Journal of Elasticity 15 (1985) 185–191.

The viscoelastic behavior of linear elastic materials with voids

STEPHEN C. COWIN

Department of Biomedical Engineering, Tulane University, New Orleans, LA 70118, USA

(Received November 7, 1983)

Abstract

It is shown that the constitutive equations for a linear elastic material with voids imply a viscoelastic stress-strain relation known as the "standard linear solid" in the case of quasi-static, homogeneous deformations in the absence of self-equilibrated body forces. It is noted that, even for deformations that are dynamic and/or inhomogeneous the viscoelastic behavior is still qualitatively similar to that predicted by the standard linear solid model.

1. Introduction

This paper concerns the viscoelastic behavior of the material described by the theory of linear elastic materials with voids. The basic equations for this theory are described in Section 2. The theory was developed by Cowin and Nunziato [1] and its intended application is to the prediction of the mechanical behavior of solid materials with small distributed voids. In all applications [1,2,3,4,5,6] of this theory, the viscoelastic effects have played a prominent role. The purpose of this paper is to show that the type of linear viscoelasticity generally displayed by linear elastic materials with voids is that of a standard linear solid [7,8,9,10]. The standard linear solid is a particular linear viscoelastic model. It will be described and its constitutive equation recorded in Section 3.

In the theory of linear elastic materials with voids the constitutive equation for stress is linearly related to the strain and to a parameter ϕ which represents the change in local volume fraction of the solid from a reference value of the local volume fraction. The value of ϕ is determined by an equation relating ϕ and its time rate of change to the value of the local strain and other parameters. The inclusion of the time rate of change of ϕ is the source of the viscoelastic type behavior in the theory. Since stress is related to strain and to ϕ it is not obvious how to classify the viscoelastic behavior of a linear elastic material with voids. Normally, a viscoelastic constitutive relation contains only stress, strain and time rates of change of stress and strain of various orders. The classification of the viscoelastic constitutive relation is by the particular orders of the temporal derivatives of stress and strain that occur in it. In this paper the parameter ϕ is removed from the constitutive relation so that a relation between stress, strain and the time rates of change of stress and strain can be obtained. It is shown here that the stress-strain relation for a linear elastic material with voids undergoing a quasi-static homogeneous deformation in the absence of a self-equilibrated body force is exactly the constitutive relation of the linear viscoelastic model called the standard linear solid. In Section 3 this is shown in the special case of uniaxial deformation of an isotropic material, and in Section 4 the same calculation is repeated for an arbitrary deformation of an anisotropic material in order to specify the conditions under which the result is true. The implications of this result are discussed in Section 5. In particular, it is noted that viscoelastic behavior of the general type characterized by the standard linear solid model has been obtained in all the inhomogeneous and dynamic problems solved in the content of the linear theory of elastic materials with pores.

2. Summary of the theory

The linear theory of elastic materials with voids deals with small changes from a reference configuration of a porous body. In this configuration, the bulk density ρ, matrix density γ, and matrix volume fraction ν are related by

$$\rho_R = \gamma_R \nu_R, \tag{2.1}$$

and the body is taken to be strain-free, although not necessarily stress-free. The independent kinematic variables in the linear theory are the displacement field $u_i(\boldsymbol{x}, t)$ from the reference configuration and the change in volume fraction from the reference volume fraction, $\phi(\boldsymbol{x}, t)$,

$$\phi(\boldsymbol{x}, t) = \nu(\boldsymbol{x}, t) - \nu_R(\boldsymbol{x}), \tag{2.2}$$

where $\boldsymbol{x}$ is the spatial position vector in cartesian coordinates and t is time. The infinitesimal strain tensor $E_{ij}(\boldsymbol{x}, t)$ is determined from the displacement field, u_i, according to

$$E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.3}$$

where the comma followed by a lower case Latin letter indicates a partial derivative with respect to the indicated coordinate.

Assuming the region occupied by the body is regular, the equations of motion governing a linear elastic continuum with voids are the balance of linear momentum,

$$\rho \ddot{u}_i = T_{ij,j} + \rho b_i, \tag{2.4}$$

and the balance of equilibrated force,

$$\rho k \ddot{\phi} = h_{i,i} + g + \rho \ell. \tag{2.5}$$

Here T_{ij} is the symmetric stress tensor, b_i is the body force vector, h_i is the equilibrated stress vector, k is the equilibrated inertia, g is the intrinsic equilibrated body force and ℓ is the extrinsic equilibrated body force.

The constitutive equations for the linear theory of elastic materials with voids relate the stress tensor T_{ij}, the equilibrated stress vector h_i and the intrinsic equilibrated body force g to the strain E_{ij}, the change in volume fraction ϕ, the time rate of change of the volume fraction $\dot{\phi}$, and the gradient of the change in volume fraction $\phi_{,i}$; thus

$$T_{ij} = C_{ijkm} E_{km} + D_{ijk} \phi_{,k} + B_{ij} \phi, \tag{2.6}$$

$$h_i = A_{ij} \phi_{,j} + D_{ijk} E_{jk} + f_i \phi_{,i}, \tag{2.7}$$

$$g = -\omega \dot{\phi} - \xi \phi - B_{ij} E_{ij} - f_i \phi_{,i} \tag{2.8}$$

where C_{ijkm}, D_{ijk}, B_{ij}, A_{ij}, f_i, ω, ξ are functions of ν_R. If the material symmetry is of a type that possesses a center of symmetry, then the tensors D_{ijk} and f_i are identically zero and the constitutive equations (2.6), (2.7), and (2.8) simplify. If, in addition, the material is isotropic in its dependence of T_{ij}, h_i and g upon E_{ij}, $\phi,_i$ and ϕ, then C_{ijkm}, A_{ij} and B_{ij} are given by

$$C_{ijkm} = \lambda\delta_{ij}\delta_{km} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}), \tag{2.9}$$
$$A_{ij} = \alpha\delta_{ij},\ B_{ij} = \beta\delta_{ij},$$

and the constitutive equations (2.6), (2.7) and (2.8) become

$$T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij} + \beta\phi\delta_{ij}, \tag{2.10}$$

$$h_i = \alpha\phi,_i, \tag{2.11}$$

$$g = -\omega\dot{\phi} - \xi\phi - \beta E_{kk}. \tag{2.12}$$

The isotropic elastic coefficients must satisfy the inequalities

$$\mu \geqslant 0, \quad \alpha \geqslant 0, \quad \xi \geqslant 0, \quad \kappa \geqslant 0,\ \kappa\xi \geqslant \beta^2, \tag{2.13}$$

where

$$\kappa = \lambda + \tfrac{2}{3}\mu. \tag{2.14}$$

The coefficient ω and the equilibrated inertia k must be non-negative in order to satisfy a dissipation inequality resulting from the second law of thermodynamics.

3. The stress-strain relation for uniaxial deformation

In this section it will be shown that the stress-strain relation governing the quasi-static, homogeneous, uniaxial deformation of a linear elastic material with voids in the absence of self-equilibrated forces is the viscoelastic stress-strain relation known as a standard linear solid. In the case of a uniaxial deformation the stress-strain relation (2.10) takes the form

$$T_{11} = \lambda E_{kk} + 2\mu E_{11} + \beta\phi, \tag{3.1}$$
$$0 = \lambda E_{kk} + 2\mu E_{22} + \beta\phi,$$
$$0 = \lambda E_{kk} + 2\mu E_{33} + \beta\phi.$$

These equations are easily solved for E_{22}, E_{33}, E_{kk} and ϕ in terms of the quantities T_{11} and E_{11} which are to appear in the stress-strain relation; thus

$$E_{22} = E_{33} = E_{11} = -\tfrac{1}{2}\mu T_{11}, \quad E_{kk} = 3E_{11} - \frac{1}{\mu}T_{11}, \tag{3.2}$$

$$\phi = \frac{1}{\beta}\left(\frac{(\lambda+\mu)}{\mu}T_{11} - (3\lambda + 2\mu)E_{11}\right). \tag{3.3}$$

In the case of a homogeneous quasi-static deformation in the absence of self-equilibrated body forces, every term in (2.5) is identically zero. Thus, in particular, g is zero and it follows from (2.12) that the equation governing ϕ in this case is

$$\omega\dot{\phi} = \xi\phi + \beta E_{kk} = 0. \tag{3.4}$$

The stress-strain relation is obtained by substitution of E_{kk} from (3.2) and ϕ from (3.3) into (3.4) and subsequent rearrangement of terms; thus

$$T_{11} = t_E \dot{T}_{11} = E_0 \left(E_{11} + t_T \dot{E}_{11} \right), \tag{3.5}$$

where

$$E_0 = \frac{3\mu(\kappa\xi - \beta^2)}{(\lambda + \mu)\xi - \beta^2}, \quad t_E = \frac{\omega(\lambda + \mu)}{\xi(\lambda + \mu) - \beta^2}, \quad t_T = \frac{\kappa\omega}{\kappa\xi - \beta^2}. \tag{3.6}$$

The quantities t_E and t_T are the times of relaxation at constant strain and stress, respectively, and E_0 is the Young's modulus of the material when t_E and t_T vanish. The inequalities (2.13) insure that t_E, t_T and E_0 are all positive. In linear viscoelastic theory a stress-strain relation of the form (3.5) is said to characterize a standard linear solid. The spring and dashpot analogue models for this stress-strain relation are shown in Fig. 1. Such analogue models are, in general, not unique and for the standard linear solid there are two.

4. General stress-strain relations

It is not difficult to manipulate the constitutive relations (2.6), (2.7), (2.8) and the balance of equilibrated force (2.5) into a general stress-strain relation for linear elastic materials with voids. The first step consists of substituting (2.7) and (2.8) into (2.5) and rearranging terms; thus

$$\omega\dot{\phi} + \xi\phi + B_{rs}E_{rs} = A_{ij}\phi_{,ij} + D_{ijk}E_{jk,i} + \rho\ell - \rho\kappa\ddot{\phi}. \tag{4.1}$$

The second step is to solve (2.6) for $B_{ij}\phi$,

$$B_{ij}\phi = T_{ij} - C_{ijkm}E_{km} - D_{ijk}\phi_{,jk}. \tag{4.2}$$

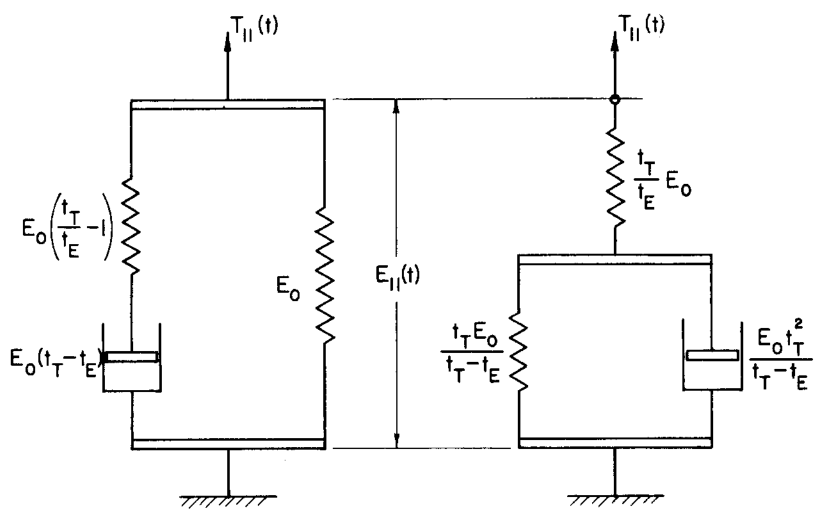

Figure 1. Illustrations of the two spring and dashpot models of the standard linear solid.

The third step is to multiply (4.1) by B_{ij} and then to replace $B_{ij}\phi$ by the right hand side of (4.2) and rearrange terms; thus

$$T_{ij} + \frac{\omega}{\xi}\dot{T}_{ij} = \left(C_{ijkm} - B_{ij}B_{km}\right)E_{km} + \frac{\omega}{\xi}C_{ijkm}\dot{E}_{km} + J_{ij}, \tag{4.3}$$

where

$$J_{ij} = \frac{1}{\xi}B_{ij}\left(A_{rm}\phi,_{rm} + D_{kmn}E_{mn},_{k} + \rho\ell - \rho\kappa\ddot{\phi}\right) + D_{ijk}\phi,_{k} + \frac{\omega}{\xi}D_{ijk}\dot{\phi},_{k}. \tag{4.4}$$

The principal conclusion can be obtained from an inspection of equations (4.3) and (4.4). Comparison of (4.3) with (3.5) shows that (4.3) is the tensor component form of the constitutive equation for a standard linear solid if the term J_{ij} is zero. Equation (4.4) shows that J_{ij} will be zero if the deformation is homogeneous and quasi-static and if the self-equilibrated body force vanishes. These are then the conditions under which a linear elastic material with pores is characterized by a stress-strain relation of the standard linear solid type.

A further result can be obtained if the material symmetry is of a type that possesses a center of symmetry and, consequently, the tensor D_{ijk} vanishes. In this case a stress-strain relation is obtained by substituting the expression for $B_{ij}\phi$ from (4.2) into (4.4) and subsequently substituting the result into (4.3) and rearranging terms; thus

$$\begin{aligned} T_{ij} + \frac{\omega}{\xi}\dot{T}_{ij} + \rho k\ddot{T}_{ij} - \frac{1}{\xi}A_{km}T_{ij},_{km} &= \left(C_{ijkm} - B_{ij}B_{km}\right)E_{km} \\ &+ \frac{\omega}{\xi}C_{ijkm}\dot{E}_{km} + \rho k C_{ijkm}\ddot{E}_{km} - \frac{1}{\xi}A_{km}C_{ijrs}E_{rs},_{km} + \frac{1}{\xi}\rho\ell. \end{aligned} \tag{4.5}$$

This rather long equation shows that, in general, in a linear elastic material with pores and with a center of symmetry, inhomogeneous and non-quasi-static deformations change the type of viscoelastic behavior. If the deformation of the material is homogeneous but not quasi-static then the terms involving the second gradients of stress and strain vanish and the stress-strain relation is of a higher order viscoelastic type than the standard linear solid. The inclusion of the terms involving the second gradients of stress and strain induce non-local effects in the stress-strain relation.

5. Discussion

It has been established that the constitutive equations for a linear elastic material with voids imply a viscoelastic stress-strain relation known as the standard linear solid in the case of quasi-static, homogeneous deformations in the absence of self-equilibrated body forces. In this section it is argued that, even for deformations that are dynamic and/or inhomogeneous the viscoelastic behavior is still qualitatively similar to that predicted by the standard linear solid model. The single attractive feature of the spring and dashpot analogue models for viscoelastic behavior is that one can gain some intuitive understanding of the viscoelastic response from thought experiments with the model. For example, the effect of the quasi-static placement of a constant load on the model can be seen to be an initial elastic deflection characterized by an initial elastic constant followed by a creeping deflection that terminates in a state characterized by a final elastic constant. In between the initial and final states the instantaneous response is characterized by an instantaneous elastic modulus that depends on the amount of creeping deformation that has occurred. These and other general features of the qualitative response of a standard linear solid have been seen in all the solutions to specific problems involving linear elastic materials with voids in quasi-static inhomogeneous deformations. This general type of viscoelastic behavior has been displayed in each of the following quasi-static non-homogeneous problem solutions: thick walled cylindrical and spherical pressure vessels [2], pure bending of a beam [4], shrink fit about an elastic cylinder [5], and a circular hole in a field of uniaxial tension [5]. In each of these problems the initial stress and strain distribution is exactly that predicted by classical elasticity theory in the same situation. This result is a consequence of the solid volume fraction field being initially homogeneous. As time increases inhomogeneity develops in the solid volume fraction field. The rate dependence development of this inhomogeneity is the causal factor for the viscoelastic effects. The developing inhomogeneity in ϕ causes the stress and strain fields to deviate from their initial, classical elastic values. As time tends to infinity the creeping motions induced by the inhomogeneous field stop and a steady solution with an inhomogeneous volume fraction field, a new strain field and, generally, a new stress field emerge. With the exception of the behavior of the volume fraction field, these qualitative features are the qualitative features of the standard linear solid. They are induced in a linear elastic material with voids by the inclusion of ϕ as an independent variable, and thus they have a specific physical origin not considered in classical viscoelasticity.

The similarity between the qualitative behavior of a linear elastic material with voids and the behavior of a standard linear solid is observed even more strikingly in the dynamic and inhomogeneous situation of harmonic wave propagation. The propagation of harmonic waves in a standard linear solid is discussed by Hillier [9] and by Kolsky [10] and in a linear elastic material with pores by Puri and Cowin [6]. For both materials the phase velocity and internal friction or specific loss is plotted as a function of frequency. These plots appear as Figure 28 of Kolsky's book [10] and in Figs. 5, 6, 7 and 8 of Puri and Cowin [6]. In both situations the internal friction or specific loss tends to zero at both low and high frequencies and in between it is characterized by a relatively sharp peak that occurs at a specific, finite value of frequency. Also, for both the standard linear solid and the linear elastic material with voids the phase velocity, identified in [6] as the elastic phase velocity, has one relatively steady value at low frequencies and a higher and relatively steady value at high frequencies. The transition from the lower to the higher phase velocity occurs relatively sharply near the frequency at which the internal friction reaches its peak value. The only apparent qualitative difference between the predictions of the two theories with regard to harmonic wave propagation is the existence in the theory of elastic materials with voids of a second wave associated with changes in the solid volume fraction. This wave however, is very heavily damped at all frequencies except very high frequencies.

There are some interesting implications of the results presented here. For example, there is a demonstration that rate dependence in a microstructural parameter such as solid volume fraction can induce a viscoelastic stress-strain relation. This result is interesting from a theoretical viewpoint and from the viewpoint of trying to identify the best constitutive models for specific real materials. There are many real materials that have, for some range of parameters, been classified as viscoelastic on the basis of their stress-strain response. The present result suggests that it might be more instructive to measure stress, strain and some microstructural parameter in order to obtain a simpler and more physical constitutive relation.

Acknowledgement

This work was partially supported by the National Science Foundation grant to Tulane University.

References

[1] S.C. Cowin and J.W. Nunziato, Linear elastic materials with voids. *J. Elasticity* 13 (1983) 125.
[2] S.C. Cowin and P. Puri, The classical pressure vessel problems for linear elastic materials with voids. *J. Elasticity* 13 (1983) 157.
[3] S.L. Passman, Stress relaxation, creep, failure and hysteresis in a linear elastic material with voids. *J. Elasticity*, 14 (1984) 201.
[4] S.C. Cowin, A note on the problem of pure bending for a linear elastic material with voids. *J. Elasticity* 14 (1984) 227.
[5] S.C. Cowin, The stresses around a hole in a linear elastic material with voids. *Quart. J. Mech. Appl. Math.*, 37 (1984) 441.
[6] P. Puri and S.C. Cowin, Plane waves in linear elastic materials with voids. *J. Elasticity* 15 (1985) 167–183.
[7] R.M. Christensen, *Theory of Viscoelasticity*. Academic Press (1971).
[8] F.J. Lockett, *Nonlinear Viscoelastic Solids*. Academic Press (1972).
[9] K.W. Hillier, A method of measuring some dynamic elastic constants and its application to the study of high polymers. *Proc. Phys. Soc.* B 62 (1949) 701.
[10] H. Kolsky, *Stress Waves in Solids*. Dover (1963).