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Plane waves in linear elastic materials with voids

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Abstract

The behavior of plane harmonic waves in a linear elastic material with voids is analyzed. There are two dilational waves in this theory, one is predominantly the dilational wave of classical linear elasticity and the other is predominantly a wave carrying a change in the void volume fraction. Both waves are found to attenuate in their direction of propagation, to be dispersive and dissipative. At large frequencies the predominantly elastic wave propagates with the classical elastic dilational wave speed, but at low frequencies it propagates at a speed less than the classical speed. It makes a smooth but relatively distinct transition between these wave speeds in a relatively narrow range of frequency, the same range of frequency in which the specific loss has a relatively sharp peak. Dispersion curves and graphs of specific loss are given for four particular, but hypothetical, materials, corresponding to four cases of the solution.

1. Introduction

The theory of linear elastic materials with voids is a modified elasticity theory in which change in volume fraction as well as strain are taken as independent kinematic variables. The theory is intended for application to solids with small distributed voids. The voids are assumed to contain nothing of mechanical or energetic significance. The linear theory was developed by Cowin and Nunziato [1] as a specialization of a non-linear theory [2].

In this paper we investigate the propagation of plane harmonic waves in a linear elastic material with voids. The response of an idealized material to propagating plane waves illustrates many of the features of the material's behavior. For linear elastic materials these waves are discussed, for example, by Kolsky [3]. For linear coupled thermoelastic materials an analysis of the plane wave behavior is given by Deresiewicz [4] and Puri [5,6]. Cowin and Nunziato [1] derived the frequency equation for plane waves in the context of the theory of linear elastic materials with voids, but they did not undertake a complete analysis of the equation. They observed that the frequency equation obtained was similar to the frequency equation for linear coupled thermoelasticity. Nunziato and Walsh [7] have discussed plane waves in a one dimensional granular medium. The theory of one dimensional granular media is a special case of the theory presented in [1] and considered in this paper. A survey of the current literature on plane waves in coupled elastic fields is given by Puri [6]. In the present article we present a complete analysis of the frequency equation for plane waves in a linear elastic material with voids. There are two dilational waves in this theory, one is predominantly the dilational wave of classical linear elasticity and the other is predominantly a wave carrying a change in the void volume fraction. Both waves are found to attenuate in their direction of propagation, to be dispersive and dissipative. At large frequencies the predominantly elastic wave propagates with the classical elastic dilational wave speed, but at low frequencies it propagates at a speed less than the classical speed. It makes a smooth but relatively distinct transition between these wave speeds in a relatively narrow range of frequency, the same range of frequency in which the specific loss has a relatively sharp peak. Dispersion curves and graphs of specific loss are given for four particular, but hypothetical, materials, corresponding to four cases of the solution.

The basic equations for linear isotropic elastic materials with voids are summarized in the next section, and material parameters that occur in the wave analysis are introduced in the subsequent section, Section 3. The major results of this paper are presented in Section 4 and concern the dispersion of plane harmonic waves propagating in an infinite medium. The amplitude ratios of these waves are analyzed in Section 5. Section 6 contains a discussion and Section 7, a summary of conclusions.

2. Summary of the theory

In this theory the basic concept is that the bulk density of the material is the product of two fields, the density field of the matrix material γ and the volume fraction field ν,

$$\rho = \gamma\nu. \tag{2.1}$$

We make the customary assumption of the linear theory that the reference configuration is free of stress and strain. Equation (2.1) can be written as $\rho_R = \gamma_R \nu_R$ for the reference configuration, where ν_R is assumed to be spatially constant.

The independent kinematic variables in the linear theory are $u_i(x, t)$ and $\phi(x, t)$, where $u_i(x, t)$ is the displacement field from the reference configuration and $\phi(x, t)$ is the change in the volume fraction from the reference volume fraction and is given by

$$\phi(x, t) = \nu(x, t) - \nu_R, \tag{2.2}$$

where x is the spatial variable in cartesian co-ordinates and t is the time. The infinitesimal strain tensor $E_{ij}(x, t)$ is determined from the displacement field u_i by the relation

$$E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.3}$$

where the comma followed by a lower case Latin letter indicates a partial derivative with respect to the indicated co-ordinate axis.

The equations of motion for the medium considered are the balance of linear momentum

$$\rho\ddot{u}_i = T_{ij,j} + \rho b_i \tag{2.4}$$

and the balance of equilibrated force

$$\rho k\phi = h_{i,i} + g + \rho\ell,$$

where T_{ij} is the symmetric stress tensor, b_i is the body force vector, h_i is the equilibrated stress vector, k is the equilibrated inertia, g is the intrinsic equilibrated body force and ℓ is the extrinsic equilibrated body force. These terms are discussed in detail in [1] and [2].

The constitutive equations for the linear isotropic theory of elastic materials with voids relate the stress tensor T_{ij}, the equilibrated stress vector h_i and the intrinsic equilibrated body force g to the strain E_{ij}, the change in volume fraction ϕ, the time rate of change of the volume fraction $\dot{\phi}$, and the gradient of the change in volume fraction $\phi,_i$; thus

$$T_{ij}=\lambda\delta_{ij}E_{kk}+2\mu E_{ij}+\beta\phi\delta_{ij}, \tag{2.6}$$

$$h_i=\alpha\phi,_i, \tag{2.7}$$

$$g=-\omega\dot{\phi}-\xi\phi-\beta E_{kk}. \tag{2.8}$$

The coefficients λ, μ, α, β, ξ and ω depend on ν_R and satisfy the following inequalities

$$\mu\geqslant 0,\quad \alpha\geqslant 0,\quad \xi\geqslant 0,\ \omega\geqslant 0,$$
$$3\lambda+2\mu\geqslant 0,\quad (3\lambda+2\mu)\xi\geqslant 3\beta^2.\ * \tag{2.9}$$

The field equations governing the displacement field $u_i(\boldsymbol{x}, t)$ and the volume fraction field $\phi(\boldsymbol{x}, t)$ are obtained by substituting the constitutive relations (2.6), (2.7) and (2.8) into the equations of motion (2.4) and (2.5) as

$$(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}+\mu\nabla^2\boldsymbol{u}+\beta\nabla\phi=\rho\ddot{\boldsymbol{u}}, \tag{2.10}$$

and

$$\alpha\nabla^2\phi-\omega\dot{\phi}-\xi\phi-\beta\nabla\cdot\boldsymbol{u}=\rho k\ddot{\phi}. \tag{2.11}$$

The boundary condition on ϕ is

$$\boldsymbol{n}\cdot\nabla\phi=0, \tag{2.12}$$

where $\boldsymbol{n}$ is the unit normal to the external boundary and the boundary conditions on u_i are those of classical elasticity.

3. Material parameters

The purpose of this section is to define and discuss the significance of the material parameters that will appear in the next section on the analysis of plane progressive waves. These material parameters are simply algebraic combinations of the material properties such as the density ρ, the elastic coefficients with dimension of stress λ, μ, β and ξ and the material coefficients α (of dimension stress times squared length) and ω (of dimension stress times time). The material parameters to be discussed fall into three categories: wave speeds, dimensionless numbers, and material coefficients with dimension of length.

The material coefficients of dimension length are defined by the expressions

$$\ell_1=\sqrt{\frac{\alpha}{\beta}},\quad \ell_2=\sqrt{\frac{\alpha}{\xi}},\quad \ell_0=\frac{\ell_1\ell_2}{\sqrt{\ell_1^2-\ell_2^2H}}, \tag{3.1}$$

* In [1], this inequality is incorrectly reported as $(3\lambda+2\mu)\xi>12\beta^2$.

where H is a dimensionless number,

$$H = \frac{\beta}{\lambda + 2\mu}. \tag{3.2}$$

The definitions of ℓ_0 and H were introduced in [1], where it was shown that ℓ_0 is always real and H positive. There are four wave velocities of interest. They are denoted by c_1 through c_4 and defined as follows:

$$c_1 = \sqrt{\frac{\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_3 = \sqrt{\frac{\alpha}{\rho\kappa}}, \quad c_4 = \frac{2\alpha}{\ell_0 \omega}. \tag{3.3}$$

The velocities c_1 and c_2 are the familiar distortional and dilational wave velocities, respectively, from classical elasticity (cf., e.g., Kolsky [3]). The velocities c_3 and c_4 will be shown to be the wave velocities of a wave carrying a change in the void fraction ϕ at high and low frequencies, respectively.

The dimensionless number N defined by

$$N \equiv \frac{\beta^2}{\xi(\lambda + 2\mu)} = \frac{\ell_2^2}{\ell_1^2} H, \qquad \text{for} \quad \xi > 0, \ \mu > 0, \tag{3.4}$$

is called the coupling number because it is a measure of the coupling between the displacement deformation $\boldsymbol{u}$ and the void volume deformation ϕ. N is a non-negative number which only vanishes when β vanishes. When N is zero the constitutive equation for stress, (2.6), does not depend upon ϕ; the constitutive equation for the self-equilibrated force g, (2.8), does not depend upon strain; and the wave equations, (2.10) and (2.11), governing the elastic waves and the void volume fraction wave uncouple. An increase in β is reflected as an increase in N and the coupling between volume fraction deformation ϕ and the displacement deformation $\boldsymbol{u}$. Furthermore, N satisfies the inequalities

$$0 \leqslant N < 1, \qquad \text{for} \quad \mu > 0, \ \xi > 0; \tag{3.5}$$

thus N must be a non-negative number less than one. In order to prove the inequality (3.5) consider the identity

$$3(\lambda + 2\mu)\xi = (3\lambda + 2\mu)\xi + 4\mu\xi. \tag{3.6}$$

From $(2.9)_1$, $(2.9)_3$, $(2.9)_6$ it follows that

$$3(\lambda + 2\mu)\xi > (3\lambda + 2\mu)\xi \geqslant 3\beta^2 \geqslant 0; \tag{3.7}$$

thus

$$\frac{(\lambda + 2\mu)\xi}{\beta^2} > 1, \tag{3.8}$$

and the restriction (3.5) follows from (3.4).

4. Dispersion of plane waves in an infinite medium

Plane acoustic waves are solutions of (2.10) and (2.11) for which the displacement vector u_i and the volume fraction difference ϕ are of the form

$$u_i = A d_i U(x, t), \quad \phi = BU(x, t), \tag{4.1}$$

where

$$U(x, t) = \mathrm{Re}(\exp(\mathrm{i}(\kappa t - \psi m_i x_i))), \tag{4.2}$$

and κ is the frequency, m_i the direction cosines of the normal to the plane of the waves, d_i the direction cosines of the displacement vector $\boldsymbol{u}(x, t)$ and ψ is a complex number such that $\mathrm{Im}\,\psi < 0$. A and B are the amplitudes of the displacement and volume fraction change waves, respectively. Substitution of (4.1) and (4.2) into (2.10) and (2.11) yields

$$\left[\{(\lambda+\mu)d_j m_j m_i + \mu d_i\}\psi^2 - \rho\kappa^2 d_i\right]A + \mathrm{i}\beta\psi m_i B = 0 \tag{4.3}$$

and

$$(\alpha\psi^2 + \mathrm{i}\omega\kappa - \rho k\kappa^2 + \xi)B - \mathrm{i}\beta\psi A = 0. \tag{4.4}$$

A partial analysis of this result was presented in [1]. It was shown there that, for equivoluminal or distortional waves characterized by $d_i m_i = 0$, the amplitude B of the volume fraction vanishes. These distortional waves propagate at the velocity c_1 defined by (3.3) without affecting the porosity of the material, and without attenuation. Longitudinal or dilational waves are characterized by $d_i m_i = 1$. Multiplying (3.2) by d_i or m_i and using the condition $d_i m_i = 1$ yields

$$\{(\lambda+2\mu)\psi^2 - \rho\kappa^2\}A + \mathrm{i}\beta\psi B = 0. \tag{4.5}$$

For the pair of equations (4.4) and (4.5) to have a non-trivial solution for A and B we must set the determinant of their coefficients equal to zero, thus

$$\{(\lambda+2\mu)\psi^2 - \rho\kappa^2\}\{\alpha\psi^2 + \mathrm{i}\omega\kappa + \xi - \rho k\kappa^2\} - \beta^2\psi^2 = 0. \tag{4.6}$$

Equation (4.6) is the dispersion relation for dilational waves. Using the notations (3.1) through (3.4), (4.6) can be rewritten as

$$\left(\psi^2 - \frac{\kappa^2}{c_2^2}\right)\left(\psi^2 - \frac{\kappa^2}{c_3^2} + \frac{1}{\ell_2^2} + \mathrm{i}\frac{\omega\kappa}{\alpha}\right) - \frac{N}{\ell_2^2}\psi^2 = 0, \tag{4.7}$$

or, equivalently,

$$\psi^4 - \left(\frac{\kappa^2}{c_2^2} + \frac{\kappa^2}{c_3^2} - \frac{1}{\ell_0^2} - \frac{\mathrm{i}\omega\kappa}{\alpha}\right)\psi^2 + \frac{\kappa^2}{c_2^2}\left(\frac{\kappa^2}{c_3^2} - \frac{1}{\ell_2^2} - \frac{\mathrm{i}\omega\kappa}{\alpha}\right) = 0, \tag{4.8}$$

where $(3.1)_3$ has been employed.

The dispersion relation (4.8) is similar to the dispersion relation for plane thermoelastic waves, and our analysis of the dispersion relation (4.8) will follow that of Puri [5] for thermoelastic waves. The information obtained from the dispersion relation includes the wave numbers, the wave speeds, the attenuation coefficients and the specific loss. The wave numbers are the real parts of ψ and the wave speeds are frequency divided by the wave number. The attenuation coefficients are the imaginary part of ψ multipled by minus one. The specific loss, $\Delta W/W$, is 4π times the absolute value of the ratio of the imaginary part of ψ to the real part of ψ,

$$\frac{\Delta W}{W} = 4\pi\left|\frac{\mathrm{Im}\,\psi}{\mathrm{Re}\,\psi}\right|.$$

Kolsky [3, p. 99] notes that specific loss is the most direct method of defining internal friction for a material. Specific loss is the ratio of the energy dissipated in taking a specimen through a stress cycle, ΔW, to the elastic energy stored in the specimen when the strain is a maximum, W. For a plane sinusoidal wave of small amplitude Kolsky [3, p. 106] shows that the specific loss $\Delta W/W$ is given by the above formula.

The two solutions of (4.8) will be designated as ψ_e^2 and ψ_v^2. It will be shown that ψ_e is associated with a wave that is predominantly an elastic wave of dilatation and ψ_v is associated with a wave that is predominantly a volume fraction wave. These two solutions are given by

$$2\psi_e^2,\, 2\psi_v^2 = D \pm F, \tag{4.9}$$

where

$$D = \frac{\kappa^2}{c_2^2} + \frac{\kappa^2}{c_3^2} - \frac{1}{\ell_0^2} - \frac{i\omega\kappa}{\alpha},$$
$$F^2 = D^2 - \frac{4\kappa^2}{c_2^2}\left(\frac{\kappa^2}{c_3^2} - \frac{1}{\ell_2^2} - \frac{i\omega\kappa}{\alpha}\right). \tag{4.10}$$

These solutions to (4.8) can be reduced to the form

$$2\psi_e,\, 2\psi_v = \pm\left(\left(D + \sqrt{D^2 - F^2}\right)^{1/2} \pm \left(D - \sqrt{D^2 - F^2}\right)^{1/2}\right), \tag{4.11}$$

where the first sign is chosen so that $\operatorname{Im}\psi < 0$. The quantity $\sqrt{D^2 - F^2}$ can be expressed as

$$\sqrt{D^2 - F^2} = \frac{2\kappa}{c_2}(a - ib), \tag{4.12}$$

where

$$a,\, b = \frac{1}{\sqrt{2}}\left[\left(\left(\frac{\kappa^2}{c_3^2} - \frac{1}{\ell_2^2}\right)^2 + \left(\frac{\omega\kappa}{\alpha}\right)^2\right)^{1/2} \pm \left(\frac{\kappa^2}{c_3^2} - \frac{1}{\ell_2^2}\right)\right]^{1/2}. \tag{4.13}$$

The detailed structure of the solution of dispersion relation (4.8) along with expressions for the wave speeds, attenuation coefficients and specific loss are all given in Table 1. In Table 1 the following notations

$$P,\, Q = \frac{1}{\sqrt{2}}\left[\left(\left(\frac{\kappa^2}{c_2^2} + \frac{\kappa^2}{c_3^2} + \frac{2\kappa a}{c_2} - \frac{1}{\ell_0^2}\right)^2 + \left(\frac{2b\kappa}{c_2} + \frac{\omega}{\alpha}\kappa\right)^2\right)^{1/2} \pm \left(\frac{\kappa^2}{c_2^2} + \frac{\kappa^2}{c_3^2} + \frac{2\kappa a}{c_2} - \frac{1}{\ell_0^2}\right)\right]^{1/2}, \tag{4.14}$$

$$L,\, M = \frac{1}{\sqrt{2}}\left[\left(\left(\frac{\kappa^2}{c_2^2} + \frac{\kappa^2}{c_3^2} - \frac{2\kappa a}{c_2} - \frac{1}{\ell_0^2}\right)^2 + \left(\frac{2b\kappa}{c_2} - \frac{\omega}{\alpha}\kappa\right)^2\right)^{1/2} \pm \left(\frac{\kappa^2}{c_2^2} + \frac{\kappa^2}{c_3^2} - \frac{2\kappa a}{c_2} - \frac{1}{\ell_0^2}\right)\right]^{1/2}, \tag{4.15}$$

Table 1. Structure of the solution to the dispersion relation.

Case	Conditions	Solution	Wave numbers	Wave speeds	Attenuation coefficients	Specific loss
1	$c_2 < c_3$ $c_2\sqrt{1-N} < c_4$	$2\psi_{e,v} = P \pm L - i(Q \mp M)$	$\frac{1}{2}(P \pm L)$	$\frac{2\kappa}{P \pm L}$	$Q \mp M$	$4\pi\left\|\frac{Q \mp M}{P \pm L}\right\|$
2	$c_2 > c_3$ $c_2\sqrt{1-N} > c_4$	$2\psi_{e,v} = P \mp L - i(Q \mp M)$	$\frac{1}{2}(P \mp L)$	$\frac{2\kappa}{P \mp L}$	$Q \mp M$	$4\pi\left\|\frac{Q \mp M}{P \mp L}\right\|$
3	$\left.\begin{array}{l} c_2\sqrt{1-N} > c_4 \\ c_2 < c_3 \end{array}\right\} \begin{array}{l} \kappa < \kappa_0 \\ \kappa > \kappa_0 \end{array}$	$2\psi_{e,v} = P \mp L - i(Q \mp M)$ $2\psi_{e,v} = P \pm L - i(Q \mp M)$	$\frac{1}{2}(P \mp L)$ $\frac{1}{2}(P \pm L)$	$\frac{2\kappa}{P \mp L}$ $\frac{2\kappa}{P \pm L}$	$Q \mp M$ $Q \mp M$	$4\pi\left\|\frac{Q \mp M}{P \mp L}\right\|$ $4\pi\left\|\frac{Q \mp M}{P \pm L}\right\|$
4	$\left.\begin{array}{l} c_2\sqrt{1-N} < c_4 \\ c_2 > c_3 \end{array}\right\} \begin{array}{l} \kappa < \kappa_0 \\ \kappa > \kappa_0 \end{array}$	$2\psi_{e,v} = P \pm L - i(Q \mp M)$ $2\psi_{e,v} = P \mp L - i(Q \mp M)$	$\frac{1}{2}(P \pm L)$ $\frac{1}{2}(P \mp L)$	$\frac{2\kappa}{P \pm L}$ $\frac{2\kappa}{P \mp L}$	$Q \mp M$ $Q \mp M$	$4\pi\left\|\frac{Q \mp M}{P \pm L}\right\|$ $4\pi\left\|\frac{Q \mp M}{P \mp L}\right\|$

are employed. Cases 3 and 4 in Table 1 include a change in the structure of the solution at a frequency κ_0. The frequency κ_0 at which the solution changes is given by

$$\kappa_0^2 = \frac{c_2^2 c_3^2}{\ell_2^2 (c_3^2 - c_2^2)} \left(\frac{c_2^2}{c_4^2}(1 - N) - 1 \right), \tag{4.16}$$

which is real if

$$c_3 \geqslant c_2 \geqslant \frac{c_4}{\sqrt{1-N}} \quad \text{or} \quad c_3 \leqslant c_2 \leqslant \frac{c_4}{\sqrt{1-N}}. \tag{4.17}$$

Although the solution to the dispersion relation (4.8) summarized in Table 1 is complex, it can be interpreted physically without a great deal of difficulty. To obtain these physical interpretations we first consider the special solutions of the dispersion relations for high and low frequency and then we consider specific numerical examples for materials that fall in the cases 1, 2, 3 and 4 of Table 1. We begin with the analysis and interpretation of the dispersion relations for high and low frequencies. For high frequencies, from (4.9), we can write

$$\psi_e^2 = \frac{\kappa^2}{c_2^2} + \frac{c_3^2 N}{\ell_2^2 (c_3^2 - c_2^2)} - \mathrm{i} \left(\frac{c_2^2 c_3^4}{c_3^2 - c_2^2} \right) \frac{N}{\ell_2^2} \frac{\omega}{\alpha \kappa} + \mathrm{O}\left(\frac{1}{\kappa^2} \right), \tag{4.18}$$

which yields

$$\psi_e = \frac{\kappa}{c_2} - \mathrm{i} \frac{c_2^3 c_3^4 N \omega}{2(c_3^2 - c_2^2)\ell_2^2 \alpha} \frac{1}{\kappa^2} + \mathrm{O}\left(\frac{1}{\kappa^3} \right) \tag{4.19}$$

and

$$\psi_v^2 = \frac{\kappa^2}{c_3^2} - \frac{1}{\ell_2^2} + \frac{c_3^2 N}{\ell_2^2 (c_3^2 - c_2^2)} - \mathrm{i} \frac{\omega}{\alpha} \kappa + \mathrm{O}(1), \tag{4.20}$$

which gives

$$\psi_v = \frac{\kappa}{c_3} - \mathrm{i} \frac{1}{2} \frac{\omega}{\alpha} c_3 + \mathrm{O}\left(\frac{1}{\kappa} \right). \tag{4.21}$$

Since c_2 is the classical elastic bulk wave speed and c_3 is a wave speed associated with the volume fraction, ψ_e represents the predominantly elastic wave and ψ_v represents the predominantly volume fraction wave. The phase speeds, attenuation coefficients and specific loss for these waves at high frequency are summarized in Table 2. These results show that the specific loss vanishes for both the waves as κ approaches infinity. In the case of low frequencies, it follows from (4.9), for small values of κ, that

$$\psi_e = \frac{\kappa}{c_2 \sqrt{1-N}} - \frac{\mathrm{i} N \ell_2 \kappa^2}{(1-N) c_2 c_4} \ldots, \tag{4.22}$$

and

$$\psi_v = \frac{\kappa}{c_4} - \frac{\mathrm{i}}{\ell_0} - \ldots. \tag{4.23}$$

Table 2. The phase speed, attenuation coefficient, specific loss and amplitude ratios for the elastic wave and the volume fraction wave at high and low frequencies.

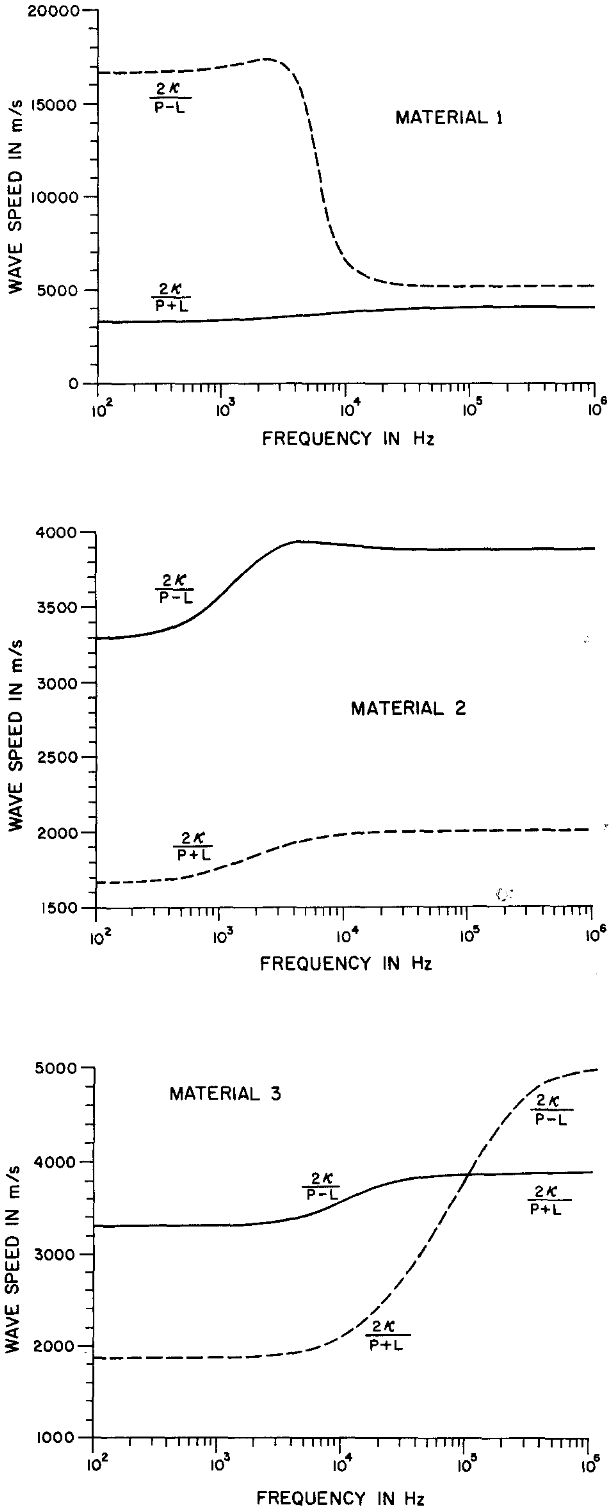
Frequency	Wave type	Phase speed	Attenuation coefficient	Specific loss	Amplitude ratio
High frequency	Elastic wave	c_2	$\frac{c_2^3 c_3^4 D^2 N}{c_4^3\left(c_3^2-c_2^2\right)^2(1-N)\ell_0}$	$\frac{4\pi c_2^4 c_3^4 D^3 N}{c_4^4\left(c_3^2-c_2^2\right)^2(1-N)}$	$\frac{2c_3^2}{c_2 c_4}\frac{1}{H\ell_0}$
	Volume fraction wave	c_3	$\frac{c_3}{c_4\ell_0}$	$4\pi\frac{c_3^2}{c_4^2}D$	∞
Low frequency	Elastic wave	$c_2\sqrt{1-N}$	$\frac{c_4 N}{(\sqrt{1-N})c_2 D^2\ell_0}$	$\frac{4\pi N}{D}$	0
	Volume fraction wave	c_4	$\frac{1}{\ell_0}$	$D=\frac{c_4}{\ell_0\kappa}$	$\frac{1}{H\ell_0}$

The phase speeds, attenuation coefficients and specific loss for these waves at low frequency are also summarized in Table 2. From Table 2 it can be seen that the phase speed for the elastic wave is not modified for large frequencies, but it is multiplied by the factor $(1-N)^{1/2}$ which is greater than zero but less than or equal to 1 for small frequencies. For the elastic wave, the attenuation is small at both large and small frequencies, and the specific loss approaches zero as κ tends to both zero and infinity. Since the specific loss is not, in general, zero it follows that it must be maximum for some value of κ. The predominantly volume fraction wave has different phase velocities, like the elastic wave, as the frequency tends to zero and infinity. However, the attenuation coefficient for this wave has a finite value at both the frequency extremes, indicating a greater attenuation than that experienced by the elastic wave. For the volume fraction wave the specific loss becomes unbounded as the frequency tends to zero and decreases to zero as the frequency becomes unbounded.

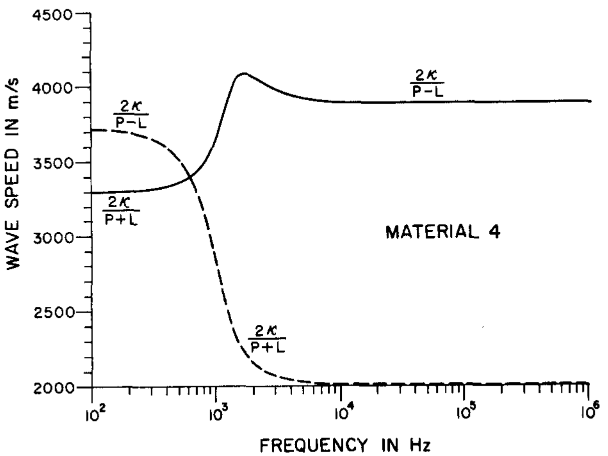
We return now to the interpretation of the general solution to the dispersion relation (4.8) summarized in Table 1, without the approximation of high or low frequencies. To give a graphical representation of the various solutions to the dispersion relation we consider four sets of material constants which specify four different hypothetical materials. The four different materials have values of λ, μ, ξ and β which are identical. These values, $\lambda = 15$ GPa, $\mu = 7.5$ GPa, $\xi = 12$ GPa and $\beta = 10$ GPa from (3.1) through (3.4), yield the following values for the indicated physical constants: $c_1 = 1937$ m/s, $c_2 = 3873$ m/s, $H = 1/3$ and $N = 0.2778$. The four different materials are distinguished by their values of c_3, ω, α and c_4, which are given in Table 3. Figures 1, 2, 3 and 4 are plots of the predominantly elastic wave and the predominantly volume

Table 3. The values of material constants for materials 1, 2, 3 and 4.

Material	c_3	ω	α	c_4
1	5000 m/s	1 M Pa s	8 GPa m^2	16,653 m/s
2	2000 m/s	10 M Pa s	8 GPa m^2	1665 m/s
3	5000 m/s	1 M Pa s	0.1 GPa m^2	1862.3 m/s
4	2000 m/s	10 M Pa s	40 GPa m^2	3724.5 m/s



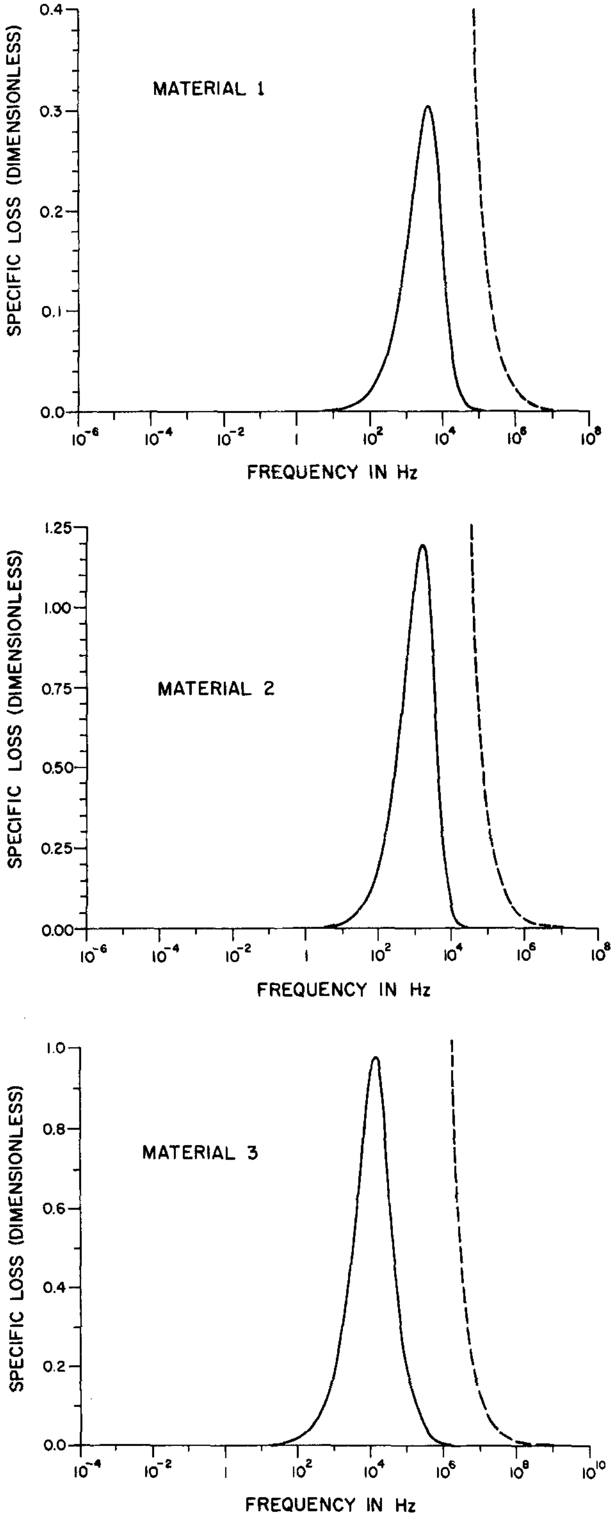
20000
15000
10000
5000
0
WAVE SPEED IN m/s
2K/P-L
2K/P+L
MATERIAL 1
10²
10³
10⁴
10⁵
10⁶
FREQUENCY IN Hz
4000
3500
3000
2500
2000
1500
WAVE SPEED IN m/s
2K/P-L
2K/P+L
MATERIAL 2
10²
10³
10⁴
10⁵
10⁶
FREQUENCY IN Hz
5000
4000
3000
2000
1000
WAVE SPEED IN m/s
MATERIAL 3
2K/P-L
2K/P+L
2K/P-L
2K/P+L
10²
10³
10⁴
10⁵
10⁶
FREQUENCY IN Hz

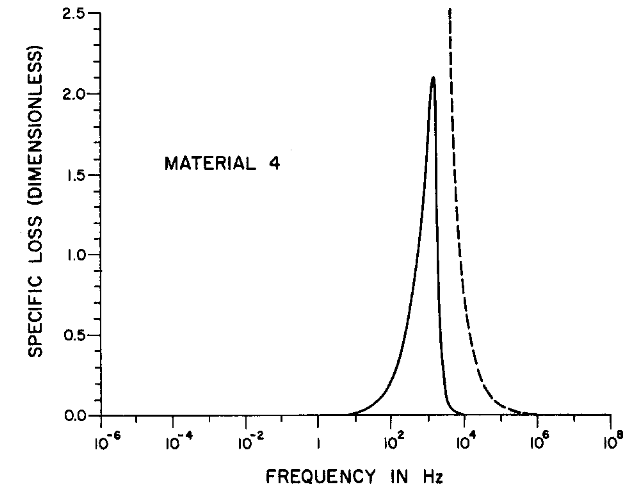


Figures 1 to 4. These are graphs of wave speeds against frequency for materials 1 through 4, respectively. The material constants $\lambda = 15$ GPa, $\mu = 7.5$ GPa, $\xi = 12$ GPa and $\beta = 10$ GPa are the same for each material and the values of c_3, ω, α and c_4 are given in Table 3 for materials 1, 2, 3 and 4. The solid curves represent the phase velocity for the predominantly elastic dilational wave and the dashed curves, the phase velocity for the predominantly void volume fraction wave. The value of the frequency at which these curves cross in Figs. 3 and 4 is given by (4.16).

fraction wave phase velocities against frequency for the materials 1, 2, 3 and 4, respectively. The materials 1, 2, 3 and 4 correspond to the cases 1, 2, 3 and 4 of Table 1 so that each solution case is represented by a material designated by the same number. Figures 5, 6, 7 and 8 are graphs of specific loss against frequency for materials 1, 2, 3 and 4, respectively. These graphs are all consistent with the limiting values obtained above for phase velocities and specific loss for large and small frequencies. In each case the predominantly elastic wave phase velocity generally, but not always, increases with frequency as shown by materials 3 and 4 in Figs. 3 and 4. The phase velocity of the predominantly volume fraction wave can be either less than or greater than the phase velocity of the predominantly elastic wave, and for materials 3 and 4 the two wave speeds cross at the frequency κ_0 given by (4.16). At this frequency the graph for the elastic wave velocity for material 3, Fig. 3, shifts smoothly from the branch $2\kappa/(P-L)$ for $\kappa < \kappa_0$ to the branch $2\kappa/(P+L)$ for $\kappa > \kappa_0$ and the situation is reversed for the void volume wave speed. An analogous, but reversed, situation pertains in the case of material 4 as illustrated by Fig. 4. These are simply the graphic representations of the analytical results for wave speeds summarized as cases 3 and 4 in Table 1. The graphs of the wave speeds in Figs. 1 through 4 clearly indicate the narrow range of frequency in which the transition from wave speeds at small frequencies to wave speeds at large frequencies takes place. Note that the frequency scales in Figs. 1 through 4 all start at 100 Hz rather than zero Hz. This representation was chosen because all the wave velocities were constant in this frequency domain and were equal to their values at 100 Hz.

The graphs of specific loss versus frequency given in Figs. 5 through 8 for materials 1 through 4, respectively, are basically quite similar to one another. For the predomi-


SPECIFIC LOSS (DIMENSIONLESS)
MATERIAL 1
0.4
0.3
0.2
0.1
0.0
10^{-6} 10^{-4} 10^{-2} 1 10^{2} 10^{4} 10^{6} 10^{8}
FREQUENCY IN Hz
SPECIFIC LOSS (DIMENSIONLESS)
MATERIAL 2
1.25
1.00
0.75
0.50
0.25
0.00
10^{-6} 10^{-4} 10^{-2} 1 10^{2} 10^{4} 10^{6} 10^{8}
FREQUENCY IN Hz
SPECIFIC LOSS (DIMENSIONLESS)
MATERIAL 3
1.0
0.8
0.6
0.4
0.2
0.0
10^{-4} 10^{-2} 1 10^{2} 10^{4} 10^{6} 10^{8} 10^{10}
FREQUENCY IN Hz



Figures 5 to 8. These are graphs of specific loss against frequency for materials 1 through 4, respectively. The material constants are as described in the captions to Figs. 1 to 4. The solid curves represent the specific loss for the predominantly elastic dilational wave and the dashed curves, the phase velocity for the predominantly void volume fraction wave.

nantly elastic wave the specific loss curve is zero at the extremes of high and low frequency and rises to a relatively sharp peak near the frequency κ_0 at which the phase velocities are most rapidly changing. For the predominantly void volume wave the specific loss is always greater than for the elastic wave and Figs. 5 through 8 suggest that this wave will be damped out at all but the highest frequencies.

5. The amplitude ratios of the plane harmonic waves

In this section, in order to obtain further insight into the nature of these waves, we determine the ratio of the amplitude of the void fraction wave to the amplitude of the elastic wave. From (4.5) we can write this ratio of amplitudes as

$$R \equiv \frac{B}{A} = \frac{\rho\kappa^2 - (\lambda + 2\mu)\psi^2}{\mathrm{i}\beta\psi} = \frac{1}{\mathrm{i}}\left(\frac{\rho\kappa^2}{\beta\psi} - \frac{\psi}{H}\right). \tag{5.1}$$

There are two amplitude ratios of the form (5.1) corresponding to the two solutions, ψ_e and ψ_v, of the dispersion relation (4.8). The complete solutions (4.1) of the displacement field $u_i(x, t)$ and the field $\phi(x, t)$ are given by

$$u_i = d_i(A_e U_e(x, t) + A_v U_v(x, t)), \qquad \phi = B_e U_e(x, t) + B_v U_v(x, t), \tag{5.2}$$

where

$$U_e(x, t) = \mathrm{Re}(\exp(\mathrm{i}(\kappa t - \psi_e m_i x_i))), \qquad U_v(x, t) = \mathrm{Re}(\exp(\mathrm{i}(\kappa t - \psi_v m_i x_i))). \tag{5.3}$$

There are, therefore, two amplitude ratios, R_e and R_v, given by

$$R_e = \frac{B_e}{A_e}, \quad R_v = \frac{B_v}{A_v}. \tag{5.4}$$

In the case of the predominantly elastic wave the amplitude ratio R_e is given by (5.1) as

$$R_e = \frac{1}{H}\left|\psi_e - \frac{c_3\psi_v}{c_2\left(1 - i\frac{\omega}{\alpha\kappa}c_3^2 - \frac{c_3^2}{\ell_2^2\kappa}\right)^{1/2}}\right|. \tag{5.5}$$

For large κ, (5.5) has the approximation

$$R_e = \frac{c_3^2}{c_2c_4}\frac{1}{\ell_0 H} + O\left(\frac{1}{\kappa}\right), \tag{5.6}$$

and for small κ,

$$R_e = \frac{\ell_2\ell_0}{\ell_1^2}\frac{\kappa}{c_2} + O(\kappa^2). \tag{5.7}$$

For the wave that is predominantly void volume fraction, (5.1) yields

$$R_v = \frac{1}{H}\left|\psi_v - \frac{c_3\psi_e}{c_2\left(1 - i\frac{\omega}{\alpha\kappa}c_3^2 - \frac{c_3^2}{\ell_2^2\kappa^2}\right)^{1/2}}\right|, \tag{5.8}$$

where the subscript indicates the wave type. For large values of κ,

$$R_v = \frac{1}{Hc_3}\left|1 - \frac{c_3^2}{c_2^2}\right|\kappa + O(1), \tag{5.9}$$

and for small κ,

$$R_v = \frac{1}{H\ell_0} + O(\kappa). \tag{5.10}$$

Thus we can conclude, for the elastic wave

$$\operatorname*{limit}_{\kappa\to 0} R_e = 0, \quad \operatorname*{limit}_{\kappa\to\infty} R_e = \frac{c_3^2}{c_2c_4}\frac{1}{H\ell_0} \tag{5.11}$$

and for the void volume wave

$$\operatorname*{limit}_{\kappa\to 0} R_v = \frac{1}{H\ell_0}, \quad \operatorname*{limit}_{\kappa\to\infty} R_v \to \infty. \tag{5.12}$$

These results have also been summarized in Table 2. They show that the void volume change amplitude of the elastic wave vanishes at zero frequency, and the displacement amplitude of void volume wave vanishes as the frequency becomes infinite. This suggests that the predominantly elastic wave is purely elastic at zero frequency and that the predominantly void volume wave is purely a void volume wave as the frequency tends to infinity. In other portions of the frequency domain these results show that both waves have non-zero amplitudes of both types.

6. Discussion

Our conclusions are summarized in the following section. In this section we discuss two topics. The first is the relation of the present work to the study of Nunziato and Walsh [7]. The second topic is the estimation of the frequency at which the specific loss reaches its maximum or peak value.

Nunziato and Walsh [7] consider the propagation of infinitesimal sinusoidal progressive waves in the context of a one dimensional theory which is identical with the theory employed here when the ω appearing in equation (2.8) is set equal to zero. Their results on harmonic waves can be obtained from the results presented here by setting $\omega = 0$. Nunziato and Walsh point out a resonance that occurs at a particular frequency. Our results show that this resonance will not occur unless we eliminate the damping in the void volume, that is to say, unless we set $\omega = 0$. The results of the present work, summarized in Table 1, show that only one wave can have a resonance and that the resonance will occur when $P = L$. A close inspection of the expressions for P and L reveals that P and L cannot be equal unless ω and a are both zero. The condition that a be zero yields the same resonance frequency as given in [7]. In [7] it appears that there is resonance in both the waves, but a closer investigation of $c^2(\omega)$ given by equation (48) in [7] yields the fact that there is resonance only in one wave and for the other wave, in the notation of [7], $c^2 = v_1^2 v_2^2/(v_2^2\omega_r^2 + \lambda_0)$ as the frequency approaches ω_r. All the results for plane waves in [7], except for the velocity of void volume wave at small frequencies, can be obtained from the results presented here by letting $\omega = 0$. The results for small frequencies can be obtained by letting $\omega = 0$ in (4.9) and (4.10).

Our second topic in this section is estimating the frequency at which the peak occurs in the specific loss versus frequency curve. This peak is illustrated in each of the Figs. 5 through 8. It would be quite useful to have a formula for the frequency in question. In theory one could take the derivative of the specific loss with respect to frequency and set it equal to zero to obtain the value at which the specific loss is an extremum. In practice, the calculation is too complex. Following Deresiewicz [4] it is possible to estimate desired frequency by assuming that the coupling number N is small. The zero order approximation of the frequency in question for small values of N is

$$\kappa = \frac{c_2^2 c_3^2}{\sqrt{6}\,|c_3^2 - c_2^2|}\left[\left\{16\left(\frac{1}{c_2^2} - \frac{1}{c_3^2}\right)^2 \frac{1}{\ell_2^4} + 4\frac{\omega^2}{\alpha^2 \ell_2^2}\left(\frac{1}{c_2^2} - \frac{1}{c_3^2}\right) + \frac{\omega^4}{\alpha^4}\right\}^{1/2} - \left\{\frac{2}{\ell_2^2}\left(\frac{1}{c_2^2} - \frac{1}{c_3^2}\right) + \frac{\omega^2}{\alpha^2}\right\}\right]^{1/2} + \mathrm{O}(N). \tag{6.1}$$

7. Conclusions

1. There are two dilational and one equivoluminal plane waves in a linear elastic material with voids. The equivoluminal wave has the same characteristics as the corresponding wave in classical elasticity. Of the two dilational waves, one is predominantly the elastic wave and the other can be identified as a predominantly void volume fraction wave. The presence of the voids introduces the second dilational wave and a dissipative mechanism associated with the voids causes both waves to attenuate. The coupling of the equations of motion also makes the waves dispersive in nature. There is no resonance unless the dissipative mechanism, denoted by ω, is taken to be zero.

2. At large frequencies the predominantly elastic dilational wave propagates with the classical dilational velocity denoted here as c_2. As the frequency approaches infinity for the elastic wave, the attenuation coefficient and consequently the specific loss are very small and approach zero. The void volume wave at large frequencies propagates with constant speed, has a constant attenuation coefficient, but a small specific loss which approaches zero as the frequency becomes infinite. These remarks are illustrated by the phase velocity versus frequency curves, Figs. 1 to 4, and the specific loss versus frequency curves, Figs. 5 to 8.

3. At small frequencies the velocity of the elastic wave is given by $c_2\sqrt{1-N}$ where $0 \leqslant N < 1$ and c_2 is the classical phase velocity of the elastic wave. For the elastic wave, the attenuation coefficient and specific loss remain small and approach zero as the frequency approaches zero. The low frequency phase velocity of the void volume wave is also different from its phase velocity at large frequencies, as can be seen in each of the Figs. 1 to 4. The attenuation coefficient of this wave is modified but remains a constant. The specific loss, however, is large and increases without limit as the frequency approaches zero. The void volume wave is so heavily damped at any but a very high frequency that it is unlikely to be observed at those frequencies. These results are illustrated by the Figs. 5 to 8.

4. The specific loss for the elastic wave approaches zero at both ends of the frequency spectrum. In general it has an extremal value between the two limits, which in Figs. 5 through 8 is a single sharp peak. The specific loss for the void volume wave has an infinite value at zero frequency and decreases monotonically to zero as the frequency approaches infinity.

5. The solutions for displacement (or stress) and the void volume fraction ϕ consist of two parts corresponding to ψ_e and ψ_v, representing elastic and void volume effects. Each part is a wave form. The amplitude of the elastic wave in the solution for ϕ is very small as compared to the amplitude of the elastic wave in the solution for u_i at small frequencies. At large frequencies the ratio of the two amplitudes is a constant. The amplitude of the void volume fraction wave in the solution for ϕ is very large as compared to the amplitude of the void volume fraction wave in the solution for u_i at large frequencies. For small frequencies the ratio of the two amplitudes is a constant.

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