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Magnetohydrodynamic flow for a non-Newtonian power-law fluid having a pressure gradient and fluid injection

M. H. COBBLE

Department of Mechanical Engineering, New Mexico State University, Las Cruces, New Mexico 88003, USA

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SUMMARY

The nonlinear partial differential equation of motion for an incompressible, non-Newtonian power-law fluid flowing over flat plate under the influence of a magnetic field and a pressure gradient, and with or without fluid injection or ejection, is transformed to a nonlinear third-order ordinary differential equation by using a stream function and a similarity transformation.

The necessary boundary conditions are developed for flow with and without fluid injection (or ejection), and a solution for four different power-law fluids, including a Newtonian fluid, is presented.

The controlling equation includes, as special cases, the Falkner-Skan equation and the Blasius equation.

1. Introduction

The current world energy shortage has served to stimulate interest in developing new methods to generate power. Magnetohydrodynamic power generation, for instance, has been underway since 1974 in Russia [1].

Interest in magnetohydrodynamic flow began in 1918, when Hartmann invented the electromagnetic pump [2]. The first papers treating the flow of an electrically conducting fluid were by Hartmann [3], and Hartmann and Lazarus [4] in 1937. Since then a large body of literature has developed.

With the exception of linear problems, there are very few nonlinear magnetohydrodynamic problems solved in the literature. Cobble [5] determined a solution for a Newtonian fluid in 1977 using similarity. Similarity problems in fluid flow have been extensively analyzed by Ames [6] and Hansen [7].

This paper develops a unique similarity differential equation for incompressible flow of a non-Newtonian power-law fluid flowing over a semi-infinite plate, in the presence of a magnetic field and a pressure gradient, with or without injection or ejection through the plate wall.

2. Theory

The motion equation for an incompressible non-Newtonian fluid flowing over a semi-infinite flat plate, see Figure 1, under the influence of a magnetic field and a pressure gradient is

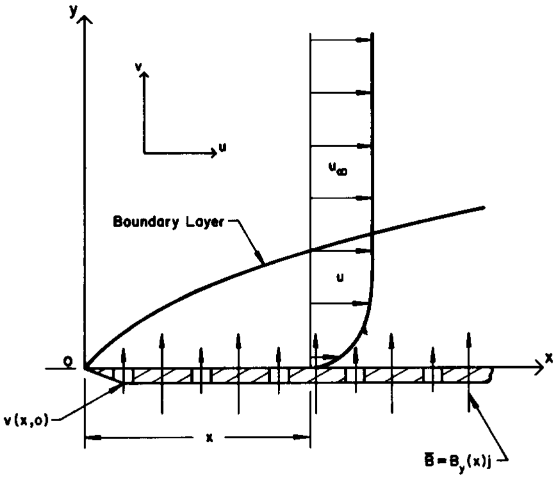

Figure 1. Magnetohydrodynamic boundary layer with fluid injection.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{g}{\rho}\frac{\partial P}{\partial x}+\frac{g}{\rho}\frac{\partial}{\partial y}(\tau_{xy})-\frac{g\sigma B_y^2(x)}{\rho}u \tag{1}$$

and the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

where, for a non-Newtonian power law fluid,

$$\tau_{xy}=\mu_0\left(\frac{\partial u}{\partial y}\right)^n, \qquad n\neq 0, \tag{3}$$

and where

u = velocity in the x-direction,
v = velocity in the y-direction,
g = acceleration of gravity,
ρ = fluid density,
P = pressure,
τ_{xy} = shear stress,
σ = electrical conductivity,
B_y = magnetic field strength,
μ_o = viscosity coefficient.

If we define a similarity variable, η, as

$$\eta = ay/x^{\alpha}, \qquad (a > 0, \alpha > 0) \tag{4}$$

and a stream function, ψ, as

$$\psi = G(x)\, f(\eta) \tag{5}$$

where

$$\psi_x = \left(\frac{\partial \psi}{\partial x}\right)_y = -v \tag{6}$$

and

$$\psi_y = \left(\frac{\partial \psi}{\partial y}\right)_x = u, \tag{7}$$

then continuity is automatically satisfied, and the motion equation, using Equation (3), becomes

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = H(x) + n\nu_0 \psi_{yy}^{n-1} \psi_{yyy} - S(x)\psi_y \tag{8}$$

where

$$H(x) = -\frac{g}{\rho}\frac{\partial P}{\partial x} \tag{9}$$

$$\nu_o = \frac{g\mu_o}{\rho} \tag{10}$$

and

$$S(x) = \frac{g\sigma B_y^2(x)}{\rho}\ . \tag{11}$$

It can be shown that

$$\psi_x = \left(\frac{\partial \psi}{\partial G}\right)_f G'(x) + \left(\frac{\partial \psi}{\partial f}\right)_G f'(\eta) \left(\frac{\partial \eta}{\partial x}\right)_y \tag{12}$$

and

$$\psi_y = \left(\frac{\partial \psi}{\partial f}\right)_G f'(\eta) \left(\frac{\partial \eta}{\partial y}\right)_x, \tag{13}$$

where

$$\left(\frac{\partial\psi}{\partial G}\right)_f = f(\eta) \tag{14}$$

$$\left(\frac{\partial\psi}{\partial f}\right)_G = G(x) \tag{15}$$

and

$$\left(\frac{\partial\eta}{\partial x}\right)_y = -\frac{\alpha\eta}{x}, \tag{16}$$

$$\left(\frac{\partial\eta}{\partial y}\right)_x = \frac{a}{x^\alpha}. \tag{17}$$

Further substitution in Equation (8) gives

$$\frac{a^2G^2(x)}{x^{2\alpha+1}}\left\{\frac{xG'(x)}{G(x)}\left[\{f'(\eta)\}^2 - f(\eta)f''(\eta)\right] - \alpha\{f'(\eta)\}^2\right\} \tag{18}$$

$$= H(x) + n\nu_o\left[G(x)\left(\frac{a}{x^\alpha}\right)^2 f''(\eta)\right]^{n-1}\left[G(x)\left(\frac{a}{x^\alpha}\right)^3 f'''(\eta)\right]$$

$$-\frac{aG(x)}{x^\alpha}S(x)f'(\eta).$$

Now, to permit a similarity solution, it must be that

$$\frac{xG'(x)}{G(x)} = \text{a constant} = \beta, \tag{19}$$

so that

$$G(x) = bx^\beta. \tag{20}$$

Further substitution in Equation (18) gives

$$a^2b^2x^{2(\beta-\alpha)-1}\left[(\beta-\alpha)\{f'(\eta)\}^2 - \beta f(\eta)f''(\eta)\right] \tag{21}$$

$$= H(x) + n\nu_o a^{1+2n}b^n x^{-\alpha+n(\beta-2\alpha)}\{f''(\eta)\}^{n-1}f'''(\eta) - abx^{\beta-a}S(x)f'(\eta).$$

Now, it must be that

$$H(x) = H_o x^\gamma \tag{22}$$

and

$$S(x) = S_o x^{\delta}. \tag{23}$$

Now, if for convenience we set

$$\gamma = 2m - 1 \tag{24}$$

then, to allow for a similarity solution, it must be that

$$\delta = m - 1 \tag{25}$$

and

$$\beta - \alpha = m \tag{26}$$

and

$$\alpha = \frac{1 + m(n-2)}{1+n} > 0 \tag{27}$$

and

$$\beta = \frac{1 + m(2n-1)}{1+n} \tag{28}$$

Additionally, if we require that

$$ab = U_o \tag{29}$$

and

$$a^2 b^2 \beta = n\nu_o a^{1+2n} b^n, \tag{30}$$

then

$$a = \left[\frac{U_o^{\,2-n}\beta}{n\nu_o}\right]^{\frac{1}{n+1}} \tag{31}$$

and

$$b = \left[\frac{U_o^{\,2n-1} n\nu_o}{\beta}\right]^{\frac{1}{n+1}} \tag{32}$$

Then Equation (21) can be written as

$$\{f''(\eta)\}^{n-1} f'''(\eta) + f(\eta) f''(\eta) - \overline{\beta}\{f'(\eta)\}^2 + H_1 - N_m f'(\eta) = 0 \tag{33}$$

where

$$H_1 = H_o/\beta U_o^2 \tag{34}$$

and

$$\overline{\beta} = \frac{m(1+n)}{1+m(2n-1)} \tag{35}$$

and where we define a dimensionless magnetic field strength number, N_m, as

$$N_m = \frac{S_o}{\beta U_o}. \tag{36}$$

It is necessary to determine three boundary conditions for Equation (33), and it is also desirable to put the equation into a more convenient form for computing.

Using the condition that

$$\lim_{y\to\infty} u(x,y) = K(x) = U_o x^m = U_\infty \left(\frac{x}{\ell}\right)^m = \lim_{\eta\to\infty} U_o x^m f'(\eta), \tag{37}$$

so that

$$\lim_{y\to\infty} u(\ell,y) = K(\ell) = U_o \ell^m = U_\infty = \lim_{\eta\to\infty} U_o \ell^m f'(\eta), \tag{38}$$

then it must be that

$$\lim_{\eta\to\infty} f'(\eta) = 1. \tag{39}$$

Using Equation (39) it can be shown that

$$H_1 = \overline{\beta} + N_m. \tag{40}$$

To find the other two necessary boundary conditions, we assume that at the wall, $y = 0$, there is no slippage for u, so that

$$u(x,0) = 0 = \lim_{\eta\to o} U_o x^m f'(\eta). \tag{41}$$

Also, using Equations (12) and (20), we see that

$$v(x,0) = \lim_{\eta \to o} \left\{ -bx^{\beta-1} \left[\beta f(\eta) - \alpha n f'(\eta)\right] \right\} = F(x) = V_o x^{\xi}. \tag{42}$$

Thus, it must be that

$$f(0) = -\frac{V_o}{b\beta} \tag{43}$$

and

$$f'(0) = 0 \tag{44}$$

and also

$$F(x) = V_o x^{\frac{m(2n-1)-n}{1+n}} \tag{45}$$

where

V_o = magnitude of velocity coefficient for injection ($V_o > 0$) or ejection ($V_o < 0$) of fluid through wall.

If $V_o = 0$, then

$$f(0) = 0 \tag{46}$$

and there is no fluid passing through the wall.

Additionally, the following must be hold:

$$P(x) = P_o - \frac{\rho H_o x^{2m}}{2\,mg}\quad, m \neq 0 \tag{47}$$

where P_o is a constant and

$$S(x) = S_o x^{m-1} = \frac{g\sigma B_{yo}^2 x^{m-1}}{\rho}\quad, \tag{48}$$

so that

$$B_y(x) = B_{yo} x^{(m-1)/2}. \tag{49}$$

Lykoudis [8] developed an expression equivalent to Equation (49) for compressible Newtonian fluids.

Using Equation (40) in Equation (33), we obtain

$$\{f''(\eta)\}^{n-1} f'''(\eta) + f(\eta) f''(\eta) + \bar{\beta}[1 - \{f'(\eta)\}^2] + N_m [1 - f'(\eta)] = 0. \tag{50}$$

The boundary conditions for $f(\eta)$ are:

1. $f(0) = -V_o/b\beta$,
2. $f'(0) = 0$,
3. $\lim_{\eta \to \infty} f'(\eta) = 1$.

Equation (50) is the general similarity differential equation controlling the effects of magnetic field, pressure gradient, and fluid injection or ejection through the wall, for a non-Newtonian power-law fluid. When $n = 1$, the equation reduces to one treated by Cobble [9]. When $n \neq 1$, and the magnetic field is zero, the equation reduces to that discussed by Schowalter [9]. If $n = 1$, and the magnetic field is zero, the equation reduces to the well-known Falkner-Skan equation. If $n = 1$, and the magnetic field is zero, and the pressure gradient is zero ($H_0 = 0$), the equation reduces to the Blasius equation.

3. Example

Given:

Equation (50) and the accompanying boundary conditions where $V_0 = 0$, (no fluid injection or ejection).

Assume:

$m = 0.1$, $N_m = 0.01$,

$n = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ (pseudoplastic fluids), 1 (Newtonian)

Figure 2 shows a plot of $f'(\eta) = \frac{u}{U_\infty}\left(\frac{\ell}{x}\right)^m$ vs. $\eta = ay/x^\alpha$, for $0 \leq \eta \leq 20$. It should be noted in Figure 2, that the η's are *different* for each n.

4. Conclusion

Using a stream function, ψ, which satisfies continuity, and a similarity variable, η, the nonlinear partial differential equation of motion for an incompressible, non-Newtonian power-law fluid flowing over a flat plate under the influence of a magnetic field and a pressure gradient, with or without fluid injection or ejection is transformed to a nonlinear third-order ordinary differential equation. The necessary boundary conditions have been established from a physical basis. The derived differential equation includes the equation developed by Cobble [5], and by Schowalter [9], the Falkner-Skan equation and the Blasius equation as special cases.

An example for a set of four power-law fluids (including Newtonian) is solved numerically, and $f'(\eta)$ is shown plotted against the similarity variable η in a common graph.

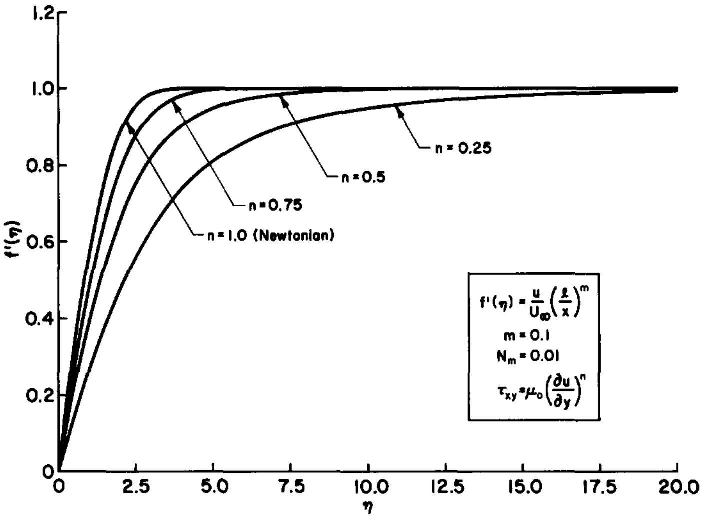

Figure 2. Magnetohydrodynamic flow with a pressure gradient.

REFERENCES

[1] A. W. Sheindlin and W. D. Jackson, MHD electrical power generation, an international status report, *Ninth World Energy Conference*, Detroit, Michigan, September 1974.
[2] V. J. Rossow, On flow of electrically conducting fluids over a flat plate in the presence of a transverse magnetic field, *NASA Technical Note* 3971, 1957.
[3] Jul. Hartmann, Hg-Dynamics I – Theory of laminar flow of an electrically conductive liquid in a homogeneous magnetic field. *Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser*, Vol. 15, No. 6, Copenhagen, 1937.
[4] Jul. Hartmann and L. Freimut, Hg-Dynamics II – Experimental investigations on the flow of mercury in a homogeneous magnetic field. *Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser*, Vol. 15, No. 7, Copenhagen, 1937.
[5] M. H. Cobble, Magnetofluiddynamic flow with a pressure gradient and fluid injection, Journal of Engineering Mathematics 11 (1977) 249-256.
[6] W. F. Ames, *Nonlinear partial differential equations in engineering*, Academic Press, 1972, pp. 87-123.
[7] A.G. Hansen, *Similarity analyses of boundary value problems in engineering*, Prentice Hall, Inc., 1964.
[8] P. S. Lykoudis, On a class of magnetic laminar boundary layers, *Proceedings of the Heat Transfer and Fluid Mechanics Institute*, 1959, pp. 176-184.
[9] R. W. Schowalter, The application of boundary-layer theory to power-law pseudoplastic fluids: similar solutions, *A.I.Ch.E. Journal* 6 (1960) 24-28.