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Internal constraints in linear elasticity

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ABSTRACT

Sufficient conditions are obtained for continuous dependence of solutions of boundary value problems of linear elasticity on internal constraints. Arbitrary hyperelastic materials with arbitrary (linear) internal constraints are included. In particular the results of Bramble and Payne, Kobelkov, Mikhlin for homogeneous, isotropic, incompressible materials are obtained as a special case. In the case of boundary value problem of place, a compatibility condition is obtained between the internal constraints and the boundary data which is necessary for the existence of solutions. With a further coercivity assumption on the compliance tensor, it is shown that the compatibility condition is also sufficient for existence. An orthogonal decomposition theorem for second order tensor fields modeled after Weyl's decomposition of solenoidal and gradient fields leads to the variational formulation of the problem and existence theorems.

Almost all the results here apply to materials both with or without internal constraints. For internally constrained materials however, the verification of certain hypothesis is surprisingly non-trivial as indicated by the computation in the appendix.

In a previous paper [17] we have obtained some results concerning the dependence of the minimizers of quadratic functions on the linear operator defining the quadratic function. Here we apply these results to the mixed boundary value problem of linear elastostatics for internally constrained materials. Our aim is to find conditions under which the solutions of the boundary value problem are continuous functions of the elastic properties of the material. For this, we develop a unified formulation of the equations of elasticity which includes both the internally constrained (e.g. incompressible) and unconstrained (i.e. standard) linearly elastic materials. The usual condition of the coercivity of the elasticity tensor is replaced with a positive-semi-definiteness condition on the compliance tensor (Section 1). We show in the Appendix that the homogeneous and isotropic materials with Poisson ratio ν in the interval $(-1, \frac{1}{2}]$ and positive shear modulus satisfy our condition. In particular, $\nu = \frac{1}{2}$ corresponds to an incompressible material. This unified theory allows direct comparison between the solutions of the boundary value problems corresponding to internally constrained and unconstrained materials.

To illustrate, consider the following standard formulation of the mixed boundary value problem where S and u are the stress and displacement fields respectively.

$$\begin{aligned} \operatorname{div} S &= f \quad &&\text{in} \quad \Omega \\ C(Eu) &= S \quad &&\text{in} \quad \Omega \\ u &= \bar{u} \quad &&\text{on} \quad \partial_1\Omega \\ Sn &= \bar{s} \quad &&\text{on} \quad \partial_2\Omega \end{aligned} \tag{E-1}$$

Here f is the body force field and $\bar{u}$ and $\bar{s}$ are the prescribed surface displacement and traction on disjoint subsets $\partial_1\Omega$ and $\partial_2\Omega$ of the boundary. We have written Eu for the strain field corresponding to the displacement u, and C is the elasticity tensor.

The equations (E-1) apply when there are no internal constraints present and should be modified for internally constrained materials. For example, for incompressible materials (E-1) should be replaced by

$$\begin{aligned} \operatorname{div} S &= f \quad &&\text{in} \quad \Omega \\ C(Eu) &= S + pI \quad &&\text{in} \quad \Omega \\ \operatorname{div} u &= 0 \quad &&\text{in} \quad \Omega \\ u &= \bar{u} \quad &&\text{on} \quad \partial_1\Omega \\ Sn &= \bar{s} \quad &&\text{on} \quad \partial_2\Omega \end{aligned} \tag{E-2}$$

where an extra unkown (p) and an extra equation ($\operatorname{div} u = 0$) are introduced. See Gurtin [7], Section 58 for derivation and Pipkin [15] for further discussion. For a general treatment of internal constraints in continuum mechanics see Truesdell and Noll [21].

The equations (E-1) and (E-2), being of different types, complicate the procedure of comparison between their respective solutions. Specifically, they do not explicitely exhibit the intuitional notion of the "closeness" of the solutions of (E-1) to those of (E-2) when the material satisfying (E-1) is only slightly compressible. This problem is eliminated when we express the constitutive equation of the material as *strain as a function of stress*, so that we obtain

$$\begin{aligned} \operatorname{div} S &= f \quad &&\text{in} \quad \Omega \\ Eu &= K(S) \quad &&\text{in} \quad \Omega \\ u &= \bar{u} \quad &&\text{on} \quad \partial_1\Omega \\ Sn &= \bar{s} \quad &&\text{on} \quad \partial_2\Omega \end{aligned} \tag{E-3}$$

The *compliance tensor* K may or may not be invertible. When K^{-1} exists, (E-3) is equivalent to (E-1), but when K has a non-trivial null space the stress fields that fall within the null space of K do not contribute to the strain, so the material exhibits an internal constraint. For example, if K is such that

$$N(K) = \{pI : p \text{ an arbitrary scalar field on } \Omega\}$$

then the material is incompressible, because clearly arbitrary pressure fields do not affect the strain. In fact, for an isotropic and incompressible material with shear modulus μ, we have

$$K(S) = \frac{1}{2\mu}[S - \tfrac{1}{3}(\text{tr } S)I]$$

whence the null space given above.

The constitutive equation in (E-3), having the same form for both constrained and unconstrained materials, is best suited for our purposes. We will always assume that the internal constraints are specified via prescribing the null space of K. In this regard, we remark that Pipkin [15] has shown that the *a priori* prescription of a constraint of the type

$$\sum_{i,j} a_{ij}(x) \frac{\partial}{\partial x_j} u_i(x) = 0, \qquad x \in \Omega$$

with a given tensor field $A = (a_{ij})$, is equivalent to the statement

$$A \in N(K).$$

To study the dependence of the solutions of the problem (E-3) on the material constants, we proceed as follows. Let $\{K_\varepsilon\}_{\varepsilon \geq 0}$ be a family of compliance tensors such that $K_\varepsilon \to K_0$ as $\varepsilon \to 0$. In general, solutions $(u_\varepsilon, S_\varepsilon)$ of (E-3) corresponding to K_ε do not converge as $\varepsilon \to 0$. (The non-uniqueness of the solutions due to non-invertibility of K_ε or some special boundary conditions is not the issue as we shall see.) The result stated in Theorem 5.1 establishes conditions on the family $\{K_\varepsilon\}_{\varepsilon \geq 0}$ which imply the convergence $u_\varepsilon \to u_0$. There we also obtain a bound on the rate at which S_ε might blow up as $\varepsilon \to 0$. For the special case of homogeneous and isotropic materials approaching incompressibility, our results coincide with the ones obtained by Bramble and Payne [1], Kobelkov [11] and Mikhlin [13].

The paper is organized as follows. In Section 1 we introduce the basic notation and definitions and state the hypothesis on positive-semi-definiteness of K. In Section 2 we obtain some necessary conditions on the data of the problem for the existence of solutions. In particular, we make precise the conditions to be satisfied by the prescribed displacement $\bar{u}$ on part $\partial_1\Omega$ of the boundary such that it is compatible with the given internal constraints. In Section 3 we prove an orthogonal decomposition theorem for the Hilbert space of second order tensor fields defined on domain Ω. This result is used in Section 4 to reformulate the boundary value problem as a variational problem, then the existence of solutions is established. In Section 5 we study the problem of the dependence of solutions on the compliance tensor and in Section 6 we examine the implications of the general theory.

1. Definitions and assumptions

Let Ω be an open, bounded domain in $\mathbb{R}^m$ with locally Lipschitz boundary $\partial\Omega$. Let $\partial_1\Omega$ and $\partial_2\Omega$ be open (relative to $\partial\Omega$) subsets of $\partial\Omega$ such that $\partial_1\Omega \cap \partial_2\Omega = \varnothing$ and

$\overline{\partial_1\Omega \cap \partial_2\Omega} = \partial\Omega$. Also assume that the boundaries of $\partial_1\Omega$ as submanifolds of $\partial\Omega$ are smooth (locally Lipschitz will be sufficient). With $H^s(\Omega)$, $s \in \mathbb{R}$, we denote the space of restriction to Ω of the distributions u in $\mathbb{R}^m$ such that $(1+|\xi|)^s \hat{u}(\xi) \in L^2(\mathbb{R}^m)$, where $\hat{u}$ is the Fourier transform of u. For $u \in (H^1(\Omega))^m$, we let $\gamma(u)$ denote the *trace* of u on $\partial\Omega$. γ is a bounded, linear map of $[H^1(\Omega)]^m$ onto $[H^{1/2}(\partial\Omega)]^m$ and $\gamma(u) = u|_{\partial\Omega}$ whenever u is continuous up to the boundary.

Let $\mathcal{M}_m$ be the space of $m \times m$ *symmetric* matrices. Let

$$H = \{S : \Omega \to \mathcal{M}_m \qquad S_{ij} \in L^2(\Omega), \qquad i, j = 1, 2, \ldots, m\}.$$

Define the divergence operator

$$\operatorname{div} : H \to [\mathcal{D}'(\Omega)]^m$$

by

$$(\operatorname{div} S)_i = \sum_{j=1}^{m} \frac{\partial}{\partial x_j} S_{ij}, \qquad i = 1, \ldots, m.$$

We also set

$$\mathcal{V} = \{S \in H : \operatorname{div} S \in [L^2(\Omega)]^m\}.$$

H is a Hilbert space with the inner product

$$(S, F) = \sum_{i,j=1}^{m} \int_\Omega S_{ij}(x) F_{ij}(x)\, dx, \qquad S, F \in H.$$

and the corresponding norm denoted by $|\cdot|$. $\mathcal{V}$ is a Hilbert space with the norm

$$\|S\|^2 = |S|^2 + |\operatorname{div} S|^2_{L^2}$$

For $S \in \mathcal{V}$ one can give a meaning to the normal component $Sn|_{\partial\Omega}$, n = unit normal to the boundary, in the sense of the distributions. In fact, it can be shown, see Hünlich and Naumann [10] for example, that there exists a bounded, linear map $\pi : \mathcal{V} \to [H^{-1/2}(\partial\Omega)]^m$ such that $\pi(S) = Sn|_{\partial\Omega}$ if S is continuous in $\bar{\Omega}$. Moreover, the following version of Green's formula holds:

$$(S, Eu) = \langle \pi(S), \gamma(u) \rangle - (\operatorname{div} S, u)_{L^2} \tag{1.1}$$

for all $S \in \mathcal{V}$, $u \in [H^1(\Omega)]^m$ where $Eu = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric part of the gradient of u and where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-s}(\partial\Omega)$ and $H^s(\partial\Omega)$ as an extension of the inner product of L^2.

Now we state the mixed boundary value problem of linear elasticity as follows:

Problem 1. Let $f \in [L^2(\Omega)]^m$, $\bar{u} \in [H_0^{1/2}(\partial_1\Omega)]^m$, $\bar{s} \in [H^{-1/2}(\partial_2\Omega)]^m$. Let K be a bounded linear operator on H. Find $S \in \mathcal{V}$ and $u \in [H^1(\Omega)]^m$ such that

$$\operatorname{div} S = f \quad \text{in} \quad \Omega \tag{1.2}$$

$$Eu = KS \quad \text{in} \quad \Omega \tag{1.3}$$

$$\gamma(u) = \bar{u} \quad \text{on} \quad \partial_1\Omega \tag{1.4}$$

$$\pi(S) = \bar{s} \quad \text{on} \quad \partial_2\Omega \tag{1.5}$$

Remark. The last equality has of course to be interpreted in the sense of the distributions.

Remark. The spaces $H_0^{1/2}(\partial_1\Omega)$ and $H^{-1/2}(\partial_2\Omega)$ are understood in the sense of Hörmander [9]. $\bar{u}\in H_0^{1/2}(\partial_1\Omega)$ means that $\bar{u}$ is a function in $H^{1/2}(\partial\Omega)$ and has its support in $\overline{\partial_1\Omega}$. $\bar{s}\in H^{-1/2}(\partial_2\Omega)$ means that $\bar{s}$ is a distribution defined on $\partial_2\Omega$ and has an extension to $\partial\Omega$ which is in $H^{-1/2}(\partial\Omega)$. With these definitions, $H^{-1/2}(\partial_1\Omega)$ and $H_0^{1/2}(\partial_1\Omega)$ are duals of each other. These spaces are different from the spaces denoted by the same symbols in Lions and Magenes [12] for example. I am indebted to Professor M. S. Baouendi for bringing this point to my attention.

In Section 4, we reformulate Problem 1 as a minimization problem. To obtain the existence of minimizers, we will assume that the operator K satisfies some additional properties. We list these properties here after the following definitions.

Let

$$V=\{S\in\mathcal{V}:\operatorname{div} S=0 \quad \text{in } \Omega,\ \pi(S)=0 \text{ on } \partial_2\Omega\} \tag{1.6}$$

It is easily verified that this is a closed subspace of H. Let $N(K)$ denote the null-space of the operator K and let W be the orthogonal complement of $V\cap N(K)$ in H:

$$[V\cap N(K)]\oplus W=H \tag{1.7}$$

Let $P:H\to W$ be the orthogonal projection onto W. We will assume throughout this paper that the compliance operators K satisfy the following hypothesis:

(K1). K is a bounded, linear, self-adjoint operator on H.

(K2). K is non-negative, i.e.

$$(S, KS)\geqq 0 \qquad \forall S\in H.$$

(K3). K is semi-definite in the following sense; there exists a $\sigma>0$ such that

$$(S, KS)\geqq \sigma\,|PS|^2 \qquad \forall S\in V$$

Remarks.

(i) The hypothesis K3 is equivalent to

(K3′). There exists a $\sigma>0$ such that

$$(S, KS)\geqq \sigma\,|S|^2 \qquad \forall S\in V\cap W$$

(ii) The semi-definiteness hypothesis K3 is consistent with the possibility of the null-space of K being non-trivial, for if $S\in V\cap W$ as in K3′ and if simultaneously $S\in N(K)$, then $S=0$ by (1.7), so this does not contradict K3′.

(iii) The condition K3 is in particular satisfied in the case of homogeneous, isotropic elastic materials for which the elasticity tensor is positive-definite. To see this, we note that for such materials the compliance tensor K has the form

$$K(S)=\frac{1}{2\mu}\left[S-\frac{\nu}{\nu+1}(\operatorname{tr} S)I\right] \tag{1.8}$$

where μ is the shear modulus and ν is the Poisson's ratio. The positive-definiteness of elasticities is equivalent to $\mu>0$ and $-1<\nu<\frac{1}{2}$ (see Gurtin [7], Section 24). Clearly $N(K)=\{0\}$, so we obtain from (1.7) that $W=H$. It is easily verified that

$$(S, KS)=\frac{1}{2\mu}\left[|S|^2-\frac{\nu}{\nu+1}|\text{tr } S|^2_{L^2}\right] \geqq \frac{1}{2\mu}\xi(\nu)\,|S|^2 \tag{1.9}$$

where $\xi(\nu)=\min[1,(1-2\nu)/(1+\nu)]$, hence the condition K3′ is satisfied.

(iv) Notice that coefficient $\sigma=(1-2\nu)/2\mu(1+\nu)$ approaches zero as $\nu\to\frac{1}{2}$ i.e., when the material becomes almost incompressible. The positive-definiteness of K is lost in the limiting case of $\nu=\frac{1}{2}$. The semi-definiteness property K3, however, persists. The proof is given in the Appendix. Here we remark only that when $\nu=\frac{1}{2}$, we have

$$N(K)=\{S\in H: S=pI, \qquad p\in L^2(\Omega)\}.$$

Hence

$$V\cap N(K)=\{S\in H: S=cI, \qquad c\in\mathbb{R}\}$$

when $\partial_2\Omega=\varnothing$, and

$$V\cap N(K)=\{0\}$$

when $\partial_2\Omega\neq\varnothing$.

Thus

$$V\cap W=\left\{S\in H: \text{div } S=0 \quad \text{in} \quad \Omega, \qquad \int_\Omega \text{tr } S(x)\,dx=0\right\} \tag{1.10}$$

when $\partial_2\Omega=\varnothing$, and

$$V\cap W=V=\{S\in H: \text{div } S=0, \qquad \pi(S)=0 \quad \text{on} \quad \partial_2\Omega\} \tag{1.11}$$

when $\partial_2\Omega\neq\varnothing$.

2. Necessary conditions for the existence of solutions

Problem 1 will not have a solution in general without further qualifications. In this section we obtain a set of necessary conditions to be satisfied by the data of the problem if a solution is to exist.

(i) *The compatibility of displacement boundary conditions with internal constraints*

In the classical boundary-value problems of elasticity the boundary displacement data are only limited by their smoothness. In the presence of internal constraints however, additional compatibility conditions should be satisfied. To illustrate, consider a boundary value problem corresponding to an incompressible material and where the displacement is to be specified on the entire boundary. (In our notation $\partial_1\Omega = \partial\Omega$, $\partial_2\Omega = \emptyset$). Since the only admissible deformations are the volume preserving ones and since the volume of the deformed configuration is determined completely by the prescribed boundary data, this limits the class of the admissible boundary data to those which map the boundary $\partial\Omega$ of Ω onto surfaces which enclose a volume equal to the volume of Ω. The class of admissible boundary data in the case of an arbitrary internal constraint and when the displacement is prescribed only on part of the boundary is no more intuitively obvious. For a complete characterization of this class we recall the definition of the subspace V in Section 1:

$$V = \{S \in \mathcal{V} : \operatorname{div} S = 0 \quad \text{in} \quad \Omega,$$
$$\pi(S) = 0 \text{ on } \partial_2\Omega \text{ in the sense of the distributions}\}. \qquad (2.1)$$

and we prove

THEOREM 2.1. *A necessary condition for the existence of solutions of the Problem* 1 *is the following:*

$$\langle \pi(S'), \bar{u} \rangle = 0 \qquad \forall S' \in V \cap N(K) \qquad (2.2)$$

Proof. Let (S, u) be a solution of the Problem 1. Then for any $S' \in V \cap N(K)$ it follows from Green's formula (1.1) that

$$\begin{aligned} 0 = (KS', S) = (S', KS) &= (S', Eu) \\ &= \langle \pi(S'), \gamma(u) \rangle - (\operatorname{div} S', u)_{L^2} \\ &= \langle \pi(S'), \bar{u} \rangle + \langle \pi(S'), \gamma(u) - \bar{u} \rangle \end{aligned}$$

The last term is zero because $\gamma(u) - \bar{u} \in H_0^{1/2}(\partial_2\Omega)$ and $\pi(S') = 0$ on $\partial_2\Omega$. This proves the Theorem.

Remark. In the absence of internal constraints, $N(K) = \{0\}$, so (2.2) is trivially satisfied.

Remark. If K corresponds to an incompressible elastic material, then $K(I) = 0$ where I is the identity stress field (i.e. the homogeneous stress field which can be represented by the identity matrix everywhere), so $I \in N(K)$. When $\partial_2\Omega = \emptyset$ we also have $I \in V$, so in this case (2.2) implies that

$$\langle \pi(I), \bar{u} \rangle = \int_{\partial\Omega} \bar{u} \cdot n \, da = 0,$$

that is, the mapping $\bar{u}$ does not change the enclosed volume, as expected.

(ii) *The compatibility of the boundary tractions with the body forces*

This condition is necessary only when $\partial_1\Omega = \emptyset$. We state it here in a somewhat general form so that our result applies to domains of any dimension. In particular for three dimensional domains the familiar formula in terms of the cross-product emerges after a straightforward manipulation. For this, let

$$\mathcal{R} = \{\rho : \Omega \to \mathbb{R}^m \mid \rho(x) = b + Wx,\ b \text{ a constant } m\text{-vector, } W \text{ a constant skew-symmetric matrix}\}.$$

This is the class of the (infinitesimal) rigid rotations of Ω. Then we have:

THEOREM 2.2. *Let $\partial_1\Omega = \emptyset$ in Problem 1. Then a necessary condition for the existence of solutions is the following:*

$$(f, \rho)_{L^2} = \langle \bar{s}, \gamma(\rho) \rangle, \qquad \forall \rho \in \mathcal{R} \tag{2.3}$$

Proof. Multiply (1.2) by ρ, use the fact that $C^\infty(\bar{\Omega})$ is dense in $\mathcal{V}$, approximate S, apply the divergence theorem and pass to the limit.

In the rest of this paper we will always assume that $\partial_1\Omega \neq \emptyset$. The development corresponding to the case $\partial_1\Omega = \emptyset$ follows in parallel lines with standard procedures established in the literature. See for example any of the following: [3], [4], [8].

3. An orthogonal decomposition theorem

In this section we prove some propositions which are needed in later sections. The Korn inequality (3.1) for mixed boundary value problems is well known, however we have included its derivation here for the sake of completeness. The main result of this section is the orthogonal decomposition formula (3.5). We start with:

PROPOSITION 3.1. *Suppose $\partial_1\Omega \neq \emptyset$. Let $u \in [H^1(\Omega)]^m$, $u = 0$ on $\partial_1\Omega$, $Eu = 0$ in Ω. Then $u = 0$ in Ω.*

Proof. It is easy to see that $Eu = 0$ implies that $u \in \mathcal{R}$ (see the footnote on page 442 of [5]), so $u = b + Wx$ for some b and W. Let

$$B = \{x \in \mathbb{R}^m : u(x) = 0\}$$

Thus $\partial_1\Omega \subset B$. Since W is skew-symmetric, its range is an even-dimensional subspace of $\mathbb{R}^m$ regardless of the parity of m, so B is a manifold of dimension $m - 2k$ for some integer k. If $k > 0$, we have $\dim B < m - 1$ so B cannot contain the $m - 1$ dimensional manifold $\partial_1\Omega$. If $k = 0$, we have $B = \mathbb{R}^m$, hence $u(x) \equiv 0$, thus proving the proposition.

PROPOSITION 3.2. *Let $\partial_1\Omega \neq \emptyset$. Then there exists an $\alpha > 0$ such that*

$$|Eu|_H \geqq \alpha\, |u|_{H^1(\Omega)} \qquad \forall u \in [H^1(\Omega)]^m \quad \text{with} \quad u = 0 \text{ on } \partial_1\Omega. \tag{3.1}$$

Proof. (3.1) follows from the general Korn inequality

$$|Eu|_H^2 + |u|_{L^2}^2 \geqq c\, |u|_{H^1(\Omega)}^2 \qquad \forall u \in [H^1(\Omega)]^m \tag{3.2}$$

(see [3], [5], [8], [14] for proof).

Indeed, if (3.1) is false, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ in $[H^1(\Omega)]^m$ with $u_n = 0$ on $\partial_1\Omega$ such that $|u_n|_{H^1(\Omega)} = 1$ while $|Eu|_H \to 0$. Passing to a subsequence, there is a $u \in [H^1(\Omega)]^m$ such that

$$u_n \rightharpoonup u \text{ weakly in } H^1(\Omega) \tag{3.3}$$

so $\gamma(u_n) \rightharpoonup \gamma(u)$ weakly in $H^{1/2}(\partial\Omega)$ and $Eu_n \rightharpoonup Eu$ weakly in H. Thus $Eu = 0$ in Ω and $u = 0$ on $\partial_1\Omega$. Thus by the Proposition 3.1 $u = 0$ in Ω. Thus (3.3) and Rellich's Lemma imply that $u_n \to 0$ in $L^2(\Omega)$. Then from (3.2) we get $u_n \to 0$ in $H^1(\Omega)$ which is a contradiction. This completes the proof.

Now let's define

$$U = \{F \in H : F = Eu, \quad u \in [H^1(\Omega)]^m, \quad u = 0 \text{ on } \partial_1\Omega\} \tag{3.4}$$

This is a closed subspace of H as it follows immediately from (3.1). Now we state

THEOREM 3.1 (The orthogonal decomposition of H). *The closed subspaces V and U of H defined in* (2.1) *and* (3.4) *are mutually orthogonal and*

$$U \oplus V = H. \tag{3.5}$$

Proof. First we prove that $U^\perp \subset V$. Let $S \in H$ be such that $S \perp U$. so

$$(S, Eu) = 0 \qquad \forall u \in [H^1(\Omega)]^m, \qquad u = 0 \text{ on } \partial_1\Omega.$$

Thus in particular for any $\phi \in C_0^\infty(\Omega)$ we have $(S, E\phi) = 0$, hence $(\operatorname{div} S, \phi) = 0$ so $\operatorname{div} S = 0$ in the sense of distributions. Hence $S \in \mathcal{V}$. and $\pi(S)$ is defined. From Green's formula (1.1) we get

$$\langle \pi(S), u \rangle = 0 \qquad \forall u \in [H^1(\Omega)]^m, \quad u = 0 \text{ on } \partial_1\Omega$$

Since u is arbitrary on $\partial_2\Omega$, this implies that $\pi(S) = 0$ on $\partial_2\Omega$, so $S \in V$. The inclusion $U^\perp \supset V$ is verified in a similar way, and since U is closed, it completes the proof.

Remark. The orthogonal decomposition (3.5) is analogous to the familiar problem in fluid mechanics of decomposing a vector field into two mutually orthogonal solenoidal and irrotational vector fields, cf. Temam [19].

COROLLARY 3.1. *Let $\bar{u} \in [H_0^{1/2}(\partial_1\Omega)]^m$ and let $F \in H$ be such that*

$$(S, F) = \langle \pi(S), \bar{u} \rangle \qquad \forall S \in V \tag{3.6}$$

Then there exists a $u \in [H^1(\Omega)]^m$ such that $u = \bar{u}$ on $\partial_1\Omega$ and $F = Eu$.

Remark. Recall that $\bar{u} \in [H_0^{1/2}(\partial_1\Omega)]^m$ means that $\bar{u} = 0$ on $\partial_2\Omega$.

Proof. Let $u' \in [H^1(\Omega)]^m$ be an extension of $\bar{u}$ to Ω. Thus for any $S \in V$ we obtain from (1.1):

$$(S, Eu') = \langle \pi(S), \bar{u} \rangle$$

Subtract this from (3.6) to get

$$(S, F - Eu') = 0 \qquad \forall S \in V$$

So $F-Eu'\in U$ by (3.5). Hence there exists a $u''\in[H^1(\Omega)]^m$ with $u''=0$ on $\partial_1\Omega$ such that $F-Eu'=Eu''$. Let $u=u'+u''$. Then $u=\bar{u}$ on $\partial_1\Omega$ and $F=Eu$ as required.

Remark. The idea behind Theorem 3.1 and the corollary is old and goes back to the nineteenth century. (See the footnote on page 117 of [7].) The formulation and proof as given here however seems to be new. Even up to relatively recently the proofs available for the theorem were more or less heuristic arguments, cf. [2], [18]. The first rigorous proof seems to have been given by Gurtin [6] under the additional assumptions of the convexity of $\partial_1\Omega$ and simple connectedness of Ω. For the case $\partial_1\Omega=\varnothing$, Ting [20] has recently extended the orthogonal decomposition (3.5) to tensor fields defined on compact Riemannian manifolds.

4. The variational formulation

The orthogonal decomposition theorem of the previous section allows us to reformulate Problem 1 as an equivalent variational problem. The formulation is standard and is known as the minimum principle for the complementary energy function (see Gurtin [7], Sections 34 and 36). We state the principle formally after the following definitions.

Let $\bar{u}\in[H_0^{1/2}(\partial_1\Omega)]^m$, $\bar{s}\in[H^{-1/2}(\partial_2\Omega)]^m$ and $f\in[L^2(\Omega)]^m$ be as in the statement of Problem 1. Define $I:\mathscr{V}\to\mathbb{R}$ by

$$I(S)=\tfrac{1}{2}(S, KS)-\langle\pi(S), \bar{u}\rangle \tag{4.1}$$

and let

$$V_{f,\bar{s}}=\{S\in :\operatorname{div} S=f \text{ in } \Omega, \qquad \pi(S)=\bar{s} \text{ on } \partial_2\Omega\}. \tag{4.2}$$

I is the *complementary energy function* for Problem 1. Also notice that $V_{f,\bar{s}}$ is a parallel translation of the subspace V defined in (1.6). Now consider:

Problem 2. For $\bar{u}$, $\bar{s}$, and f given as above find a minimizer of the restriction of I to $V_{f,s}$.

The next two propositions show that Problem 1 and Problem 2 are "equivalent".

PROPOSITION 4.1. *Let* (S, u) *be a solution of Problem* 1. *Then* S *is a solution of Problem* 2.

Proof. For an arbitrary $S'\in V_{f,\bar{s}}$ we have

$$I(S')-I(S)=\tfrac{1}{2}(S'-S, K(S'-S)) + (S'-S, KS)-\langle\pi(S'-S), \bar{u}\rangle \tag{4.3}$$

The last two terms are equal by Green's Theorem (1.1) because $KS=Eu$, so

$$I(S')-I(S)=\tfrac{1}{2}(S'-S, K(S'-S)) \qquad \forall S'\in V_{f,\bar{s}}.$$

This is non-negative by the hypothesis K2, hence S is a minimizer.

PROPOSITION 4.2. *Let S be a solution of Problem* 2. *Then there exists a function $u \in [H^1(\Omega)]^m$ such that the pair (S, u) is a solution of Problem* 1.

Proof. For an arbitrary $S' \in V_{f,\bar{s}}$ we have from (4.3) that:

$$\tfrac{1}{2}(S-S, K(S'-S)) + (S'-S, KS) - \langle \pi(S'-S), \bar{u} \rangle \geqq 0.$$

Hence the linear part is zero:

$$(S'-S, KS) = \langle \pi(S'-S), \bar{u} \rangle \qquad \forall S' \in V_{f,\bar{s}}$$

Now $S'-S$ is an arbitrary element of the subspace V, so by the Corollary 3.1, there exists a function $u \in [H^1(\Omega)]^m$ with $u = \bar{u}$ on $\partial_1\Omega$, such that $KS = Eu$. Thus the pair (S, u) is a solution of Problem 1.

Propositions (4.1) and (4.2) show the equivalence of Problems 1 and 2. Thus in the following we will limit our discussion to the variational problem only.

The minimization in Problem 2 is carried out over the set $V_{f,\bar{s}}$ which is not a subspace. By a change of variables, we can have a problem of minimization over the subspace V which is more convenient for our purposes. For this we let S^* be the point in $V_{f,\bar{s}}$ with the smallest H-norm. Then any point $S \in V_{f,\bar{s}}$ can be written as

$$S = S^* + S', \qquad S' \in V. \tag{4.4}$$

Thus yielding:

$$I(S) = I(S^*) + \tfrac{1}{2}(S', KS') - \langle \pi(S'), \bar{u} \rangle + (S', KS^*), \qquad S \in V_{f,\bar{s}}, \qquad S' \in V$$

The linear functional $S' \mapsto \langle \pi(S'), \bar{u} \rangle$ is continuous when restricted to the subspace V of H as it follows from the continuity of the trace operator:

$$\begin{aligned} |\langle \pi(S'), \bar{u} \rangle| &\leqq |\pi(S')|_{H^{-1/2}(\partial\Omega)} |\bar{u}|_{H^{1/2}(\partial\Omega)} \\ &\leqq c[|S'|_H^2 + |\text{div } S'|_{L^2}^2]^{1/2} |\bar{u}|_{H_0^{1/2}(\partial_1\Omega)} \\ &= c\, |\bar{u}|_{H_0^{1/2}(\partial_1\Omega)} |S'|_H \end{aligned}$$

We have let div $S' = 0$ because $S' \in V$. Now by Riecz's representation Theorem, there exists an $\eta \in V$ such that

$$\langle \pi(S'), \bar{u} \rangle = (S', \eta), \qquad S' \in V. \tag{4.5}$$

Let

$$\xi = \eta - KS^* \tag{4.6}$$

to get

$$I(S) = I(S^*) + \tfrac{1}{2}(S', KS') - (S', \xi).$$

Define $J : V \to \mathbb{R}$ by

$$J(S') = \tfrac{1}{2}(S', KS') - (S', \xi), \qquad S' \in V \tag{4.7}$$

Hence

$$I(S) = I(S^*) + J(S'). \tag{4.8}$$

Hence the problem of the minimization of I on $V_{f,\bar{s}}$ reduces to the problem of the minimization of J on V. Thus we introduce:

Problem 3. Let $J: V \to \mathbb{R}$ be defined as above. Find the minimizers of J on V.

Obviously $S' \in V$ is a minimizer of J in V if and only if $S = S^* + S'$ is a minimizer of I in $V_{f,\bar{s}}$. Now we can state:

THEOREM 4.1. *Let the operator K satisfy the properties* K1, K2, K3 *and suppose that $\bar{u}$ satisfies the compatibility condition* (2.2). *Then there exists a unique $u \in [H^1(\Omega)]^m$ and a unique $S_0 \in V_{f,\bar{s}} \cap W$ such that $\{u, S_0\}$ is a solution of Problem* 1. *Moreover the general solution of Problem* 1 *is given by $\{u, S_0 + \hat{S}\}$ where $\hat{S} \in V \cap N(K)$ is arbitrary.*

Proof. Since we have established the equivalence of the Problems 1, 2, and 3, it will suffice to demonstrate the existence of solutions of Problem 3. Theorem (3.1) of [17] is directly applicable to this case. The only hypothesis to verify is that ξ in (4.6) lies in the subspace W, that is, it has to be shown that

$$(S, \xi) = 0 \qquad \forall S \in V \cap N(K). \tag{4.9}$$

But from (2.2) we have

$$\langle \pi(S), \bar{u} \rangle = 0 \qquad \forall S \in V \cap N(K).$$

Hence by (4.5),

$$(S, \eta) = 0 \qquad \forall S \in V \cap N(K).$$

Then by the self-adjointness of K and that $S \in N(K)$:

$$(S, \eta - KS^*) = 0$$

Substitute ξ from (4.6) to obtain (4.9). Thus by Theorem 3.1 of [17] there exists a unique $S' \in V \cap W$ which minimizes J on $V \cap W$. All other minimizers of J in V are of the form $S' + \hat{S}$, $\hat{S} \in V \cap N(K)$. Let $S_0 = S' + S^*$. Since $S^* \perp V \cap N(K)$, then S_0 is the unique minimizer of I on $V_{f,\bar{s}} \cap W$. Hence all minimizers of I in $V_{f,\bar{s}}$ are of the form $S = S_0 + \hat{S}$. By Proposition 4.2, corresponding to each minimizer S of I there exists a function u such that the pair $\{u, S\}$ is a solution of Problem 1. The function u is always determined uniquely regardless of the non-uniqueness of S (recall that we are considering the case $\partial_1\Omega \neq \emptyset$). To see this, let $\{u, S_1\}$ and $\{u_2, S_2\}$ be two solutions of Problem 1. Then $S_1 = S_0 + \hat{S}_1$ and $S_2 = S_0 + \hat{S}_2$ with $\hat{S}_1$ and $\hat{S}_2 \in V \cap N(K)$ as proved above. Hence $S_1 - S_2 = \hat{S}_1 - \hat{S}_2 \in V \cap N(K)$. Now by (1.3), $E(u_1 - u_2) = K(S_1 - S_2) = 0$ and since $u_1 - u_2$ on $\partial_1\Omega$, Proposition 3.1 implies that $u_1 = u_2$.

Remark. Let $\{u, S_0\}$ be the particular solution defined above and let $\{u, S\}$ be any other solution. Then it is easy to see that $(S - S_0, S_0) = 0$. Therefore S_0 has the smallest norm among all solutions of the problem. We will refer to the solution $\{u, S_0\}$ as the *principal solution* of Problem 1, and to S_0 as the *principal stress*. In the next section we will investigate the dependence of the principal solution on the compliance operator K.

Remark. The theorem above and in fact most of the propositions in this paper are stated in somewhat a more general context than is actually required for problems of elasticity. The compliance tensor K in elasticity is a pointwise acting operator, that is, in terms of components, K can be written as follows:

$$(KS)_{ij}(x) = \sum_{k,l} K_{ijkl}(x) S_{ij}(x) \tag{4.10}$$

$K_{ijk1}(x)$ being fourth order tensors defined at each point $x \in \Omega$. Our treatment of K as a linear operator on the Hilbert space H includes (4.10) as well as operators which are not defined through their pointwise action.

The non-unique solutions for the stress field are characterized in Theorem 4.1. In particular, we have the immediate

COROLLARY 4.1. *The solutions of Problem* 1 *are unique if and only if* $V \cap N(K) = \{0\}$.

Next, observe that

$$V \cap N(K) = \{S \in H : \operatorname{div} S = 0, \qquad \pi(S) = 0 \text{ on } \partial_2\Omega, \qquad KS = 0\}.$$

Consider the standard case of elasticity where the operator K is defined pointwise as in (4.10). Let A be a given second order symmetric tensor field on Ω and let $A \in L^\infty(\Omega)$. Suppose that the internal constraint is specified as an *a priori* restriction of the admissible strain fields by

$$\sum_{i,j} A_{ij}(x)(Eu)_{ij}(x) = 0 \quad \text{a.e. in } \Omega$$

Pipkin [15] has shown that in this case

$$\sum_{k,l} K_{ijkl}(x) A_{kl}(x) = 0 \quad \text{a.e. in } \Omega \tag{4.11}$$

hence $A \in N(K)$. Thus

$$N(K) = \{S \in H : S(x) = \phi(x)A(x) \quad \text{a.e. in } \Omega\}$$

where $\phi \in L^2(\Omega)$ is an arbitrary scalar field on Ω.
Hence we have

COROLLARY 4.2. *When K is as* (4.10) *and* (4.11), *Problem* 1 *has a unique solution if and only if the system*

$$\begin{aligned} \operatorname{div}(\phi A) &= 0 \quad \text{in} \quad \Omega \\ \pi(\phi A) &= 0 \quad \text{on} \quad \partial_2\Omega \end{aligned}$$

has only the trivial solution $\phi \equiv 0$.

Remark. $A(x) \equiv I$ defines an incompressible material. Corollary 4.2 in this case yields the expected conclusion that the solutions of Problem 1 are unique if and only if $\partial_2\Omega \neq \varnothing$.

5. Dependence of solutions on the compliance tensor

The question of dependence of solutions of Problem 1 on the data f, $\bar{s}$, $\bar{u}$ and the compliance operator K is not well posed because of the non-uniqueness of the component S of the solution $\{u, S\}$. The question becomes meaningful however if we regard S as an element in the quotient space $V_{f,\bar{s}}/[V \cap N(K)]$. By the construction of S in Theorem 4.1 it suffices to concentrate only on the uniquely defined principal solution of the problem. It is easy to show that the principal solution $\{u, S_0\}$ depends continuously on the boundary conditions $\bar{u}$ and $\bar{s}$ and the body force f (see Proposition 5.1 below). The dependence of $\{u, S_0\}$ on the compliance operator K is not so trivial and in fact is not continuous in general. We will apply some results from [17] to obtain the continuity results stated below. For the sake of completeness we also include the following proposition which is an immediate consequence of the Closed Graph Theorem.

PROPOSITION 5.1. *Consider the Hilbert spaces* $\mathcal{A} = [H_0^{1/2}(\partial_1\Omega)]^m \times [H^{-1/2}(\partial_2\Omega)]^m \times [L_2(\Omega)]^m$ *and* $\mathcal{B} = [H^1(\Omega)]^m \times H$ *both with the corresponding product topologies. The set* $\mathcal{A}'$ *of all* $\{\bar{u}, \bar{s}, f\} \in \mathcal{A}$ *such that Problem* 1 *has a solution for a fixed field K is a closed subspace of* $\mathcal{A}$ *(because* $\bar{u}$ *has to satisfy the compatibility condition* (2.2)). *Define the linear map* $T: \mathcal{A}' \to \mathcal{B}$ *by* $T\{\bar{u}, \bar{s}, f\} = \{u, S\}$ *where* $\{u, S\}$ *is the principal solution of Problem* 1. *Then T is continuous.*

To study the dependence of the principal solution on the operator K we follow the idea outlined in the introductory section of the paper. Thus let $\{K_\varepsilon\}_{\varepsilon \geq 0}$ be a family of compliance tensors, each K_ε satisfying the conditions K1, K2, and K3 of Section 1. Suppose that $K_\varepsilon \to K_0$ as $\varepsilon \to 0$. Let $\{u_\varepsilon, S_\varepsilon\}$ denote the principal solution of Problem 1 for each ε. The main purpose in this section is to obtain sufficient conditions on the family $\{K_\varepsilon\}$ which imply that $u_\varepsilon \to u_0$. We first restate the conditions equivalent to K1–K3 of Section 1 as they apply to each K_ε. Thus analogous to (1.7), for each $\varepsilon \geqq 0$ we let W_ε be the orthogonal complement of $V \cap N(K_\varepsilon)$ in H and let P_ε be the orthoprojection onto W. Also let $Q_\varepsilon = I - P_\varepsilon$. We assume that for each $\varepsilon \geqq 0$ there exists a $\sigma_\varepsilon > 0$ such that

$$(S, K_\varepsilon S) \geqq \sigma_\varepsilon\ |P_\varepsilon S|^2 \qquad \forall S \in V \tag{5.1}$$

This, together with the compatibility condition:

$$\langle \pi(S), \bar{u}\rangle = 0, \qquad \forall S \in V \cap N(K_\varepsilon), \qquad \forall \varepsilon \geqq 0 \tag{5.2}$$

fulfills the requirements of Theorem (4.1) thus ensuring the existence of solutions. Next, we introduce additional hypothesis as they are required by Theorem 4.1 of [17].

The coefficients σ_ε in (5.1) are not bounded away from zero in general as $\varepsilon \to 0$. A better estimate on the order of magnitude of σ_ε can possibly be obtained however, when S is restricted to $V \cap N(K_0)$. Thus let $\lambda_\varepsilon > 0$ be such that

$$(S, K_\varepsilon S) \geqq \lambda_\varepsilon\ |P_\varepsilon S|^2 \qquad \forall S \in V \cap N(K_0) \tag{5.3}$$

(Of course λ_ε might also approach zero as $\varepsilon \to 0$, but the rate of decay might possibly be slower than that of the σ_ε.)

Also notice that $K_0 Q_0 = 0$ by the definition of the projection Q_0. So for any $S \in V$ we have

$$|S', K_\varepsilon Q_0 S| \leqq \|K_\varepsilon - K_0\| \, |S'| \, |Q_0 S| \tag{5.4}$$

In particular, when S' is restricted to be $S' = P_0 S$, this inequality might be sharpened by replacing the coefficient $\|K_\varepsilon - K_0\|$ by one of a faster decay. So let τ_ε be such that

$$|(P_0 S, K_\varepsilon Q_0 S)| \leqq \tau_\varepsilon \, |P_0 S| \, |Q_0 S| \qquad \forall S \in V \tag{5.5}$$

The notation and definitions of σ_ε and τ_ε are given exactly in such a way that the Theorem 4.1 of [17] applies. We state here the conclusion:

THEOREM 5.1. *Let $\{K_\varepsilon\}_{\varepsilon \geqq 0}$ be the family of compliance tensor operators as defined above. Let $u \in [H_0^{1/2}(\partial_1 \Omega)]^m$ satisfy* (5.2) *and let $\{u_\varepsilon, S_\varepsilon\}$ be the principal solution of Problem* 1 *for each $\varepsilon \geqq 0$. Then if*

$$\tau_\varepsilon^2 / \lambda_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{5.6}$$

the solutions of the mixed boundary value problem of elasticity converge in the following sense.

$$u_\varepsilon \to u_0 \quad \text{in} \quad H^1(\Omega) \tag{5.7}$$

$$P_0 S_\varepsilon \to S_0 \quad \text{in} \quad H \tag{5.8}$$

$$\lambda_\varepsilon^{1/2} Q_0 S_\varepsilon \text{ is bounded independently of } \varepsilon. \tag{5.9}$$

Remark. (5.9) establishes an upper bound on the growth of the stress field. We can equivalently write

$$\|Q_0 S_\varepsilon\| \leqq \lambda_\varepsilon^{-1/2}$$

or

$$\|S_\varepsilon\| \leqq \lambda_\varepsilon^{-1/2}.$$

6. Applications

Here we consider a few specific problems of elasticity and examine the implications of Theorem 5.1.

(i) *Elasticity in the absence of internal constraints*

In this case each member of the family $\{K_\varepsilon\}_{\varepsilon \geqq 0}$ of compliance operator is inventible, so $K_\varepsilon \to K_0$ implies that $K_\varepsilon^{-1} \to K_0^{-1}$. Since

$$(S, KS) \geqq \frac{1}{\|K_\varepsilon^{-1}\|} |S|^2, \qquad \forall S \in H,$$

in comparison with (5.3) we take

$$\lambda_\varepsilon = \|K_\varepsilon^{-1}\|^{-1}$$

Also in comparison with (5.4) and (5.5) we take

$$\tau_\varepsilon = \|K_\varepsilon - K_0\|$$

Then obviously the condition $\tau_\varepsilon^2/\lambda_\varepsilon \to 0$ of Theorem 5.1 is satisfied, so the displacement field converges according to (5.7). Also observe that in this case $N(K_\varepsilon) = \{0\}$, $W_\varepsilon = H$ and the projection P_ε is the identity operator. Thus (5.8) implies that

$$S_\varepsilon \to S_0 \quad \text{in} \quad H$$

i.e. the stress field also converges in the case. Of course these results are well known from the standard theory of the linear elasticity.

(ii) *Approximation of internally constrained materials with materials which are not internally constrained*

Suppose that K_0 is the compliance tensor for an internally constrained material and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compliance tensor approximating K_0, i.e. $K_\varepsilon \to K_0$ as $\varepsilon \to 0$. Suppose that each K_ε with $\varepsilon > 0$ is invertible, so it corresponds to a material which is not internally constrained. (For an explicit construction of such a family see Rostamian [16].) Then the condition (5.6) of Theorem (5.1) is satisfied if

$$\|K_\varepsilon^{-1}\| \, \|K_\varepsilon - K_0\|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{6.1}$$

That is, the solutions will converge in the sense of Theorem (5.1) provided that the convergence of the approximating operators is faster than the rate at which $K_\varepsilon^{1/2}$ becomes singular.

(iii) *Non-negative perturbations of a singular operator*

It has been established in [17] that if the family $\{K_\varepsilon\}_{\varepsilon>0}$ of the elasticity operators consists of non-negative perturbations of the operator K_ε, in the sense that

$$(S, (K_\varepsilon - K_0)S) \geqq 0 \qquad \forall S \in H, \tag{6.2}$$

then the solutions of the problem of elasticity will converge in the sense of (5.7), (5.8), and (5.9). No invertibility conditions are required of the operators K_ε in this case.

(iv) *Isotropic and homogeneous materials*

The compliance operator for an isotropic and homogeneous material is given by

$$K_\nu(S) = \frac{1}{2\mu}\left[S - \frac{\nu}{\nu+1}(\operatorname{tr} S)I\right], \qquad S \in H \tag{6.3}$$

where μ and ν are the shear modulus and the Poisson ratio respectively (see Gurtin [7], Section 22). Suppose that $\mu > 0$. Then K_ν is non-negative if and only if

$-1<\nu \leqq \frac{1}{2}$. Actually K_ν is positive-definite when $-1<\nu<\frac{1}{2}$. $K_{1/2}$ is not invertible, as we have

$$N(K_{1/2})=\{pI : p\in L^2(\Omega)\}. \tag{6.4}$$

which expresses the fact that for the Poisson ratio $\nu=\frac{1}{2}$, the hydrostatic pressure does not produce a strain in the material. The internal constraint in this case is the constraint of *incompressibility*. The positive-semi-definiteness condition (K3) is satisfied for $K_{1/2}$ as it is shown in the Appendix.

Suppose that we approximate an incompressible material ($\nu=\frac{1}{2}$) with a family of compressible materials with $\nu\in(-1,\frac{1}{2})$. We see that

$$K_\nu \to K_0 \quad \text{as} \quad \nu\to\tfrac{1}{2}-0 \tag{6.5}$$

where the convergence is in the operator norm of the space H. The convergence of the corresponding solutions of the elasticity problem follows from the general results of either part (ii) or (iii) of this section. For example, the condition of part (iii) is satisfied because K_ν is a non-negative perturbation of $K_{1/2}$. To see this, we write

$$K_{1/2}(S)=\frac{1}{2\mu}\left[S-\tfrac{1}{3}\operatorname{tr}(S)I\right] \tag{6.6}$$

which is obtained from (6.3) by letting $\nu=\frac{1}{2}$. Thus we get

$$(K_\nu-K_{1/2})(S)=\frac{1-2\nu}{6\mu(1+\nu)}(\operatorname{tr} S)I.$$

Hence

$$(S,(K_\nu-K_{1/2})S)=\frac{1-2\nu}{6\mu(1+\nu)}|\operatorname{tr} S|^2_{L^2(\Omega)}\geqq 0.$$

Thus (6.2) is satisfied, so the conclusions of Theorem (5.1) follow.

The conditions of part (ii) is also fulfilled in this case. Notice that from (6.6) we get

$$\|K_\nu-K_{1/2}\|=\frac{1-2\nu}{6\mu(1+\nu)}.$$

On the other hand, for $\nu\neq\frac{1}{2}$ we have

$$K_\nu^{-1}(S)=2\mu\left[S+\frac{\nu}{1-2\nu}(\operatorname{tr} S)I\right]$$

so

$$\|K_\nu^{-1}\|=\frac{2\mu(1+\nu)}{1-2\nu}$$

Hence

$$\|K_\nu^{-1}\|\,\|K_\nu-K_{1/2}\|^2=\frac{1-2\nu}{18\mu(1+\nu)}\to 0 \quad \text{as} \quad \nu\to\tfrac{1}{2}.$$

Thus the condition (6.1) is satisfied and the conclusions of Theorem 5.1 follow. We obtain

$$u_\nu \to u_{1/2} \quad \text{as} \quad \nu \to \tfrac{1}{2} \tag{6.7}$$

$$P_{1/2} S_\nu \to S_{1/2} \quad \text{as} \quad \nu \to \tfrac{1}{2} \tag{6.8}$$

When the traction is specified at least on part of the boundary, i.e. when $\partial_2\Omega \neq \varnothing$, we see from (1.11) that $P_{1/2}$ is the identity map, hence (6.8) implies

$$S_\nu \to S_{1/2} \quad \text{as} \quad \nu \to \tfrac{1}{2}$$

that is, the stress fields S_ν converge to the unique stress field corresponding to the incompressible material. In the case of the displacement boundary value problem, i.e. when $\partial_2\Omega = \varnothing$, we see from (1.10) that

$$P_{1/2}(S) = S - \frac{1}{3}\left[\int_\Omega \operatorname{tr}(S)\, dx\right] I$$

Hence (6.8) in this case becomes

$$S_\nu - \frac{1}{3}\left[\int_\Omega \operatorname{tr}(S_\nu)\, dx\right] I \to S_{1/2}$$

where $S_{1/2}$ is the principal solution corresponding to the incompressible material. We remark that $S_{1/2}$ being the principal solution, it satisfies $P_{1/2}(S_{1/2}) = S_{1/2}$, hence

$$\int_\Omega \operatorname{tr} S_{1/2}(x)\, dx = 0.$$

The problem of the dependence of solutions of boundary value problems of elasticity on the Poisson ratio ν with ν near $\frac{1}{2}$ has been studied with special methods in the papers of Bramble and Payne [1], Kobelkov [11], Mikhlin [13], Pipkin [15], and Rostamian [16]. The discussion in the first three papers is exclusively developed for the case of isotropic and homogeneous materials where the only possible internal constraint is the incompressibility. Because of the special nature of the constitutive equations, some of the results there are stronger than what we have obtained here from the general theory. In [1], in addition to the convergence in $H^1(\Omega)$ of the displacement field, the pointwise convergence is also proved. In [11], it is shown that in unbounded domains, the displacement field converges in the sense of $W^{1,q}(\Omega)$, $1 \leqq q < 2$. Moreover, when Ω is the square $0 \leqq x_1, x_2 \leqq \pi$, the convergence is actually in $W^{2,q}(\Omega)$, $1 < q < \infty$. In [13], the convergence in $H^1(\Omega)$ of the solutions is obtained via the spectral analysis of the elasticity operator. Moreover it is explicitly demonstrated that the corresponding stress fields do not converge in general.

Appendix

Here we verify the positive-semi-definiteness condition (K3) of Section 1 for an isotropic, homogeneous, and incompressible material. The compliance operator is obtained from (1.8) by letting $\nu=\frac{1}{2}$:

$$K(S)=\frac{1}{2\mu}[S-\tfrac{1}{3}(\operatorname{tr} S)I]. \tag{A.1}$$

In view of (1.10) and (1.11), to verify (K3) we have only to show that

PROPOSITION. *There exists a constant $\sigma>0$ such that*

$$|S|^2-\tfrac{1}{3}|\operatorname{tr} S|^2_{L^2}\geqq\sigma|S|^2 \tag{A.2}$$

whenever

$$S\in\left\{S\in H:\operatorname{div} S=0 \quad in \quad \Omega, \qquad \int_\Omega \operatorname{tr} S(x)\,dx=0\right\} \tag{A.3}$$

if $\partial_2\Omega=\varnothing$, *or*

$$S\in\{S\in H:\operatorname{div} S=0 \quad in \quad \Omega, \qquad \pi(S)=0 \quad on \quad \partial_2\Omega\} \tag{A.4}$$

if $\partial_2\Omega\neq\varnothing$.

Proof. We prove by contradiction. Suppose (A.2) does not hold. Then there exists a sequence $\{S^n\}_{n=1}^\infty$ in H such that

(i) $|S^n|^2-\frac{1}{3}|\operatorname{tr} S^n|^2\leqq\dfrac{1}{n}|S^n|^2$

(ii) $\operatorname{div} S^n=0$

(iii) $\pi(S^n)=0 \quad \text{on} \quad \partial_2\Omega \quad \text{if} \quad \partial_2\Omega\neq\varnothing$

or

$$\int_\Omega \operatorname{tr} S^n(x)\,dx=0 \quad \text{if} \quad \partial_2\Omega=\varnothing.$$

Without loss of generality we take $|S^n|=1$, and if necessary, pass to a subsequence to have $S^n\rightharpoonup S^0$, where the convergence is weakly in H. From (ii) and (iii) we get

$$\operatorname{div} S^0=0$$

and

$$\pi(S^0)=0 \quad \text{on} \quad \partial_2\Omega \quad \text{if} \quad \partial_2\Omega\neq\varnothing$$

or

$$\int_\Omega \operatorname{tr} S^0(x)\,dx=0 \quad \text{if} \quad \partial_2\Omega=\varnothing.$$

Next, we observe that

$$|S^n|^2-\tfrac{1}{3}|\operatorname{tr} S^n|^2=\frac{1}{3}\int_\Omega[(S^n_{11}-S^n_{22})^2+(S^n_{22}-S^n_{33})^2+(S^n_{33}-S^n_{11})^2]\,dx$$
$$+2\int_\Omega[(S^n_{12})^2+(S^n_{23})^2+(S^n_{31})^2]\,dx.$$

Hence (i) implies that

$$S^n_{ij} \to 0 \quad \text{in} \quad L^2(\Omega) \quad \text{when} \quad i \neq j \tag{A.5}$$

$$S^n_{ii} - S^n_{jj} \to 0 \quad \text{in} \quad L^2(\Omega) \quad \text{when} \quad i \neq j \tag{A.6}$$

Thus

$$S^0_{12} = S^0_{23} = S^0_{31} = 0$$

and

$$S^0_{11} = S^0_{22} = S^0_{33} = p \quad \text{for some} \quad p \in L^2(\Omega).$$

Hence $S^0 = pI$. Then div $S^0 = 0$ implies that $\nabla p = 0$, so $p = \text{const}$. Then from (iii) it follows that $p = 0$, hence $S^0 = 0$. Next, we prove that the convergence $S^n \to S^0$ is actually *strong* in H, and thus will reach a contradiction. For this, we write div $S^n = 0$ in the expanded form

$$S^n_{11,1} + S^n_{12,2} + S^n_{13,3} = 0$$
$$S^n_{12,1} + S^n_{22,2} + S^n_{23,3} = 0$$
$$S^n_{13,1} + S^n_{23,2} + S^n_{33,3} = 0$$

from which we obtain

$$S^n_{11,1} = -S^n_{12,2} - S^n_{13,3}$$
$$S^n_{11,2} = -S^n_{12,1} - S^n_{23,3} + (S^n_{11} - S^n_{22})_{,2}$$
$$S^n_{11,3} = -S^n_{13,1} - S^n_{23,2} + (S^n_{11} - S^n_{33})_{,3}$$

Now the right hand side converges to zero strongly in $H^{-1}(\Omega)$ by (A.5) and (A.6). The left hand side is the gradient of S_{11}. Hence

$$\nabla S^n_{11} \to 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{A.7}$$

Since we have already shown that $S^n \rightharpoonup 0$ in H, by the compact imbedding of H into $H^{-1}(\Omega)$ we get

$$S^n_{11} \to 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{A.8}$$

(A.7) and (A.8) together imply that

$$S^n_{11} \to 0 \quad \text{in} \quad L^2(\Omega).$$

This is because the expression $\|u\|^2_{H^{-1}} + \|\nabla u\|^2_{H^{-1}}$ defines a norm on $L_2(\Omega)$ which is equivalent to the usual norm.

In a similar way, we obtain

$$S^n_{22} \to 0 \quad \text{and} \quad S^n_{33} \to 0 \quad \text{in} \quad L^2(\Omega).$$

hence by (A.5)

$$S^n \to 0 \quad \text{in} \quad L^2(\Omega).$$

This is in contradiction with $|S^n| = 1$, thus completing the proof of the proposition.

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