Journal of Elasticity 33: 1–26, 1993.

Variational theory for spatial rods

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Received 24 September 1991

Abstract. The simplest theory of spatial rods is presented in a variational setting and certain necessary conditions for minimizers of the potential energy are derived. These include the Weierstrass and Legendre inequalities, which require that the vector describing curvature and twist belong to a domain of convexity of the strain energy function.

1. Introduction

In this work we present a modern reformulation of the classical Kirchhoff-Clebsch theory for three-dimensional deformations of elastic rods. Our purpose is to elucidate the structure of the simplest purely mechanical theory of inextensible rods that models both flexural and torsional response. Specifically, we postulate a particular form of the potential energy for conservatively loaded rods and derive certain necessary conditions for energy-minimizing configurations. Thus we assume at the outset that at least one minimizer of the requisite smoothness exists.

Our principal contribution is the derivation of the Weierstrass and Legendre necessary conditions for rods [1]. These conditions require that the vector describing curvature and twist of the rod belong to a domain of convexity of the strain energy function at every material point in a minimizing configuration. If the strain energy is non-convex on some part of its domain, then energy-minimizers may contain discontinuities in the curvatures and twist. Fosdick and James [2] and James [3] have studied configurations of this kind in the context of the theory of plane inextensible elasticae. The latter theory can be posed as a problem in the calculus of variations, and the appropriate Weierstrass and Legendre inequalities can be deduced by appealing to established results. The situation for three-dimensional deformations is much less clear-cut, however. Much of the motivation for the present paper derives from the search for the appropriate forms of these inequalities.

The literature on rod theory is of course far too voluminous to recount here. We briefly mention only those works that we have found particularly illuminating. Love's treatise [4] should be consulted for historical references and for an extensive account of the basic elements of the theory. An accessible discussion of the theory can also be found in the book by Landau and Lifshitz [5].

Most modern developments are based on the notion of a one-dimensional continuum, endowed with sufficient kinematical and constitutive structure to represent the most important components of rod-like behaviour. Typically this structure is conferred by a set of *directors* (e.g. [6–12]), which are required to satisfy balance laws deduced from appropriate statements of virtual work or conservation of energy, together with suitable constitutive hypotheses. Reissner [13] described an interesting alternative approach whereby the equilibrium equations, deduced from elementary considerations, are used with a postulated virtual work principle to obtain a general set of strain-displacement relations.

Our treatment is based on a version of the Bernoulli-Euler hypothesis, which is modelled by assigning a triad of embedded orthonormal vectors and a position function to every material point. The rate of change of this triad with respect to arc length along the rod is described by a skew tensor that measures curvature and twist. The associated axial vector is analogous to the Darboux vector of the Serret-Frenet theory of spatial curves (e.g. Stoker [14]). Our triad cannot be identified in all configurations with the Serret-Frenet triad consisting of the unit tangent, principal normal and binormal vectors, however, as the latter vectors are determined solely by the shape of the centreline of the rod, and, with the exception of the tangent, are not embedded in the material. This distinction is rarely made explicit in formulations based on the use of the Serret-Frenet theory ([5, 15, 16]).

Our kinematical and constitutive descriptions are similar to those used by Kafadar [17], who also considered the thermoelastic theory. Green and Laws [18] demonstrated that Kafadar's theory can be obtained as a special case of their director theory. By introducing suitable kinematic constraints into a general director theory, Naghdi and Rubin [19] derived a systematic procedure for obtaining rod theories of various degrees of kinematical refinement. The most severely constrained of these corresponds to the theory considered here.

Antman's many contributions (e.g. [11, 12, 20–22]) together comprise the most penetrating and far-reaching investigation into the mathematical structure of general rod theories. Among the most important of these is an existence theorem for energy-minimizers based, in part, on constitutive inequalities derived from the well known strong ellipticity condition of three-dimensional elasticity [21, 22]. This condition is the analogue of the strict Legendre inequality discussed in the present paper. Its adoption as a constitutive inequality rules out behaviour of the kind analysed by Fosdick and James [2, 3]. Similar inequalities were invoked by Maddocks [23] in a study of the stability of planar equilibria.

The kinematical basis of the theory is described in Section 2. There we also introduce a constitutive framework by specifying the manner in which the strain-energy function depends on the curvatures and twist. Invariance under superposed rigid motions is used as a constitutive hypothesis to obtain a restriction on this function that is central to the further development of the theory. This is used in Section 3 to obtain the Euler-Lagrange equations for configurations with continuous curvatures and twist. These are simply the classical equilibrium equations. They are included here because our derivation is apparently not entirely standard. The equilibrium theory for configurations with a finite number of discontinuities in the curvatures and twist is developed in Section 4. There we derive the Weierstrass-Erdmann conditions that must be satisfied at the points of discontinuity. These generalize similar restrictions obtained by Fosdick and James [2, 3] for planar elasticae.

The Weierstrass inequality is obtained in Section 5 by considering a certain class of kinematically admissible variations with piecewise continuous curvatures and twist and demanding that the associated potential energy be no less than the energy furnished by the minimizer. The Legendre condition for invariant strain energies is derived directly from the Weierstrass inequality. Our analysis is similar to Graves's [24] treatment of multiple integral variational problems, which in turn has been adapted to a theory of elastic plates by Hilgers and Pipkin [25]. Finally, in Section 6 we interpret the Legendre condition for transversely isotropic strain energies. These energies furnish an idealization of actual thin rods with circular cross-sections formed from homogeneous and isotropic materials [5].

Certain operational considerations pertaining to the solution of problems, such as the use of Euler angles, have no bearing on the basic structure of the theory and are not discussed. Such matters are thoroughly treated elsewhere (e.g. [4, 12, 26]). We also confine our attention to the simplest types of configuration-independent loading, as the precise nature of the loading is immaterial insofar as our main results are concerned. Potentials for various classes of configuration-dependent conservative loads are discussed in [27, 28].

We use standard Cartesian tensor notation throughout, with the usual summation convention in effect for Latin and Greek indices ranging over $\{1, 2, 3\}$ and $\{2, 3\}$, respectively.

2. Kinematical and constitutive hypotheses

Consider a spatial rod of total arc length L. A configuration of the rod is defined by a mapping of the arc length parameter $s \in [0, L]$ onto $\{\mathbf{r}(s), \mathbf{e}_i(s)\}$; $i = 1, 2, 3$, where $\mathbf{r}(\cdot)$ is the position function of points on the rod relative to a fixed origin and $\mathbf{e}_i(\cdot)$ are fields of embedded vectors which specify the orientations of the *cross-sections* $s = \text{const}$. We use a prime to denote differentiation with respect to s, and the notation $\{\mathbf{e}_i\} = \{\mathbf{t}, \mathbf{e}_\alpha\}$; $\alpha = 2, 3$, will sometimes be used to emphasize the distinguished role played by $\mathbf{e}_1$ in the subsequent development.

The material rod is identified with a particular configuration in which the functions $\mathbf{r}(s)$ and $\mathbf{e}_i(s)$ take the values $\mathbf{x}(s)$ and $\mathbf{E}_i(s)$, respectively, for each $s \in [0, L]$. This configuration will serve as reference for the measurement of all kinematical quantities. In particular, we choose $\{\mathbf{E}_i(s)\}$ to be an orthonormal basis with $\mathbf{E}_1 = \mathbf{x}'(s)$ and $\mathbf{E}_1 \cdot \mathbf{E}_2 \times \mathbf{E}_3 = 1$. Then for every $s \in [0, L]$, $\{\mathbf{E}_2, \mathbf{E}_3\}$ spans the planes normal to the space curve defined by $\mathbf{x}(s)$.

We invoke the Bernoulli-Euler hypotheses: First we suppose that cross-sections remain plane, suffer no strain, and are normal to the space curves $\mathbf{r}(s)$ in every configuration. We further assume that deformations from $\{\mathbf{x}, \mathbf{E}_i\}$ to $\{\mathbf{r}, \mathbf{e}_i\}$ are inextensional and orientation-preserving. These hypotheses are equivalent to

$$\mathbf{e}_i(s) \cdot \mathbf{e}_j(s) = \delta_{ij}, \; \mathbf{e}_i(s) \cdot \mathbf{e}_j(s) \times \mathbf{e}_k(s) = e_{ijk}, \; \forall s \in [0, L], \tag{2.1}$$

where δ_{ij} is the Kronecker delta and e_{ijk} is the permutation symbol, together with the nonholonomic constraint

$$\mathbf{r}'(s) = \mathbf{t}(s), \; s \in [0, L], \tag{2.2}$$

which further implies that s measures arc length in every configuration. Because $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_i\}$ are right-handed orthonormal bases, it follows that

$$\mathbf{e}_i(s) = \mathbf{R}(s)\mathbf{E}_i(s), \tag{2.3}$$

where $\mathbf{R} = \mathbf{e}_j \otimes \mathbf{E}_j$ is a rotation, i.e. $\det \mathbf{R} = 1$, $\mathbf{R}^T = \mathbf{R}^{-1}$.

We postulate that rods furnish elastic resistance to changes in curvature and relative twist between sections s and $s + ds$. These quantities are measured by the functions $\mathbf{e}_i'(s)$. The tensor

$$\mathbf{G}(s) = \mathbf{e}_i' \otimes \mathbf{E}_i; \; \mathbf{e}_j' = \mathbf{G}\mathbf{E}_j \tag{2.4}$$

accounts for the part of the rate of change of any embedded vector that is due to curvature and twist of the rod in the configuration $\{\mathbf{r}, \mathbf{e}_i\}$. Thus we idealize the rod as a one-dimensional continuum with a strain energy $W(\mathbf{G}; \mathbf{G}^0, s)$ per unit arc length, where $\mathbf{G}^0$ is the value of $\mathbf{G}$ in the configuration $\{\mathbf{x}, \mathbf{E}_i\}$: $\mathbf{G}^0 = \mathbf{E}_i' \otimes \mathbf{E}_i$. In most of what follows we write the strain energy as $W(\mathbf{G})$ and suppress reference to $\mathbf{G}^0$ and to explicit s-dependence, if any.

Differentiation of (2.1a) gives $\mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' = 0$. If we set $\Omega_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j'$, then

$\Omega_{ij} = -\Omega_{ji}$; and it follows that

$$\mathbf{e}_i' = \Omega_{ji}\mathbf{e}_j = \mathbf{W}\mathbf{e}_i,$$

where

$$\mathbf{W} = \Omega_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \tag{2.5}$$

is *skew*, i.e. $\mathbf{W}^T = -\mathbf{W}$. From (2.3-5) we have $\mathbf{G}\mathbf{E}_i = \mathbf{W}\mathbf{R}\mathbf{E}_i$, and therefore, since $\{\mathbf{E}_i\}$ is a basis for 3-space, we conclude that

$$\mathbf{G} = \mathbf{W}\mathbf{R};\ \mathbf{W} = \mathbf{G}\mathbf{R}^T = \mathbf{e}_i' \otimes \mathbf{e}_i. \tag{2.6}$$

Evidently $\mathbf{G}^0$ is the value of $\mathbf{W}$ in the configuration $\{\mathbf{x}, \mathbf{E}_i\}$. Hence $\mathbf{G}^0$ is skew. In particular, we note the easily derived relation

$$\mathbf{W} = \mathbf{R}'\mathbf{R}^T + \mathbf{R}\mathbf{G}^0\mathbf{R}^T. \tag{2.7}$$

Following standard practice, we stipulate that $W(\cdot)$ be invariant with respect to rigid transformations $\mathbf{e}_i \to \mathbf{e}_i^* = \mathbf{Q}\mathbf{e}_i$, where $\mathbf{Q}$ is an arbitrary *constant* rotation. For these transformations we find that $\mathbf{R} \to \mathbf{R}^* = (\mathbf{Q}\mathbf{e}_i) \otimes \mathbf{E}_i = \mathbf{Q}\mathbf{R}$ and $\mathbf{G} \to \mathbf{G}^* = (\mathbf{Q}\mathbf{e}_i)' \otimes \mathbf{E}_i = \mathbf{Q}\mathbf{G}$. Thus we require that the strain energy satisfy the pointwise restriction

$$W(\mathbf{G}) = W(\mathbf{Q}\mathbf{G}) \tag{2.8}$$

for all $\mathbf{G}$. To solve this equation we note that the particular rotation $\mathbf{Q} = \mathbf{R}^T$ furnishes the necessary condition $W(\mathbf{G}) = W(\boldsymbol{\Omega})$, where

$$\boldsymbol{\Omega} \equiv \mathbf{R}^T\mathbf{G} = \mathbf{R}^T\mathbf{W}\mathbf{R} = \Omega_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = -\boldsymbol{\Omega}^T. \tag{2.9}$$

Evidently the components of $\boldsymbol{\Omega}$ relative to the basis $\{\mathbf{E}_i \otimes \mathbf{E}_j\}$ are the same as those of $\mathbf{W}$ relative to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$. For future reference we note in passing that

$$\mathbf{R}^T\mathbf{R}' = \boldsymbol{\Omega} - \mathbf{G}^0 \tag{2.10}$$

as a consequence of (2.7) and (2.9). For arbitrary rigid transformations we have $\boldsymbol{\Omega} \to \boldsymbol{\Omega}^* = (\mathbf{Q}\mathbf{R})^T\mathbf{Q}\mathbf{G} = \boldsymbol{\Omega}$, and therefore

$$W(\mathbf{G}^*) = W(\boldsymbol{\Omega}^*) = W(\boldsymbol{\Omega}) = W(\mathbf{G}). \tag{2.11}$$

Thus (2.8) is satisfied for arbitrary rotations $\mathbf{Q}$ if and only if $W(\mathbf{G}) = W(\mathbf{\Omega})$.

The skew tensors $\mathbf{W}$ and $\mathbf{\Omega}$ are equivalent to their axial vectors $\mathbf{w} = w_i\mathbf{e}_i$ and $\boldsymbol{\kappa} = \kappa_i\mathbf{E}_i$ in the sense that $\mathbf{Wu} = \mathbf{w}\times\mathbf{u}$ and $\mathbf{\Omega u} = \boldsymbol{\kappa}\times\mathbf{u}$ for every vector $\mathbf{u}$. Then from (2.5) it follows that

$$\mathbf{e}_i' = \mathbf{w}\times\mathbf{e}_i. \tag{2.12}$$

We also have the well known connections

$$\kappa_i = \tfrac{1}{2}e_{ijk}\Omega_{kj} \quad \text{and} \quad \Omega_{kj} = \kappa_i e_{ijk}, \tag{2.13}$$

from which it can be inferred that $w_i = \kappa_i$ and $\mathbf{w} = \mathbf{R}\boldsymbol{\kappa}$. Thus we can write the strain energy in the form $W = w(\boldsymbol{\kappa}; \boldsymbol{\kappa}^0, s)$, where $\boldsymbol{\kappa}^0 = \kappa_i^0\mathbf{E}_i$ and $\kappa_i^0 = \frac{1}{2}e_{ijk}\mathbf{E}_k\cdot\mathbf{E}_j'$. As usual, we suppress reference to $\boldsymbol{\kappa}^0$ and s and write $W = w(\boldsymbol{\kappa})$. We assume that $w(\cdot)$ is twice differentiable on its domain of definition.

According to (2.5) and (2.13), the components of $\boldsymbol{\kappa}$ are determined by the vectors $\mathbf{e}_i$ and $\mathbf{e}_i'$. Thus the strain energy can be expressed as a function $W = U(\mathbf{e}_i, \mathbf{e}_i')$, tacitly dependent on $\mathbf{E}_i$, $\mathbf{E}_i'$ and s. This form is convenient for the considerations that follow. When the strain energy is expressed in this way, the dependence on $\mathbf{e}_i$ and $\mathbf{e}_i'$ is not arbitrary, but must be restricted in accordance with the invariance rule

$$U(\mathbf{e}_i, \mathbf{e}_i') = U(\mathbf{Qe}_i, \mathbf{Qe}_i') \tag{2.14}$$

for all constant rotations $\mathbf{Q}$.

To derive a necessary condition for this consider the one-parameter family of constant rotations

$$\mathbf{Q}(\varepsilon) = \exp(\varepsilon\mathbf{V}) = \mathbf{I} + \sum_{n=1}^{\infty}(\varepsilon^n/n!)\mathbf{V}^n, \tag{2.15}$$

where $\mathbf{V}$ is a fixed skew tensor and $\mathbf{I}$ is the unit tensor. Let $\mathbf{v}$ be the axial vector of $\mathbf{V}$. Then for small ε we have

$$\mathbf{Qe}_i = \mathbf{e}_i + \varepsilon\mathbf{v}\times\mathbf{e}_i + o(\varepsilon), \; \mathbf{Qe}_i' = \mathbf{e}_i' + \varepsilon\mathbf{v}\times\mathbf{e}_i' + o(\varepsilon). \tag{2.16}$$

For arbitrary vectors $\mathbf{a}, \mathbf{b}$ and scalar-valued functions $F(\mathbf{a}, \mathbf{b})$, we define $\partial F/\partial\mathbf{a} = (\partial F/\partial a_i)\mathbf{c}_i$ and $\partial F/\partial\mathbf{b} = (\partial F/\partial b_i)\mathbf{c}_i$, where $\{\mathbf{c}_i\}$ is an arbitrary orthonormal basis and $a_i = \mathbf{a}\cdot\mathbf{c}_i$, $b_i = \mathbf{b}\cdot\mathbf{c}_i$. Then (2.14) and (2.16) give

$$\varepsilon(\mathbf{v}\times\mathbf{e}_i\cdot\partial U/\partial\mathbf{e}_i + \mathbf{v}\times\mathbf{e}_i'\cdot\partial U/\partial\mathbf{e}_i') + o(\varepsilon) = 0, \tag{2.17}$$

where the derivatives are evaluated at $\varepsilon = 0$. Now divide by ε and pass to the limit to obtain

$$\mathbf{v}\cdot(\mathbf{e}_i \times \partial U/\partial \mathbf{e}_i + \mathbf{e}'_i \times \partial U/\partial \mathbf{e}'_i) = 0. \tag{2.18}$$

Because $\mathbf{v}$ is arbitrary, we conclude that U must satisfy the restriction

$$\mathbf{e}_i \times \partial U/\partial \mathbf{e}_i + \mathbf{e}'_i \times \partial U/\partial \mathbf{e}'_i = \mathbf{0}. \tag{2.19}$$

Of course, this equation is an identity when the strain energy is expressed as a function of $\boldsymbol{\kappa}$ because it is just the differential form of the invariance statement (2.8). To see this explicitly we write

$$\partial U/\partial \mathbf{e}_i = (\partial w/\partial \kappa_j)\partial \kappa_j/\partial \mathbf{e}_i, \tag{2.20}$$

with a similar formula for $\partial U/\partial \mathbf{e}'_i$. Then with $\kappa_j = \frac{1}{2}e_{jkl}\mathbf{e}_l \cdot \mathbf{e}'_k$ we derive

$$\partial \kappa_j/\partial \mathbf{e}_i = \tfrac{1}{2}e_{jki}\mathbf{e}'_k, \; \partial \kappa_j/\partial \mathbf{e}'_i = \tfrac{1}{2}e_{jik}\mathbf{e}_k. \tag{2.21}$$

The left hand side of (2.19) is the sum of three terms with coefficients of the form

$$e_{jki}\mathbf{e}_i \times \mathbf{e}'_k + e_{jik}\mathbf{e}'_i \times \mathbf{e}_k = e_{jki}(\mathbf{e}_i \times \mathbf{e}'_k + \mathbf{e}'_k \times \mathbf{e}_i); \; j = 1, 2, 3. \tag{2.22}$$

Each of these vanishes so that (2.19) is trivially satisfied.

In the next section we will identify the vector-valued function

$$\mathbf{M}(s) = \mathbf{e}_i \times \partial U/\partial \mathbf{e}'_i \tag{2.23}$$

as the resultant moment exerted by the material in $(s, L]$ on the material in $[0, s]$. The foregoing results can be used to show that

$$\mathbf{M} = \tfrac{1}{2}(\partial w/\partial \kappa_j)e_{jik}\mathbf{e}_i \times \mathbf{e}_k = \tfrac{1}{2}(\partial w/\partial \kappa_j)e_{ikj}e_{ikl}\mathbf{e}_l. \tag{2.24}$$

One of the well known $e - \delta$ identities then furnishes the simple formula

$$\mathbf{M} = (\partial w/\partial \kappa_i)\mathbf{e}_i. \tag{2.25}$$

3. Potential energy and equilibrium

Our treatment of rod theory is based on a potential energy functional $E[\mathbf{r}, \mathbf{e}_i]$ of the form

$$E[\mathbf{r}, \mathbf{e}_i] = S[\mathbf{e}_i] - P[\mathbf{r}, \mathbf{e}_i], \tag{3.1}$$

where

$$S[\mathbf{e}_i] = \int_0^L U(\mathbf{e}_i, \mathbf{e}_i')\,\mathrm{d}s \tag{3.2}$$

is the total strain energy of the deformed rod and $P[\mathbf{r}, \mathbf{e}_i]$ is a potential associated with a system of conservative loads.

For the sake of illustration we first consider a dead force $\mathbf{f}$, applied at the end $s = L$, together with a dead distributed force $\mathbf{b}(s)$ per unit length of the rod. The associated potential is

$$P[\mathbf{r}, \mathbf{e}_i] = L[\mathbf{r}] \equiv \int_0^L \mathbf{b}\cdot\mathbf{r}\,\mathrm{d}s + \mathbf{f}\cdot\mathbf{r}(L). \tag{3.3}$$

We assume that $\mathbf{b}(\cdot)$ is at least piecewise continuous. We further assume that $\mathbf{r}(0)$ and $\mathbf{e}_i(0)$ are assigned, and that no kinematical data are prescribed at $s = L$. This limited class of problems is chosen to avoid inessential detail in the ensuing manipulations. Certain generalizations are discussed at the end of this section.

Stable equilibria are defined to be minimizers of $E[\cdot,\cdot]$ in some suitable class of competing configurations. In particular, if $\{\mathbf{r}, \mathbf{e}_i\}$ is stable then we require that

$$E[\mathbf{r}, \mathbf{e}_i] \leqslant E[\mathbf{r}^*, \mathbf{e}_i^*], \tag{3.4}$$

where

$$\mathbf{r}^* = \mathbf{r}(s) + \varepsilon\mathbf{u}(s) + o(\varepsilon),\ \mathbf{e}_i^* = \mathbf{e}_i(s) + \varepsilon\mathbf{v}_i(s) + o(\varepsilon);\ \varepsilon \to 0. \tag{3.5}$$

Here we consider $\{\mathbf{u}(\cdot), \mathbf{v}_i(\cdot)\} \in C^1$ for $0 \leqslant s \leqslant L$ with $\mathbf{u}(0) = \mathbf{v}_i(0) = \mathbf{0}$. In the present section we also assume that $\{\mathbf{e}_i(\cdot)\} \in C^2$, piecewise. This restriction is relaxed in Section 4.

We admit only those competitors $\{\mathbf{e}_i^*\}$ that comply with the constraints (2.1).

From (2.1a) and (3.5b) it follows that

$$\delta_{ij} = \mathbf{e}_i^* \cdot \mathbf{e}_j^* = \delta_{ij} + \varepsilon(\mathbf{e}_i \cdot \mathbf{v}_j + \mathbf{e}_j \cdot \mathbf{v}_i) + o(\varepsilon). \tag{3.6}$$

If we divide by ε and let $\varepsilon \to 0$, we find the restriction

$$\mathbf{e}_i \cdot \mathbf{v}_j + \mathbf{e}_j \cdot \mathbf{v}_i = 0 \tag{3.7}$$

on admissible variations $\mathbf{v}_i$. The general solution of this equation is obtained by defining $\alpha_{ji} = \mathbf{e}_j \cdot \mathbf{v}_i$ so that $\mathbf{v}_i = \alpha_{ji}\mathbf{e}_j = \boldsymbol{\alpha}\mathbf{e}_i$, where $\boldsymbol{\alpha} = \alpha_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. It follows from (3.7) that $\boldsymbol{\alpha}$ is skew, and thus there is a vector $\mathbf{a}(s)$ such that

$$\mathbf{v}_i = \mathbf{a} \times \mathbf{e}_i. \tag{3.8}$$

Conversely, if there is a vector $\mathbf{a}$ such that (3.8) is valid, then (3.7) becomes

$$\mathbf{e}_i \cdot \mathbf{a} \times \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{a} \times \mathbf{e}_i = \mathbf{a} \cdot (\mathbf{e}_i \times \mathbf{e}_j + \mathbf{e}_j \times \mathbf{e}_i), \tag{3.9}$$

which vanishes identically. Thus for arbitrary $\mathbf{a}$, (3.8) furnishes the general solution of the variational constraint (3.7). It is easily demonstrated that the $\mathbf{v}_i$ automatically satisfy the variational form of the constraint (2.1b). Moreover, the result (3.8) can be used with (3.5) to establish the variational version of the constraint (2.2):

$$\mathbf{u}' = \mathbf{a} \times \mathbf{t}. \tag{3.10}$$

According to the *multiplier rule* of the calculus of variations [1, 29], an admissible configuration that renders $E[\cdot,\cdot]$ stationary is also a stationary configuration for the functional

$$\hat{E}[\mathbf{r}, \mathbf{e}_i] = E[\mathbf{r}, \mathbf{e}_i] + \int_0^L \mathbf{F} \cdot (\mathbf{r}' - \mathbf{t})\,\mathrm{d}s, \tag{3.11}$$

where $\mathbf{F}(s)$ is a vector of Lagrange multipliers. For fixed $\{\mathbf{r}, \mathbf{e}_i\}$ and $\{\mathbf{u}, \mathbf{v}_i\}$, let $F(\varepsilon) = \hat{E}[\mathbf{r}^*, \mathbf{e}_i^*]$, where $\mathbf{r}^*$ and $\mathbf{e}_i^*$ are defined by (3.5). Then (3.4) is satisfied only if $dF/d\varepsilon|_{\varepsilon=0} = 0$, where

$$dF/d\varepsilon|_{\varepsilon=0} = \int_0^L [\mathbf{v}_i' \cdot \partial U/\partial \mathbf{e}_i' + \mathbf{v}_i \cdot \partial U/\partial \mathbf{e}_i - \mathbf{b} \cdot \mathbf{u} + \mathbf{F} \cdot (\mathbf{u}' - \mathbf{a} \times \mathbf{t})]\,\mathrm{d}s - \mathbf{f} \cdot \mathbf{u}(L), \tag{3.12}$$

and the derivatives of $U(\cdot,\cdot)$ are evaluated at $\varepsilon = 0$.

After integrating by parts, invoking (3.8) and recalling the definition (2.23), we obtain

$$\begin{aligned} dF/d\varepsilon|_{\varepsilon=0} &= \mathbf{a}(L)\cdot\mathbf{M}(L) + \mathbf{u}(L)\cdot[\mathbf{F}(L) - \mathbf{f}] \\ &- \int_0^L \mathbf{a}\cdot\{\mathbf{e}_i \times [(\partial U/\partial \mathbf{e}_i')' - \partial U/\partial \mathbf{e}_i] - \mathbf{F}\times\mathbf{t}\}\,ds \\ &- \int_0^L \mathbf{u}\cdot(\mathbf{F}' + \mathbf{b})\,ds. \end{aligned} \tag{3.13}$$

Then $\{\mathbf{r}, \mathbf{e}_i\}$ is stable only if $\mathbf{M}(L)$ and $\mathbf{F}(L)$ satisfy the natural end conditions

$$\mathbf{M}(L) = \mathbf{0},\ \mathbf{F}(L) = \mathbf{f} \tag{3.14}$$

and only if the Euler-Lagrange equations

$$\mathbf{F}\times\mathbf{t} = \mathbf{e}_i \times (\partial U/\partial \mathbf{e}_i')' - \mathbf{e}_i \times \partial U/\partial \mathbf{e}_i, \tag{3.15}$$

$$\mathbf{F}' + \mathbf{b} = \mathbf{0} \tag{3.16}$$

are satisfied for $0 < s < L$. From definition (2.23), eq. (3.15) can be written as

$$\mathbf{F}\times\mathbf{t} = \mathbf{M}' - (\mathbf{e}_i \times \partial U/\partial \mathbf{e}_i + \mathbf{e}_i' \times \partial U/\partial \mathbf{e}_i'), \tag{3.17}$$

and for invariant strain energies (2.19) furnishes the result

$$\mathbf{F}\times\mathbf{t} = \mathbf{M}'. \tag{3.18}$$

Equation (3.14b) identifies $\mathbf{F}(L)$ as the force supplied at the end $s = L$ by an external agency. From (3.16) and (3.18) it is apparent that $\mathbf{F}(s)$ and $\mathbf{M}(s)$ are, respectively, the force and moment exerted by the segment $(s, L]$ on the part $[0, s]$. Equation (3.14a) then requires that the moment vanish at the unrestrained end, in accordance with our expectations. With these interpretations of $\mathbf{F}$ and $\mathbf{M}$, eqs. (3.16) and (3.18) are the classical equilibrium equations of rod theory [4, 5].

A generalization of the classical *energy integral* ([4], arts. 260, 261) can be obtained by combining the foregoing results with

$$U' = \mathbf{e}_i'\cdot\partial U/\partial \mathbf{e}_i + \mathbf{e}_i''\cdot\partial U/\partial \mathbf{e}_i' + U_s. \tag{3.19}$$

The derivative U_s accounts for the s-dependence of the energy that is not attributable to $\mathbf{e}_i$ and $\mathbf{e}_i'$. For invariant strain energies $U = w(\boldsymbol{\kappa}; \boldsymbol{\kappa}^0, s)$, this extra dependence is due to the explicit appearance of s and to the presence of $\kappa_i^0(s)$. In particular, U_s vanishes identically if w is not explicitly dependent on s and if $\kappa_i^0 = \text{const.}$, so that the rod is uniformly curved and twisted in the configuration $\{\mathbf{x}, \mathbf{E}_i\}$. In this case the rod is said to be homogeneous.

For homogeneous rods, differentiation of (2.12) and substitution into (3.19) yields

$$U' = \mathbf{w}\cdot(\mathbf{e}_i \times \partial U/\partial \mathbf{e}_i + \mathbf{e}_i' \times \partial U/\partial \mathbf{e}_i') + \mathbf{w}'\cdot\mathbf{M}, \tag{3.20}$$

where the definition (2.23) has been applied. The equilibrium equation (3.18) can be used to write the last term as $(\mathbf{w}\cdot\mathbf{M})' + \mathbf{F}\cdot\mathbf{w}\times\mathbf{t}$. Now (2.12) and (3.16) give $\mathbf{w}\times\mathbf{t} = \mathbf{t}'$ and $\mathbf{F}\cdot\mathbf{t}' = (\mathbf{t}\cdot\mathbf{F})' + \mathbf{t}\cdot\mathbf{b}$. Thus for invariant strain energies, (3.20) becomes

$$(U - \mathbf{w}\cdot\mathbf{M} - \mathbf{t}\cdot\mathbf{F})' = \mathbf{t}\cdot\mathbf{b}, \tag{3.21}$$

and in the important case $\mathbf{b}(s) = \text{const.}$ this furnishes the integral

$$U - \mathbf{w}\cdot\mathbf{M} - \mathbf{t}\cdot\mathbf{F} - \mathbf{r}\cdot\mathbf{b} = \text{const.} \tag{3.22}$$

In addition to the dead distributed and end forces treated thus far, we also consider fixed couples $\mathbf{m}_i$; $i = 1, 2, 3$ applied at the end $s = L$, with a potential of the form $\mathbf{m}_i\cdot\mathbf{e}_i(L)$. If $\mathbf{e}_j(L)$ is prescribed for fixed j, then the jth term is omitted from the sum. With this proviso, the total load potential is

$$P[\mathbf{r}, \mathbf{e}_i] = L[\mathbf{r}] + \mathbf{m}_i\cdot\mathbf{e}_i(L), \tag{3.23}$$

where $L[\mathbf{r}]$ is defined in (3.3). It is then a simple matter to verify that all of our previous results remain valid with the exception of (3.14a), which now reads

$$\mathbf{a}(L)\cdot[\mathbf{M}(L) - \mathbf{e}_i(L)\times\mathbf{m}_i] = 0. \tag{3.24}$$

If no member of $\{\mathbf{e}_i(L)\}$ is prescribed, then the virtual rotation $\mathbf{a}(L)$ is arbitrary, and (3.24) requires that

$$\mathbf{M}(L) = \mathbf{e}_i(L)\times\mathbf{m}_i. \tag{3.25}$$

It is important to realize that this does not furnish a prescription of a dead moment at $s = L$. Indeed, Ziegler [30] has shown that there is no load potential associated with a dead moment applied at an unrestrained end. Such loadings are non-conservative. Instead, (3.25) places a restriction on the values of $\mathbf{e}_i(L)$ delivered by solutions $\mathbf{e}_i(s)$ of the equilibrium problem defined by (3.14b), (3.16), (3.18) and (2.25).

Alternatively, if the tangent to the rod at $s = L$ is prescribed, i.e. $\mathbf{t}(L) = \mathbf{e}$ (say), then (3.8) implies that $\mathbf{a}(L) \times \mathbf{e} = \mathbf{0}$. The general representation $\mathbf{a} = (\mathbf{a}\cdot\mathbf{e})\mathbf{e} + \mathbf{e}\times(\mathbf{a}\times\mathbf{e})$ then requires that $\mathbf{a}(L) = a\mathbf{e}$ for some scalar a. In this case (3.24) becomes

$$a\mathbf{e}\cdot[\mathbf{M}(L) - \mathbf{e}_i(L)\times\mathbf{m}_i] = 0, \tag{3.26}$$

and therefore, since a is arbitrary, it follows that the axial end torque is

$$\mathbf{e}\cdot\mathbf{M}(L) = \mathbf{e}\cdot\mathbf{e}_\alpha(L)\times\mathbf{m}_\alpha;\ \alpha = 2, 3. \tag{3.27}$$

Again, this is not a prescription of the end torque, but rather a restriction on the values of $\mathbf{e}_\alpha(L)$ delivered by solutions of the equilibrium problem. We note that this restriction does not involve $\mathbf{m}_1$. This is to be expected because $\mathbf{e}_1(L)$ is prescribed.

4. Discontinuities and the Weierstrass–Erdmann conditions

The analysis of the previous section was limited to energy-minimizing configurations for which the $\mathbf{e}_i(\cdot)$ are piecewise twice-differentiable. Here we relax this smoothness assumption and assume that the $\mathbf{e}_i(\cdot)$ may have discontinuous derivatives at a finite number of points in the interval $(0, L)$. Thus we assume that $\mathbf{e}_i(\cdot)\in C^1$, piecewise. We use the terminology of the calculus of variations and refer to points of discontinuity of the derivatives $\mathbf{e}_i'$ as *corners*. This terminology is a bit misleading in the present context as the configurations considered have continuously turning tangents. The corners are points of discontinuity of the curvatures and twist.

Our objective in the present section is to establish the jump conditions that minimizing configurations must satisfy at corners. To this end we reconsider the stationarity condition (3.12). Between corners, the $\mathbf{e}_i'$ are continuous and (3.8) gives $\mathbf{v}_i' = \mathbf{a}'\times\mathbf{e}_i + \mathbf{a}\times\mathbf{e}_i'$. Then with

$$\begin{aligned} \mathbf{v}_i\cdot\partial U/\partial\mathbf{e}_i &= \mathbf{a}\cdot\mathbf{e}_i\times\partial U/\partial\mathbf{e}_i, \\ \mathbf{v}_i'\cdot\partial U/\partial\mathbf{e}_i' &= \mathbf{a}'\cdot\mathbf{M} + \mathbf{a}\cdot\mathbf{e}_i'\times\partial U/\partial\mathbf{e}_i', \end{aligned} \tag{4.1}$$

and (2.19), we can express (3.12) in the form

$$\int_0^L (\mathbf{a}'\cdot\mathbf{M} - \mathbf{a}\cdot\mathbf{t}\times\mathbf{F})\,\mathrm{d}s + \int_0^L (\mathbf{u}'\cdot\mathbf{F} - \mathbf{u}\cdot\mathbf{b})\,\mathrm{d}s - \mathbf{u}(L)\cdot\mathbf{f} - \mathbf{a}(L)\cdot\mathbf{e}_i(L)\times\mathbf{m}_i = 0, \tag{4.2}$$

where allowance has been made for the application of fixed couples $\mathbf{m}_i$ at $s = L$.

The relations

$$\begin{aligned}\mathbf{u}\cdot\mathbf{b} &= \left(\mathbf{u}\cdot\int_L^s \mathbf{b}\,\mathrm{d}x\right)' - \mathbf{u}'\cdot\int_L^s \mathbf{b}\,\mathrm{d}x,\\ \mathbf{a}\cdot\mathbf{t}\times\mathbf{F} &= \left(\mathbf{a}\cdot\int_L^s \mathbf{t}\times\mathbf{F}\,\mathrm{d}x\right)' - \mathbf{a}'\cdot\int_L^s \mathbf{t}\times\mathbf{F}\,\mathrm{d}x\end{aligned} \tag{4.3}$$

can be used with the fixity conditions $\mathbf{u}(0) = \mathbf{a}(0) = \mathbf{0}$ to reduce (4.2) to

$$\begin{aligned}&\int_0^L \mathbf{a}'\cdot\left[\mathbf{M}(s) + \int_L^s \mathbf{t}\times\mathbf{F}\,\mathrm{d}x - \mathbf{e}_i(L)\times\mathbf{m}_i\right]\mathrm{d}s\\ &\quad+ \int_0^L \mathbf{u}'\cdot\left[\mathbf{F}(s) + \int_L^s \mathbf{b}\,\mathrm{d}x - \mathbf{f}\right]\mathrm{d}s = 0.\end{aligned} \tag{4.4}$$

The fundamental lemma of the calculus of variations requires that both integrals vanish separately. For variations that satisfy the additional conditions $\mathbf{u}(L) = \mathbf{a}(L) = \mathbf{0}$, we can invoke the lemma of du Bois Reymond [31] to obtain the necessary conditions

$$\mathbf{F}(s) + \int_L^s \mathbf{b}\,\mathrm{d}x = \mathbf{c}, \qquad \mathbf{M}(s) - \int_L^s \mathbf{F}\times\mathbf{t}\,\mathrm{d}x = \mathbf{d}, \tag{4.5}$$

where $\mathbf{c}$ and $\mathbf{d}$ are constant vectors.

It is evident from (2.25) that $\mathbf{M}$ could conceivably be discontinuous at corners where the κ_i are discontinuous. The continuity of $\mathbf{F}$ follows from (4.5a) if $\mathbf{b}$ is at least piecewise continuous. Then since $\mathbf{t}$ is presumed to be continuous, (4.5b) implies that in energy-minimizing configurations $\mathbf{M}$ is in fact continuous at corners, i.e.

$$\Delta\mathbf{M} = \mathbf{0}, \tag{4.6}$$

where Δ is used to denote the difference of the limiting values on either side of a corner.

We derive a second jump condition by considering variations in the variable used to parametrize a minimizing configuration $\{\mathbf{r}(\cdot), \mathbf{e}_i(\cdot)\}$. Here we follow the procedure that Bliss [1] used to obtain a similar result in the calculus of variations. Let the equations of an energy minimizer be written in the parametric form

$$s = t,\ \mathbf{r} = \mathbf{r}(t),\ \mathbf{e}_i = \mathbf{e}_i(t);\ 0 \leqslant t \leqslant L. \tag{4.7}$$

Consider the re-parametrization of this configuration defined by

$$s = \xi(t),\ \mathbf{r} = \boldsymbol{\rho}(t),\ \mathbf{e}_i = \boldsymbol{\varepsilon}_i(t);\ t_1 \leqslant t \leqslant t_2, \tag{4.8}$$

with $\dot{\xi} > 0$ for $t \in [t_1, t_2]$. We use a superposed dot to denote differentiation with respect to t. Then the energy attributed to the configuration $\{\mathbf{r}, \mathbf{e}_i\}$ is the functional of ξ defined by

$$I[\xi] = \int_{t_1}^{t_2} H(\xi, \dot{\xi};\ \boldsymbol{\rho}, \dot{\boldsymbol{\rho}};\ \boldsymbol{\varepsilon}_i, \dot{\boldsymbol{\varepsilon}}_i)\, \dot{\xi}\, dt - \mathbf{f}\cdot\boldsymbol{\rho}(t_2) - \mathbf{m}_i\cdot\boldsymbol{\varepsilon}_i(t_2), \tag{4.9}$$

where

$$H = U(\boldsymbol{\varepsilon}_i, \dot{\xi}^{-1}\dot{\boldsymbol{\varepsilon}}_i;\ \xi) + \mathbf{F}(\xi)\cdot(\dot{\xi}^{-1}\dot{\boldsymbol{\rho}} - \boldsymbol{\varepsilon}_1) - \mathbf{b}(\xi)\cdot\boldsymbol{\rho}. \tag{4.10}$$

The third argument in the strain energy U accounts for the dependence on arc length that is not due to $\mathbf{e}_i$ and $\mathbf{e}_i'$.

We stipulate that the value of the total potential energy be invariant with respect to parameter transformations. To make this notion precise we consider variations of the form $\xi \to \xi^* = \xi(t) + \varepsilon u(t)$, with $u(\cdot)$ piecewise continuously differentiable for $t_1 \leqslant t \leqslant t_2$ and $u(t_1) = u(t_2) = 0$. The induced change in H is

$$H \to H^* = H + \varepsilon(u\partial H/\partial\xi + \dot{u}\partial H/\partial\dot{\xi}) + o(\varepsilon), \tag{4.11}$$

where the derivatives are evaluated at $\varepsilon = 0$. From this we obtain

$$H\dot{\xi} \to H^*\dot{\xi}^* = H\dot{\xi} + \varepsilon[u(\dot{\xi}\partial H/\partial\xi) + \dot{u}(H + \dot{\xi}\partial H/\partial\dot{\xi})] + o(\varepsilon). \tag{4.12}$$

Then invariance requires that

$$\int_{t_1}^{t_2} [\dot{u}(H + \dot{\xi}\partial H/\partial\dot{\xi}) + u(\dot{\xi}\partial H/\partial\xi)]\, dt = 0. \tag{4.13}$$

Now

$$u(\dot{\xi}\partial H/\partial\xi) = \left(u\int_{t_1}^{t}\dot{\xi}\partial H/\partial\xi\,\mathrm{d}x\right)^{\cdot} - \dot{u}\int_{t_1}^{t}\dot{\xi}\partial H/\partial\xi\,\mathrm{d}x, \tag{4.14}$$

and since $u(t_2) = 0$, (4.13) becomes

$$\int_{t_1}^{t_2}\left(H + \dot{\xi}\partial H/\partial\dot{\xi} - \int_{t_1}^{t}\dot{\xi}\partial H/\partial\xi\,\mathrm{d}x\right)\dot{u}\,\mathrm{d}t = 0. \tag{4.15}$$

Because $u(t_1)$ also vanishes, we can appeal to the du Bois Reymond lemma to conclude that

$$H + \dot{\xi}\partial H/\partial\dot{\xi} = \int_{t_1}^{t}\dot{\xi}\partial H/\partial\xi\,\mathrm{d}x + c, \tag{4.16}$$

where c is a constant.

According to (4.8) and (4.10),

$$\partial H/\partial\xi = U_s + \mathbf{F}'(s)\cdot(\mathbf{r}' - \mathbf{t}) - \mathbf{b}'(s)\cdot\mathbf{r}$$

and

$$\partial H/\partial\dot{\xi} = -\dot{\xi}^{-2}(\dot{\mathbf{e}}_i\cdot\partial U/\partial\mathbf{e}_i' + \dot{\boldsymbol{\rho}}\cdot\mathbf{F}). \tag{4.17}$$

Then

$$\dot{\xi}\partial H/\partial\dot{\xi} = -(\mathbf{e}_i'\cdot\partial U/\partial\mathbf{e}_i' + \mathbf{r}'\cdot\mathbf{F}) \tag{4.18}$$

so that (4.16) is equivalent to

$$\begin{aligned} &U + \mathbf{F}\cdot(\mathbf{r}' - \mathbf{t}) - \mathbf{b}\cdot\mathbf{r} - \mathbf{e}_i'\cdot\partial U/\partial\mathbf{e}_i' - \mathbf{F}\cdot\mathbf{r}' \\ &\quad = \int_0^s [U_s + \mathbf{F}'\cdot(\mathbf{r}' - \mathbf{t}) - \mathbf{b}'\cdot\mathbf{r}]\,\mathrm{d}x + c. \end{aligned} \tag{4.19}$$

For invariant strain energies $U = w(\boldsymbol{\kappa})$, (2.13) and (2.21) can be used to obtain

$$\mathbf{e}_i'\cdot\partial U/\partial\mathbf{e}_i' = \tfrac{1}{2}(\partial w/\partial\kappa_j)e_{jik}\mathbf{e}_i'\cdot\mathbf{e}_k = \kappa_j\partial w/\partial\kappa_j. \tag{4.20}$$

We also have $\mathbf{b}'\cdot\mathbf{r} = (\mathbf{b}\cdot\mathbf{r})' - \mathbf{r}'\cdot\mathbf{b}$. Then for configurations that satisfy the constraint (2.2), (4.19) reduces to

$$U - \mathbf{w}\cdot\mathbf{M} - \mathbf{t}\cdot\mathbf{F} - \int_0^s (U_s + \mathbf{t}\cdot\mathbf{b})\,\mathrm{d}x = \text{const.}, \tag{4.21}$$

where $\mathbf{w} = \kappa_j \mathbf{e}_j$. Our continuity hypothesis ensures that the integral is a continuous function of s. The continuity of $\mathbf{F}(s)$ follows from (4.5a). Then from (4.6) we conclude that

$$\Delta U = \mathbf{M}\cdot\Delta\mathbf{w} \tag{4.22}$$

at corners, where ΔU is the jump in strain energy, $\Delta\mathbf{w} = (\Delta\kappa_j)\mathbf{e}_j$ is the jump in the curvatures and twist, and $\mathbf{M}$ is the common value of the limiting moments on either side of the point of discontinuity. Equations (4.6) and (4.22) together constitute the Weierstrass–Erdmann *corner conditions* of variational calculus. Between corners, eqs. (4.5a, b) can be differentiated to yield the Euler–Lagrange eqs. (3.16) and (3.18). For homogeneous rods, (3.21) follows from differentiation of (4.21).

5. The Weierstrass and Legendre inequalities

In this section we derive a version of the well known Weierstrass necessary condition for configurations $\{\mathbf{r}, \mathbf{e}_i\}$ that minimize the energy with respect to *strong variations*, i.e.

$$E[\mathbf{r}, \mathbf{e}_i] \leqslant E[\mathbf{r}^*, \mathbf{e}_i^*] \tag{5.1}$$

for all $\mathbf{r}^*$ and $\mathbf{e}_i^*$ such that $\mathbf{r}^*(0) = \mathbf{r}(0)$, $\mathbf{e}_i^*(0) = \mathbf{e}_i(0)$ and $|\mathbf{r}^* - \mathbf{r}| + \Sigma_i |\mathbf{e}_i^* - \mathbf{e}_i| < \delta$ for some $\delta > 0$ and for all $s \in [0, L]$. Such configurations are called *strong relative minimizers.*

Proofs of the Weierstrass condition for unconstrained variational problems can be found in the books by Bliss [1] and Ewing [31]. These proofs presuppose that the Euler–Lagrange equations are satisfied. For constrained problems, extensions of these proofs based on the multiplier rule have been given by Graves [32] and McShane [33]. Here we give a direct proof by considering only those perturbations that satisfy the constraints identically. Our derivation closely parallels Graves's proof [24] of the so-called rank-one convexity condition for unconstrained minimizers of functionals defined by multiple integrals. In particular, we do not invoke the Euler–Lagrange equations or the multiplier rule.

For simplicity's sake, we first limit ourselves to the case in which the end $s = L$ is free. We further suppose that the distributed force $\mathbf{b}(s)$ vanishes. For this case the potential energy reduces to the strain energy $S[\mathbf{e}_i]$ defined by (3.2), and (5.1) then becomes

$$S[\mathbf{e}_i] \leqslant S[\mathbf{e}_i^*]. \tag{5.2}$$

The perturbations are not arbitrary, but must be restricted in accordance with eqs. (2.1), i.e.

$$\mathbf{e}_i^* \cdot \mathbf{e}_j^* = \delta_{ij}, \; \mathbf{e}_i^* \cdot \mathbf{e}_j^* \times \mathbf{e}_k^* = e_{ijk}. \tag{5.3}$$

To satisfy the first requirement we set $\mathbf{e}_i^*(s) = \mathbf{Q}(s)\mathbf{e}_i(s)$ where $\mathbf{Q}$ is an arbitrary orthogonal tensor: $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. The second of (5.3) then gives

$$e_{ijk} = (\det \mathbf{Q})\mathbf{Q}\mathbf{e}_i \cdot \mathbf{Q}(\mathbf{e}_j \times \mathbf{e}_k) = (\det \mathbf{Q})e_{ijk} \tag{5.4}$$

and therefore $\det \mathbf{Q} = 1$. Thus $\mathbf{Q}(s)$ is a rotation and (5.2) implies that

$$S[\mathbf{e}_i] \leqslant S[\mathbf{Q}\mathbf{e}_i] \tag{5.5}$$

for all continuous rotations $\mathbf{Q}(s)$. We note that these rotations are not limited to rigid motions. Such motions are recovered by setting $\mathbf{Q}(s) = \text{const.}$

Consider three points s_i; $i = 1, 2, 3$ with $0 < s_1 < s_2 < s_3 < L$. Suppose the s_i do not coincide with points of discontinuity (corners) of the $\mathbf{e}_i'$, if any. We take the $\mathbf{e}_i^*$ to be continuous, but allow discontinuities in the $(\mathbf{e}_i^*)'$ at s_1, s_2 and s_3. In particular, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be fixed skew tensors and set

$$\mathbf{Q}(s) = \begin{cases} \mathbf{I} & \text{for } s \in [0, L] - (s_1, s_3) \\ \mathbf{A}(s) & \text{for } s_1 < s < s_2 \\ \mathbf{B}(s) & \text{for } s_2 < s < s_3, \end{cases} \tag{5.6}$$

where $\mathbf{A}(s)$ and $\mathbf{B}(s)$ are the rotations defined by

$$\mathbf{A}(s) = \exp[(s - s_1)\boldsymbol{\alpha}], \; \mathbf{B}(s) = \exp[(s - s_3)\boldsymbol{\beta}]. \tag{5.7}$$

Note that $\mathbf{A}(s_1) = \mathbf{I}$ and $\mathbf{B}(s_3) = \mathbf{I}$. Thus $\mathbf{Q}$ is continuous at $s = s_1, s_3$. To ensure continuity at s_2 we take

$$\boldsymbol{\beta} = \left(\frac{s_2 - s_1}{s_2 - s_3}\right)\boldsymbol{\alpha}. \tag{5.8}$$

Then the $\mathbf{e}_i^*(\cdot)$ are continuous for all $s \in [0, L]$. It follows from the definition of the exponential function that $\mathbf{A}' = \boldsymbol{\alpha}\mathbf{A}$ and $\mathbf{B}' = \boldsymbol{\beta}\mathbf{B}$, and therefore

$$(\mathbf{e}_i^*)' = \begin{cases} \mathbf{e}_i', & s \in [0,\ L] - (s_1,\ s_3) \\ \boldsymbol{\alpha}\mathbf{A}\mathbf{e}_i + \mathbf{A}\mathbf{e}_i', & s_1 < s < s_2 \\ \boldsymbol{\beta}\mathbf{B}\mathbf{e}_i + \mathbf{B}\mathbf{e}_i', & s_2 < s < s_3. \end{cases} \tag{5.9}$$

From this it is evident that the $(\mathbf{e}_i^*)'$ suffer jump discontinuities at $s = s_1$, s_2 and s_3.

We define

$$\Delta = s_3 - s_1,\ \Delta_1 = s_2 - s_1 = \theta\Delta,\ \Delta_2 = s_3 - s_2 = (1 - \theta)\Delta,$$

where

$$\theta = (s_2 - s_1)/(s_3 - s_1) \in (0, 1). \tag{5.10}$$

Then $(s_2 - s_1)/(s_2 - s_3) = -\theta/(1 - \theta)$, so that (5.8) can be written

$$\boldsymbol{\beta} = -\theta/(1 - \theta)\boldsymbol{\alpha}. \tag{5.11}$$

Now divide (5.5) by Δ and use (3.2), (5.6), (5.9) and (5.10) to obtain

$$\begin{aligned} 0 \leqslant \frac{\theta}{\Delta_1} \int_{s_1}^{s_2} U(\mathbf{A}\mathbf{e}_i,\ \boldsymbol{\alpha}\mathbf{A}\mathbf{e}_i + \mathbf{A}\mathbf{e}_i')\,\mathrm{d}s + \frac{1-\theta}{\Delta_2} \int_{s_2}^{s_3} U(\mathbf{B}\mathbf{e}_i,\ \boldsymbol{\beta}\mathbf{B}\mathbf{e}_i + \mathbf{B}\mathbf{e}_i')\,\mathrm{d}s \\ - \frac{1}{\Delta} \int_{s_1}^{s_3} U(\mathbf{e}_i,\ \mathbf{e}_i')\,\mathrm{d}s. \end{aligned} \tag{5.12}$$

The definitions (5.7) imply that $\mathbf{A}(s), \mathbf{B}(s) \to \mathbf{I}$ as Δ_1 and $\Delta_2 \to 0$. Then if we let $\Delta \to 0$ with θ fixed and use the mean value theorem together with (5.11), we get

$$\theta U(\mathbf{e}_i,\ \mathbf{e}_i' + \boldsymbol{\alpha}\mathbf{e}_i) + (1 - \theta)U(\mathbf{e}_i,\ \mathbf{e}_i' - \theta/(1 - \theta)\boldsymbol{\alpha}\mathbf{e}_i) - U(\mathbf{e}_i, \mathbf{e}_i') \geqslant 0 \tag{5.13}$$

for all $\theta \in (0, 1)$ and for all skew tensors $\boldsymbol{\alpha}$. The arbitrariness in the choice of the points s_1, s_2, s_3 implies that inequality (5.13) must be satisfied at each point $s \in (0, L)$ of the energy minimizer $\{\mathbf{e}_i(\cdot)\}$. Moreover, (5.13) holds at corners of the minimizer provided that the $\mathbf{e}_i'$ are interpreted as the appropriate left or right limits [1].

For small θ we have

$$\frac{1-\theta}{\theta} U\left(\mathbf{e}_i, \mathbf{e}_i' - \frac{\theta}{1-\theta}\boldsymbol{\alpha}\mathbf{e}_i\right) = \frac{1}{\theta} U(\mathbf{e}_i, \mathbf{e}_i') - U(\mathbf{e}_i, \mathbf{e}_i') \\ - \boldsymbol{\alpha}\mathbf{e}_i \cdot \partial U/\partial \mathbf{e}_i' + (1-\theta)o(\theta)/\theta, \tag{5.14}$$

where the derivatives are evaluated at $\theta = 0$. We substitute this into (5.13) and let $\theta \to 0^+$ to arrive at the Weierstrass inequality:

$$U(\mathbf{e}_i, \mathbf{e}_i' + \boldsymbol{\alpha}\mathbf{e}_i) - U(\mathbf{e}_i, \mathbf{e}_i') - \boldsymbol{\alpha}\mathbf{e}_i \cdot \partial U/\partial \mathbf{e}_i' \geqslant 0, \ \forall \text{skew } \boldsymbol{\alpha}. \tag{5.15}$$

This inequality is also valid if $\mathbf{e}_i'$ and $\mathbf{e}_i' + \boldsymbol{\alpha}\mathbf{e}_i$ are taken to be the left and right limits of $\mathbf{e}_i'(s)$ at a corner, and vice versa. The resulting inequalities can be used with moment-continuity (see (4.6)) to obtain the Weierstrass–Erdmann condition (4.22) (see [31]). The latter condition is therefore necessary for configurations with corners that minimize the energy with respect to strong variations. This type of variation is implicit in the derivation of Section 4; the parameter transformations considered there induce translations of the functions $\boldsymbol{\rho}$ and $\boldsymbol{\varepsilon}_i$ with respect to arc length s along the rod. This amounts to a strong variation for fixed values of the arc length.

The condition (4.22) is *not* necessary if the minimum is *weak* in the sense that $|\mathbf{r}^* - \mathbf{r}| + \Sigma_i|\mathbf{e}_i^* - \mathbf{e}_i| + \Sigma_i|(\mathbf{e}_i^*)' - \mathbf{e}_i'| < \delta$ for all $s \in [0, L]$. The moment continuity condition (4.6) remains valid, however, as weak variations were used in the development leading to (4.5). Because the class of weak variations is smaller than the class of strong variations, it follows that strong minima are also weak minima, but the converse is not valid.

To interpret the Weierstrass inequality (5.15) for invariant strain energies $U = w(\boldsymbol{\kappa})$, we use the definition $\kappa_i = \frac{1}{2}e_{ijk}\mathbf{e}_k \cdot \mathbf{e}_j'$ (see (2.13)) to infer that

$$U(\mathbf{e}_i, \mathbf{e}_i' + \boldsymbol{\alpha}\mathbf{e}_i) = w(\hat{\boldsymbol{\kappa}}), \tag{5.16}$$

where

$$\hat{\boldsymbol{\kappa}} = \hat{\kappa}_i \mathbf{E}_i; \ \hat{\kappa}_i = \kappa_i + \tfrac{1}{2}e_{ijk}\mathbf{e}_k \cdot \boldsymbol{\alpha}\mathbf{e}_j. \tag{5.17}$$

Next we use (2.21) to write

$$\boldsymbol{\alpha}\mathbf{e}_i \cdot \partial U/\partial \mathbf{e}_i' = \tfrac{1}{2}(\partial w/\partial \kappa_j)e_{jik}\mathbf{e}_k \cdot \boldsymbol{\alpha}\mathbf{e}_i. \tag{5.18}$$

Since $\boldsymbol{\alpha}$ is skew, it follows from (5.17) that $\mathbf{e}_k \cdot \boldsymbol{\alpha}\mathbf{e}_i = e_{ikl}(\hat{\kappa}_l - \kappa_l)$, and therefore the Weierstrass inequality (5.15) is equivalent to

$$w(\hat{\boldsymbol{\kappa}}) - w(\boldsymbol{\kappa}) - (\hat{\kappa}_j - \kappa_j)\partial w/\partial \kappa_j \geqslant 0, \ \forall s \in (0, L), \tag{5.19}$$

where the derivatives are evaluated at the function $\boldsymbol{\kappa}(s)$ furnished by the minimizing configuration. Thus for every $s \in (0, L)$, $\boldsymbol{\kappa}(s)$ belongs to a domain of convexity of the function $w(\cdot)$.

For small $|\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}|$ we obtain

$$w(\hat{\boldsymbol{\kappa}}) = w(\boldsymbol{\kappa}) + (\hat{\kappa}_j - \kappa_j)\partial w/\partial \kappa_j + \tfrac{1}{2}C^0_{ij}(\hat{\kappa}_i - \kappa_i)(\hat{\kappa}_j - \kappa_j) + o(|\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}|^2), \tag{5.20}$$

where C^0_{ij} is the value of the *stiffness*

$$C_{ij} = \partial^2 w/\partial \kappa_i \partial \kappa_j \tag{5.21}$$

furnished by the minimizer. Substituting (5.20) into (5.19), dividing by $|\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}|^2$ and passing to the limit $\hat{\boldsymbol{\kappa}} \to \boldsymbol{\kappa}$, we find that the Weierstrass inequality implies the Legendre condition [1]:

$$C^0_{ij} a_i a_j \geqslant 0, \ \forall a_i. \tag{5.22}$$

It can be shown by direct methods that this condition is also necessary for configurations that minimize the energy in the weak sense [31]. Thus a configuration is a strong or weak minimizer of the energy only if the associated stiffness is positive semi-definite.

Our development so far has been based on the premise that minimizing configurations contain at most a finite number of points (corners) at which $\boldsymbol{\kappa}(s)$ may be discontinuous. We show that if the domain D of $w(\cdot)$ is convex and if the strict Legendre condition is satisfied as a constitutive inequality, i.e.

$$C_{ij}(\boldsymbol{\kappa}) a_i a_j > 0, \ \forall a_i \neq 0, \tag{5.23}$$

for all $\boldsymbol{\kappa} \in D$, then $\boldsymbol{\kappa}(s)$ is continuous and there are no corners. Let $\boldsymbol{\kappa}_+$ and $\boldsymbol{\kappa}_-$ be the limiting values of $\boldsymbol{\kappa}(s)$ on either side of a corner. Then the jump $\Delta\boldsymbol{\kappa} = \boldsymbol{\kappa}_+ - \boldsymbol{\kappa}_-$ must satisfy the Weierstrass–Erdmann conditions (4.6), (4.22), which can be expressed in the forms

$$(\partial w/\partial \kappa_i)_+ - (\partial w/\partial \kappa_i)_- = 0$$

and

$$w(\boldsymbol{\kappa}_+) - w(\boldsymbol{\kappa}_-) - \Delta\kappa_i(\partial w/\partial\kappa_i)_\pm = 0, \tag{5.24}$$

respectively. The second of these restrictions need not be satisfied if the minimum is weak. Since D is convex, the line segment connecting $\boldsymbol{\kappa}_+$ and $\boldsymbol{\kappa}_-$ is contained in D, and each of eqs. (5.24) implies that

$$\Delta\kappa_i\Delta\kappa_j C_{ij}[\boldsymbol{\kappa}_- + \theta(\boldsymbol{\kappa}_+ - \boldsymbol{\kappa}_-)] = 0 \tag{5.25}$$

for some $\theta \in (0, 1)$. This contradicts (5.23) unless $\boldsymbol{\kappa}(s)$ is continuous, i.e. $\Delta\boldsymbol{\kappa} = \mathbf{0}$.

Finally, we demonstrate that the Weierstrass and Legendre inequalities remain valid for the dead loading problem. To this end we show that with $\mathbf{Q}$ defined by (5.6) and Δ defined by (5.10a), we have

$$\Delta^{-1}(P[\mathbf{r}^*, \mathbf{e}_i^*] - P[\mathbf{r}, \mathbf{e}_i]) \to 0 \quad \text{as} \quad \Delta \to 0^+, \tag{5.26}$$

and therefore the presence of the load potential does not affect the development leading to (5.13). For the potential given by (3.3) and (3.23), we find

$$P[\mathbf{r}^*, \mathbf{e}_i^*] - P[\mathbf{r}, \mathbf{e}_i] = \int_0^L \mathbf{b}\cdot(\mathbf{r}^* - \mathbf{r})\,\mathrm{d}s + \mathbf{f}\cdot[\mathbf{r}^*(L) - \mathbf{r}(L)] + \mathbf{m}_i\cdot[\mathbf{Q}(L) - \mathbf{I}]\mathbf{e}_i(L). \tag{5.27}$$

The terms involving the end couples vanish because $\mathbf{Q}(L) = \mathbf{I}$, according to (5.6).

In view of (2.2), the displacement $\mathbf{r}^* - \mathbf{r}$ induced by the perturbation (5.6) is

$$\mathbf{r}^*(s) - \mathbf{r}(s) = \int_0^s [\mathbf{Q}(x) - \mathbf{I}]\mathbf{t}(x)\,\mathrm{d}x. \tag{5.28}$$

This vanishes if $0 \leqslant s \leqslant s_1$. Elsewhere, it reduces to

$$\mathbf{r}^*(s) - \mathbf{r}(s) = \begin{cases} \displaystyle\int_{s_1}^{s} [\mathbf{Q}(x) - \mathbf{I}]\mathbf{t}(x)\,\mathrm{d}x, & s_1 < s < s_3 \\ \displaystyle\int_{s_1}^{s_3} [\mathbf{Q}(x) - \mathbf{I}]\mathbf{t}(x)\,\mathrm{d}x, & s_3 \leqslant s \leqslant L. \end{cases} \tag{5.29}$$

Then for all $s \in [0, L]$,

$$|\mathbf{r}^*(s) - \mathbf{r}(s)| \leqslant \Delta\left[\max_{s\in(s_1,s_3)} |(\mathbf{Q}(s) - \mathbf{I})\mathbf{t}(s)|\right]. \tag{5.30}$$

According to (5.6) and (5.7),

$$\mathbf{Q}(s) - \mathbf{I} = \begin{cases} (s - s_1)\boldsymbol{\alpha} + o(|s - s_1|), & s_1 < s < s_2 \\ (s - s_3)\boldsymbol{\beta} + o(|s - s_3|), & s_2 < s < s_3 \end{cases} \tag{5.31}$$

for small Δ. Thus $|(\mathbf{Q}(s) - \mathbf{I})\mathbf{t}(s)| = O(\Delta)$ and it follows from (5.30) and (5.27) that $P[\mathbf{r}^*, \mathbf{e}_i^*] - P[\mathbf{r}, \mathbf{e}_i] = o(\Delta)$, so that (5.26) is valid.

6. The Legendre inequality for transversely isotropic rods

To give a concrete interpretation of the Legendre necessary condition, we consider its implications for a rod that is transversely isotropic in a sense to be defined. The theory of transversely isotropic rods includes as a special case the classical theory for rods with equal principal flexural stiffnesses [5]. As in the classical theory ([4], art. 259), we assume that the strain energy $w(\boldsymbol{\kappa}; \boldsymbol{\kappa}^0, s)$ depends on $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^0$ through the vector difference

$$\boldsymbol{\gamma} = \boldsymbol{\kappa} - \boldsymbol{\kappa}^0 = \gamma_i \mathbf{E}_i; \; \gamma_i = \kappa_i - \kappa_i^0. \tag{6.1}$$

This is simply the axial vector of the tensor $\mathbf{R}^T\mathbf{R}'$ (see (2.10)) that measures relative curvatures and twist. We denote the strain energy by $w(\gamma_i)$ and suppress reference to explicit s-dependence due to non-uniformity of the material properties.

Consider the basis transformation $\mathbf{E}_i \to \mathbf{E}_i^* = \mathbf{P}^T\mathbf{E}_i$, where $\mathbf{P} = P_{ij}\mathbf{E}_i \otimes \mathbf{E}_j$ is orthogonal. For such transformations we have $\gamma_i \to \gamma_i^* = P_{ij}\gamma_j$, and we define transverse isotropy by the requirement

$$w(\gamma_i) = w(P_{ij}\gamma_j), \; \mathbf{P} \in SG, \tag{6.2}$$

for all $\boldsymbol{\gamma}$, where SG is the symmetry group

$$SG = \{\pm\mathbf{P}_1, \pm\mathbf{P}_2\} \tag{6.3}$$

consisting of all orthogonal transformations with axis $\mathbf{E}_1$:

$$\mathbf{P}_1 = \mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{S}(\theta), \mathbf{P}_2 = \mathbf{E}_1 \otimes \mathbf{E}_1 - \mathbf{S}(\theta); \tag{6.4}$$
$$\mathbf{S}(\theta) = \cos\theta(\mathbf{E}_2 \otimes \mathbf{E}_2 + \mathbf{E}_3 \otimes \mathbf{E}_3) + \sin\theta(\mathbf{E}_3 \otimes \mathbf{E}_2 - \mathbf{E}_2 \otimes \mathbf{E}_3), \theta \in [0, 2\pi].$$

We use the notation $\phi = \gamma_1$ for the relative twist. Then

$$\gamma_i = \delta_{i1}\phi + \delta_{i\alpha}\gamma_\alpha, \tag{6.5}$$

where $\gamma_\alpha (\alpha = 2, 3)$ are the relative curvatures, and for $\mathbf{P} = \mathbf{P}_1$ or $\mathbf{P}_2$ we find that (6.2) reduces to

$$w(\phi, \gamma_\alpha) = w(\phi, \pm S_{\alpha\beta}\gamma_\beta). \tag{6.6}$$

This is valid for all γ_α if and only if there is a function F such that

$$w(\phi, \gamma_\alpha) = F(\phi, \psi); \; \psi = (\gamma_\alpha\gamma_\alpha)^{1/2}. \tag{6.7}$$

The only additional restriction imposed by (6.2-4) is $F(\phi, \psi) = F(-\phi, \psi)$. Thus the rod is transversely isotropic if and only if

$$w(\gamma_i) = F(\phi, \psi), \tag{6.8}$$

where $F(\cdot, \psi)$ is an even function.

A straightforward calculation based on (6.1) and (6.7) yields

$$\partial w/\partial \kappa_i = F_\phi \delta_{i1} + \psi^{-1} F_\psi \delta_{i\alpha}\gamma_\alpha, \tag{6.9}$$

where subscripts ϕ and ψ are used to denote partial derivatives. Then from (2.25) we derive the constitutive relation

$$\mathbf{M} = F_\phi \mathbf{t} + \psi^{-1} F_\psi \gamma_\alpha \mathbf{e}_\alpha, \tag{6.10}$$

which implies that $|F_\phi|$ and $|F_\psi|$ are the magnitudes of the torsional and flexural moments, respectively.

The stiffness can be obtained from (5.21), (6.1) and (6.9):

$$\begin{aligned} C_{ij} = {} & F_{\phi\phi}\delta_{i1}\delta_{j1} + \psi^{-1}F_\psi(\delta_{i\alpha}\delta_{j\alpha} - \psi^{-2}\delta_{i\alpha}\delta_{j\beta}\gamma_\alpha\gamma_\beta) \\ & + \psi^{-2}F_{\psi\psi}\delta_{i\alpha}\delta_{j\beta}\gamma_\alpha\gamma_\beta + \psi^{-1}F_{\phi\psi}(\delta_{i1}\delta_{j\alpha}\gamma_\alpha + \delta_{j1}\delta_{i\alpha}\gamma_\alpha). \end{aligned} \tag{6.11}$$

The associated quadratic form $C_{ij}a_i a_j$ can be written

$$\begin{aligned} C_{ij}a_i a_j = {} & F_{\phi\phi}a^2 + \psi^{-1}F_\psi[b_\alpha b_\alpha - \psi^{-2}(b_\alpha\gamma_\alpha)(b_\beta\gamma_\beta)] \\ & + \psi^{-2}F_{\psi\psi}(b_\alpha\gamma_\alpha)(b_\beta\gamma_\beta) + 2a\psi^{-1}F_{\phi\psi}(b_\alpha\gamma_\alpha), \end{aligned} \tag{6.12}$$

where $a = a_1$ and $b_\alpha = a_\alpha$. We define unit vectors m_α and n_α by $\gamma_\alpha = \psi m_\alpha$ and $b_\alpha = bn_\alpha$. Then

$$C_{ij}a_i a_j = F_{\phi\phi}a^2 + [(1-\theta^2)\psi^{-1}F_\psi + \theta^2 F_{\psi\psi}]b^2 + 2ab\theta F_{\phi\psi}, \tag{6.13}$$

where $\theta \equiv m_\alpha n_\alpha \in [-1, 1]$. For fixed θ this is a quadratic form in (a, b), and is non-negative if and only if

$$F_{\phi\phi} \geqslant 0,\ \theta^2 F_{\psi\psi} + (1-\theta^2)\psi^{-1}F_\psi \geqslant 0$$

and

$$F_{\phi\phi}[\theta^2 F_{\psi\psi} + (1-\theta^2)\psi^{-1}F_\psi] \geqslant \theta^2(F_{\phi\psi})^2. \tag{6.14}$$

The necessary conditions

$$\psi^{-1}F_\psi \geqslant 0,\ F_{\psi\psi} \geqslant 0 \tag{6.15}$$

follow from the second of inequalities (6.14) by setting $\theta = 0$ and $\theta = \pm 1$, respectively. These are clearly sufficient also, since $\theta^2 \in [0, 1]$. The third inequality in (6.14) is equivalent to

$$(1-\theta^2)\psi^{-1}F_\psi F_{\phi\phi} + \theta^2[F_{\phi\phi}F_{\psi\psi} - (F_{\phi\psi})^2] \geqslant 0. \tag{6.16}$$

In view of (6.14a) and (6.15a), this is satisfied for fixed (ϕ, ψ) and for all $\theta^2 \in [0, 1]$ if and only if

$$F_{\phi\phi}F_{\psi\psi} - (F_{\phi\psi})^2 \geqslant 0. \tag{6.17}$$

Thus the quadratic form $C_{ij}a_i a_j$ is positive semi-definite if and only if

$$\psi^{-1}F_\psi \geqslant 0 \tag{6.18}$$

and F is jointly convex in its arguments:

$$F_{\phi\phi} \geqslant 0,\ F_{\psi\psi} \geqslant 0,\ F_{\phi\phi}F_{\psi\psi} - (F_{\phi\psi})^2 \geqslant 0. \tag{6.19}$$

The classical theory [4] is based on quadratic strain energies of the form

$$F(\phi, \psi) = \tfrac{1}{2}A\phi^2 + \tfrac{1}{2}B\psi^2, \tag{6.20}$$

where $A(s)$ and $B(s)$ are the torsional and flexural rigidities, respectively. Our results imply that the stiffness is positive semi-definite if and only if the rigidities are non-negative. This in turn implies that the strain energy is positive semi-definite.

Returning to the general theory, it follows from the results of the previous section that a configuration minimizes the energy only if the associated values of $\phi(s)$ and $\psi(s)$ satisfy inequalities (6.18) and (6.19) for every $s \in (0, \mathrm{L})$. We note that (6.18) and (6.19) need not be interpreted as constitutive inequalities, however. Thus if they fail on some subdomain of the strain energy function, then values of ϕ and ψ in that subdomain cannot be present at any point in a minimizing configuration. The objectionable values are excluded by allowing for the presence of jump discontinuities (corners) in $\phi(s)$ and $\psi(s)$. This sort of behaviour has been investigated by Fosdick and James [2, 3] for problems involving plane deformation without twist.

It is evident from (6.10) and (6.18) that in energy-minimizing configurations the flexural part of the moment either vanishes or has the same sense as the relative curvature vector $\gamma_\alpha \mathbf{e}_\alpha$. It is interesting that the Legendre inequality implies nothing about the sense of the torsional moment. If we adopt the mild constitutive assumption that $F_\phi \geqslant 0$ when $\phi \geqslant 0$, then the evenness and smoothness of the function $F(\cdot, \psi)$ imply that the torsional moment either vanishes or has the same sense as the twist. We do not know if this is necessary for stability, however.

Acknowledgement

We gratefully acknowledge support provided by the Natural Sciences and Engineering Research Council of Canada through grants OGP0041743 (D.J.S.) and OGP A7514 (M.G.F.).

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