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Conservation laws in the dynamics of rods

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Abstract. Conservation laws and associated integrals of motion for the dynamics of rods are derived. The classic conservation laws are those of total linear and angular momentum, and, for hyperelastic rods, conservation of energy. It will here be shown that an additional conservation law arises in each of two cases. The first case is that of uniform, hyperelastic rods, the second is that of a class of transversely isotropic rods. AMS(MOS) 73C50, 73K05.

1. Introduction

This note describes two conservation laws that are implied by the equations of motion of a rod in the presence of a continuous symmetry group. For our purposes a conservation law means an expression of the form

$$D_t = F_s, \tag{1.1}$$

where D, the *density*, and F, the (negative of) the *flux*, are each functions of the (scalar) independent variables s and t, and a (vector) dependent variable $\mathbf{y}$ and its derivatives. The actual system of partial differential equations that model the dynamics of a rod is equivalent to the two conservation laws (2.12) and (2.14) below, that express balance of linear and angular momentum. If the rod is hyperelastic (2.12) and (2.14) imply a third conservation law (2.19) that expresses conservation of energy. The additional conservation laws that are of primary concern here arise in the cases of a uniform, hyperelastic rod, where the symmetry is translation in arc-length, and in a class of transversely isotropic rods, where the symmetry is rotation about the centerline of the rod. The precise forms of the conservation laws are detailed in equation (3.2) for the case of a uniform hyperelastic rod, and in equation (4.5) for the case of isotropy.

One reason that conservation laws are of interest is that for boundary conditions such that the flux evaluated at the boundaries vanishes, each

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conservation law (1.1) can be integrated over the space variable s to yield an associated integral of motion

$$\frac{d}{dt}\int D\,ds = 0. \tag{1.2}$$

In the case of a uniform, hyperelastic rod the integral will be called the *impulse* (3.3). The impulse is associated with the symmetry of translation in arc-length, and is not simply related to total linear momentum, which, for rod dynamics, is the integral generated by translational symmetry in space. In the case of an isotropic rod the integral is the total component of the angular momentum along the local axis of transverse isotropy, and will therefore be called the *isotropic angular momentum* (4.9).

The bulk of the presentation concerns conservation laws for the dynamics of a shearable, extensible rod deforming in three dimensions. In Section 2 the dynamical system governing the motion of rods is described. In Section 3 the conservation law and associated integral for uniform, hyperelastic rods are described, and in Section 4 the analogous quantities are derived in the appropriate (sub-)case of transverse isotropy. The particular manifestations of the conservation laws in the special cases of string models, of inextensible, unshearable rods, and of planar motion are outlined in Section 5. Existing literature is described in the remainder of the Introduction.

The existence of extra conservation laws in the presence of additional symmetry is unsurprising when the system governing the motion of rods is viewed within the context of Noether's Theorem as described by, for example, Olver [(11, Chpts. 4 and 7]. Moreover the flux in each conservation law for system of partial differential equations describing the time dynamics of a rod necessarily generates an integral of the corresponding system of ordinary differential equations describing the equilibrium configurations. (However, as in the case of conservation of energy (2.19), the static integral may be trivial.) Thus it is not coincidental that the two symmetries that are here used to generate conservation laws for the dynamics, are exactly cases in which integrals of the static equations are known. Similarly the special cases of the conservation laws (3.2) and (4.5) that arise for the system of ordinary differential equations describing travelling waves in a rod were previously found by Antman & Liu [1, eqs. (3.8) and (3.13)]. Because of these relations to known special cases, it is comparatively easy to conjecture the appropriate form of the conservation laws for the case of general dynamics. Accordingly we found it simpler to couch our presentation in terms of the a posteriori verification of various identities. Nevertheless it should be recognized that for hyperelastic rods we could have exploited Noether's Theorem to find the conservation laws (3.2) and (4.5) constructively. On the other hand conservation law (4.5) persists for a class of rods that are not hyperelastic. As the dynamics of non-hyperelastic rods are not associated with an action principle, Noether's Theorem is not applicable, and there is no immediate and general connection between symmetries and conservation laws. This fact is another motivation for the nonconstructive arguments presented here, which are equally valid in both the hyperelastic and elastic cases.

Despite the above remarks, to our knowledge neither of the conservation laws (3.2) and (4.5) have been described previously for the generality of dynamic rod models that are adopted here. However each conservation law certainly has its antecedents in the literature. For a hyperelastic, shearable, extensible rod, Simo et al. [12, eq. (6.26)] found the conserved quantity (4.9) as a consequence of their Hamiltonian formulation of the rod dynamics. As previously remarked our direct derivation of the conservation law (4.5) and associated integral (4.9) is slightly more general because we need not assume hyperelasticity, which is required for the Hamiltonian formulation to be valid. Perhaps more relevantly our derivation also implies (4.5) in the case of an inextensible, unshearable rod. It is a somewhat surprising fact that while the system governing the statics of an inextensible, unshearable rod is a considerable simplification of the system governing the statics of an extensible, shearable rod, the dynamics in the supposedly special case are in many ways much more complicated. In particular, even if the rod is assumed to be hyperelastic, the Hamiltonian formulation of the dynamics, as described in [12], does not specialize in any obvious and simple fashion. The difficulty is associated with the fact that the net force in the rod, $\mathbf{n}$ say, is no longer determined via a constitutive relation, but instead plays the role of an additional dependent variable (or Lagrange multiplier maintaining the constraints of inextensibility and absence of shear), with no explicit equation governing its time evolution, cf. [4], [5], [6], [8], [9], and [13]. We also record the fact that Coleman et al. [6] independently discovered conservation of isotropic angular momentum for an inextensible, unshearable, linearly elastic rod using a formulation of the rod dynamics in terms of certain Euler angles.

The explicit manifestation (5.12) of the impulse in the case of the planar dynamics of an inextensible rod was only recently found in [8]. In that work the impulse was also used in the variational characterization and stability analysis of planar solitary waves. Our terminology 'impulse' for the conserved quantity associated with translation in arc-length is motivated by analogy with similar problems arising in the analysis of one-dimensional wave motions in fluid mechanics, cf. [3]. Subsequently the particular form (5.8) of the impulse conservation law for three-dimensional motion within the context of a certain string model was discovered by Healey [10] and was exploited in the analysis of relative equilibria. In the case of these classic string dynamics some simplifications arise, cf. the two expressions (3.3) and (5.9) for the impulse. Generalization from the planar case also led Coleman et al. [6] (independently of this work) to discovery of the impulse for three-dimensional motions of an inextensible, unshearable, linearly elastic rod, again using an analysis involving Euler angles. It seems almost certain that the impulse (3.3) will prove useful in the variational characterization and stability analysis of steady motions of rods in three dimensions, (such as the three-dimensional travelling waves that have been discussed in [1] and more recently in [7]), but those issues will be investigated elsewhere.

2. Rod dynamics

The derivation of the equations of motion will only be outlined here, as the rod model that will be adopted is a comparatively standard one (cf. e.g. [1] or [2]). The dependent variables are a vector function $\mathbf{r}(s,t) \in \mathscr{R}^3$ and an orthonormal frame of *directors* $\mathbf{d}_i(s,t) \in \mathscr{R}^3$, $i = 1, 2, 3$. (Hereafter subscripts will be taken to run from 1 to 3, and repeated indices should be summed unless stated to the contrary.) The independent variables are two scalars, namely undeformed arc-length s, and time t. At each time t, the curve $\mathbf{r}(\cdot, t)$ can be interpreted as a material line in a long, slender elastic body. The triad $\{\mathbf{d}_i\}$ can be interpreted as providing information concerning the orientation of the material cross-section of the rod.

The kinematics of the rod are encapsulated in the relations

$$\mathbf{r}' = \mathbf{v}, \tag{2.1}$$

$$\mathbf{d}_i' = \mathbf{u} \times \mathbf{d}_i, \tag{2.2}$$

and

$$\dot{\mathbf{d}}_i = \boldsymbol{\omega} \times \mathbf{d}_i. \tag{2.3}$$

Here $\mathbf{d}_i'$ denotes the partial derivative with respect to arc-length s, while $\dot{\mathbf{d}}_i$ denotes the partial derivative with respect to t, etc. The components $v_i \equiv \mathbf{v} \cdot \mathbf{d}_i$ of the vector $\mathbf{v}(s,t)$ with respect to the triad $\{\mathbf{d}_i\}$ are the strains associated with stretching and shear, the components $u_i \equiv \mathbf{u} \cdot \mathbf{d}_i$ of the (Darboux) vector $\mathbf{u}$ are the strains associated with bending and twist, and the components $\omega_i \equiv \boldsymbol{\omega} \cdot \mathbf{d}_i$ of the vector $\boldsymbol{\omega}(s,t)$ can be regarded as the body components of the angular velocity of the material cross-section at s when viewed as a rigid body. Because v_i, u_i and ω_i are components with respect to a variable frame we obtain relations such as

$$\boldsymbol{\omega}' = \omega_i'\mathbf{d}_i + \mathbf{u} \times \boldsymbol{\omega}, \tag{2.4}$$

and

$$\mathbf{r}'' = \mathbf{v}' = v_i'\mathbf{d}_i + \mathbf{u} \times \mathbf{v}. \tag{2.5}$$

(By convention expressions such as v_i' involve the derivative of the ith component of $\mathbf{v}$ with respect to the triad $\{\mathbf{d}_j\}$, which is of course not the same as the ith component of $\mathbf{v}'$, etc.) We shall exploit the compatibility conditions

$$\dot{\mathbf{u}} = \omega_i'\mathbf{d}_i, \tag{2.6}$$

and

$$\boldsymbol{\omega}' = \dot{u}_i\mathbf{d}_i, \tag{2.7}$$

which are implied by use of equations (2.2), (2.3) and (2.4) in the identities

$$\frac{\partial^2}{\partial s \partial t}\mathbf{d}_i = \frac{\partial^2}{\partial t \partial s}\mathbf{d}_i.$$

The densities of linear and angular momentum are, respectively,

$$\mathbf{p}(s, t) \equiv \rho(s)\dot{\mathbf{r}}, \tag{2.8}$$

and

$$\boldsymbol{\pi}(s, t) \equiv I_{ij}(s)\omega_j\mathbf{d}_i. \tag{2.9}$$

Here $\rho(s)$ is the mass per unit arc-length associated with the material cross-section at the point s, which is determined entirely by the reference configuration. The quantities $I_{ij}(s)$ are the components of the inertia tensor of the material cross-section at s expressed with respect to the triad $\{\mathbf{d}_i\}$. It will be assumed that these inertia coefficients are independent of the strains u_i and v_i. Thus the quantities $I_{ij}(s)$ are also completely determined by the reference configuration and will therefore be regarded as known. In particular the symmetry conditions

$$I_{ij} = I_{ji} \tag{2.10}$$

are valid. It is consistent to restrict to the case in which the cross-section is modelled as a lamina with principal axes of inertia $\mathbf{d}_1$ and $\mathbf{d}_2$, so that

$$I_{ij} = 0, \quad i \neq j, \quad I_{33} = I_{11} + I_{22}, \tag{2.11}$$

but our analysis does not require this delimitation.

In a rod model the stresses acting across each material cross-section are reduced to a net force $\mathbf{n}(s, t)$ and moment $\mathbf{m}(s, t)$ (of the material on the side s^+ acting on the material on the side s^- at the point $\mathbf{r}(s, t)$). Then balance of linear and angular momentum yield the equations

$$\dot{\mathbf{p}} = \mathbf{n}', \tag{2.12}$$

and

$$\dot{\boldsymbol{\pi}} = \mathbf{m}' + \mathbf{r}' \times \mathbf{n}. \tag{2.13}$$

The system is closed by specification of a constitutive law relating the stresses $\mathbf{n}$ and $\mathbf{m}$ to the strains $\mathbf{u}$ and $\mathbf{v}$. However it may immediately be observed that, trivially and independent of the constitutive relation, (2.12) is in the form of a conservation law with density $\mathbf{p}$ and flux $\mathbf{n}$. Moreover, using (2.8) and (2.12), equation (2.13) can be rewritten in the conservation form

$$(\boldsymbol{\pi} + \mathbf{r} \times \mathbf{p})_t = (\mathbf{m} + \mathbf{r} \times \mathbf{n})_s. \tag{2.14}$$

Consequently, for appropriate boundary conditions, such as the periodic ones pertinent for a closed loop, the total linear momentum $\int \mathbf{p}\,ds$ and total angular momentum $\int (\boldsymbol{\pi} + \mathbf{r} \times \mathbf{p})\,ds$ about a fixed origin are integrals of the motion, as could have been surmised at the outset.

We next demonstrate that for a hyperelastic rod the equations of motion (2.12) and (2.13) imply a third (well-known) conservative law, namely conservation of the total energy. A rod is said to be hyperelastic if there exists a scalar valued strain energy density function $W(v_i, u_i, s)$, dependent upon the six strains and arc-length, with the property that the components of stress $n_i \equiv \mathbf{n} \cdot \mathbf{d}_i$ and $m_i \equiv \mathbf{m} \cdot \mathbf{d}_i$ satisfy the constitutive relations

$$n_i = W_{v_i}, \tag{2.15}$$

and

$$m_i = W_{u_i}, \tag{2.16}$$

Here W_{v_i} denotes the partial derivative of W with respect to the argument v_i, etc. Because the constitutive laws specify the components of $\mathbf{n}$ and $\mathbf{m}$ with respect to the variable basis $\{\mathbf{d}_i\}$, the kinematic relations

$$\mathbf{n}' = n_i'\mathbf{d}_i + \mathbf{u} \times \mathbf{n}, \tag{2.17}$$

and

$$\mathbf{m}' = m_i'\mathbf{d}_i + \mathbf{u} \times \mathbf{m} \tag{2.18}$$

are implied. Conservation of energy is expressed by the relation

$$(\tfrac{1}{2}\mathbf{p}\cdot\dot{\mathbf{r}} + \tfrac{1}{2}\boldsymbol{\pi}\cdot\boldsymbol{\omega} + W)_t = (\mathbf{m}\cdot\boldsymbol{\omega} + \mathbf{n}\cdot\dot{\mathbf{r}})_s. \tag{2.19}$$

Identity (2.19) can be verified by expansion of the derivative on the right-hand side to obtain

$$\mathbf{m}'\cdot\boldsymbol{\omega} + \mathbf{m}\cdot\boldsymbol{\omega}' + \mathbf{n}'\cdot\dot{\mathbf{r}} + \mathbf{n}\cdot\dot{\mathbf{r}}'$$

which, by (2.1), (2.7), (2.12) and (2.13), is the same as

$$(\dot{\boldsymbol{\pi}} - \mathbf{v} \times \mathbf{n})\cdot\boldsymbol{\omega} + m_i\dot{u}_i + \dot{\mathbf{p}}\cdot\dot{\mathbf{r}} + \mathbf{n}\cdot\dot{\mathbf{v}}$$

or

$$\dot{\pi}_i\omega_i + \dot{\mathbf{p}}\cdot\dot{\mathbf{r}} + m_i\dot{u}_i + n_i\dot{v}_i$$

which, with constitutive relations (2.8), (2.9), (2.15), and (2.16), can be seen to coincide with the left-hand side of (2.19).

3. A conservation law for uniform, hyperelastic rods

It will now be demonstrated that for a uniform, hyperelastic rod, the equations (2.12) and (2.13) governing the dynamics imply another conservation law and associated first integral. A rod is said to be uniform if the coefficients ρ and I_{ij} are actually constants,

$$\rho' \equiv I_{ij}' \equiv 0, \tag{3.1}$$

and, in addition, the constitutive law has no explicit dependence on the arc-length s. For a hyperelastic rod to be uniform the strain energy density function W must have no explicit dependence upon arc-length s.

The following conservation law is then a consequence of the governing equations (2.12) and (2.13) that express balance of linear and angular momentum:

$$(\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{v})_t = (\mathbf{m} \cdot \mathbf{u} + \mathbf{n} \cdot \mathbf{v} - W + \tfrac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\pi} + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p})_s. \tag{3.2}$$

To verify (3.2), the derivative on the left-hand side can be expanded, using (2.1), as

$$\dot{\boldsymbol{\pi}} \cdot \mathbf{u} + \boldsymbol{\pi} \cdot \dot{\mathbf{u}} + \dot{\mathbf{p}} \cdot \mathbf{v} + \mathbf{p} \cdot \dot{\mathbf{r}}',$$

which, upon elimination of $\dot{\mathbf{u}}$, $\mathbf{p}$, $\boldsymbol{\pi}$, $\dot{\mathbf{p}}$ and $\dot{\boldsymbol{\pi}}$ by use of (2.6), (2.8), (2.9), (2.12) and (2.13), can be seen to be the same as

$$(\mathbf{m}' + \mathbf{r}' + \mathbf{n}) \cdot \mathbf{u} + \omega_i I_{ij} \omega_j' + \mathbf{n}' \cdot \mathbf{v} + \rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}'.$$

Then by the kinematic relations (2.1), (2.5), (2.17), (2.18) and symmetry condition (2.10), we obtain

$$m_i' u_i + n_i' v_i + (\tfrac{1}{2}\omega_i I_{ij} \omega_j)_s + (\tfrac{1}{2}\rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}})_s,$$

or, with constitutive laws (2.15) and (2.16),

$$(u_i W_{u_i} + v_i W_{v_i} - W + \tfrac{1}{2}\omega_i I_{ij} \omega_j + \tfrac{1}{2}\rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}})_s,$$

which can be seen to coincide with the right-hand side of (3.2) because of the constitutive relations (2.8), (2.9), (2.15) and (2.16).

It should be remarked that it has nowhere been assumed that the constitutive relations (2.15) and (2.16) are such that the stresses $(\mathbf{n}, \mathbf{m})$ vanish for vanishing strains $(\mathbf{v}, \mathbf{u})$. Consequently the analysis presented here is valid for the dynamics of uniform hyperelastic rods whose unstressed state is helical.

The quantity

$$\int (\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{v})\, \mathrm{d}s \tag{2.3}$$

will be called the *impulse*, where it is understood that the range of integration is the entire length of the rod. The impulse is clearly a finite quantity if the rod is of finite length (and the evolution is assumed to be smooth), but the impulse will also be finite for an infinite rod if either the strains $(\mathbf{v}, \mathbf{u})$ or the momenta $(\mathbf{p}, \boldsymbol{\pi})$ decay sufficiently rapidly.

The impulse is actually a time-invariant quantity, i.e., a conserved quantity or integral of the dynamics, for systems with appropriate boundary conditions. For example in a finite closed loop of rod, all the dependent variables must be periodic functions of s, and the impulse is therefore a conserved quantity. For an infinite rod the impulse can also be a (finite) conserved quantity. For example the impulse will be a constant of the motion if the momenta satisfy $\lim_{s\to\pm\infty}(\mathbf{p}, \boldsymbol{\pi}) = 0$ and the strains satisfy $\lim_{s\to\pm\infty}(v_i, u_i) = (\hat{v}_i, \hat{u}_i)$ for some constants $(\hat{v}_i, \hat{u}_i)$.

Notice that the flux appearing in the conservation law (3.2) is the Legendre transform of the density of total energy of the rod. As such it is reminiscent of the Eshelby Energy-Momentum tensor which arises in a conservation law for the elasto-statics of a homogeneous elastic material in several space dimensions. This problem is discussed among other places, by Olver [11], Example 4.32, p. 281, with a brief discussion of related issues and literature on p. 288.

It should also be remarked that linear combinations of the conservation laws (2.12), (2.14), (2.19) and (3.2) can be taken to yield new conservation laws. In some circumstances such a construction may have nontrivial implications. For example the boundary conditions may be such that a linear combination of the fluxes may vanish and so provide a conserved quantity, while the individual conservation laws do not imply any integral of motion. Moreover in the case of an infinite rod some linear combination of the densities may provide a finite conserved quantity while none of the individual conservation laws have associated convergent integrals. This last phenomenon is related to the choice of frame from which the rod dynamics is viewed.

4. Isotropic rods

We now drop the hypotheses that the rod is uniform and hyperelastic and instead assume that the rod is *transversely isotropic*. By transversely isotropic it is meant that the constitutive relations for the stresses and momenta are symmetric to rotations about one of the directors, $\mathbf{d}_3$ say. As full isotropy, i.e., independence of the constitutive relation to arbitrary rotations of the basis $\{\mathbf{d}_i\}$, is not a pertinent concept in the context of rods, we shall henceforth allow the adjective transversely to stand by implication. Antman [2, Theorem 9.16] demonstrates that the hypothesis of isotropic elastic response is equivalent to the specialized constitutive relations

$$\mathbf{n} = N_1\mathbf{v} + N_2\mathbf{u} + N_3\mathbf{d}_3, \tag{4.1}$$

and

$$\mathbf{m} = M_1\mathbf{v} + M_2\mathbf{u} + M_3\mathbf{d}_3, \tag{4.2}$$

where the N_i and M_i are each scalar functions of the six scalar arguments $|\mathbf{u}|$, $|\mathbf{v}|$, $\mathbf{u}\cdot\mathbf{v}$, u_3, v_3 and s. Isotropy of the dynamic properties of the rod follows when the expression (2.9) for the angular momentum is specialized to

$$\boldsymbol{\pi} \equiv I_1\boldsymbol{\omega} + (I_3 - I_1)\omega_3\mathbf{d}_3. \tag{4.3}$$

Here $\mathbf{d}_3$ is a principal axis of inertia with associated principal inertia $I_3(s)$, and the other two principal inertias both equal $I_1(s)$ (cf. (2.11) with $I_1 = I_2$). The general expression (2.8) for the linear momentum is already isotropic.

With constitutive relations (4.1), (4.2) and (4.3), and the additional restriction

$$M_1(|\mathbf{u}|, \mathbf{u}\cdot\mathbf{v}, |\mathbf{v}|, u_3, v_3, s) \equiv N_2(|\mathbf{u}|, \mathbf{u}\cdot\mathbf{v}, |\mathbf{v}|, u_3, v_3, s), \tag{4.4}$$

the conservation law

$$(\boldsymbol{\pi}\cdot\mathbf{d}_3)_t = (\mathbf{m}\cdot\mathbf{d}_3)_s, \tag{4.5}$$

is a consequence of the angular momentum balance (2.13). In particular the derivation of (4.5) does not exploit the linear momentum balance (2.12) so the conservation law remains valid even for systems with external body forces. The delimitation (4.4) is actually implied in the case of an isotropic hyperelastic rod where there is a strain energy function of the form $W(|\mathbf{u}|, \mathbf{u}\cdot\mathbf{v}, |\mathbf{v}|, u_3, v_3, s)$. Another plausible special case is $M_1 \equiv N_2 \equiv 0$.

The validity of (4.5) can be seen after three preliminary calculations. First we note that because of hypothesis (4.2)

$$\mathbf{m}\cdot(\mathbf{u} \times \mathbf{d}_3) = M_1\mathbf{v}\cdot(\mathbf{u} \times \mathbf{d}_3). \tag{4.6}$$

Similarly by (4.3)

$$\boldsymbol{\pi}\cdot(\boldsymbol{\omega} \times \mathbf{d}_3) = 0. \tag{4.7}$$

Thirdly equations (2.1) and (4.1) are used to write the balance of angular momentum (2.13) in the form

$$\mathbf{m}' + \mathbf{v} \times (N_2\mathbf{u} + N_3\mathbf{d}_3) = \dot{\boldsymbol{\pi}},$$

from which it may be concluded that

$$\mathbf{m}'\cdot\mathbf{d}_3 + N_2(\mathbf{v} \times \mathbf{u})\cdot\mathbf{d}_3 = \dot{\boldsymbol{\pi}}\cdot\mathbf{d}_3. \tag{4.8}$$

Now we use (2.2), (2.3), (4.4), (4.6), (4.7) and (4.8) to calculate that

$$(\boldsymbol{\pi}\cdot\mathbf{d}_3)_t - (\mathbf{m}\cdot\mathbf{d}_3)_s = \dot{\boldsymbol{\pi}}\cdot\mathbf{d}_3 + \boldsymbol{\pi}\cdot(\boldsymbol{\omega}\times\mathbf{d}_3) - \mathbf{m}'\cdot\mathbf{d}_3 - \mathbf{m}\cdot(\mathbf{u}\times\mathbf{d}_3) = 0,$$

as is required to validate (4.5).

The quantity

$$\int \boldsymbol{\pi}\cdot\mathbf{d}_3\,\mathrm{d}s \qquad (4.9)$$

will be called the (component of) *isotropic angular momentum*. The isotropic angular momentum is an integral of the motion for appropriate boundary conditions, for example the periodic conditions associated with a closed loop. The remarks made at the end of Section 3 concerning infinite rods and linear combinations of conservation laws apply with equal validity here.

5. Strings, inextensible rods and planar dynamics

In this section we shall consider the conservation laws that arise in certain degenerate cases of the rod model described in Section 2. The treatment of special cases is by no means exhaustive, but it does relate the conservation laws (3.2) and (4.5) to quantities found in previous works.

A model of a string is obtained if the strain $\mathbf{v}$ is declared to satisfy $v_1 = v_2 = 0$, so that (2.1) reduces to

$$\mathbf{r}' = v_3\mathbf{d}_3, \qquad (5.1)$$

and the constitutive laws for the force and moment are taken to be

$$n_1 = n_2 = 0, \quad n_3 = N_3, \qquad (5.2)$$

and

$$m_1 = m_2 = 0, \quad m_3 = M_3. \qquad (5.3)$$

Here N_3 and M_3 are scalar functions of the arguments (v_3, u_1, u_2, u_3, s). Conditions (5.1) and (5.2) require that the force in the string is always tangential. Condition (5.3) states that there are no bending moments, although the string can support a twisting moment about the unit tangent vector $\mathbf{d}_3$.

For this string model the balance of angular momentum (2.13) simplifies to

$$\dot{\pi} = \mathbf{m}', \tag{5.4}$$

which is trivially in conservation form. Thus for the dynamics of any such string there are always three conservation laws, namely (2.12), (5.4), and, because (2.14) is still valid,

$$(\mathbf{r} \times \mathbf{p})_t = (\mathbf{r} \times \mathbf{n})_s. \tag{5.5}$$

It could be argued that (5.5) is not independent of (2.12), and (5.4), but because it is expressed in terms of different variables it has the potential to yield integrals of motion under different sets of boundary conditions.

The string is uniform and hyperelastic if there is a strain energy density function $W(v_3, u_3)$ such that $N_3 = W_{v_3}$ and $M_3 = W_{u_3}$. (W must be independent of u_1 and u_2 to be consistent with the requirements $n_1 = n_2 = 0$.) For such a string the conservation of impulse (3.2) takes the form

$$(\pi \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{r}')_t = (m_3 u_3 + n_3 v_3 - W + \tfrac{1}{2}\omega \cdot \pi + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p})_s. \tag{5.6}$$

Notice that the constitutive laws for a hyperelastic string automatically satisfy the constitutive isotropy assumptions (4.1), (4.2), and (4.4) (with $N_1 \equiv N_2 \equiv M_1 \equiv M_2 \equiv 0$), so if the dynamic isotropy consition (4.3) is also satisfied we obtain the law of conservation of isotropic angular momentum (4.5) as before. Furthermore the conservation laws (5.6) and (4.5) are truly independent. In the absence of the dynamic isotropy condition (4.3), equation (4.5) does not hold. A final special feature arises when for a uniform, hyperelastic, isotropic string, it is further assumed, as in classic string models, that the strain energy density function is of the decoupled form

$$W(v_3, u_3) = W_1(v_3) + W_2(u_3). \tag{5.7}$$

Then the balance of linear momentum (2.12) decouples from the balance of angular momentum (2.13), and the additional conservation law

$$(\mathbf{p} \cdot \mathbf{r}')_t = (n_3 v_3 - W_1 + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p})_s \tag{5.8}$$

is implied. The associated impulse is

$$\int \mathbf{p} \cdot \mathbf{r}' \, ds \left(= \rho \int \dot{\mathbf{r}} \cdot \mathbf{r}' \, ds \right). \tag{5.9}$$

The conservation law (5.8) was found and exploited by Healey [10] in an analysis of the dynamics of closed loops of string. Healey also remarked that for uniform closed loops the impulse (5.9) is a conserved quantity that can be interpreted as a total circulation.

A different special case arises in the example of an inextensible, unshearable rod. In this model we prescribe

$$v_1 = v_2 = 0, \quad v_3 = 1,$$

so that $\mathbf{r}' = \mathbf{d}_3$. The constitutive relations (2.15) are discarded and the force $\mathbf{n}(s, t)$ assumes the role of an additional dependent variable. The equations of motion are unaffected, and so in the uniform hyperelastic case the conservation laws (2.12) and (3.2) imply that

$$(\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot (\mathbf{d}_3 - \mathbf{e}))_t = (\mathbf{m} \cdot \mathbf{u} + \mathbf{n} \cdot (\mathbf{d}_3 - \mathbf{e}) - W + \tfrac{1}{2}\boldsymbol{\omega} \cdot \pi + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p})_s. \tag{5.10}$$

Here $\mathbf{e} \in \mathscr{R}^3$ is any constant vector. The case of an infinite rod with conditions at infinity requiring that the rod be at rest and $\mathbf{d}_3 \to \mathbf{e}$ provides an example where the conservation law (5.10) implies the integral of motion

$$\int (\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot (\mathbf{d}_3 - \mathbf{e}))\, \mathrm{d}s \tag{5.11}$$

while, if $\mathbf{n}(\infty) \neq \mathbf{n}(-\infty)$, neither the total linear momentum $\int \mathbf{p}\, \mathrm{d}s$ nor the impulse (3.3) are individually invariants of the motion.

A further special case arises when the planar dynamics of an inextensible, unshearable rod with a linear constitutive relation between bending moment and curvature are considered. This model is the dynamic version of the classic elastica of Euler. Then we can set $\mathbf{r}' = \mathbf{d}_3 = (\cos\phi, \sin\phi, 0)$, and calculate that $\mathbf{u} = (0, 0, \phi_s)$ and (after nondimensionalization) $\boldsymbol{\pi} = (0, 0, \phi_t)$. For the planar elastica, (5.11) reduces to

$$\int (\phi_t \phi_s + \mathbf{p} \cdot (\mathbf{d}_3 - \mathbf{e}))\, \mathrm{d}s, \tag{5.12}$$

which is precisely the integral of motion discovered and exploited in [8]. Motivated by (5.12), but independent of the present work, Coleman et al. [6, eq. 3.32a] found the invariant

$$\int (\dot{\mathbf{r}} \cdot \mathbf{d}_3 + \dot{\mathbf{d}}_1 \cdot \mathbf{d}'_1 + \dot{\mathbf{d}}_2 \cdot \mathbf{d}'_2)\, \mathrm{d}s$$

which is the equivalent of a nondimensional form of (5.11) when there is a linear bending law, the inertia is of the particular form (2.11), and $\mathbf{e} = 0$.

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