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Some theorems in the theory of elastic materials with voids

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Abstract

The linear theory of elastic materials with voids is considered. Some basic theorems concerning the existence and uniqueness of solution, the reciprocity relations and the variational characterization of the solution are presented.

1. Introduction

In [1], Nunziato and Cowin gave a nonlinear theory of elastic materials with voids. The intended applications of the theory are to geological material like rock and soils and to manufactured porous materials.

The linear theory of elastic materials with voids was established by Cowin and Nunziato [2]. In this paper we consider the linear theory of elastic materials with voids and establish some basic theorems concerning the existence and uniqueness of solution, the reciprocity relations and the variational characterization of the solution.

2. Basic equations

We refer the motion of the continuum to a fixed system of rectangular Cartesian axes Ox_k $(k = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts (unless otherwise specified) are understood to range over the integers (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

Let $\bar{B}$ be a regular region of three-dimensional space occupied by an elastic material with voids. Let B be the interior of $\bar{B}$. We call ∂B the boundary of $\bar{B}$, and designate by n_i the components of the outward unit normal to ∂B.

Let u_i denote the components of the displacement vector field. Then the components of the infinitesimal strain field are given by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

We denote by φ the change in volume fraction from the reference volume fraction [2]. We consider an elastic material with voids which possesses a reference configuration in which the volume fraction is constant.

The constitutive equations for the linear theory of elastic materials with voids are [2]

$$\begin{aligned} t_{ij} &= C_{ijrs}e_{rs} + D_{ijs}\varphi_{,s} + B_{ij}\varphi, \\ h_i &= A_{ij}\varphi_{,j} + D_{rsi}e_{rs} + b_i\varphi, \\ g &= -\xi\varphi - B_{ij}e_{ij} - b_i\varphi_{,i}, \end{aligned} \tag{2.2}$$

where t_{ij} are the components of the stress tensor, h_i are the components of the equilibrated stress vector, g is the intrinsic equilibrated body force and C_{ijrs}, D_{ijk}, A_{ij}, B_{ij}, b_i, ξ are characteristic coefficients of the material. We have assumed that g does not depend on $\dot{\varphi}$. Obviously, this restriction involves only the quasi-static or dynamic theory. The results presented in Section 4 for the dynamic theory can be easily extended to the case when g depends on $\dot{\varphi}$. The material coefficients obey the symmetry relations

$$\begin{aligned} &C_{ijrs} = C_{rsij} = C_{jirs}, \quad B_{ij} = B_{ji}, \\ &A_{ij} = A_{ji}, \quad D_{ijr} = D_{jir}. \end{aligned} \tag{2.3}$$

The equations of motion governing a continuum with voids are the balance of linear momentum

$$t_{ji,j} + \rho f_i = \rho \ddot{u}_i, \tag{2.4}$$

and the balance of equilibrated force

$$\rho k \ddot{\varphi} = h_{i,i} + g + \rho \ell. \tag{2.5}$$

Here f_i are the components of the body force vector, ρ is the density in the reference configuration, k is the equilibrated inertia and ℓ is the extrinsic equilibrated body force.

Let us consider the subsets Σ_i $(i = \overline{1,\ldots,4})$ of ∂B such that $\bar{\Sigma}_1 \cup \Sigma_2 = \bar{\Sigma}_3 \cup \Sigma_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$. We consider the following boundary conditions

$$\begin{aligned} &u_i = \tilde{u}_i \quad \text{on} \quad \Sigma_1 \times [0, t_0), \qquad t_i \equiv t_{ji}n_j = \tilde{t}_i \quad \text{on} \quad \Sigma_2 \times [0, t_0), \\ &\varphi = \tilde{\varphi} \quad \text{on} \quad \Sigma_3 \times [0, t_0), \qquad h \equiv h_i n_i = \tilde{h} \quad \text{on} \quad \Sigma_4 \times [0, t_0), \end{aligned} \tag{2.6}$$

where $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}$, $\tilde{h}$ are prescribed functions and t_0 is some instant that may be infinite.

To the system of field equations we adjoin the boundary conditions (2.6) and the initial conditions

$$\begin{aligned} &u_i(x, 0) = u_i^0(x), \quad \dot{u}_i(x, 0) = v_i^0(x), \quad \varphi(x, 0) = \varphi_0(x), \\ &\dot{\varphi}(x, 0) = \nu_0(x), \quad x \in \bar{B}, \end{aligned} \tag{2.7}$$

where u_i^0, v_i^0, φ_0, ν_0 are prescribed functions.

3. Equilibrium theory

In the case of equilibrium the equations (2.4) and (2.5) become

$$t_{ji,j} + \rho f_i = 0, \tag{3.1}$$

and

$$h_{i,i} + g + \rho\ell = 0, \tag{3.2}$$

respectively. The basic equations of elastostatics are: the equilibrium equations (3.1) and (3.2); the constitutive equations (2.2); the geometrical equations (2.1). To the system of field equations we add the boundary conditions (2.6) where the boundary data are independent of time.

Let us consider the body subjected to two different systems of loadings $L^{(\alpha)} = \{f_i^{(\alpha)}, \ell^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{\varphi}^{(\alpha)}, \tilde{h}^{(\alpha)}\}$, $(\alpha = 1, 2)$ and the corresponding states $C^{(\alpha)} = \{u_i^{(\alpha)}, \varphi^{(\alpha)}, e_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}, h_i^{(\alpha)}, g^{(\alpha)}\}$. We introduce the notations

$$t_i^{(\alpha)} = t_{ji}^{(\alpha)} n_j, \quad h^{(\alpha)} = h_i^{(\alpha)} n_i. \tag{3.3}$$

Let us prove the following reciprocal theorem.

THEOREM 3.1: *If an elastic material with voids is subjected to two different systems of loadings $L^{(\alpha)}$ $(\alpha = 1, 2)$, then between the corresponding states $C^{(\alpha)}$ there is the following relation*

$$\begin{aligned}&\int_B \rho\left(f_i^{(1)} u_i^{(2)} + \ell^{(1)}\varphi^{(2)}\right)\mathrm{d}x + \int_{\partial B}\left(t_i^{(1)} u_i^{(2)} + h^{(1)}\varphi^{(2)}\right)\mathrm{d}x \\ &\quad = \int_B \rho\left(f_i^{(2)} u_i^{(1)} + \ell^{(2)}\varphi^{(1)}\right)\mathrm{d}x + \int_{\partial B}\left(t_i^{(2)} u_i^{(1)} + h^{(2)}\varphi^{(1)}\right)\mathrm{d}x.\end{aligned} \tag{3.4}$$

PROOF: On the basis of the relations (2.3), from (2.2) we get

$$\begin{aligned}&\left(t_{ij}^{(1)} - D_{ijk}\varphi_{,k}^{(1)} - B_{ij}\varphi^{(1)}\right)e_{ij}^{(2)} = \left(t_{ij}^{(2)} - D_{ijk}\varphi_{,k}^{(2)} - B_{ij}\varphi^{(2)}\right)e_{ij}^{(1)}, \\ &\left(h_i^{(1)} - D_{rsi}e_{rs}^{(1)} - b_i\varphi^{(1)}\right)\varphi_{,i}^{(2)} = \left(h_i^{(2)} - D_{rsi}e_{rs}^{(2)} - b_i\varphi^{(2)}\right)\varphi_{,i}^{(1)}, \\ &\quad -\left(g^{(1)} + \xi\varphi^{(1)} + B_{ij}e_{ij}^{(1)} + b_i\varphi_{,i}^{(1)}\right)\varphi^{(2)} = -\left(g^{(2)} + \xi\varphi^{(2)} + B_{ij}e_{ij}^{(2)} + b_i\varphi_{,i}^{(2)}\right)\varphi^{(1)}.\end{aligned}$$

Adding up these relations we obtain

$$t_{ij}^{(1)}e_{ij}^{(2)} + h_i^{(1)}\varphi_{,i}^{(2)} - g^{(1)}\varphi^{(2)} = t_{ij}^{(2)}e_{ij}^{(1)} + h_i^{(2)}\varphi_{,i}^{(1)} - g^{(2)}\varphi^{(1)}. \tag{3.5}$$

If we introduce the notation

$$I_{\alpha\beta} = \int_B\left(t_{ij}^{(\alpha)}e_{ij}^{(\beta)} + h_i^{(\alpha)}\varphi_{,i}^{(\beta)} - g^{(\alpha)}\varphi^{(\beta)}\right)\mathrm{d}x, \quad (\alpha, \beta = 1, 2), \tag{3.6}$$

then, from (3.5), we have

$$I_{12} = I_{21}. \tag{3.7}$$

By using the relations (2.1), (3.1)–(3.3) and the divergence theorem, we obtain

$$I_{\alpha\beta} = \int_{\partial B}\left(t_i^{(\alpha)}u_i^{(\beta)} + h^{(\alpha)}\varphi^{(\beta)}\right)\mathrm{d}x + \int_B\left(f_i^{(\alpha)}u_i^{(\beta)} + \ell^{(\alpha)}\varphi^{(\beta)}\right)\mathrm{d}x. \tag{3.8}$$

From (3.7) and (3.8) we obtain (3.4). Using Theorem 3.1 we can obtain relations of Somigliana type.

Let us note that if we take $L^{(1)} = \{f_i, \ell, \tilde{u}_i, \tilde{t}_i, \tilde{\varphi}, \tilde{h}\}$ and $C^{(1)} = \{u_i, \varphi, e_{ij}, t_{ij}, h_i, g\}$, then, from (3.6) and (2.2), we obtain

$$I_{11} = \int_B\left(t_{ij}e_{ij} + h_i\varphi_{,i} - g\varphi\right)\mathrm{d}x = 2\int_B W\mathrm{d}x, \tag{3.9}$$

where

$$W=\tfrac{1}{2}C_{ijrs}e_{ij}e_{rs}+\tfrac{1}{2}\xi\varphi^2+\tfrac{1}{2}A_{ij}\varphi_{,i}\varphi_{,j}+B_{ij}\varphi e_{ij}+D_{ijk}e_{ij}\varphi_{,k}+b_i\varphi\varphi_{,i} \tag{3.10}$$

is the potential energy density associated with strain and void volume distortion. The relation (3.8) becomes

$$I_{11}=\int_{\partial B}(t_iu_i+h\varphi)\mathrm{d}x+\int_B\rho(f_iu_i+\ell\varphi)\mathrm{d}x. \tag{3.11}$$

Thus, from (3.9) and (3.11) we obtain

$$2\int_B W\mathrm{d}x=\int_{\partial B}(t_iu_i+h\varphi)\mathrm{d}x+\int_B\rho(f_iu_i+\ell\varphi)\mathrm{d}x. \tag{3.12}$$

This relations leads to the following uniqueness theorem.

THEOREM 3.2: *Suppose that the potential energy density is a positive definite form. Then any two solutions of the problem are equal modulo a rigid displacement. Moreover, if* Σ_1 *is non-empty, then the mixed problem has at most one solution.*

PROOF: Let $\{u_i, \varphi, e_{ij}, t_{ij}, h_i, g\}$ and $\{\bar{u}_i, \bar{\varphi}, \bar{e}_{ij}, \bar{t}_{ij}, \bar{h}_i, \bar{g}\}$ be solutions of the boundary value problem, and let $u_i^0=u_i-\bar{u}_i$, $\varphi^0=\varphi-\bar{\varphi},\ldots, g^0=g-\bar{g}$. According to the linearity of the problem $\{u_i^0, \varphi^0, e_{ij}^0, t_{ij}^0, h_i, g^0\}$ is a solution corresponding to $f_i=0$, $\ell=0$. Moreover $t_i^0u_i^0+h^0\varphi^0=0$ on ∂B. We conclude from (3.12) that

$$\int_B W^0\mathrm{d}x=0, \tag{3.13}$$

where W^0 is the potential energy density corresponding to e_{ij}^0 and φ^0. Since W^0 is a positive definite quadratic form, from (3.13) we obtain $e_{ij}^0=0$, $\varphi^0=0$ and therefore

$$u_i^0=a_i^0+\epsilon_{ijk}b_j^0x_k,\quad \varphi^0=0, \tag{3.14}$$

where a_i^0 and b_i^0 are arbitrary constants. If Σ_1 is non-empty we obtain $u_i^0=0$, $\varphi^0=0$ in B.

The field equations of the static theory can be written in the form

$$Au=f, \tag{3.15}$$

where $u=(u_1, u_2, u_3)\equiv(u_i, \varphi)$, $f=(\rho f_i, \rho\ell)$ and Au has the components

$$A_iu=-\left(C_{ijrs}u_{r,s}+D_{ijk}\varphi_{,k}+B_{ij}\varphi\right)_{,j},$$
$$A_4u=-\left(D_{rsi}u_{r,s}+A_{ij}\varphi_{,j}+b_i\varphi\right)_{,i}+B_{ij}u_{i,j}+b_i\varphi_{,i}+\xi\varphi. \tag{3.16}$$

If $a=(a_1, a_2, a_2, a_4)$ and $b=(b_1, b_2, b_3, b_4)$ are two vectors, then we will denote by ab the scalar product $ab=\Sigma_{i=1}^4 a_ib_i$.

If we introduce the notations

$$u=\left(u_i^{(1)}, \varphi^{(1)}\right),\quad v=\left(u_i^{(2)}, \varphi^{(2)}\right), \tag{3.17}$$
$$t_i(u)=t_i^{(1)},\quad h(u)=h^{(1)},\quad p(u)=(t_i(u), h(u)),$$

then the relation (3.4) can be expressed as

$$\int_B(uAv-vAu)\mathrm{d}x=\int_{\partial B}(vp(u)-up(v))\mathrm{d}x. \tag{3.18}$$

From (3.18) we conclude that in the case of homogeneous boundary conditions the operator A is symmetric. Moreover, in this case (3.12) becomes

$$\int_B uAu\,\mathrm{d}x = 2\int_B W\,\mathrm{d}x. \tag{3.19}$$

Let us study the existence of the solution of the equation (3.15) with the boundary condition

$$p(u) = 0 \quad \text{on} \quad \partial B. \tag{3.20}$$

We assume that the domain B is C^∞-smooth [3] and the body forces and the coefficients of the material belong to C^∞. We consider only a "C^∞-theory" but it is possible to get a classical solution of the problem for more general domains and more general assumptions of regularity for the above functions (see [3,4]). In what follows we establish an existence theorem using results from [3]. We assume that W is a positive definite quadratic form, so that

$$2W \geqslant c\left(e_{ij}e_{ij} + \varphi_{,i}\varphi_{,i} + \varphi^2\right), \quad c > 0\ (c = \text{const.}). \tag{3.21}$$

To prove existence of the solution of the boundary value problem (3.15), (3.20), as in [3], we consider the system

$$Au + qu = f, \tag{3.22}$$

where q is an arbitrarily fixed positive constant. First we consider the boundary value problem (3.22), (3.20). Using (3.19) it follows [3, p. 62] that the inequality to be proven in this case is the following

$$2\int_B W\,\mathrm{d}x + q\int_B u^2\,\mathrm{d}x \geqslant c_0\|u\|_1^2, \quad c_0 > 0\ (c_0 = \text{const.}) \tag{3.23}$$

for any $u \in H_1(B)$. $H_1(B)$ is the Hilbert function space obtained by the functional completion of $C^1(\bar{B})$ with respect to the scalar product

$$(u, v) = \int_B D^s u D^s v\,\mathrm{d}x, \quad (s = 0, 1).$$

Using (3.21) one easily sees that the inequality (3.23) is implied by the following inequality

$$\int_B \left(e_{ij}e_{ij} + \varphi_{,i}\varphi_{,i} + \varphi^2\right)\mathrm{d}x + \int_B u^2\,\mathrm{d}x \geqslant c_1\|u\|_1^2, \quad c_1 > 0\ (c_1 = \text{const.}). \tag{3.24}$$

Using the second Korn's inequality we can write

$$\int_B e_{ij}e_{ij}\,\mathrm{d}x + \int_B \left(u^{(1)}\right)^2\mathrm{d}x \geqslant c_1\|u^{(1)}\|_1^2, \quad c_2 > 0\ (c_2 = \text{const.}), \tag{3.25}$$

where $u^{(1)} = (u_i, 0)$. If we denote $u^{(2)} = (0, 0, 0, \varphi)$ we have

$$\int_B \left(\varphi_{,i}\varphi_{,i} + \varphi^2\right)\mathrm{d}x + \int_B \left(u^{(2)}\right)^2\mathrm{d}x > \|u^{(2)}\|_1^2. \tag{3.26}$$

From (3.25), (3.26) follows (3.23). Thus, the boundary value problem (3.22), (3.20) has only one solution which is C^∞ in $\bar{B}$. The differential operator is formally self-adjoint, so that a C^∞ solution in $\bar{B}$ of the system

$$Au + qu - \lambda u = f, \tag{3.27}$$

with the boundary condition (3.20) exists when and only when

$$\int_B fu^* \mathrm{d}x = 0,$$

where u^* is any solution belonging to C^∞ of the problem (3.27), (3.20) with $f=0$. In the case when $\lambda = q$ the only C^∞ solution of the homogeneous system is (3.14). Thus we have the following theorem of existence.

THEOREM 3.3: *The boundary value problem* (3.15), (3.20) *has solutions belonging to* $C^\infty(\bar{B})$ *if and only if*

$$\int_B \rho f_i \mathrm{d}x = 0, \quad \int_B \rho \epsilon_{ijk} x_j f_k \mathrm{d}x = 0. \tag{3.28}$$

In the above relations ϵ_{ijk} is the alternating symbol. It is easy to show that in the case of the inhomogeneous condition $p(u) = (\tilde{t}_i, \tilde{h})$ on ∂B, the conditions (3.28) are replaced by

$$\begin{aligned} &\int_B \rho f_i \mathrm{d}x + \int_{\partial B} \tilde{t}_i \mathrm{d}x = 0, \\ &\int_B \rho \epsilon_{ijk} x_j f_k \mathrm{d}x + \int_{\partial B} \epsilon_{ijk} x_j \tilde{t}_k \mathrm{d}x = 0. \end{aligned} \tag{3.29}$$

We say that $S = \{u_i, \varphi, e_{ij}, t_{ij}, h_i, g\}$ is a kinematically admissible state if
(i) $u_i \in C^2(B)$, $\varphi \in C^2(B)$, $u_i \in C^1(\bar{B})$, $\varphi \in C^1(\bar{B})$,
(ii) the functions u_i, φ, e_{ij}, t_{ij}, h_i, g satisfy the equations (2.1), (2.2) and the boundary conditions imposed on Σ_1 and Σ_3. We have the following theorem of minimum potential energy.

THEOREM 3.4: *Let K denote the set of all kinematically admissible states, and let Λ be the functional on K defined by*

$$\Lambda(S) = \int_B W \mathrm{d}x - \int_B \rho(f_i u_i + \ell\varphi)\mathrm{d}x - \int_{\Sigma_2} \tilde{t}_i u_i \mathrm{d}x - \int_{\Sigma_4} \tilde{h}\varphi \mathrm{d}x,$$

for every $S = \{u_i, \varphi, e_{ij}, t_{ij}, h_i, g\} \in K$, where W is given by (3.10). *Let S be the solution of the mixed problem. Then*

$$\Lambda(S^*) \leqslant \Lambda(S),$$

for every $S \in K$, and equality holds only if $S = S^*$ modulo a rigid displacement.

The proof of this theorem can be made by the procedure used in the classical theory of elasticity [5].

4. The dynamic theory

Let f be a function of position and time defined on $\bar{B} \times [0, t_0)$. We say that $f \in C^{M,N}$ if

$$\frac{\partial^m}{\partial x_i \partial x_j \ldots \partial x_s}\left(\frac{\partial^n f}{\partial t^n}\right)$$

exists and is continuous on $\bar{B}\times[0, t_0)$ for $m=0, 1, \ldots, M$; $n=0, 1, \ldots, N$, and $m+n \leqslant \max(M, N)$. We introduce the notion of admissible state $S=\{u_i, \varphi, e_{ij}, t_{ij}, h_i, g\}$ by which we mean an ordered array of functions u_i, φ, e_{ij}, t_{ij}, h_i, g defined on $\bar{B}\times[0,t_0)$ with the following properties

$$u_i\in C^{1,2},\quad \varphi\in C^{1,2},\quad e_{ij}\in C^{1,0},\quad h_i\in C^{1,0},\quad g\in C^{0,0},\quad e_{ij}=e_{ji},\quad t_{ij}=t_{ji}.$$

Clearly, the set of all admissible states is a vector space provided we define addition and scalar multiplication in the natural manner $S+S'=\{u_i+u_i', \ldots, g+g'\}$, $\lambda S=\{\lambda u_i, \ldots, \lambda g\}$. By a solution of the mixed problem we mean an admissible state which satisfies the field equations (2.1), (2.2), (2.4), (2.5), the boundary conditions (2.6) and the initial condition (2.7). The uniqueness of the solution has been established in [2]. In what follows we derive the reciprocal theorem and variational theorems of Gurtin type [7]. First, we will present an alternative formulation of the boundary-initial-value problem. Let u and v be functions defined on $\bar{B}\times[0, t_0)$, continuous on $[0, t_0)$ with respect to time t for each $x\in\bar{B}$. We denote by $u * v$ the convolution of u and v

$$[u * v](x, t)=\int_0^t u(x, t-\tau)v(x, \tau)\mathrm{d}\tau.$$

Let us introduce the notations

$$\begin{aligned}&\gamma(t)=t,\quad F_i=\rho\left(\gamma * f_i+tv_i^0+u_i^0\right),\\&G=\rho\left[\gamma * \ell+k\left(t\nu_0+\varphi_0\right)\right].\end{aligned}\tag{4.1}$$

Following [6,7] one can prove the following theorem.

THEOREM 4.1: *The functions u_i, φ, t_{ij}, h_i, g satisfy the equations* (2.4), (2.5) *and the initial conditions* (2.7) *if and only if*

$$\gamma * t_{ji,j}+F_i=\rho u_i,\quad \gamma * (h_{i,i}+g)+G=\rho k\varphi.\tag{4.2}$$

This theorem enables us to give an alternate formulation of the boundary-initial-value problem in which the initial conditions are incorporated into the field equations. Thus, the admissible state S is a solution of the boundary-initial-value problem if and only if S satisfies the equations (2.1), (2.2), (4.2) and the boundary conditions (2.6).

Let us consider two systems of loadings $L^{(\alpha)}=\{f_i^{(\alpha)}, \ell^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{\varphi}^{(\alpha)}, \tilde{h}^{(\alpha)}, u_i^{0(\alpha)}, v_i^{0(\alpha)}, \varphi_0^{(\alpha)}, \nu_0^{(\alpha)}\}$ $(\alpha=1, 2)$ and the two corresponding solutions $S^{(\alpha)}=\{u_i^{(\alpha)}, \varphi^{(\alpha)}, e_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}, h_i^{(\alpha)}, g^{(\alpha)}\}$. We introduce the notations

$$F_i^{(\alpha)}=\rho\left(\gamma * f_i^{(\alpha)}+tv_i^{0(\alpha)}+u_i^{0(\alpha)}\right),\quad G^{(\alpha)}=\rho\left[\gamma * \ell^{(\alpha)}+k\left(t\nu_0^{(\alpha)}+\varphi_0^{(\alpha)}\right]\right..$$

THEOREM 4.2: *If an elastic solid is subjected to two systems of loading $L^{(\alpha)}$ $(\alpha=1, 2)$, then between the corresponding solutions $S^{(\alpha)}$ there is the following reciprocity relation*

$$\begin{aligned}&\int_B\left[F_i^{(1)} * u_i^{(2)}+G^{(1)} * \varphi^{(2)}\right]\mathrm{d}x+\int_{\partial B}\gamma * \left[t_i^{(1)} * u_i^{(2)}+h^{(1)} * \varphi^{(2)}\right]\mathrm{d}x\\&\quad=\int_B\left[F_i^{(2)} * u_i^{(1)}+G^{(2)} * \varphi^{(1)}\right]\mathrm{d}x+\int_{\partial B}\gamma * \left[t_i^{(2)} * u_i^{(1)}+h^{(2)} * \varphi^{(1)}\right]\mathrm{d}x.\end{aligned}\tag{4.3}$$

Proof. We will use the method given in [8,9]. On the basis of the relations (2.3), from (2.2) we get

$$t_{ij}^{(1)} * e_{ij}^{(2)} + h_i^{(1)} * \varphi_{,i}^{(2)} - g^{(1)} * \varphi^{(2)} = t_{ij}^{(2)} * e_{ij}^{(1)} + h_i^{(2)} * \varphi_{,i}^{(1)} - g^{(2)} * \varphi^{(1)}. \tag{4.4}$$

If we introduce the notations

$$I_{\alpha\beta} = \int_B \gamma * \left[t_{ij}^{(\alpha)} * e_{ij}^{(\beta)} + h_i^{(\alpha)} * \varphi_{,i}^{(\beta)} - g^{(\alpha)} * \varphi^{(\beta)} \right] \mathrm{d}x, \tag{4.5}$$

then from (4.4) we have

$$I_{12} = I_{21}. \tag{4.6}$$

Using (2.1) and (4.2) we obtain

$$\begin{aligned} \gamma * &\left[t_{ij}^{(\alpha)} * e_{ij}^{(\beta)} + h_i^{(\alpha)} * \varphi_{,i}^{(\beta)} - g^{(\alpha)} * \varphi^{(\beta)} \right] \\ &= \gamma * \left(t_{ij}^{(\alpha)} * u_i^{(\beta)} \right)_{,j} - \gamma * t_{ji,j}^{(\alpha)} * u_i^{(\beta)} + \gamma * \left(h_i^{(\alpha)} * \varphi^{(\beta)} \right)_{,i} \\ &\quad - \gamma * \left(h_{i,i}^{(\alpha)} + g^{(\alpha)} \right) * \varphi^{(\beta)} \\ &= \gamma * \left(t_{ij}^{(\alpha)} * u_j^{(\beta)} + h_i^{(\alpha)} * \varphi^{(\beta)} \right)_{,i} + F_i^{(\alpha)} * u_i^{(\beta)} + G^{(\alpha)} * \varphi^{(\beta)} \\ &\quad - \rho u_i^{(\alpha)} * u_i^{(\beta)} - \rho k \varphi^{(\alpha)} * \varphi^{(\beta)}, \end{aligned} \tag{4.7}$$

so that

$$\begin{aligned} I_{\alpha\beta} &= \int_B \left(F_i^{(\alpha)} * u_i^{(\beta)} + G^{(\alpha)} * \varphi^{(\beta)} \right) \mathrm{d}x \\ &\quad + \int_{\partial B} \gamma * \left(t_i^{(\alpha)} * u_i^{(\beta)} + h^{(\alpha)} * \varphi^{(\beta)} \right) \mathrm{d}x \\ &\quad - \int_B \rho \left(u_i^{(\alpha)} * u_i^{(\beta)} + k \varphi^{(\alpha)} * \varphi^{(\beta)} \right) \mathrm{d}x. \end{aligned} \tag{4.8}$$

From (4.6) and (4.8) we obtain the relation (4.3).

By using (2.1) and (2.2), the equations (4.2) can be expressed in terms of displacement and volume fraction. The resulting equations can be written in the form

$$Lu = F,$$

where $u = (u_i, \varphi)$, $F = (F_i, G)$ and the components of the operator L can be found easily. The relation (4.3) becomes

$$\int_B (v * Lu - u * Lv) \mathrm{d}x = \int_{\partial B} \gamma * [v * p(u) - u * p(v)] \mathrm{d}x.$$

We conclude that in the case of homogeneous boundary conditions the operator L is symmetric in convolution. Let us denote by D_L the domain of the definition of the operator L. Following [10,11] we have the following variational theorem.

Theorem 4.3: *Let $M \subset D_L$ be the set of all admissible vectors u which satisfy the homogeneous boundary conditions, and for each $t \in [0, t_0)$ define the functional $\Gamma_t\{\cdot\}$ on M by*

$$\Gamma_t\{u\} = \int_B (u * Lu - 2u * F)(x, t) \mathrm{d}x, \tag{4.9}$$

for every $u \in M$. Then

$$\delta\Gamma_t\{u\} = 0, \quad t \in [0, t_0),$$

at $u \in M$ if and only if u is a solution of the boundary-initial-value problem with homogeneous boundary conditions.

With the help of (2.2), (4.5) and (4.8) we can express the functional $\Gamma_t\{\cdot\}$ in the form

$$\Gamma_t\{u\} = \int_B \gamma * \left(C_{ijrs}u_{r,s} * u_{i,j} + 2D_{ijk}u_{i,j} * \varphi_{,k} + A_{ij}\varphi_{,i} * \varphi_{,j}\right.$$
$$\left. + \xi\varphi * \varphi + 2b_i\varphi_{,i} * \varphi + 2B_{ij}u_{i,j} * \varphi\right)\mathrm{d}x$$
$$+ \int_B \rho\left(u_i * u_i + k\varphi * \varphi\right)\mathrm{d}x - 2\int_B \rho\left(f_i * u_i + \ell * \varphi\right)\mathrm{d}x. \tag{4.10}$$

The following variational theorem fully characterizes the solution of the mixed problem.

THEOREM 4.4: *Let $\mathscr{M}$ be the set of all admissible states, and for each $t \in [0, t_0)$ define the functional $\Omega_t\{\cdot\}$ on $\mathscr{M}$ by*

$$\Omega_t\{S\} = \int_B \Big\{ \gamma * \left(\tfrac{1}{2}C_{ijrs}e_{rs} * e_{ij} + D_{ijk}e_{ij} * \varphi_{,k} + \tfrac{1}{2}A_{ij}\varphi_{,i} * \varphi_{,j}\right.$$
$$\left. + \tfrac{1}{2}\xi\varphi * \varphi + b_i\varphi_{,i} * \varphi + B_{ij}e_{ij} * \varphi\right) - \gamma * \left(t_{ij} * e_{ij} + h_i * \varphi_{,i}\right.$$
$$\left. - g * \varphi\right) - \left(\gamma * t_{ji,j} + F_i\right) * u_i - \left[\gamma * (h_{i,i} + g) + G\right] * \varphi$$
$$+ \tfrac{1}{2}\rho\left(u_i * u_i + k\varphi * \varphi\right)\Big\}\mathrm{d}x + \int_{\Sigma_1} \gamma * t_i * \tilde{u}_i\mathrm{d}x + \int_{\Sigma_2} \gamma * (t_i - \tilde{t}_i) * u_i\mathrm{d}x$$
$$+ \int_{\Sigma_3} \gamma * h * \tilde{\varphi}\mathrm{d}x + \int_{\Sigma_4} \gamma * (h - \tilde{h}) * \varphi\mathrm{d}x. \tag{4.11}$$

Then

$$\delta\Omega_t\{S\} = 0 \qquad \text{over } \mathscr{M},\ t \in [0, t_0), \tag{4.12}$$

at an admissible process S if and only if S is a solution of the mixed problem.

PROOF: Let $S' = \{u_i', \varphi', e_{ij}', t_{ij}', h_i', g'\} \in \mathscr{M}$ from which it follows that $S + \lambda S' \in \mathscr{M}$ for every real λ. Then (4.12), together with (2.3), the divergence theorem and the properties of the convolution, implies

$$\delta_{S'}\Omega_t\{S\} = \int_B \Big\{ \gamma * \Big[\left(C_{ijrs}e_{rs} + D_{ijs}\varphi_{,s} + B_{ij}\varphi - t_{ij}\right) * e_{ij}'$$
$$+ \left(A_{ij}\varphi_{,j} + D_{rsi}e_{rs} + b_i\varphi - h_i\right) * \varphi_{,i}'$$
$$+ \left(\xi\varphi + B_{ij}e_{ij} + b_i\varphi_{,i} + g\right) * \varphi' + \tfrac{1}{2}\left(u_{i,j} + u_{j,i} - 2e_{ij}\right) * t_{ij}'\Big]$$
$$- \left(\gamma * t_{ji,j} + F_i - \rho u_i\right) * u_i' - \left[\gamma * (h_{i,i} + g) + G - \rho k\varphi\right] * \varphi'\Big\}\mathrm{d}x$$
$$+ \int_{\Sigma_1} \gamma * (\tilde{u}_i - u_i) * t_i'\mathrm{d}x + \int_{\Sigma_2} \gamma * (t_i - \tilde{t}_i) * u_i'\mathrm{d}x$$
$$+ \int_{\Sigma_3} \gamma * (\tilde{\varphi} - \varphi) * t_i'\mathrm{d}x + \int_{\Sigma_4} \gamma * (h - \tilde{h}) * \varphi'\mathrm{d}x, \quad t \in [0, t_0). \tag{4.13}$$

If S is a solution of the mixed problem then (4.13) implies (4.12). As in [7] we can prove that the "only if" part is also true.

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