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Static equilibria of planar, rigid bodies: is there anything new?*

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1. Introduction

The equilibrium of a planar, rigid body is one of the first problems to be discussed in any book on statics. However, we are not aware that the simple facts described in this note are generally known and we hope that we can offer some new aspects to the understanding of this old and fundamental topic.

We consider a planar, rigid body in the presence of a uniform, vertical gravity field, supported by rolling contact with a straight horizontal surface. We consider two types of bodies: (1) a slab-like body with constant density (weight per unit area) and (2) a wire-like body with constant boundary density (weight per unit of boundary arc length).

Typically such a rigid body will have at least one stable equilibrium position since the potential energy is a periodic function which is bound to have at least one minimum. (By "typically" we mean that constant and non-smooth potentials are excluded.) We claim, however, that if the body is convex then there exists a *second* stable equilibrium position. We will show that the homogeneity and convexity requirements are essential, since in their absence examples with only one stable equilibrium can be constructed.

In Section 2 we formulate our claims more precisely, Section 3 deals with the proof for the planar body, Section 4 with the proof for the wire, in Section 5 we discuss non-convex bodies and in Section 6 we mention possible generalizations.

2. Notations and claims

Let us consider a planar, rigid, strictly convex (curvature bounded away from zero) and homogeneous body $\mathscr{B}$ with center of gravity at point G, lying on a

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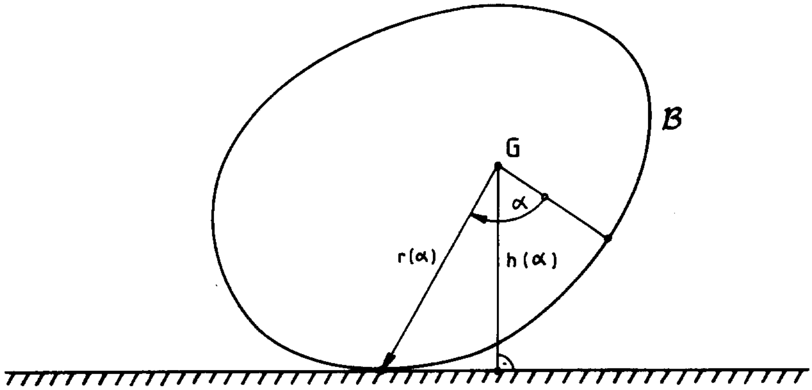

Fig. 1. The body $\mathscr{B}$, center of gravity G with height $h(\alpha)$ and radius at contact point $r(\alpha)$.

horizontal plane. The circumference of $\mathscr{B}$ will be described as a function $r(\alpha)$, $\alpha \in [0, 2\pi)$ in a polar coordinate system fixed to $\mathscr{B}$, with origin at G. (see Fig. 1.)

The function $r(\alpha)$ will be assumed to have isolated stationary points.

PROPOSITION 1. *$\mathscr{B}$ has always at least two stable equilibria.*

Let us now consider a homogeneous wire $\mathscr{W}$ on the circumference of $\mathscr{B}$.

PROPOSITION 2. *$\mathscr{W}$ has always at least two stable equilibria.*

The function $h(\alpha)$ will be interpreted as the height of the center of gravity G at the value of α when the point of contact is marked by $r(\alpha)$. (see Fig. 1.)

3. Proof of Proposition 1

First of all we would like to show that Proposition 1 is equivalent to

COROLLARY 1. *The function $r(\alpha)$ has at least two local minima.*

In order to show this equivalence we will rely on

LEMMA 1. *$\mathscr{B}$ is in a stable equilibrium configuration if and only if $r(\alpha)$ has a local minimum at the contact point.*

Proof of Lemma 1. We take as understood that a stable equilibrium implies a local minimum of $h(\alpha)$, (and vice versa). Thus we have to show that all local minima of $h(\alpha)$ and $r(\alpha)$ coincide. From the kinematics of monotonic rolling (monotonicity is a consequence of the convexity of $\mathscr{B}$) we learn that those and only those points of $\mathscr{B}$ move locally in horizontal direction which lie above the point of contact, since the rolling motion is locally equivalent to a rotation around the point of contact (see Fig. 2.).

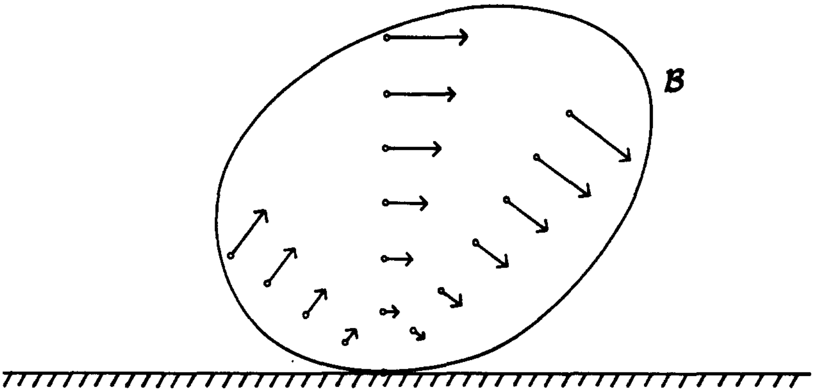

Fig. 2. Kinematics of monotonic rolling (velocity field)

From this we conclude that if G has a stationary height $h(\alpha)$ it has to lie above the point of contact, i.e.

$$h(\alpha) = r(\alpha) \quad \text{if and only if } h(\alpha) \text{ is stationary.} \tag{1}$$

On the other hand we observe that the perimeter of $\mathscr{B}$ is tangent to the circle with center G and radius $r(\alpha)$ if and only if $r(\alpha)$ is stationary. In this case the normal of the perimeter passes through G. (see Fig. 3).

From this fact we conclude that

$$r(\alpha) = h(\alpha) \quad \text{if and only if } r(\alpha) \text{ is stationary.} \tag{2}$$

From (1) and (2) we learn that the functions $r(\alpha)$ and $h(\alpha)$ (both having period 2π) have stationary points at the same values of α, moreover, at all stationary points the function values coincide. Thus far we did not claim that the corresponding stationary points of $h(\alpha)$ and $r(\alpha)$ are of the same type, i.e. the second derivative has the same sign. However, we observe that if any of the

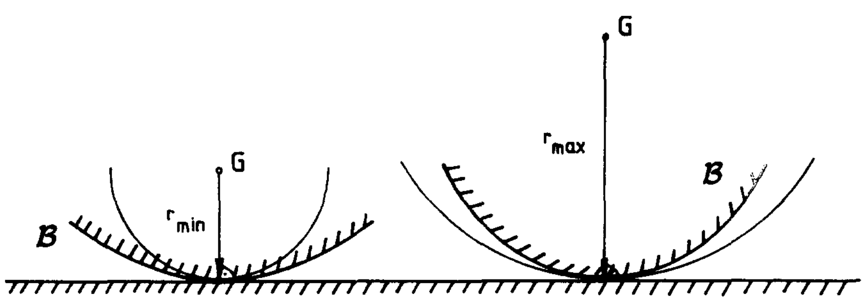

Fig. 3. Stationary value of the radius $r(\alpha)$.

corresponding stationary points would be of a different type (for example $h(\alpha)$ had a maximum at the same value of α when $r(\alpha)$ had a minimum or horizontal inflexion) at least one of the neighbour stationary points the function values could not coincide, thus we would contradict either (1) or (2). Therefore we conclude that the type of the stationary points is identical, i.e. $h(\alpha)$ has a local minimum if and only if $r(\alpha)$ has one, thus $\mathscr{B}$ has a stable equilibrium if and only if $r(\alpha)$ has a local minimum. □

Now we can proceed to deliver

Proof of Corollary 1

Assume that the statement is false. In this case $r(\alpha)$ has exactly one minimum, and consequently, one maximum in $\alpha \in [0, 2\pi)$. Let us denote the corresponding values of α by $\alpha_{\min}$ and $\alpha_{\max}$, respectively. Now let $\beta = \alpha - \alpha_{\min}$; the function $r(\beta)$ is depicted in Figure 4.

Let us draw a horizontal tangent to $r(\beta)$ at $\beta = \alpha_{\max} - \alpha_{\min}$ and lower this horizontal (r = *constant*) line continuously. The segment of the horizontal line lying below $r(\beta)$ will grow monotonically from O to 2π, thus it has to pass the value of π. Denote the corresponding value of r by r_0, the corresponding values of β by β_1 and β_2. From this definition of β_1 and β_2 it follows that

$$r(\beta) < r_0 < r(\beta + \pi) \text{ if } \beta \in (\beta_1, \beta_2). \tag{3}$$

Since $\beta_2 - \beta_1 = \pi$, (3) implies that there exists a straight cut c of $\mathscr{B}$ passing through G such that any radius in one part is greater than any radius in the other part. Since G is the center of gravity of $\mathscr{B}$

$$\int_{\beta^*}^{\beta^* + 2\pi} \sin(\beta - \beta^*) r^3(\beta)\, d\beta = 0$$

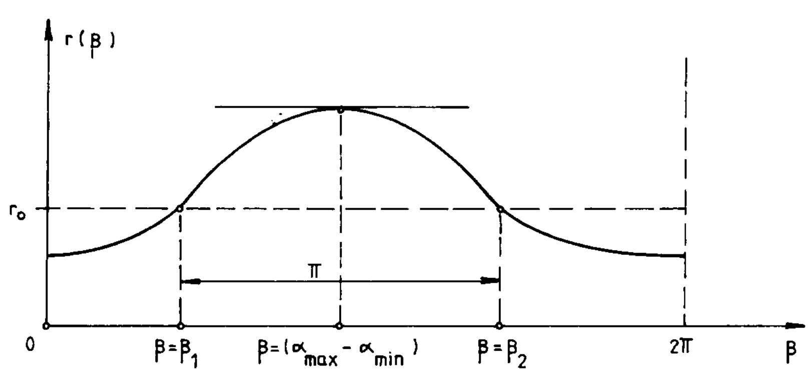

Fig. 4. The function $r(\beta)$ and the horizontal cut.

for all β^*. To balance moments about the line c is expressed by setting $\beta^* = \beta_1$ yielding

$$\int_{\beta_1}^{\beta_2} \sin(\beta - \beta_1)(r^3(\beta) - r^3(\beta + \pi))\, d\beta = 0, \tag{4}$$

which contradicts (3). Thus we have proved Corollary 1. □

Since Lemma 1 tells us that Corollary 1 is equivalent to Proposition 1, we proved Proposition 1 as well: A uniform, convex solid has at least 2 stable balance points. □

4. Proof of Proposition 2

This proof coincides with the previous one up to (3), i.e. we have again a straight cut c passing through G, dividing $\mathcal{W}$ into two parts $\mathcal{W}_1$ and $\mathcal{W}_2$ in such a way that any radius in $\mathcal{W}_1$ is smaller than any radius in $\mathcal{W}_2$. However, this fact does not provide direct evidence for contradiction, since the balance of moments about c depends now not only on $r(\beta)$ but on the derivative $\dot{s}(\beta)$ of the arclength, as well (the operator $d/d\beta(\ )$ is denoted by $(\dot{\ })$). The analogous equation to (4) can be formulated as

$$\int_{\beta_1}^{\beta_2} \sin(\beta - \beta_1)(r(\beta)\dot{s}(\beta) - r(\beta + \pi)\dot{s}(\beta + \pi))\, d\beta = 0, \tag{5}$$

and nothing guarantees that $\dot{s}(\beta) < \dot{s}(\beta + \pi)$. In order to eliminate this difficulty we define a mutually one-to-one mapping M between points of $\mathcal{W}_1$ and $\mathcal{W}_2$ in such a way that *both* the corresponding values of $r(\beta)$ *and* the corresponding values of $\dot{s}(\beta)$ are related in the same way.

Let us reflect the segment $\mathcal{W}_2$ around c to obtain $\bar{\mathcal{W}}_2$. Now we have two line segments on the same side of c, $\mathcal{W}_1$ lying inside $\bar{\mathcal{W}}_2$ (See Figure 5.)

We introduce the arclengths on the inner $\mathcal{W}_1$ and outer $\bar{\mathcal{W}}_2$ segment respectively by

$$\begin{aligned} s_1(\beta) &= s(\beta) - s(\beta_1), \\ s_2(\beta) &= s(\beta_1) - s(2\beta_1 - \beta), \end{aligned} \tag{6}$$

and the mapping $M: s_1(\beta) \leftrightarrow s_2(\beta)$ is defined by projecting the inner line segment along its normal onto the outer line segment (Cao, personal communication). Since the inner line segment is convex, the (local) triangle inequality

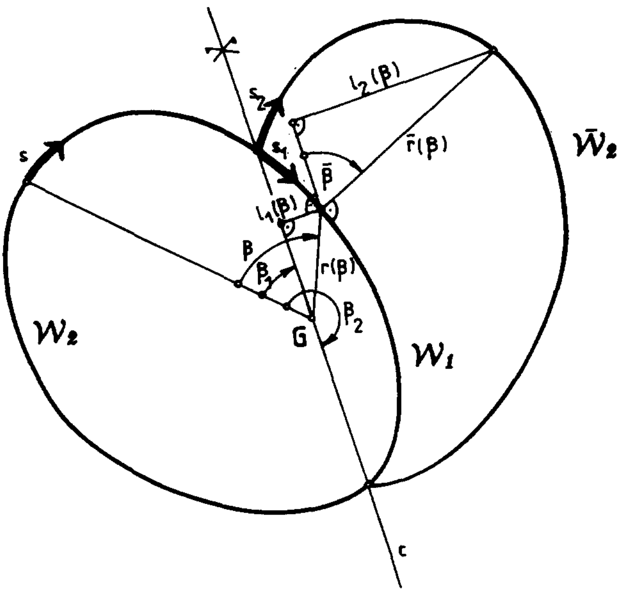

Fig. 5. Mapping between the segments of the wire.

always implies $(ds_2/ds_1) > 1$ from which

$$\dot{s}_2(\beta) > \dot{s}_1(\beta) \qquad (\beta \in (\beta_1, \beta_2)) \tag{7}$$

follows by the chain rule. The distance between the inner and outer line along the normal will be denoted by $\bar{r}(\beta)$, the angle between $\bar{r}(\beta)$ and c by $\bar{\beta}$. Using these notations the level arms (distances from c) $l_1(\beta)$ and $l_2(\beta)$ of corresponding points on the inner $\mathscr{W}_1$ and outer $\bar{\mathscr{W}}_2$ line can be expressed as

$$\begin{aligned} l_1(\beta) &= r(\beta)\sin(\beta - \beta_1), \\ l_2(\beta) &= l_1 + \bar{r}(\beta)\sin(\bar{\beta}). \end{aligned} \tag{8}$$

Since $0 < \bar{\beta} < \pi$ we have always

$$l_1(\beta) < l_2(\beta) \qquad (\beta \in (\beta_1, \beta_2)). \tag{9}$$

The expression (5) for the balance of linear momentum can be re-formulated now as

$$\int_{\beta_1}^{\beta_2} (l_1(\beta)\dot{s}_1(\beta) - l_2(\beta)\dot{s}_2(\beta))\,\mathrm{d}\beta = 0, \tag{10}$$

which is a contradiction because of (7) and (9). Thus we proved Proposition 2: A homogeneous convex wire has at least 2 stable balance points.

5. Non-convex bodies

Our results thus far relied on convexity. We will now investigate what statements are possible for non-convex bodies.

The rolling of non-convex rigid bodies can be interpreted in two different ways. The physical (and more realistic) interpretation assumes that non-convex bodies are rolling on their convex hull. Based on this interpretation it is easy to construct a non-convex body for which our propositions fail, i.e. which has only one stable equilibrium. Figure 6 depicts one example for the homogeneous slab and one for the homogeneous wire. (G is the center of gravity, C is the center of the bounding circle).

However, there exists an (at least mathematically) plausible interpretation of rolling for non-convex bodies. In this interpretation the body rolls strictly on it's circumference, which implies that material overlap with the supporting plane has to be admitted. Moreover the contact force may change sign, as well. This type of "rolling" is depicted in Figure 7.

If we accept this interpretation then we claim that the body has to have at least two stable equilibria. In order to show this we observe that in this type of rolling the minimum of $r(\alpha)$ is still sufficient but not necessary condition for stable equilibrium, since at the inflexion points of the circumference the direction of rolling changes sign, and therefore every inflexion point means a (non-smooth) extremal position for the center of gravity G. Since inflexion points always appear in pairs, moreover there exist always two smooth extrema for $r(\alpha)$ (and, as a consequence, for $h(\alpha)$) there will be always four extremal positions for the center of gravity G, which implies necessarily two local minima. We remark that at the mentioned non-smooth extrema G is generally not above the point of contact, hence a moment at the contact point is needed as well to maintain equilibrium.

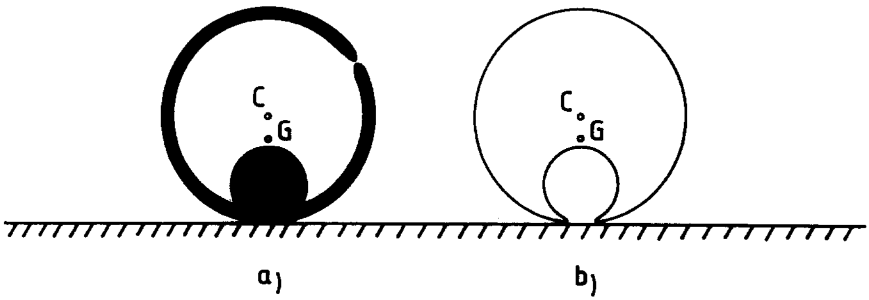

Fig. 6. Non-convex bodies with only one stable equilibrium; (a) slab, (b) wire.

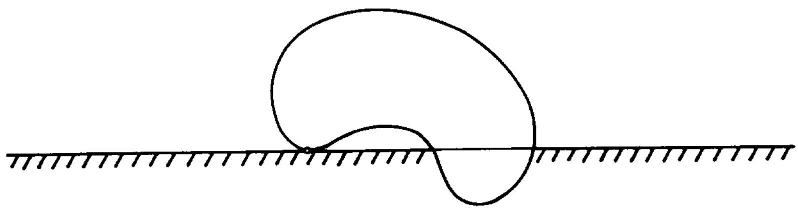

Fig. 7. An alternative interpretation for non-convex rolling.

6. Concluding remarks

We have shown that planar, rigid, homogeneous bodies with strictly convex boundary having isolated critical points always have two stable equilibria. This statement still holds if: (1) the body is non-homogeneous but the mass distribution depends only on $r(\alpha)$ (2) the boundary is convex but not strictly convex, i.e. we admit straight segments. Both generalizations can be proven by using the same line of thought, though more technical details are needed. This was the reason why we showed the proof only of the homogeneous, smooth case. Generalization (2) is valid for the planar wire, as well.

In the case of non-convex bodies we showed that by adopting the "traditional" interpretation of rolling there are cases when the body has only one stable equilibrium. However, if we accept the "alternative" interpretation for rolling then the body still has at least two stable equilibria but for different reasons as the convex body.

Acknowledgement

We thank J. Cao for suggesting that the wire can be parameterized by its normal directions.