf{A}]\delta\mathbf{E} + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon[\mathbf{A} \underline{\bar{\otimes}} \mathbf{I} + \mathbf{I} \underline{\bar{\otimes}} \mathbf{A}]\delta\mathbf{E} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

$$\nabla_{\mathbf{E}}(\mathbf{A}\mathbf{E}+\mathbf{E}\mathbf{A}) = \mathbf{A} \underline{\bar{\otimes}} \mathbf{I} + \mathbf{I} \underline{\bar{\otimes}} \mathbf{A}. \tag{A.2}$$

Appendix B: Linear elasticity: explicit representations for usual symmetries

For an *orthotropic* material, $d = 3$ and, using $\mathbf{C} = \mathbf{I} - \mathbf{A} - \mathbf{B}$ instead of $\mathbf{I}$ in the isotropy column of Table 1, the following explicit expressions which involve 9 constants are obtained:

$$\begin{aligned} W(\mathbf{E}) = {} & \frac{\lambda_{AA}}{2}\operatorname{tr}^2(\mathbf{AE}) + \frac{\lambda_{BB}}{2}\operatorname{tr}^2(\mathbf{BE}) + \frac{\lambda_{CC}}{2}\operatorname{tr}^2(\mathbf{CE}) \\ & + \lambda_{AB}\operatorname{tr}(\mathbf{AE})\operatorname{tr}(\mathbf{BE}) + \lambda_{BC}\operatorname{tr}(\mathbf{BE})\operatorname{tr}(\mathbf{CE}) + \lambda_{CA}\operatorname{tr}(\mathbf{CE})\operatorname{tr}(\mathbf{AE}) \\ & + \mu_A\operatorname{tr}(\mathbf{AE}^2) + \mu_B\operatorname{tr}(\mathbf{BE}^2) + \mu_C\operatorname{tr}(\mathbf{CE}^2); \end{aligned} \tag{B.1}$$

$$\begin{aligned} \mathbf{S}(\mathbf{E}) = {} & \lambda_{AA}\operatorname{tr}(\mathbf{AE})\mathbf{A} + \lambda_{BB}\operatorname{tr}(\mathbf{BE})\mathbf{B} + \lambda_{CC}\operatorname{tr}(\mathbf{CE})\mathbf{C} \\ & + \lambda_{AB}[\operatorname{tr}(\mathbf{AE})\mathbf{B} + \operatorname{tr}(\mathbf{BE})\mathbf{A}] + \lambda_{BC}[\operatorname{tr}(\mathbf{BE})\mathbf{C} + \operatorname{tr}(\mathbf{CE})\mathbf{B}] \\ & + \lambda_{CA}[\operatorname{tr}(\mathbf{CE})\mathbf{A} + \operatorname{tr}(\mathbf{AE})\mathbf{C}] + \mu_A[\mathbf{AE} + \mathbf{EA}] \\ & + \mu_B[\mathbf{BE} + \mathbf{EB}] + \mu_C[\mathbf{CE} + \mathbf{EC}]; \end{aligned} \tag{B.2}$$

$$\begin{aligned} \mathbb{S} = {} & \lambda_{AA}\mathbf{A}\otimes\mathbf{A} + \lambda_{BB}\mathbf{B}\otimes\mathbf{B} + \lambda_{CC}\mathbf{C}\otimes\mathbf{C} \\ & + \lambda_{AB}[\mathbf{A}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{A}] + \lambda_{BC}[\mathbf{B}\otimes\mathbf{C} + \mathbf{C}\otimes\mathbf{B}] + \lambda_{CA}[\mathbf{C}\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{C}] \\ & + \mu_A[\mathbf{A}\,\bar{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{A}] + \mu_B[\mathbf{B}\,\bar{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{B}] + \mu_A[\mathbf{C}\,\bar{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{C}]. \end{aligned} \tag{B.3}$$

For a *transotropic* (i.e. transverse isotropic) material, $d = 2$ and the formulas involve 5 constants

$$W(\mathbf{E}) = \frac{\lambda}{2}\operatorname{tr}^2(\mathbf{E}) + \frac{\lambda_{AA}}{2}\operatorname{tr}^2(\mathbf{AE}) + \lambda_A\operatorname{tr}(\mathbf{E})\operatorname{tr}(\mathbf{AE}) + \mu\operatorname{tr}(\mathbf{E}^2) + \mu_A\operatorname{tr}(\mathbf{AE}^2); \tag{B.4}$$

$$\mathbf{S}(\mathbf{E}) = \lambda\operatorname{tr}(\mathbf{E})\mathbf{I} + \lambda_{AA}\operatorname{tr}(\mathbf{AE})\mathbf{A} + \lambda_A[\operatorname{tr}(\mathbf{E})\mathbf{A} + \operatorname{tr}(\mathbf{AE})\mathbf{I}] + 2\mu\mathbf{E} + \mu_A[\mathbf{AE} + \mathbf{EA}]; \tag{B.5}$$

$$\mathbb{S} = \lambda\mathbf{I}\otimes\mathbf{I} + \lambda_{AA}\mathbf{A}\otimes\mathbf{A} + \lambda_A[\mathbf{A}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{A}] + 2\mu\mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{I} + \mu_A[\mathbf{A}\,\bar{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{A}]. \tag{B.6}$$

For an *isotropic* material, $d = 1$ and formulas (2.19) to (2.24) further simplify into

$$W(\mathbf{E}) = \frac{\lambda}{2}\operatorname{tr}^2(\mathbf{E}) + \mu\operatorname{tr}(\mathbf{E}^2), \qquad W^*(\mathbf{S}) = \frac{-\nu}{2\varepsilon}\operatorname{tr}^2(\mathbf{S}) + \frac{1+\nu}{2\varepsilon}\operatorname{tr}(\mathbf{S}^2); \tag{B.7}$$

$$\mathbf{S}(\mathbf{E}) = \lambda\operatorname{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \qquad \mathbf{E}(\mathbf{S}) = \frac{-\nu}{\varepsilon}\operatorname{tr}(\mathbf{S})\mathbf{I} + \frac{1+\nu}{\varepsilon}\mathbf{S}; \tag{B.8}$$

$$\mathbb{S} = \lambda\mathbf{I}\otimes\mathbf{I} + 2\mu\mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{I}, \qquad \mathbb{E} = \frac{-\nu}{\varepsilon}\mathbf{I}\otimes\mathbf{I} + \frac{1+\nu}{\varepsilon}\mathbf{I}\,\bar{\underline{\otimes}}\,\mathbf{I}. \tag{B.9}$$

where λ and μ are the 2 Lamé constants, ε is Young's modulus and ν Poisson's ratio.

REMARKS. (i) In (B.9), the elasticity tensor can also be written as

$$\mathbb{S} = \kappa[\mathbf{I} \otimes \mathbf{I}] + 2\mu[\mathbf{I} \,\bar{\underline{\otimes}}\, \mathbf{I} - \tfrac{1}{3}\mathbf{I} \otimes \mathbf{I}], \quad \kappa = (3\lambda + 2\mu)/3;$$

where κ is the bulk modulus.

(ii) If (full rank) convexity is relaxed into rank-one convexity and, equivalently, positive definiteness into ellipticity, i.e. if $\mathbf{H} - \mathbf{E}$ is replaced by $\mathbf{h} \otimes \mathbf{e}$ in (2.6), (2.7) and (2.8), then the requirement $3\lambda + 2\mu > 0$ relaxes into $\lambda + 2\mu > 0$.

Appendix C: Half-spacewise linear elasticity: commutative symmetries

It can be shown that, for isotropic, transotropic and orthotropic materials, *the two elasticity tensors commute if and only if the bulk coefficient matrices* $\Lambda_+ = [\lambda_{+ab}]$ *and* $\Lambda_- = [\lambda_{-ab}]$ *commute*:

$$\mathbb{S}_+\mathbb{S}_- = \mathbb{S}_-\mathbb{S}_+ \Rightarrow \Lambda_+\Lambda_- = \Lambda_-\Lambda_+ \Rightarrow \lambda_{+ac}\lambda_{-cb} = \lambda_{-ac}\lambda_{+cb}. \tag{C.1}$$

In this case, $\mathbb{S}_+$ and $\mathbb{S}_-$ have the same eigentensors. Moreover the following holds.

LEMMA. *If* $\mathbb{S}_+$ *and* $\mathbb{S}_-$ *commute and if they differ by a rank one tensor* $\mathbf{N} \otimes \mathbf{N}$, *then* $\mathbf{N}$ *is a common eigentensor of* $\mathbb{S}_+$ *and* $\mathbb{S}_-$.

Proof. Assume that $\mathbb{S}_+$ and $\mathbb{S}_-$ satisfy the jump condition (4.5) and that they *commute*

$$\mathbb{S}_+ - \mathbb{S}_- = s\mathbf{N} \otimes \mathbf{N}, \tag{C.2}$$

$$\mathbb{S}_+\mathbb{S}_- = \mathbb{S}_-\mathbb{S}_+. \tag{C.3}$$

The jump condition (C.2) and the commutativity condition (C.3) yield

$$\mathbb{S}_+\mathbf{N} \otimes \mathbf{N} = \mathbf{N} \otimes \mathbf{N}\mathbb{S}_+, \tag{C.4}$$

$$\mathbb{S}_-\mathbf{N} \otimes \mathbf{N} = \mathbf{N} \otimes \mathbf{N}\mathbb{S}_+. \tag{C.5}$$

This implies in turn that

$$\mathbb{S}_+\mathbf{N} = (\mathbf{N} : \mathbb{S}_+\mathbf{N})\mathbf{N}, \tag{C.6}$$

$$\mathbb{S}_-\mathbf{N} = (\mathbf{N} : \mathbb{S}_-\mathbf{N})\mathbf{N}. \tag{C.7}$$

It follows that $\mathbf{N}$ is an eigentensor of both $\mathbb{S}_+$ and $\mathbb{S}_-$.

Due to their symmetry, the 5 other eigentensors are orthogonal to $\mathbf{N}$ and therefore they span the interface hyperplane $\mathbf{N} : \mathbf{E} = 0$.