Journal of Elasticity, Vol. 10, No. 1, January 1980

On a unified boundary-integral equation method[1]

NICHOLAS J. ALTIERO and STEVEN D. GAVAZZA[2]

Department of Metallurgy, Mechanics and Materials Science, Michigan State University, East Lansing, Michigan 48824, U.S.A.

(Received August 7, 1978)

ABSTRACT

In this paper the connection is developed between the direct and indirect boundary-integral equation methods of linear elastostatics from both a physical and a mathematical viewpoint. It is shown that the indirect method in its various forms, like the direct method, can be derived from Somigliana's identity, and one particular indirect formulation is presented which reduces the mixed problem of elastostatics to a system of Cauchy singular integral equations.

RÉSUMÉ

Dans ce texte la relation est developpe entre la direct et l'indirect méthode d'équations intégrales de la frontiére de l'elasticite linear des deux points de vue mathematique et physique.

Il est demonstre que la methode indirecte dans ses differents aspects resemble a la methode direct pouvant etre tire de l'identite de Somigliana. Et une formulation particuliere indirecte a ete presente. Elle reduit le problemes mixte de l'elasticite a un systeme d'equation d'integration singulier de Cauchy.

1. Introduction

In recent years increasing attention has been given to boundary-integral equation methods in mathematical physics, primarily because of the computational advantages which they enjoy over finite-difference and finite-element methods for linear boundary-value problems. Rizzo [1] based his formulation of the so-called direct BIE method for linear elasticity on the classical method of Somigliana. To solve the traction problem, for example, Rizzo obtains a system of singular integral equations for the unknown boundary values of the displacements in terms of the known boundary tractions. Once the boundary displacements have been determined, the interior elastic fields are computed in terms of both the boundary tractions and displacements by means of integrations. A seemingly different approach, the so-called indirect method, has been put forth by Massonnet [2] and more recently by

[1] Presented at Eighth National Congress of Applied Mechanics, Los Angeles, California.

[2] Now at Department of Mechanics, Civil and Industrial Engineering, California State University, Northridge, California 91330.

Altiero and Sikarskie [3], in which the region of interest is imbedded in another region of the same material for which the Green's function or point force solution is known. Typically the matrix, or region in which the imbedding is done, is taken to be infinite, but this need not be so. A layer of body force, or "fictitious tractions", is then applied to the imbedded boundary, and this layer is adjusted to produce the desired solution in the imbedded region. This method also involves the solution of a system of singular integral equations, but the unknown quantity is now the "fictitious" distribution of body force as opposed to the "real" boundary displacements in the direct formulation.

The direct and indirect formulations will be summarized in the next section, and the main differences between them will be made clear. A common feature of these formulations, however, is that each may encounter difficulty when applied to certain displacement or mixed problems of elastostatics. The source of this difficulty is not entirely clear. A possible remedy might be a formulation which yields singular integral equations of the Cauchy type, such as that presented in section 3.

Another type of indirect method, suggested first by Louat [4] and developed primarily by Lardner [5], utilizes distributions of dislocations on the imbedded boundary to produce the required solution within. This method is also best suited for the traction problem, although *certain* displacement and mixed problems can be solved as well. Recently Maiti, Das, and Palit [6] have shown that Lardner's method may be derived from Somigliana's method, and implicit in their work is the fact that the indirect method of Altiero and Sikarskie can also be derived from Somigliana's method.

Although Somigliana's method can be used to arrive at both the indirect and direct BIE methods, significant differences in the two approaches remain, primarily the distinction between solving for the "real" versus "fictitious" quantities. Moreover, these differences may be important, for the existence of the two indirect methods described above suggests the possibility of numerous indirect methods in which combinations of fictitious tractions, dislocation distributions, and related quantities are sought. In particular, one such formulation for the mixed problem, which utilizes fictitious tractions and infinitesimal dislocation loops (dipoles in two-dimensions), will be presented here. Proper choice of these parameters always leads to singular integral equations. This formulation is an extension of the method of Maiti, Das, and Palit.

The authors hasten to point out that the various results which are used to obtain the extended version of the BIE method are not new. In fact, Eshelby [7] attributes some of these ideas to Gebbia. Consequently, much of the discussion leading up to the new formulation presented in section 3 is expository in nature, but necessary for complete understanding of the method.

Numerical implementation of any boundary-integral scheme is a somewhat involved process, and it seems best to defer a discussion of the method presented here until such time as sufficient comparisons among the various methods can be made. Such work is in fact the subject of an ongoing investigation and will be reported elsewhere.

2. Basic equations and existing formulations

In this section we list the equation and boundary conditions associated with the standard boundary-value problems of elastostatics and discuss briefly the boundary integral equation method as formulated by Rizzo [1] and by Altiero and Sikarskie [3]. We wish to point out some of the similarities and differences between the two formulations and also to have them written out explicitly for reference in the following sections.

In the absence of body force the equations of equilibrium are

$$C_{ijkl}u_{k,lj} = 0 \tag{2.1}$$

where u_k denotes the displacement vector, the comma notation denotes partial differentiation with respect to the coordinate variable whose subscript follows the comma, and summation is implied over the values 1 to 3 when an index is repeated. C_{ijkl} denotes the elastic constants of the material which is anisotropic in general. We are concerned with the standard boundary-value problems of elastostatics. That is, we wish to solve (2.1) in a domain D subject to one of the following conditions on the boundary B of D:

i) u_i a prescribed function on B.
ii) $t_i = C_{ijkl}u_{k,l}n_j$ a prescribed function on B.
iii) u_i a prescribed function on B_u, t_i a prescribed function on B_t, $B = B_u + B_t$.

Here t_i represents the surface tractions on B and n_i is the unit outward normal to B.

The BIE method of Rizzo is based on Somigliana's identity:

$$u_i(\boldsymbol{\xi}) = \int_B \{t_j(\boldsymbol{x})U_j^i(\boldsymbol{x}, \boldsymbol{\xi}) - u_j(\boldsymbol{x})T_j^i(\boldsymbol{x}, \boldsymbol{\xi})\}\, ds \qquad \boldsymbol{\xi} \in D, \tag{2.2}$$

where $U_j^i(\boldsymbol{x}, \boldsymbol{\xi})$ denotes the jth component of displacement at $\boldsymbol{x}$ due to a unit force in the ith direction at $\boldsymbol{\xi}$ in an infinite medium, and $T_j^i(\boldsymbol{x}, \boldsymbol{\xi})$ are the surface tractions which correspond to the displacements $U_j^i(\boldsymbol{x}, \boldsymbol{\xi})$. In three dimensions ds denotes an element of area of B, in two dimensions, an element of arc length.

If $\boldsymbol{\xi}$ approaches a point $\boldsymbol{x}'$ on B from the inside, then (2.2) becomes

$$\tfrac{1}{2}u_i(\boldsymbol{x}') + \int_B u_j(\boldsymbol{x})T_j^i(\boldsymbol{x}, \boldsymbol{x}')\, ds = \int_B t_j(\boldsymbol{x})U_j^i(\boldsymbol{x}, \boldsymbol{x}')\, ds, \tag{2.3}$$

where the integral on the left-hand side is to be interpreted in the Cauchy principal-value sense. For a boundary condition of type (ii), i.e. traction prescribed everywhere, the right side of (2.3) is known, and one solves a system of singular integral equations for the boundary displacements u_i. The displacements within B can then be found by integration using (2.2). On the other hand, if the boundary condition is of type (i), i.e. displacement prescribed everywhere, then the left side of (2.3) is known, but the resulting integral equations for t_i are not singular. This arises because $U_j^i(\boldsymbol{x})$ is $O(\ln|\boldsymbol{x}|)$ in two dimensions and is $O(|\boldsymbol{x}|^{-1})$ in three dimensions, and these singularites are integrable. It follows that for boundary conditions of type (iii), i.e. for the mixed boundary-value problem, not all of the integral equations are singular equations of the Cauchy type.

If a layer of body force f_i is distributed over a surface B in an infinite medium, the stresses and displacements throughout the medium are determined according to the principle of superposition by

$$\sigma_{ij}(\boldsymbol{\xi}) = \int_B \Sigma_{ij}^k(\boldsymbol{\xi}, x) f_k(\boldsymbol{x})\, ds \tag{2.4a}$$

$$u_i(\boldsymbol{\xi}) = \int_B U_i^j(\boldsymbol{\xi}, \boldsymbol{x}) f_j(\boldsymbol{x})\, ds, \tag{2.4b}$$

where $\Sigma_{ij}^k(\boldsymbol{\xi}, \boldsymbol{x})$ are the stresses associated with $U_i^k(\boldsymbol{\xi}, \boldsymbol{x})$. Note that $T_i^j(\boldsymbol{\xi}, \boldsymbol{x}) = \Sigma_{ik}^j(\boldsymbol{\xi}, \boldsymbol{x}) n_k(\xi)$ for $\boldsymbol{\xi}$ on B. As before, let $\boldsymbol{\xi}$ in (2.4) approach $\boldsymbol{x}'$ on B. Then, upon multiplying (2.4a) by $n_k(\boldsymbol{x}')$ we obtain

$$\tfrac{1}{2} f_i(\boldsymbol{x}') + \int_B T_i^j(\boldsymbol{x}', \boldsymbol{x}) f_j(\boldsymbol{x})\, ds = t_i(\boldsymbol{x}'), \tag{2.5a}$$

$$\int_B U_i^j(\boldsymbol{x}', \boldsymbol{x}) f_j(\boldsymbol{x})\, ds = u_i(x'). \tag{2.5b}$$

The integral in (2.5a) is to be interpreted in the Cauchy principal value sense. These equations embody the formulation of Altiero and Sikarskie. For boundary conditions of type (ii), the right side of (2.5a) is known for all $\boldsymbol{x}'$ on B, and one solves a system of singular integral equations for the body force distribution of "fictitious traction" f_i. The elastic fields within B are then computed from equation (2.4). For boundary conditions of type (i), the right side of (2.5b) is known for all $\boldsymbol{x}'$ on B and, as with Rizzo's method, the resulting integral equations are not singular. Similarly, with $t_i(\boldsymbol{x}')$ known for $\boldsymbol{x}'$ on B_t and $u_i(\boldsymbol{x}')$ known for $\boldsymbol{x}'$ on B_u, not all of the integral equations in (2.5) are Cauchy singular. Singular integral equations of the Cauchy type may be desirable because they lead to a diagonally dominant system of linear algebraic equations which can be solved by iteration techniques as opposed to direct elimination schemes. This might provide a computational savings.

3. The unified formulation

We begin our discussion of the extended BIE method by recalling what is meant by a Somigliana dislocation or elastic inclusion. In an elastic continuum make a cut over a surface S (not necessarily closed), and give the two faces of the cut, S^+ and S^-, a relative displacement

$$b_i(\boldsymbol{x}) = u_i^-(\boldsymbol{x}) - u_i^+(\boldsymbol{x}), \tag{3.1}$$

where u_i^- and u_i^+ are the displacements of points on S^- and S^+, respectively. Note that $b_i(\boldsymbol{x})$ may vary from point to point on S. Any gaps which are created are to be filled in with material, and interpenetration of the faces of the cut is prevented by removal of material where appropriate. The faces of the cut are then welded together, the result being that S is now a surface across which the displacement vector suffers the discontinuity given in (3.1). In addition, the continuum is internally stressed, and the tractions across S are continuous. When S is a closed surface, the material within is often called an elastic inclusion.

Following Volterra [8], we may write the displacements due to the dislocation S in an infinite medium as

$$u_i(\boldsymbol{\xi}) = -\int_S b_j(\boldsymbol{x}) T_j^i(\boldsymbol{x}, \boldsymbol{\xi})\, ds \tag{3.2}$$

If (2.4b) is added to (3.2) with $S = B$, we obtain

$$u_i(\boldsymbol{\xi}) = \int_B \{f_j(\boldsymbol{x}) U_i^j(\boldsymbol{\xi}, \boldsymbol{x}) - b_j(\boldsymbol{x}) T_j^i(\boldsymbol{x}, \boldsymbol{\xi})\}\, ds. \tag{3.3}$$

Comparing (3.3) and (2.2), noting that $U_i^j(\boldsymbol{\xi}, \boldsymbol{x}) = U_j^i(\boldsymbol{x}, \boldsymbol{\xi})$, we obtain immediately the result that the elastic fields in a region D generated by boundary displacements u_i and corresponding boundary tractions t_i are identical to the elastic fields within an elastic inclusion occupying the same region D in an infinite medium, if the displacement discontinuity b_i is equal to u_i and if the inclusion is acted upon by a layer of body force f_i equal to t_i. Depending upon the boundary conditions, some combination of f_i and b_i (i.e. t_i and u_i) must be determined, and this leads directly to the singular integral equation (2.3) of Rizzo. The elastic fields outside the inclusion are zero. This can be seen physically as follows. Cut out the region D from the undeformed infinite medium and apply to B^- the surface tractions t_i, causing displacements u_i, and generating the desired solution in D. Remove or add material to B^+ (i.e. the surface of the cavity created by removal of D) until this surface matches the deformed surface B^-. This causes *no* deformation outside B^+. Insert the removed material back into the cavity and weld the surfaces together. The elastic fields inside B are given by (2.2) and outside B they are clearly zero. Eshelby [7] attributes this result, which provides a physical interpretation of Somgliana's identity, to Gebbia. A mathematical treatment of this result is given in the discussion leading to equation (3.11).

It is also possible to generate the solution associated with (2.2) by taking one or the other of f_i or b_i equal to zero in (3.3). As before, make a cut over B and apply the surface tractions t_i to B^-. Now apply to B^+ whatever surface tractions t_i' are required to match the deformed shape of B^+ to that of B^-, and then weld the faces of the cut back together. The surface B is now subjected to a body force layer $t_i - t_i'$ since the "outward" normal on B^+ opposes that on B^-, and the final displacements associated with these operations are given by

$$u_i(\boldsymbol{\xi}) = \int_B (t_j(\boldsymbol{x}) - t_j'(\boldsymbol{x})) U_i^j(\boldsymbol{\xi}, \boldsymbol{x})\, ds \tag{3.4}$$

The displacements given by (3.4) are continuous throughout the medium and are identical to those of (2.2) within B. The tractions across B are discontinuous since $t_j \neq t_j'$, and if (3.4) is compared to (2.4b), it is clear that the quantities $t_i - t_i'$ are simply the fictitious tractions of Altiero and Sikarskie.

Alternatively make a cut over B in the infinite undeformed medium and apply the tractions t_i to both B^- and B^+. This causes the points on B^+ to displace by some amount u_i', but in general this will *not* close the gap between B^- and B^+. Fill in or remove material as needed and reweld the faces of the cut, creating a Somigliana dislocation with discontinuity $u_i - u_i'$ along B. Taking into account once again the difference in normals on B^- and B^+, the final displacements associated with these operations, according to (3.1), are given by

$$u_i(\boldsymbol{\xi}) = -\int_B (u_j(\boldsymbol{x}) - u_j'(\boldsymbol{x}))T_j^i(\boldsymbol{x}, \boldsymbol{\xi})\, ds \tag{3.5}$$

These displacements are identical to those of (2.2) within B, and the tractions associated with them are continuous across B. The quantity $u_i - u_i'$ may be thought of as a "fictitious dislocation distribution" b_i analogous to the fictitious tractions f_i of Altiero and Sikarskie. It is now apparent that while the notion of a fictitious traction is useful for solving the traction problem of elastostatics, because it leads to the singular integral equation (2.5a), the notion of a fictitious dislocation distribution is equally useful for solving the displacement problem of elastostatics, because it leads to the singular integral equation

$$\tfrac{1}{2}b_i(\boldsymbol{x}') - \int_B b_j(\boldsymbol{x})T_j^i(\boldsymbol{x}, \boldsymbol{x}')\, ds = u_i(\boldsymbol{x}'), \tag{3.6}$$

which is similar to (2.5a).

It is now clear how one may generalize the foregoing in order to obtain singular integral equations regardless of the boundary conditions. Wherever tractions are specified on B, apply a body force layer to the same part of B in an infinite medium, and wherever displacements are specified on B, create a Somigliana dislocation over that part of B in an infinite medium. The displacements and stresses associated with these operations are:

$$u_i(\boldsymbol{\xi}) = \int_{B_t} f_j(\boldsymbol{x})U_i^j(\boldsymbol{\xi}, \boldsymbol{x})\, ds - \int_{B_u} b_j(\boldsymbol{x})T_j^i(\boldsymbol{x}, \boldsymbol{\xi})\, ds, \tag{3.7a}$$

$$\sigma_{ij}(\boldsymbol{\xi}) = \int_{B_t} f_k(\boldsymbol{x})\Sigma_{ij}^k(\boldsymbol{\xi}, \boldsymbol{x})\, ds - \int_{B_u} C_{ijkl}b_m(\boldsymbol{x})\frac{\sigma}{\sigma\xi_l}T_m^k(\boldsymbol{x}, \boldsymbol{\xi})\, ds. \tag{3.7b}$$

Recall that $T_m^k(\boldsymbol{x}, \boldsymbol{\xi}) = \Sigma_{mp}^k(\boldsymbol{x}, \boldsymbol{\xi})n_p(\boldsymbol{x})$. To obtain the integral equations, let $\boldsymbol{x}'$ be a point on B, multiply (3.7b) by $n_j(\boldsymbol{x}')$ and take the limit as $\boldsymbol{\xi} \to \boldsymbol{x}'$ from the inside. The only integrals in equations (3.7) which are singular are the second of (3.7a), if $\boldsymbol{x}' \in B_u$, and the first of (3.7b), if $\boldsymbol{x}' \in B_t$. The limiting procedure therefore yields the following singular integral equations:

$$u_i(\boldsymbol{x}') = \int_{B_t} f_j(\boldsymbol{x}) U_i^j(\boldsymbol{x}', \boldsymbol{x})\, ds - \int_{B_u} b_j(\boldsymbol{x}) T_j^i(\boldsymbol{x}, \boldsymbol{x}')\, ds + \tfrac{1}{2} b_i(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in B_u, \tag{3.8a}$$

$$t_i(\boldsymbol{x}') = \tfrac{1}{2} f_i(\boldsymbol{x}') + \int_{B_t} f_j(\boldsymbol{x}) T_i^j(\boldsymbol{x}', \boldsymbol{x})\, ds - \int_{B_u} C_{ijkl} b_m(\boldsymbol{x}) n_j(\boldsymbol{x}') \frac{\sigma}{\sigma x_l'} T_m^k(\boldsymbol{x}, \boldsymbol{x}')\, ds, \qquad \boldsymbol{x}' \in B_t, \tag{3.8b}$$

where the second integral of (3.8a) and the first integral of (3.8b) are Cauchy principal value integrals.

Equations (3.8) obviously reduce to equations (2.5) if B_u is empty, and so we recover the formulation of Altiero and Sikarskie. On the other hand, if B_t is empty, they reduce to equations which are similar to (2.5), one of which is (3.6). Thus the various "fictitious" formulations are contained in (3.7). In a sense the "real" formulation of Rizzo is also contained in (3.8), if one notes that if $f_i = t_i$ and $b_i = u_i$ then (3.7a) cannot hold unless both B_t and B_u are replaced by B.

Equations (3.8) can also be obtained directly from Somigliana's identity rewritten in the form

$$u_i(\boldsymbol{\xi}) = \int_{B^-} \{\sigma_{jk}(\boldsymbol{x}) U_j^i(\boldsymbol{x}, \boldsymbol{\xi}) - u_j(\boldsymbol{x}) \Sigma_{jk}^i(\boldsymbol{x}, \boldsymbol{\xi})\} n_k(\boldsymbol{x})\, ds \tag{3.9}$$

The use of B^- here reminds us that u_i and σ_{ij} are the interior elastic fields. An application of the divergence theorem to (3.9) gives

$$u_i(\boldsymbol{\xi}) = \int_{D} \{\sigma_{jk}(\boldsymbol{x}) U_{j,k}^i(\boldsymbol{x}, \boldsymbol{\xi}) - u_{j,k}(\boldsymbol{x}) \Sigma_{jk}^i(\boldsymbol{x}, \boldsymbol{\xi}) - u_j(\boldsymbol{x}) \Sigma_{jk,k}^i(\boldsymbol{x}, \boldsymbol{\xi})\}\, dv, \tag{3.10}$$

where the differentiation is with respect to $\boldsymbol{x}$ and $\sigma_{jk,k} = 0$ for equilibrium under no body force. Let D_e denote the exterior of $D + B$. If $\boldsymbol{\xi} \in D_e$, then $\Sigma_{jk,k}^i(\boldsymbol{x}, \boldsymbol{\xi}) = 0$ for all $\boldsymbol{x} \in D$ by equilibrium. Moreover, for $\boldsymbol{\xi} \in D_e$, the quantities $U_{j,k}^i(\boldsymbol{x}, \boldsymbol{\xi})$ and $\Sigma_{jk}^i(\boldsymbol{x}, \boldsymbol{\xi})$ are regular in D, and so by the reciprocal theorem

$$\sigma_{jk}(\boldsymbol{x}) U_{j,k}^i(\boldsymbol{x}, \boldsymbol{\xi}) = u_{j,k}(\boldsymbol{x}) \Sigma_{jk}^i(\boldsymbol{x}, \boldsymbol{\xi})$$

for $\boldsymbol{x} \in D$. Accordingly, $u_i(\boldsymbol{\xi})$ given by (3.9) vanishes for $\boldsymbol{\xi} \in D_e$, i.e.

$$\int_{B^-} \{t_j(\boldsymbol{x}) U_j^i(\boldsymbol{x}, \boldsymbol{\xi}) - u_j(\boldsymbol{x}) T_j^i(\boldsymbol{x}, \boldsymbol{\xi})\}\, ds = \begin{cases} u_i(\boldsymbol{\xi}), & \boldsymbol{\xi} \in D \\ 0, & \boldsymbol{\xi} \in D_e \end{cases} \tag{3.11}$$

Now consider the infinite region exterior to B^+ and subject to boundary displacements u_i' and boundary tractions t_i'. If $u_i \sim O(r^{-1})$ and $\sigma_{ij} \sim O(r^{-2})$ as $r \to \infty$, then the same argument shows that

$$\int_{B^+} \{t_j'(\boldsymbol{x})U_j^i(\boldsymbol{x}, \boldsymbol{\xi}) - u_j'(\boldsymbol{x})T_j^i(\boldsymbol{x}, \boldsymbol{\xi})\}\, ds = \begin{cases} u_i(\boldsymbol{\xi}), \boldsymbol{\xi} \in D_e \\ 0, \boldsymbol{\xi} \in D \end{cases} \tag{3.12}$$

Adding (3.11) to (3.12) we obtain

$$\int_{B} \{(t_j - t_j')U_j^i - (u_j - u_j')T_j^i\}\, ds = u_i(\boldsymbol{\xi}) \tag{3.13}$$

This result was obtained by Maiti and Makan [9] in their discussion of elastic inclusions. To obtain (3.8), we merely require that $u_i = u_i'$ wherever tractions are prescribed, $t_i = t_i'$ wherever displacements are prescribed, and take the limit as $\boldsymbol{\xi} \to B^-$.

4. Discussion

The relationship between the direct and indirect boundary-integral equation methods has long been of interest to researchers in this field. In particular many have felt that while the direct formulation has a rigorous foundation, the indirect methods could only be justified on intuitive grounds. We have shown that this is not the case and that the indirect methods, like the direct method, can be derived from Somigliana's identity. This leaves aside the question of the numerical effectiveness of the various methods, and in this regard one notes that, while there is but a single form of the direct method, there are many possible versions of the indirect method, and a proper numerical study in which the various formulations are compared should be undertaken.

We have presented, in Section 3, one of the many possible forms of the indirect method. This form was constructed so as to produce Cauchy singular integral equations for any mixed problem of elastostatics. This seems to be a reasonable approach for the solution of mixed problems, but it is not yet known whether there are other formulations which might be better. It is possible that no one formulation is best for all problems but that various classes of problems, based on geometry, etc., might be individually suited to distinct formulations of the method. These questions need to be answered.

Acknowledgement

The authors wish to acknowledge Professor J. D. Eshelby for valuable discussions regarding the physical interpretation of Somigliana's identity.

REFERENCES

[1] Rizzo, F. J., An integral equation approach to boundary value problems of classical elastostatics, *Quarterly of Applied Mathematics* 40 (1967) 83–95.

[2] Massonnet, C. E., Numerical use of integral procedures, *Stress Analysis: Recent Developments in Numerical and Experimental Methods* (ed. O. C. Zienkiewicz and G. S. Holister), Wiley, London 1965.

[3] Altiero, N. J. and Sikarskie, D. L., An integral equation method applied to penetration problems in rock mechanics, *Boundary-Integral Equation Method: Computational Applications in Applied Mechanics* (ed. T. A. Cruse and F. J. Rizzo), ASME AMD-Vol. 11, New York 1975.

[4] Louat, N., Solution of boundary value problems in plane strain, *Nature* 196 (1962) 1081–1082.

[5] Lardner, R. W., Dislocation layers and boundary value problems of plane elasticity, *Quarterly Journal of Mechanics and Applied Mathematics* 25 (1972) 45–61.

[6] Maiti, M., Das, B., and Palit, S. S., Somigliana's method applied to plane problems of elastic half-spaces, *Journal of Elasticity* 6 (1976) 429–439.

[7] Eshelby, J. D., The elastic field of a crack extending non-uniformly under general anti-plane loading, *Journal of the Mechanics and Physics of Solids* 17 (1969) 177–199.

[8] Volterra, V., Sur l'eq́uilibre des corps élastiques multiplement connexes, *Anns École Normal Supérieure* 24 (1907) 401–517.

[9] Maiti, M., and Makan, G. R., Somigliana's method applied to elastic inclusions and dislocations, *Journal of Elasticity* 3 (1973) 45–49.