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The Use of a Virtual Configuration in Formulating Constitutive Equations for Residually Stressed Elastic Materials

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Abstract. Residual stress is the stress present in the unloaded equilibrium configuration of a body. Because residual stresses can significantly affect the mechanical behavior of a component, the measurement of these stresses and the prediction of their effect on mechanical behavior are important objectives in many engineering problems. Common methods for the measurement of residual stresses include various destructive experiments in which the body is cut to relieve the residual stress. The resulting strain is measured and used to approximate the original residual stress in the intact body. In order to predict the mechanical behavior of a residually stressed body, a constitutive model is required that includes the influence of the residual stress.

In this paper we present a method by which the data obtained from standard destructive experiments can be used to derive constitutive equations that describe the mechanical behavior of elastic residually stressed bodies. The derivation is based on the idea that for each infinitesimal neighborhood in a residually stressed body, there exists a corresponding stress free configuration. We refer to this stress free configuration as the 'virtual' configuration of the infinitesimal neighborhood. The derivation requires that the constitutive equation for the stress free material be known and invertible; it is used to relate the residual stress to the deformation of the virtual configuration into the residually stressed configuration. Although the concept of the virtual configuration is central to the derivation, the geometry of this configuration need not be determined explicitly, and it need not be achievable experimentally, in order to construct the constitutive equation for the residually stressed body.

The general mathematical forms of constitutive equations valid for residually stressed elastic materials have been derived previously for a number of cases. These general forms contain numerous unknown material-response functions or material constants that must be determined experimentally. In contrast, the method presented here results in a constitutive equation that is an explicit function of residual stress and includes only the material parameters required to describe the stress free material.

After presenting the method for the derivation of constitutive equations, we explore the relationship between destructive experiments and the theory used in the derivation. Specifically, we discuss the use of the theory to improve the design of destructive experiments, and the use of destructive experiments to obtain the data required to construct the constitutive equation for a particular material.

Key words: residual stress, constitutive equation, elastic material, destructive experiment

1. Introduction

Residual stress is the stress supported by a body that is in mechanical equilibrium in the absence of external forces. These stresses are a major concern for numerous industries because they can significantly affect the mechanical properties of a component. For example, residual stresses in silicon semiconductor films can cause delamination, cracking, and stress migration of atoms, all of which degrade the stability of microelectronic devices [e.g., 1, 2]. During manufacture, composite materials often develop residual stresses that can cause cracking in the material even before it enters service. This damage clearly reduces the fatigue life of the final product [3]. Residual stresses in castings are a problem as well. Produced by thermal gradients during cooling, they often cause excessive distortion when the castings are machined.

It should be noted, however, that not all residual stresses are detrimental. Compressive residual stresses are designed into ceramic components [4] and steel roller bearings [5] to increase fatigue life and tensile failure limits. Some biological tissues, such as the heart, arteries, veins, and trachea, support residual stresses [6]. The residual stresses in the heart and arteries are thought to minimize the peak stresses experienced by these tissues in vivo [7].

There are two central issues in work dealing with residually stressed bodies. The first issue is the determination of the residual stress field supported by a given body, and the second issue is the derivation of a constitutive equation with which to describe the mechanical behavior of the residually stressed material. Because residual stresses are a common and potentially critical phenomenon, significant efforts have been made to develop methods for the detection and measurement of residual stress fields. Although there are a number of non-destructive techniques with varying applicability and effectiveness [e.g., 8, 9], we concentrate here on destructive and semi-destructive experiments [e.g., 8, 10] because they are the methods most commonly used to estimate residual stresses in elastic bodies. These experiments include sectioning the body into separate pieces, hole drilling, ring coring, and surface grinding. In all cases the body is cut in a way that is supposed to relieve residual stress, if any is present. The objective is to obtain a stress free configuration of a part of the body, and to measure the displacement that results from the relief of the residual stress. Then, in order to estimate the residual stress originally supported by that part of the body, the constitutive equation for the stress free material is evaluated with the strain calculated from the measured displacement data. (This process is discussed in more detail in Section 6.)

Prediction of the mechanical behavior of the material in the intact residually stressed body requires a constitutive model that includes the influence of the residual stress. Such a constitutive equation can be difficult to construct because the experimental approach used to measure properties of stress free materials cannot be applied to residually stressed materials. The mechanical properties of the material in a *non*-residually stressed body can be measured with experiments performed on a geometrically simple piece of the material. A constitutive law formulated for this specimen can then be used for any body composed of the same material. In contrast, the effective elastic properties of the material in a residually stressed body depend on the residual stress [11]. If a portion of the body is excised, some or all of the residual stress in that portion is relieved. Therefore, the material properties measured with this excised specimen typically will not represent the properties of the material in the intact residually stressed body. This is the case even for a residually stressed body composed of a material that would be homogeneous and isotropic if it supported no residual stress.

The question, then, is how to include the effects of the residual stress in the constitutive equation for the material in an elastic residually stressed body. In this paper we demonstrate how to derive a constitutive equation that (1) depends explicitly on the residual stress, (2) includes only the material properties required to describe the stress free state of the material, and (3) is especially suited to model the destructive experiments described above. By the method presented here, the constitutive equation can be derived for a point in an elastic residually stressed body if the constitutive equation for the stress free material associated with that point is known and invertible. Constitutive equations can be derived that are appropriate for either infinitesimal or finite deformations of the residually stressed body, given the appropriate constitutive equation for the stress free material. Knowledge of the process that produced the residual stress is not required.

The derivation of the constitutive equation is based on the idea that for each infinitesimal neighborhood in the residually stressed body there exists a corresponding stress free configuration. The derivation is presented first for a special class of residually stressed bodies that can be cut, as in a destructive experiment, into a collection of completely stress free parts with finite volume. By definition, the residually stressed bodies in this special class possess the necessary stress free configurations. In order to derive the constitutive equation for a more general residually stressed body, we establish conditions sufficient for the existence of a stress free configuration of the material in the neighborhood of a point. We show that in the general case the stress free configuration is attained in the limit as the volume of the neighborhood approaches zero, so this stress free configuration can be thought of as a point. When the residual stress and the properties of the stress free material are sufficiently smooth functions of position in the neighborhood, this stress free configuration can be used to derive the constitutive equation for the corresponding point in the residually stressed body.

In this paper we will refer to the stress free configuration of a part of a body as the 'virtual configuration' of that part. The virtual configuration is employed only to give a physical interpretation of the mathematics used in the derivation; we use the adjective 'virtual' to emphasize that this configuration is a conceptualization. The geometry of the virtual configuration need not be determined explicitly, and the virtual configuration need not be achievable experimentally. To be consistent with standard terminology, we will refer to material that supports no stress as 'natural material'; thus, a virtual configuration is composed of natural material. This includes the case where the virtual configuration is attained in the limit as the volume approaches zero.

The mathematical preliminaries employed in this paper are recalled in Section 2, and the method for the derivation of the constitutive equation is presented in Section 3. In Sections 4 and 5, constitutive equations are formulated for specific residually stressed bodies. For each of these examples, the residual stress field is given and the constitutive equation for the natural material is known and invertible. The residually stressed body in the example of Section 4 is a member of the special class described previously, for which the virtual configurations have finite volume. A more typical residually stressed body is considered in Section 5. In Section 6 we focus on the relationship between destructive experiments and the virtual configuration. We first determine approximate virtual configurations of specific parts of the example residually stressed body from Section 5. Then, since the objective of destructive experiments is to obtain an approximate virtual configuration experimentally, we explore the use of a mathematical approximate virtual configuration in the design of these destructive experiments.

To appreciate the advantages of the method presented here, one need only review the constitutive equations available prior to this work. Constitutive equations have been derived for elastic residually stressed materials under deformations with small displacement gradients [12, 13, 14], and deformations with small strains and arbitrary rotations [15]. These constitutive equations are derived by the linearization of the finite elastic constitutive equation for a stress free material, so each contains a fourth-order elasticity tensor that depends implicitly on the residual stress. The functional form of this dependence on residual stress is not known explicitly in most cases, which limits the usefulness of these constitutive equations. A method has also been developed to obtain the most general form of the constitutive equation for any hyperelastic residually stressed material with known material symmetry. The method is demonstrated in the context of transversely isotropic residually stressed materials for both finite [16] and infinitesimal [17] deformations. The explicit dependence of these general forms on residual stress is known, but they contain numerous unknown material-response functions or material constants that must be determined experimentally. Hence, these constitutive equations are impractical to use in cases where the opportunity for experiments is limited. None of the above derivations requires knowledge of the process that produced the residual stress.

In contrast, for a constitutive equation derived by the method presented here, the explicit functional form of the dependence on residual stress is known and the equation contains only the material properties required to describe the natural material. This method can be used to obtain the constitutive equation for any residually stressed material that behaves elastically in deformations from the residually stressed configuration. This includes all of the examples described at the beginning of this section. In addition, the data required to construct the constitutive equation for a specific residually stressed body can be obtained from standard destructive experiments on that body. Therefore, in many situations constitutive equations derived by the method presented in this paper will be more useful than those previously available.

2. Preliminaries

In this section we introduce the notation that will be used throughout the paper. The principal invariants of a tensor $\mathbf{A}$ are

$$I_{\mathbf{A}} = \operatorname{tr} \mathbf{A},$$
$$II_{\mathbf{A}} = \tfrac{1}{2}[(\operatorname{tr} \mathbf{A})^2 - \operatorname{tr} \mathbf{A}^2], \tag{2.1}$$
$$III_{\mathbf{A}} = \det \mathbf{A} = \tfrac{1}{6}[(\operatorname{tr} \mathbf{A})^3 - 3(\operatorname{tr} \mathbf{A})(\operatorname{tr} \mathbf{A}^2) + 2\operatorname{tr} \mathbf{A}^3].$$

By the Cayley-Hamilton theorem every tensor satisfies its own characteristic equation; thus,

$$\mathbf{A}^3 - I_{\mathbf{A}}\mathbf{A}^2 + II_{\mathbf{A}}\mathbf{A} - III_{\mathbf{A}}\mathbf{1} = \mathbf{0}. \tag{2.2}$$

Consider a body that is in equilibrium in an unloaded reference configuration. The deformation $\mathbf{f}$ is a smooth, one-to-one mapping of each point $\mathbf{p}$ in the body into a corresponding point $\mathbf{x} = \mathbf{f}(\mathbf{p})$ in the deformed configuration of the body. Since $\mathbf{f}$ is one-to-one, there exists an inverse function $\mathbf{f}^{-1}$ such that $\mathbf{p} = \mathbf{f}^{-1}(\mathbf{x})$. It is assumed that the deformation gradient $\mathbf{F} = \nabla\mathbf{f}$ satisfies $\det \mathbf{F} > 0$, so that by the polar decomposition theorem, $\mathbf{F}$ has the unique representation

$$\mathbf{F} = \mathbf{VR},$$

where the left stretch tensor $\mathbf{V}$ is positive definite symmetric and the rotation $\mathbf{R}$ is proper orthogonal. We will find it convenient to use the left Cauchy-Green strain tensor $\mathbf{B}$, which is defined by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \tag{2.3}$$

A residual stress $\overset{\circ}{\mathbf{T}}$ is a stress field supported by a body that is in mechanical equilibrium in the absence of surface tractions and body forces. The residually stressed configuration of the body will be denoted by $\mathcal{B}_0$. Clearly, a residual stress field must satisfy

$$\operatorname{div} \overset{\circ}{\mathbf{T}} = \mathbf{0}$$

and (2.4)

$$\overset{\circ}{\mathbf{T}} = \overset{\circ}{\mathbf{T}}^T$$

in the configuration $\mathcal{B}_0$, and the zero-traction condition

$$\overset{\circ}{\mathbf{T}}\mathbf{n} = \mathbf{0} \tag{2.5}$$

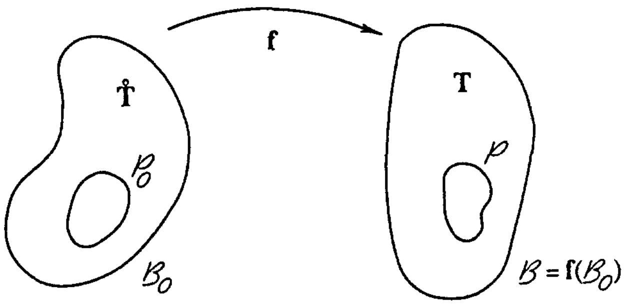

Fig. 1. The unloaded configuration $\mathcal{B}_0$, containing the part $\mathcal{P}_0$, supports a residual stress field $\overset{\circ}{\mathbf{T}}$. The deformation $\mathbf{f}$ maps $\mathcal{B}_0$ into configuration $\mathcal{B}$, which supports the Cauchy stress $\mathbf{T}$. The response of $\mathcal{B}_0$ to $\mathbf{f}$ is elastic.

on the surface $\partial\mathcal{B}_0$, where $\mathbf{n}$ is the outward unit normal.

One consequence of the zero-traction condition (2.5) is that the residual stress depends on the shape of $\mathcal{B}_0$. This condition restricts the form that a residual stress field can have in a given body; such restrictions will play a role in the example of Section 5. Another consequence of (2.5) is that the residual stress is necessarily inhomogeneous [13]. Therefore, since the response of the material in $\mathcal{B}_0$ depends on the residual stress, the constitutive equation appropriate for deformations out of configuration $\mathcal{B}_0$ must also depend on position.

For an elastic body that supports no residual stress, the stress is a function only of the current configuration, so at each point $\mathbf{x}$ in the deformed body, the Cauchy stress can be expressed in terms of a response function $\tilde{\mathfrak{T}}$ of the deformation gradient as

$$\mathbf{T}(\mathbf{x}) = \tilde{\mathfrak{T}}(\mathbf{F}(\mathbf{x}), \mathbf{x}). \tag{2.6}$$

Equation (2.6) is the constitutive equation for the material in a stress free (natural) configuration. We will use the term 'natural material' to refer to material in the natural configuration.

For the special case of an *isotropic* elastic material, the combination of material symmetry and the principle of material frame indifference gives a constitutive equation equivalent to (2.6) for the Cauchy stress in terms of $\mathbf{B}$ [18]:

$$\mathbf{T}(\mathbf{x}) = \bar{\mathfrak{T}}(\mathbf{B}(\mathbf{x}), \mathbf{x}). \tag{2.7}$$

Through the remainder of the paper, dependence on position will not always be shown explicitly. However, the fact that these constitutive equations are defined pointwise will play a central role in some of the arguments.

3. Derivation of the Constitutive Equation

Consider a body that supports a residual stress field $\overset{\circ}{\mathbf{T}}$ in the undeformed configuration $\mathcal{B}_0$, as shown in Fig. 1, and assume that the body responds elastically to

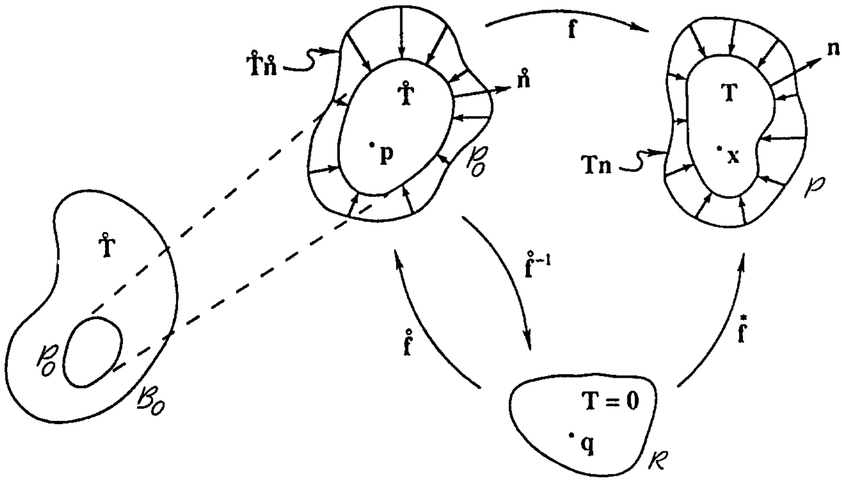

Fig. 2. On release of the surface tractions $\mathring{T}\mathring{n}$, the part $\mathcal{P}_0$ relaxes into the stress free configuration $\mathcal{R}$. This deformation is described by the mapping $\mathring{f}^{-1}(p)$.

a deformation **f** out of this residually stressed configuration. Our objective is to derive a constitutive equation with which to model the deformation of the body from configuration $\mathcal{B}_0$ into configuration $\mathcal{B} = \mathbf{f}(\mathcal{B}_0)$. The dependence of the constitutive equation on residual stress will be determined explicitly, and the constitutive equation will include only the material response functions or material constants required to describe the natural material. Thus, given the residual stress field and the properties of the natural material, this constitutive equation can be used to describe the mechanical behavior of the residually stressed material. It is assumed in the derivation that the constitutive equation for the natural material is known and invertible. However, knowledge of the process that produced the residual stress is not required.

In Section 3.1 we present the derivation in the context of a special class of residually stressed bodies for which the approach is transparent. The derivation for a more general class of residually stressed bodies is presented in Section 3.2. That derivation uses the basic method of Section 3.1, but requires additional technical details.

3.1. A SPECIAL CASE

Consider the special class of residually stressed bodies defined as follows. Suppose $\mathcal{B}_0$ can be cut into a finite number of parts such that each part is entirely free of residual stress. That is, if a part $\mathcal{P}_0$ were removed from $\mathcal{B}_0$ and relieved of the tractions imposed by the rest of the body, it would deform elastically into a stress free region. This deformation is denoted by $\mathring{\mathbf{f}}^{-1}$ in Fig. 2, where $\mathcal{R}$ is a stress free configuration of the part $\mathcal{P}_0$. (In contrast, the removal and unloading of a portion of a typical residually stressed body will not necessarily result in a stress free region.)

We now turn to the derivation of the constitutive equation for the material at a point $\mathbf{p}$ in $\mathcal{P}_0$. The constitutive equation will be appropriate for a deformation $\mathbf{f}$ out of the residually stressed configuration $\mathcal{B}_0$. We will show that the Cauchy stress $\mathbf{T}$ in the deformed configuration is given by a response function $\hat{\mathfrak{T}}$ of the deformation gradient $\mathbf{F}(\mathbf{p}) = \nabla\mathbf{f}(\mathbf{p})$ and the residual stress $\overset{\circ}{\mathbf{T}}(\mathbf{p})$:

$$\mathbf{T} = \hat{\mathfrak{T}}(\mathbf{F}, \overset{\circ}{\mathbf{T}}). \tag{3.1}$$

By this derivation, the explicit functional form of $\hat{\mathfrak{T}}$ will be obtained for a point in the body in terms of the residual stress and the mechanical properties of the natural material associated with that point.

Because the material in $\mathcal{R}$ is elastic and unstressed, the appropriate constitutive equation for any deformation out of $\mathcal{R}$ is the constitutive equation for the natural material, equation (2.6). In particular, the deformation $\overset{\circ}{\mathbf{f}}$ shown in Fig. 2 is the deformation of the region $\mathcal{R}$ that would be required to produce the residual stress $\overset{\circ}{\mathbf{T}}$ in the part $\mathcal{P}_0$. Hence, $\overset{\circ}{\mathbf{T}}$ is given by (2.6) evaluated at $\overset{\circ}{\mathbf{F}} = \nabla\overset{\circ}{\mathbf{f}}$:

$$\overset{\circ}{\mathbf{T}} = \tilde{\mathfrak{T}}(\overset{\circ}{\mathbf{F}}). \tag{3.2}$$

Similarly, for the deformation $\overset{*}{\mathbf{f}}$, which maps $\mathcal{R}$ into a configuration $\mathcal{P} = \mathbf{f}(\mathcal{P}_0)$, the stress $\mathbf{T}$ in $\mathcal{P}$ is given by

$$\mathbf{T} = \tilde{\mathfrak{T}}(\overset{*}{\mathbf{F}}). \tag{3.3}$$

Next, note that the deformation $\overset{*}{\mathbf{f}}$ is the composition of $\mathbf{f}$ and $\overset{\circ}{\mathbf{f}}$,

$$\overset{*}{\mathbf{f}} = \mathbf{f} \circ \overset{\circ}{\mathbf{f}},$$

so the deformation gradients are related by

$$\overset{*}{\mathbf{F}} = \mathbf{F}\overset{\circ}{\mathbf{F}}. \tag{3.4}$$

Substitution of (3.4) into (3.3) gives the Cauchy stress in $\mathcal{B}$ as

$$\mathbf{T} = \tilde{\mathfrak{T}}(\mathbf{F}\overset{\circ}{\mathbf{F}}). \tag{3.5}$$

To obtain a constitutive equation in the form of (3.1), we need only express $\overset{\circ}{\mathbf{F}}$ explicitly in terms of $\overset{\circ}{\mathbf{T}}$. We assume that (3.2), i.e., $\overset{\circ}{\mathbf{T}} = \tilde{\mathfrak{T}}(\overset{\circ}{\mathbf{F}})$, is invertible, so there exists a function $\tilde{\mathfrak{f}}$ such that

$$\overset{\circ}{\mathbf{F}} = \tilde{\mathfrak{F}}(\overset{\circ}{\mathbf{T}}). \tag{3.6}$$

Equations (3.5) and (3.6) combine to give

$$\mathbf{T} = \tilde{\mathfrak{T}}(\mathbf{F}\tilde{\mathfrak{F}}(\overset{\circ}{\mathbf{T}})), \tag{3.7}$$

which is of the desired form, $\mathbf{T} = \hat{\mathfrak{T}}(\mathbf{F}, \overset{\circ}{\mathbf{T}})$.

Note that the deformation $\overset{\circ}{\mathbf{f}}$ is not needed for the derivation of the constitutive equation; only the gradient $\overset{\circ}{\mathbf{F}}$ is required, and is obtained from the residual stress through (3.6). As a consequence, the configuration $\mathcal{R}$ need not be determined explicitly in the derivation. For this special class of residually stressed bodies, the derivation requires only that the natural material in $\mathcal{R}$ respond elastically under the deformations $\overset{\circ}{\mathbf{f}}$ and $\overset{*}{\mathbf{f}}$, and that the constitutive equation for the natural material be invertible.

3.2. The General Case

In the derivation presented in Section 3.1, we inverted the constitutive equation for the natural material to obtain the deformation gradient $\overset{\circ}{\mathbf{F}}$ as a function of the residual stress $\overset{\circ}{\mathbf{T}}$. This step can be accomplished without a physical interpretation of $\overset{\circ}{\mathbf{F}}$. However, to communicate the physical meaning of $\overset{\circ}{\mathbf{F}}$, and $\overset{*}{\mathbf{F}}$ as well , we incorporated the idea of the stress free region $\mathcal{R}$ with non-zero volume into the derivation. As we have already mentioned, such a region $\mathcal{R}$ does not exist for residually stressed bodies that do not belong to the special class treated in Section 3.1. Therefore, we will identify a counterpart to the region $\mathcal{R}$ that will provide a physical interpretation of the derivation for a general class of residually stressed bodies. We will see that for a typical residually stressed body, the stress free configuration of the material in the neighborhood of a point is attained in the limit as the volume of the neighborhood approaches zero. This stress free configuration can therefore be thought of as a point. Furthermore, since the properties of the material in the excised configuration of the neighborhood approach those of the natural material as the volume (and the stress) approaches zero, the constitutive equation for the associated point in the residually stressed body can be derived using the basic approach of Section 3.1.

Of course, every constitutive equation is derived in terms of the material properties at a specific point in a body. However, this idea is of central importance here

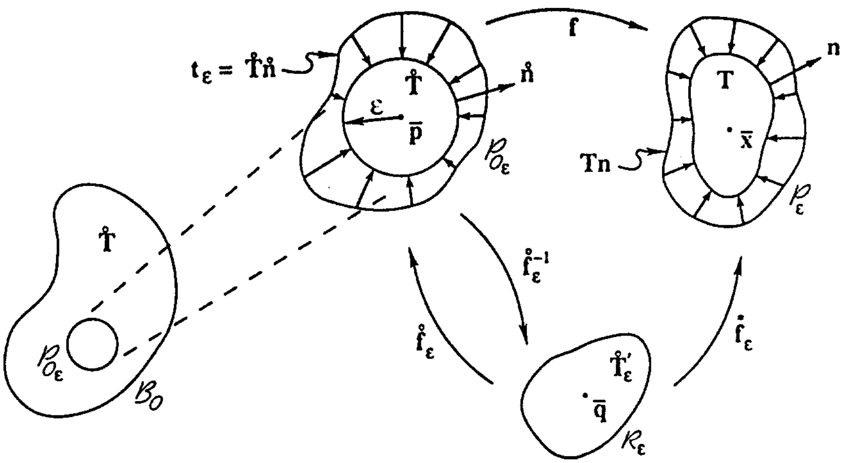

Fig. 3. The part $\mathcal{P}_{0_\varepsilon}$ is a spherical neighborhood with radius ε of the point $\bar{\mathbf{p}}$. On release of the surface tractions $\overset{\circ}{\mathbf{T}}\overset{\circ}{\mathbf{n}}$, the part $\mathcal{P}_{0_\varepsilon}$ relaxes into the residually stressed configuration $\mathcal{R}_\varepsilon$.

because residually stressed bodies are necessarily inhomogeneous. Therefore, the functional form of the constitutive equation may change with position, even if the underlying natural material is homogeneous. Our use of an infinitesimal neighborhood of a point, and the limit as the volume of the neighborhood approaches zero, is in the same spirit as the use of a similar limit in the proof of Cauchy's theorem [18]. That is, information about the stress state at a point is obtained by first considering the conditions on a finite volume of material containing the point, and then taking the limit as the volume decreases to the point.

The Stress Free Configuration

We define a part $\mathcal{P}_{0_\varepsilon}$ of the residually stressed body to be a spherical neighborhood with radius ε of a point $\bar{\mathbf{p}}$ in $\mathcal{B}_0$. Let the boundary of $\mathcal{P}_{0_\varepsilon}$ be denoted by $\partial\mathcal{P}_{0_\varepsilon}$. A typical $\mathcal{P}_{0_\varepsilon}$ is shown in Fig. 3 as it would be if it were removed from the residually stressed body, and the tractions $\mathbf{t}_\varepsilon = \overset{\circ}{\mathbf{T}}\overset{\circ}{\mathbf{n}}$ imposed on $\partial\mathcal{P}_{0_\varepsilon}$ by the rest of the body were maintained. (The vector $\overset{\circ}{\mathbf{n}}$ is the outward unit normal to $\partial\mathcal{P}_{0_\varepsilon}$.) If the tractions $\mathbf{t}_\varepsilon$ were then removed, the part $\mathcal{P}_{0_\varepsilon}$ would deform into the configuration $\mathcal{R}_\varepsilon$; this deformation is denoted by $\overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$. Note that the configuration $\mathcal{R}_\varepsilon$ supports the residual stress field $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$, which in general is not zero.

To identify a stress free configuration of the material in an infinitesimal neighborhood of $\bar{\mathbf{p}}$, we will establish that the residual stress $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ in $\mathcal{R}_\varepsilon$ vanishes in the limit as ε approaches zero. In order to be clear, we make the following observations regarding the meaning of this limit. As ε approaches zero, $\mathcal{P}_{0_\varepsilon}$ represents successively smaller spherical neighborhoods of the point $\overline{\mathbf{p}}$. The residual stress field $\overset{\circ}{\mathbf{T}}$ is independent of ε, so each successive $\mathcal{P}_{0_\varepsilon}$ contains a subset of the residual stress field supported by a larger neighborhood of $\overline{\mathbf{p}}$. The tractions $\mathbf{t}_\varepsilon$ on $\partial\mathcal{P}_{0_\varepsilon}$ depend on ε because $\partial\mathcal{P}_{0_\varepsilon}$ includes a different set of points for each choice of ε. Finally, since a different portion of both the residual stress field and the body is included in each successive neighborhood of $\overline{\mathbf{p}}$, the region $\mathcal{R}_\varepsilon$ and the deformations $\overset{\circ}{\mathbf{f}}_\varepsilon$ and $\overset{*}{\mathbf{f}}_\varepsilon$ are also different for each ε. The significance of the fact that these functions defined on $\mathcal{P}_{0_\varepsilon}$ and $\mathcal{R}_\varepsilon$ are changing as ε approaches zero is that the residual stress $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ in $\mathcal{R}_\varepsilon$ typically changes at all interior points of $\mathcal{R}_\varepsilon$ as ε approaches zero.

For convenience, we define $\overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$ for each ε such that the image of the point $\overline{\mathbf{p}}$ is always the same point in space. We denote that point as $\overline{\mathbf{q}}$ (Fig. 3) and write

$$\overline{\mathbf{q}} = \overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}(\overline{\mathbf{p}}) \tag{3.8}$$

for all ε. This can be accomplished with a superposed rigid motion for each ε and results in no loss of generality. By (3.8) it is clear that the region $\mathcal{R}_\varepsilon$ degenerates to a point at $\overline{\mathbf{q}}$ in the limit as ε approaches zero.

With (3.8) in mind, we resume our discussion of the changes in $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ as ε approaches zero. In the appendix we show that if $\overset{\circ}{\mathbf{T}}$ and the properties of the natural material are both C^1 functions of position in $\mathcal{P}_{0_\varepsilon}$, and if the deformation $\overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$ is a C^2 function of position* in $\mathcal{P}_{0_\varepsilon}$, then the stress $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ at $\overline{\mathbf{q}}$ approaches zero as ε approaches zero, i.e.,

$$\lim_{\varepsilon\to 0} \overset{\circ}{\mathbf{T}}{}'_\varepsilon(\overline{\mathbf{q}}) = \mathbf{0}. \tag{3.9}$$

The material at the point $\overline{\mathbf{q}}$ is a stress free configuration** of the residually stressed material at the point $\overline{\mathbf{p}}$, in the limit as ε approaches zero, and can therefore be used to derive the constitutive equation for $\overline{\mathbf{p}}$ as in Section 3.1.

We offer the above conditions as sufficient for the existence of the stress free configuration as defined by (3.9). In fact, the derivation presented below can be performed at $\overline{\mathbf{p}}$ as long as (3.9) holds and the constitutive equation for the natural material at $\overline{\mathbf{p}}$ is invertible. It is possible that less restrictive conditions exist for which these requirements are met.

We will refer to the stress free configuration of a part of $\mathcal{B}_0$ as the *virtual configuration* of that part, and denote it by $\mathcal{P}_v$. In addition, we denote the virtual

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* The assumption that $\overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$ is a C^2 function of position is motivated on physical grounds in the appendix.

** This idea of pointwise stress free configurations is not new. For example, a similar stress free configuration is defined in [19] and used to solve boundary value problems in [19] and [20].

configuration of $\mathcal{B}_0$ by $\overline{\mathcal{B}}_v$, and define it as the closure of the union of the virtual configurations $\mathcal{P}_{v_i}$:

$$\overline{\mathcal{B}}_v \equiv \bigcup_{i=1}^{n} \mathcal{P}_{v_i}.$$

Of course, n is finite for residually stressed bodies in the special class defined in Section 3.1, and infinite for the typical residually stressed body. When n is infinite the virtual configuration $\mathcal{P}_v$ has no volume and therefore cannot exist physically. We adopt the adjective 'virtual' to emphasize that the virtual configuration is a mathematical construct used to provide a physical interpretation of the derivation; it need not be determined explicitly and it need not be achievable experimentally. However, such a virtual configuration can be *approximated* experimentally. This topic is explored in the context of destructive experiments on residually stressed bodies in Section 6.

Derivation of the Constitutive Equation

Since the elastic material at $\overline{\mathbf{q}}$ is stress free in the limit as ε approaches zero, the constitutive equation appropriate for the deformations $\overset{\circ}{\mathbf{f}}_\varepsilon$ and $\overset{*}{\mathbf{f}}_\varepsilon$ is the constitutive equation for the natural material, equation (2.6). In order to use (2.6), we must first define the deformation gradients $\nabla \overset{\circ}{\mathbf{f}}_\varepsilon(\overline{\mathbf{q}})$ and $\nabla \overset{*}{\mathbf{f}}_\varepsilon(\overline{\mathbf{q}})$ in the limit as ε approaches zero. For a given $\varepsilon > 0$, let

$$\overset{\circ}{\mathbf{F}}_\varepsilon \equiv \nabla \overset{\circ}{\mathbf{f}}_\varepsilon(\overline{\mathbf{q}})$$

and

$$\overset{*}{\mathbf{F}}_\varepsilon \equiv \nabla \overset{*}{\mathbf{f}}_\varepsilon(\overline{\mathbf{q}}).$$

In rectangular Cartesian coordinates, the components of $\overset{\circ}{\mathbf{F}}_\varepsilon$ can be written as

$$(\overset{\circ}{\mathbf{F}}_\varepsilon)_{ij} = \lim_{\delta q \to 0} \frac{\overset{\circ}{\mathbf{f}}_{\varepsilon_i}(\overline{\mathbf{q}} + \delta q \mathbf{e}_j) - \overset{\circ}{\mathbf{f}}_{\varepsilon_i}(\overline{\mathbf{q}})}{\delta q}. \tag{3.10}$$

Since the right hand side of (3.10) is well defined as ε approaches zero, we can define

$$\overset{\circ}{\mathbf{F}} \equiv \lim_{\varepsilon \to 0} \overset{\circ}{\mathbf{F}}_\varepsilon. \tag{3.11}$$

Similarly,

$$\overset{*}{\mathbf{F}} \equiv \lim_{\varepsilon \to 0} \overset{*}{\mathbf{F}}_\varepsilon, \tag{3.12}$$

where $(\overset{*}{\mathbf{F}}_\varepsilon)_{ij}$ is given by an expression analogous to (3.10).

As in Section 3.1, the gradients $\overset{\circ}{\mathbf{F}}$ and $\overset{*}{\mathbf{F}}$ defined by (3.11) and (3.12) have physical meaning. Here $\overset{\circ}{\mathbf{F}}$ is the deformation gradient (relative to the virtual configuration) that is required to produce the residual stress $\overset{\circ}{\mathbf{T}}$ at the point $\overline{\mathbf{p}}$ in $\mathcal{B}_0$. The gradient $\overset{*}{\mathbf{F}}$ has similar physical meaning in relation to the stress $\mathbf{T}$ at the point $\overline{\mathbf{x}} = \overset{*}{\mathbf{f}}_\varepsilon(\overline{\mathbf{q}})$ in $\mathcal{B}$, in the limit as ε approaches zero.

Evaluation of (2.6) with $\overset{\circ}{\mathbf{F}}$ and $\overset{*}{\mathbf{F}}$, respectively, gives

$$\overset{\circ}{\mathbf{T}} = \tilde{\mathfrak{T}}(\overset{\circ}{\mathbf{F}})$$

and (3.13)

$$\mathbf{T} = \tilde{\mathfrak{T}}(\overset{*}{\mathbf{F}}).$$

In order to obtain a constitutive equation of the form $\mathbf{T} = \hat{\mathfrak{T}}(\mathbf{F}, \overset{\circ}{\mathbf{T}})$ from $(3.13)_2$, we need an expression for $\overset{*}{\mathbf{F}}$ in terms of $\mathbf{F}$ and $\overset{\circ}{\mathbf{T}}$. For $\varepsilon > 0$, the deformation $\overset{*}{\mathbf{f}}_\varepsilon$ is the composition of $\mathbf{f}$ and $\overset{\circ}{\mathbf{f}}_\varepsilon$ for all points in $\mathcal{R}_\varepsilon$. That is,

$$\overset{*}{\mathbf{f}}_\varepsilon = \mathbf{f} \circ \overset{\circ}{\mathbf{f}}_\varepsilon,$$

and therefore

$$\nabla \overset{*}{\mathbf{f}}_\varepsilon = \nabla \mathbf{f} \nabla \overset{\circ}{\mathbf{f}}_\varepsilon$$

in $\mathcal{R}_\varepsilon$. By taking the limit of this expression as ε approaches zero (see (3.11) and (3.12)), we have

$$\overset{*}{\mathbf{F}} = \mathbf{F}\,\overset{\circ}{\mathbf{F}} \tag{3.14}$$

at the point $\overline{\mathbf{q}}$. The derivation now proceeds exactly as in Section 3.1. An expression for $\overset{\circ}{\mathbf{F}}$ in terms of $\overset{\circ}{\mathbf{T}}$ is obtained by the inversion of $(3.13)_1$ at the point $\overline{\mathbf{p}}$:

$$\overset{\circ}{\mathbf{F}} = \tilde{\mathfrak{F}}(\overset{\circ}{\mathbf{T}}). \tag{3.15}$$

Then, substitution of (3.14) and (3.15) into $(3.13)_2$ gives the constitutive equation for the point $\overline{\mathbf{p}}$ in $\mathcal{B}_0$ for a deformation $\mathbf{f}$:

$$\mathbf{T} = \tilde{\mathfrak{T}}(\mathbf{F}\,\tilde{\mathfrak{F}}(\overset{\circ}{\mathbf{T}})), \tag{3.16}$$

which is of the desired form, $\mathbf{T} = \widehat{\mathfrak{T}}(\mathbf{F}, \overset{\circ}{\mathbf{T}})$.

Because we assume that both $\overset{\circ}{\mathbf{T}}$ and the properties of the natural material are C^1 functions of position, the constitutive equation (3.16) will depend smoothly on position $\mathbf{p}$ in $\mathcal{B}_0$. That is, the constitutive (3.16) will change smoothly with respect to position, even though it is derived separately at each point in $\mathcal{B}_0$ using a discrete virtual configuration for each point. Of course, the functional form of $\widetilde{\mathfrak{T}}$ at a point, and therefore of $\widetilde{\mathfrak{F}}(\overset{\circ}{\mathbf{T}})$, will depend on the natural material at that point.

Finally we note that, as in Section 3.1, the deformation $\overset{\circ}{\mathbf{f}}{}_{\varepsilon}^{-1}$ (and hence $\mathcal{R}_{\varepsilon}$) need not be determined in order to derive (3.16). The derivation requires only that $\lim_{\varepsilon \to 0} \overset{\circ}{\mathbf{T}}{}'_{\varepsilon}(\overline{\mathbf{q}}) = \mathbf{0}$, and that the constitutive equation for the natural material be invertible. Furthermore, a qualitative discussion of the relationship between $\mathcal{P}_{0_\varepsilon}$ and $\mathcal{R}_{\varepsilon}$ is sufficient for a conceptual understanding of the derivation. In experimental situations, however, it can be very useful to quantitatively determine a region $\mathcal{R}_{\varepsilon}$ for a small value of ε, which is an approximation of the virtual configuration. The design of destructive experiments for residually stressed bodies is an example of such a situation, and is discussed in Section 6.

3.3. Isotropic Natural Material

In the important special case where the natural material is isotropic, the constitutive equation for deformations from configuration $\mathcal{R}_{\varepsilon}$, in the limit as ε approaches zero, can be expressed in the form of (2.7). Evaluation of (2.7) with $\overset{\circ}{\mathbf{B}} = \overset{\circ}{\mathbf{F}}\overset{\circ}{\mathbf{F}}{}^T$, where $\overset{\circ}{\mathbf{F}}$ is defined by (3.11), gives the residual stress at $\overline{\mathbf{p}}$ in configuration $\mathcal{B}_0$ as

$$\overset{\circ}{\mathbf{T}} = \overline{\mathfrak{T}}(\overset{\circ}{\mathbf{B}}). \tag{3.17}$$

Evaluation of (2.7) with $\overset{*}{\mathbf{B}} = \overset{*}{\mathbf{F}}\overset{*}{\mathbf{F}}{}^T$ gives the Cauchy stress $\mathbf{T}$ in $\mathcal{B}$ as

$$\mathbf{T} = \overline{\mathfrak{T}}(\overset{*}{\mathbf{B}}) = \overline{\mathfrak{T}}(\overset{*}{\mathbf{F}}\overset{*}{\mathbf{F}}{}^T). \tag{3.18}$$

Then, substitution of (3.14) into (3.18) gives

$$\mathbf{T} = \overline{\mathfrak{T}}(\mathbf{F}\overset{\circ}{\mathbf{B}}\mathbf{F}^T), \tag{3.19}$$

where $\overline{\mathfrak{T}}$ depends on the residual stress through $\overset{\circ}{\mathbf{B}}$. Finally, for natural materials such that $\overset{\circ}{\mathbf{T}} = \overline{\mathfrak{T}}(\overset{\circ}{\mathbf{B}})$ has the inverse $\overline{\mathfrak{B}}$, we have

$$\overset{\circ}{\mathbf{B}} = \overline{\mathfrak{B}}(\overset{\circ}{\mathbf{T}}), \tag{3.20}$$

with which (3.19) can be rewritten in the form $\mathbf{T} = \widehat{\mathfrak{T}}(\mathbf{F}, \overset{\circ}{\mathbf{T}})$.

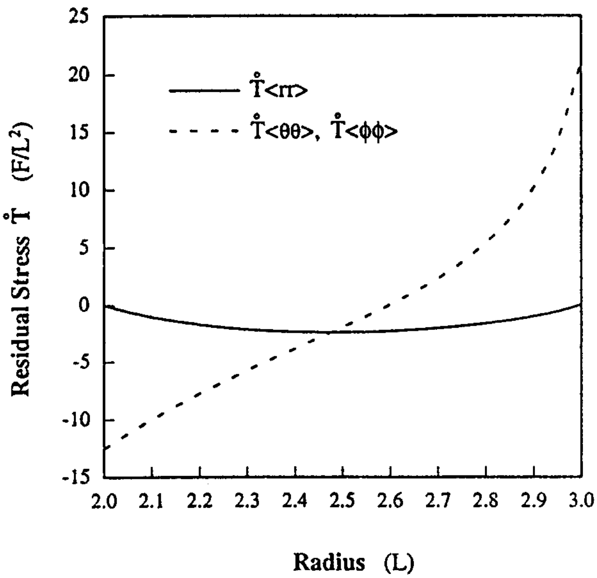

Fig. 4. Physical components of the residual stress field in the spherical shell of Example 1.

This result for the special case of an isotropic natural material will be used in the examples presented in Sections 4 and 5.

4. Example 1

Here we use the method presented in Section 3 to formulate a constitutive equation for a specific residually stressed body. In this example the geometry of the residually stressed body is known and the residual stress is given. We also assume that the constitutive equation for the natural material is known and invertible. With this information it is possible to construct an explicit constitutive law for the body which is valid for deformations from the residually stressed configuration.

The residually stressed configuration $\mathcal{B}_0$ in this example is a thick-walled incompressible spherical shell that supports the residual stress shown in Fig. 4. The internal and external radii are $r_i = 2.0L$ and $r_e = 3.0L$, respectively. The natural material is a Mooney–Rivlin material, for which the constitutive equation* is

$$\mathbf{T} = -p\,\mathbf{1} + 2c_1\,\mathbf{B} - 2c_2\,\mathbf{B}^{-1}. \tag{4.1}$$

* Equation (4.1) is meant to be a general expression of the constitutive equation for a Mooney–Rivlin material. That is, the symbol **T** for the Cauchy stress and the symbol **B** for the left Cauchy–Green strain in (4.1) do not refer to any specific physical quantities in Figs 1, 2, or 3.

For this example the constants* are $c_1 = 1.70FL/L^3$ and $c_2 = 0.25FL/L^3$, where F and L are units of force and length, respectively. The constitutive equation for a Mooney–Rivlin material is invertible for all magnitudes of strain [22, 23].

Equation (4.1) is valid for deformations from the stress free configuration, so the residual stress at a point $\mathbf{p} \in \mathcal{B}_0$ is given by

$$\overset{\circ}{\mathbf{T}}(\mathbf{p}) = -\overset{\circ}{p}(\mathbf{p})\mathbf{1} + 2c_1 \overset{\circ}{\mathbf{B}}(\mathbf{p}) - 2c_2 \overset{\circ}{\mathbf{B}}{}^{-1}(\mathbf{p}). \tag{4.2}$$

Similarly, the Cauchy stress at a point $\mathbf{x}$ in an arbitrary deformed configuration $\mathcal{B}$ is given by

$$\mathbf{T}(\mathbf{x}) = -\overset{*}{p}(\mathbf{x})\mathbf{1} + 2c_1 \overset{*}{\mathbf{B}}(\mathbf{x}) - 2c_2 \overset{*}{\mathbf{B}}{}^{-1}(\mathbf{x}). \tag{4.3}$$

With $\overset{*}{\mathbf{F}} = \mathbf{F}\overset{\circ}{\mathbf{F}}$ we can write $\overset{*}{\mathbf{B}} = \overset{*}{\mathbf{F}}\overset{*}{\mathbf{F}}{}^T = \mathbf{F}\overset{\circ}{\mathbf{B}}\mathbf{F}^T$, from which we obtain the constitutive equation

$$\mathbf{T} = -p\,\mathbf{1} + 2c_1\mathbf{F}\overset{\circ}{\mathbf{B}}\mathbf{F}^T - 2c_2\,\mathbf{F}^{-T}\overset{\circ}{\mathbf{B}}{}^{-1}\mathbf{F}^{-1} \tag{4.4}$$

for each point $\mathbf{x} \in \mathcal{B}$. Recall that we seek an equation of the form $\mathbf{T} = \hat{\mathfrak{T}}(\mathbf{F}, \overset{\circ}{\mathbf{T}})$. Because the constitutive equation for a Mooney–Rivlin material is invertible, we can obtain expressions for $\overset{\circ}{\mathbf{B}}$ and $\overset{\circ}{\mathbf{B}}{}^{-1}$ in terms of $\overset{\circ}{\mathbf{T}}$ from the inversion of (4.2). These expressions can then be substituted into (4.4) to obtain the desired form of the constitutive equation.

Details of the inversion of (4.2) are presented in [11]. Results from [11] that are needed here are displayed below. In terms of $\overset{\circ}{\mathbf{T}}$ and the first two principal invariants** of $\overset{\circ}{\mathbf{B}}$ (i.e., $I_{\overset{\circ}{\mathbf{B}}}$ and $II_{\overset{\circ}{\mathbf{B}}}$), the strain $\overset{\circ}{\mathbf{B}}$ is given by

$$\overset{\circ}{\mathbf{B}} = \psi_0\mathbf{1} + \psi_1\overset{\circ}{\mathbf{T}} + \psi_2\overset{\circ}{\mathbf{T}}{}^2 \tag{4.5}$$

where

$$\begin{aligned}
\psi_0 &= \psi_2\left(\overset{\circ}{p}{}^2 + 8c_1c_2 + 4c_1^2 II_{\overset{\circ}{\mathbf{B}}} + \frac{2\overset{\circ}{p}c_1^2}{c_2} + 4c_2^2 I_{\overset{\circ}{\mathbf{B}}} + 2\overset{\circ}{p}c_2 II_{\overset{\circ}{\mathbf{B}}}\right),\\
\psi_1 &= \psi_2\left(2\overset{\circ}{p} + \frac{2c_1^2}{c_2} + 2c_2 II_{\overset{\circ}{\mathbf{B}}}\right),\\
\psi_2 &= \frac{1}{4c_1^2 I_{\overset{\circ}{\mathbf{B}}} + \dfrac{4c_1^3}{c_2} + 4c_2^2 + 4c_1c_2 II_{\overset{\circ}{\mathbf{B}}}}.
\end{aligned} \tag{4.6}$$

---

* These values for c_1 and c_2 coincide with the properties obtained by Rivlin and Saunders in their experiments on rubber materials [21]. They use units of kilograms for the force F and units of centimeters for the length L.

** Because this is an incompressible material, $III_{\overset{\circ}{\mathbf{B}}} = 1$.

All that remains is to determine the invariants of $\overset{\circ}{\mathbf{B}}$ in terms of the invariants of $\overset{\circ}{\mathbf{T}}$. Once the invariants $I_{\overset{\circ}{\mathbf{B}}}$ and $II_{\overset{\circ}{\mathbf{B}}}$ have been determined, $\overset{\circ}{\mathbf{B}}$ can be written entirely in terms of $\overset{\circ}{\mathbf{T}}$ by use of (4.5) and (4.6).

The invariants of $\overset{\circ}{\mathbf{T}}$ can be expressed in terms of $I_{\overset{\circ}{\mathbf{B}}}$ and $II_{\overset{\circ}{\mathbf{B}}}$ by substitution of (4.2) into (2.1). The result is a system of three nonlinear equations* in the three unknowns $I_{\overset{\circ}{\mathbf{B}}}$, $II_{\overset{\circ}{\mathbf{B}}}$, and $\overset{\circ}{p}$:

$$I_{\overset{\circ}{\mathbf{T}}} = -3\overset{\circ}{p} + 2c_1 I_{\overset{\circ}{\mathbf{B}}} - 2c_2 II_{\overset{\circ}{\mathbf{B}}}, \tag{4.7}$$

$$\begin{aligned} II_{\overset{\circ}{\mathbf{T}}} &= 3\overset{\circ}{p}{}^2 + 4c_1^2 II_{\overset{\circ}{\mathbf{B}}} + 4c_2^2 I_{\overset{\circ}{\mathbf{B}}} - 4\overset{\circ}{p}c_1 I_{\overset{\circ}{\mathbf{B}}} + \\ &\quad + 4\overset{\circ}{p}c_2 II_{\overset{\circ}{\mathbf{B}}} - 4c_1c_2(I_{\overset{\circ}{\mathbf{B}}} II_{\overset{\circ}{\mathbf{B}}} - 3), \end{aligned} \tag{4.8}$$

$$\begin{aligned} III_{\overset{\circ}{\mathbf{T}}} &= -\overset{\circ}{p}{}^3 + 8c_1^3 - 8c_2^3 - 4\overset{\circ}{p}c_1^2 II_{\overset{\circ}{\mathbf{B}}} - 4\overset{\circ}{p}c_2^2 I_{\overset{\circ}{\mathbf{B}}} + 2\overset{\circ}{p}{}^2 c_1 I_{\overset{\circ}{\mathbf{B}}} - \\ &\quad - 2\overset{\circ}{p}{}^2 c_2 II_{\overset{\circ}{\mathbf{B}}} - 8c_1^2 c_2(II_{\overset{\circ}{\mathbf{B}}}^2 - 2I_{\overset{\circ}{\mathbf{B}}}) + \\ &\quad + 8c_1c_2^2(I_{\overset{\circ}{\mathbf{B}}}^2 - 2II_{\overset{\circ}{\mathbf{B}}}) + 4\overset{\circ}{p}c_1c_2(I_{\overset{\circ}{\mathbf{B}}} II_{\overset{\circ}{\mathbf{B}}} - 3). \end{aligned} \tag{4.9}$$

This system cannot be solved algebraically for $I_{\overset{\circ}{\mathbf{B}}}$, $II_{\overset{\circ}{\mathbf{B}}}$, and $\overset{\circ}{p}$; a numerical solution is necessary at each point in the body.

Because the constitutive equation for a Mooney–Rivlin material is invertible, we know not only that (4.5) defines a unique $\overset{\circ}{\mathbf{B}}$, but also that such a $\overset{\circ}{\mathbf{B}}$ exists. It follows that, at each point in the body, there will always be only one solution of the system (4.7) – (4.9) which gives the correct values of $I_{\overset{\circ}{\mathbf{B}}}$, $II_{\overset{\circ}{\mathbf{B}}}$, and $\overset{\circ}{p}$. Note that since the left Cauchy–Green strain tensor is positive definite, the values for the invariants $I_{\overset{\circ}{\mathbf{B}}}$ and $II_{\overset{\circ}{\mathbf{B}}}$ must be positive. The solution can be obtained as follows.

Equation (4.7) is first solved for $I_{\overset{\circ}{\mathbf{B}}}$ as a function of $\overset{\circ}{p}$ and $II_{\overset{\circ}{\mathbf{B}}}$ with the result

$$I_{\overset{\circ}{\mathbf{B}}} = \frac{I_{\overset{\circ}{\mathbf{T}}}}{2c_1} + \frac{3}{2c_1}\overset{\circ}{p} + \frac{c_2}{c_1} II_{\overset{\circ}{\mathbf{B}}}. \tag{4.10}$$

Substitution of (4.10) into (4.8) yields a quadratic equation for $II_{\overset{\circ}{\mathbf{B}}}$ in which the coefficients are functions of $\overset{\circ}{p}$:

* The quantity $(II_{\overset{\circ}{\mathbf{B}}}^2 - 2I_{\overset{\circ}{\mathbf{B}}})$ in (4.9) is incorrectly given as $(II_{\overset{\circ}{\mathbf{B}}} - 2I_{\overset{\circ}{\mathbf{B}}})$ in (3.14) of [11].

$$\begin{aligned}II^2_{\underset{\mathbf{B}}{\circ}} &+ \frac{3c_1c_2\mathring{p} + c_1c_2 I_{\underset{\mathbf{T}}{\circ}} - 2(c_1^3 + c_2^3)}{2c_1c_2^2} II_{\underset{\mathbf{B}}{\circ}} + \\ &+ \frac{3c_1\mathring{p}^2 + (2c_1 I_{\underset{\mathbf{T}}{\circ}} - 6c_2^2)\mathring{p} + c_1 II_{\underset{\mathbf{T}}{\circ}} - 2c_2^2 I_{\underset{\mathbf{T}}{\circ}} - 12c_1^2c_2}{4c_1c_2^2} = 0. \qquad (4.11)\end{aligned}$$

This quadratic equation has two solutions:

$$II^+_{\underset{\mathbf{B}}{\circ}} = \frac{-3c_1c_2\mathring{p} - c_1c_2 I_{\underset{\mathbf{T}}{\circ}} + 2(c_1^3 + c_2^3)}{4c_1c_2^2} + \frac{1}{4c_1c_2^2}\sqrt{\sigma\mathring{p}^2 + \zeta\mathring{p} + \delta}, \qquad (4.12)$$

where

$$\begin{aligned}\sigma &= -3c_1^2c_2^2, \\ \zeta &= 12(c_1c_2^4 - c_1^4c_2) - 2c_1^2c_2^2 I_{\underset{\mathbf{T}}{\circ}}, \\ \delta &= c_1^2c_2^2(I^2_{\underset{\mathbf{T}}{\circ}} - 4 II_{\underset{\mathbf{T}}{\circ}}) + 4(c_1c_2^4 - c_1^4c_2) I_{\underset{\mathbf{T}}{\circ}} + 56c_1^3c_2^3 + 4(c_1^6 + c_2^6),\end{aligned}$$

and a similar solution $II^-_{\underset{\mathbf{B}}{\circ}}$ associated with the negative square root.

Substitution of $II^+_{\underset{\mathbf{B}}{\circ}}$ into (4.10) gives an equation for $I_{\underset{\mathbf{B}}{\circ}}$ in terms of $\mathring{p}$ and $I_{\underset{\mathbf{T}}{\circ}}$. We will call the solution to this equation $I^+_{\underset{\mathbf{B}}{\circ}}$. Finally, substitution of the equations for $I^+_{\underset{\mathbf{B}}{\circ}}$ and $II^+_{\underset{\mathbf{B}}{\circ}}$ into (4.9) gives a single nonlinear equation for the unknown $\mathring{p}^+$:

$$\begin{aligned}(\mathring{p}^+)^3 &+ \left[3\left(\frac{c_1^2}{c_2} - \frac{c_2^2}{c_1}\right) + I_{\underset{\mathbf{T}}{\circ}}\right](\mathring{p}^+)^2 + \\ &+ \left[-12\left(\frac{c_1^4}{c_2^2} + \frac{c_2^4}{c_1^2}\right) + 2 I_{\underset{\mathbf{T}}{\circ}}\left(\frac{c_1^2}{c_2} - \frac{c_2^2}{c_1}\right) - 24c_1c_2 + II_{\underset{\mathbf{T}}{\circ}}\right]\mathring{p}^+ - \\ &- 3\left(\frac{c_1}{c_2^2} + \frac{c_2}{c_1^2}\right)\mathring{p}^+\sqrt{\sigma(\mathring{p}^+)^2 + \zeta\mathring{p}^+ + \delta} + \\ &+ \left[2\left(\frac{c_1^3}{c_2^3} - \frac{c_2^3}{c_1^3}\right) - I_{\underset{\mathbf{T}}{\circ}}\left(\frac{c_1}{c_2^2} + \frac{c_2}{c_1^2}\right)\right]\sqrt{\sigma(\mathring{p}^+)^2 + \zeta\mathring{p}^+ + \delta} + \\ &+ 4\left(\frac{c_1^6}{c_2^3} - \frac{c_2^6}{c_1^3}\right) - 4 I_{\underset{\mathbf{T}}{\circ}}\left(\frac{c_1^4}{c_2^2} + \frac{c_2^4}{c_1^2}\right) + 20(c_1^3 - c_2^3) + \\ &+ (I^2_{\underset{\mathbf{T}}{\circ}} - 2II_{\underset{\mathbf{T}}{\circ}})\left(\frac{c_1^2}{c_2} - \frac{c_2^2}{c_1}\right) - 8c_1c_2 I_{\underset{\mathbf{T}}{\circ}} + III_{\underset{\mathbf{T}}{\circ}} = 0. \qquad (4.13)\end{aligned}$$

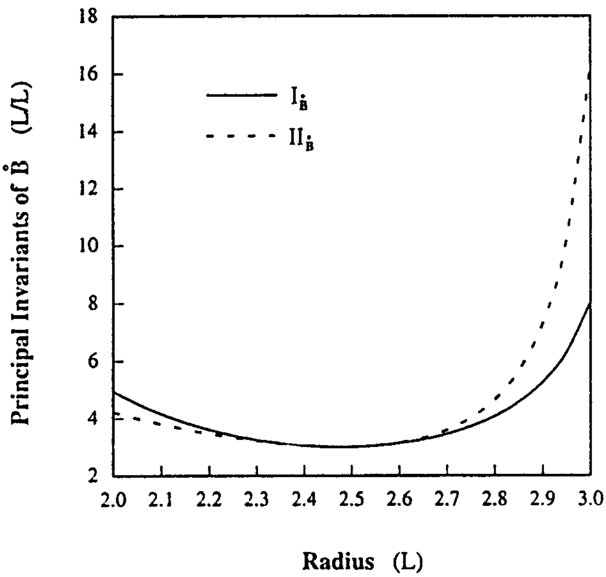

Fig. 5a. The solution $I_{\overset{\circ}{\mathbf{B}}}(r)$ and $II_{\overset{\circ}{\mathbf{B}}}(r)$ for Example 1.

At a specific point in the shell, any real roots $\overset{\circ}{p}^{+}$ of (4.13) are found*, and each root is used to compute the associated numerical values of $I^{+}_{\overset{\circ}{\mathbf{B}}}$ and $II^{+}_{\overset{\circ}{\mathbf{B}}}$. This gives a complete solution to the set of equations (4.7) through (4.9) for each root.

Since the physically meaningful solution to equations (4.7) through (4.9) is unique, the search for a root $\overset{\circ}{p}^{+}$ need proceed only until a root is found that corresponds to positive values of $I^{+}_{\overset{\circ}{\mathbf{B}}}$ and $II^{+}_{\overset{\circ}{\mathbf{B}}}$. If no such root $\overset{\circ}{p}^{+}$ exists, then similar steps are taken for $I^{-}_{\overset{\circ}{\mathbf{B}}}$ and $II^{-}_{\overset{\circ}{\mathbf{B}}}$ to obtain a counterpart to (4.13) that can have a set of real roots $\overset{\circ}{p}^{-}$. Associated with each root $\overset{\circ}{p}^{-}$ is a set $\overset{\circ}{p}^{-}$, $I^{-}_{\overset{\circ}{\mathbf{B}}}$, and $II^{-}_{\overset{\circ}{\mathbf{B}}}$ that is also a solution to equations (4.7) through (4.9). The search for real roots $\overset{\circ}{p}^{-}$ is continued until a root is found such that $I^{-}_{\overset{\circ}{\mathbf{B}}}$ and $II^{-}_{\overset{\circ}{\mathbf{B}}}$ are positive. Since the existence and uniqueness of $\overset{\circ}{\mathbf{B}}$ in (4.5) is assured for a Mooney–Rivlin material, one and only one root from the set that contains all of the $\overset{\circ}{p}^{+}$ and $\overset{\circ}{p}^{-}$ roots will correspond to a physically meaningful solution.

For the example under consideration, where the residual stress is given by Fig. 4, the values of the invariants $I_{\overset{\circ}{\mathbf{B}}}$ and $II_{\overset{\circ}{\mathbf{B}}}$ are shown in Fig. 5a as a function of radial position in the shell. The pressure $\overset{\circ}{p}$ is displayed in Fig. 5b. These values, together

* Note that the solutions for $II_{\overset{\circ}{\mathbf{B}}}$ (e.g., equation (4.12)) contain the square root of a quadratic in $\overset{\circ}{p}$. This quadratic must be positive in order that $II_{\overset{\circ}{\mathbf{B}}}$ be real. Thus, the search for real roots $\overset{\circ}{p}^{+}$ of (4.13) can be limited to the range for which this holds.

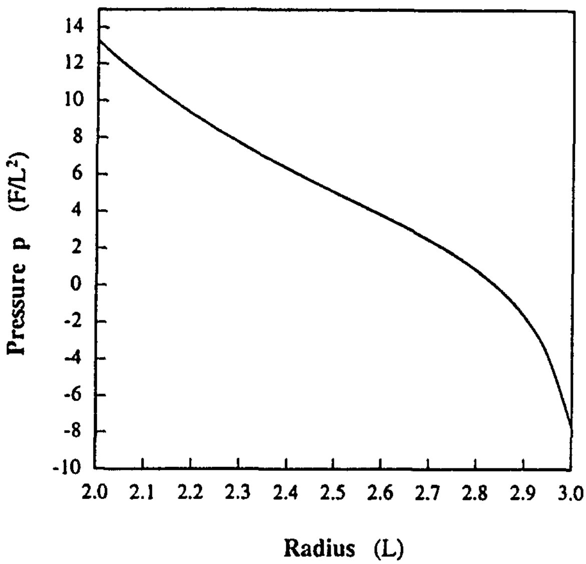

Fig. 5b. The solution $p(r)$ for Example 1.

with (4.5) and (4.6), give the tensor $\overset{\circ}{\mathbf{B}}$ associated with the residual stress field in the shell. The components of $\overset{\circ}{\mathbf{B}}$ are shown in Fig. 6, from which the components of $\overset{\circ}{\mathbf{B}}{}^{-1}$ can easily be obtained.

We now have the desired constitutive law for deformations out of the residually stressed configuration: equation (4.4), with $\overset{\circ}{\mathbf{B}}$ and $\overset{\circ}{\mathbf{B}}{}^{-1}$ computed from $\overset{\circ}{\mathbf{T}}$ at each point as described above. Note that the constitutive equation (4.4) is an explicit function of residual stress through $\overset{\circ}{\mathbf{B}}, \overset{\circ}{\mathbf{B}}{}^{-1}$, and equations (4.5) through (4.13). Equation (4.4) also contains only the mechanical properties that describe the natural material, which are the Mooney–Rivlin constants c_1 and c_2.

We chose this as our first example for the following reason. Given the residual stress, the left Cauchy–Green strain $\overset{\circ}{\mathbf{B}}$ must be determined as part of the derivation. The residual stress in Fig. 4 can be produced by the eversion of a spherical shell [24, 11], which is a known global elastic deformation. Therefore, this example provides an ideal setting in which to present and validate the method, since the deformation related to this residual stress is known a-priori. It is straightforward to show that the eversion deformation produces the left Cauchy–Green strain tensor $\overset{\circ}{\mathbf{B}}$ computed above and shown in Fig. 6.

The everted spherical shell is an example of a residually stressed body from the special class defined in Section 3.1. Recall that for a body in this class, one can always obtain a virtual configuration with a finite number of parts with non-zero volume. A discussion of the possible virtual configurations for this spher-

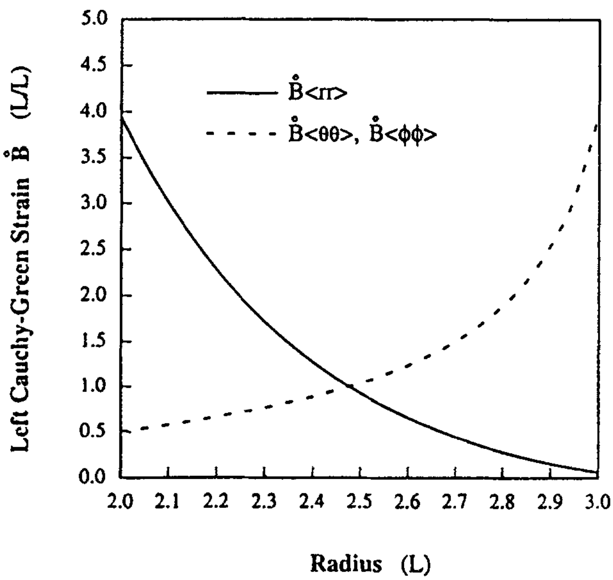

Fig. 6. Physical components of the left Cauchy–Green strain tensor in the spherical shell of Example 1.

ical shell can be found in Section 6.2. In the next section we formulate a constitutive equation for the material in a residually stressed body that is not of this special class.

5. Example 2

The constitutive equation for the material at a point in a residually stressed body can be derived by the method of Section 3 if the residual stress and the properties of the natural material are smooth functions of position, and if the constitutive equation for the natural material at that point is invertible. The method is applied in this section to a body which meets these criteria, but, in contrast with Example 1, contains no parts for which a virtual configuration with finite volume can be obtained.

Note that, given a residual stress field $\overset{\circ}{\mathbf{T}}$ (i.e., a stress field that is symmetric, satisfies equilibrium on a specific unloaded body, and satisfies the zero traction condition on the body surface), the residual stress

$$\overset{\circ}{\mathbf{T}}_\alpha \equiv \alpha\overset{\circ}{\mathbf{T}} \tag{5.1}$$

for any real α also satisfies these conditions. In this section we will exploit this fact to construct a residually stressed body.

As in the example of Section 4, we consider a thick-walled incompressible spherical shell with internal and external radii $r_i = 2.0L$ and $r_e = 3.0L$. The

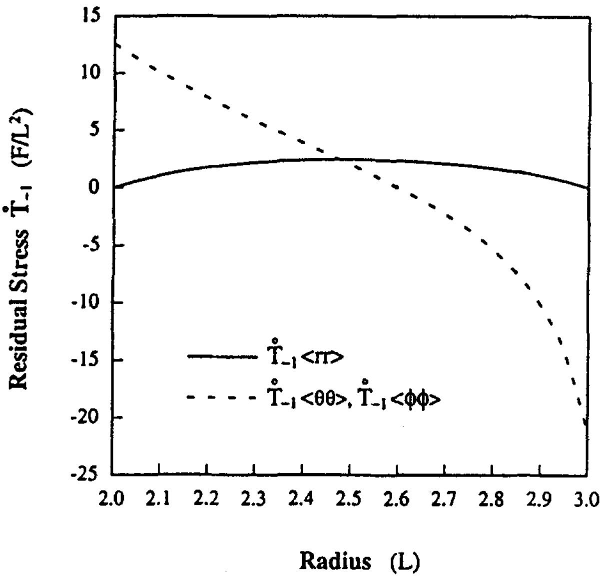

Fig. 7. Physical components of the residual stress field in the spherical shell of Example 2, for which $\alpha = -1$.

natural material is a Mooney–Rivlin material with constants $c_1 = 1.70FL/L^3$ and $c_2 = 0.25FL/L^3$, and the residual stress $\overset{o}{\mathbf{T}}_\alpha$ in the shell is constructed from the residual stress $\overset{o}{\mathbf{T}}$ in Fig. 4 and the multiplier $\alpha = -1$. This results in the residual stress $\overset{o}{\mathbf{T}}_{-1}$ shown in Fig. 7. In Section 6 it will be established that this residually stressed spherical shell is not in the special class defined in Section 3.1.

The conditions for the use of the derivation in Section 3 are satisfied in this example; the constitutive equation for the natural material is invertible and, at each point in the body, the residual stress and properties of the natural material are smooth functions of position.

The appropriate constitutive equation for deformations from the residually stressed configuration will have the form

$$\mathbf{T} = -p\mathbf{1} + 2c_1\,\mathbf{F}\,\overset{o}{\mathbf{B}}_{-1}\,\mathbf{F}^T - 2c_2\mathbf{F}^{-T}(\overset{o}{\mathbf{B}}_{-1})^{-1}\mathbf{F}^{-1}. \tag{5.2}$$

At each point along the radius of the shell, $\overset{o}{\mathbf{B}}_{-1}$ is computed from the residual stress by the method described in Section 4. The components of $\overset{o}{\mathbf{B}}_{-1}$ are shown in Fig. 8 as a function of radial position in the shell. Similar results can be obtained for any residual stress $\alpha\overset{o}{\mathbf{T}}$, with $\overset{o}{\mathbf{T}}$ given by Fig. 4.

Equation (5.2) together with the strain $\overset{o}{\mathbf{B}}_{-1}$ in Fig. 8 gives the desired constitutive equation for $\mathbf{T}$ in terms of $\mathbf{F}$ and $\overset{o}{\mathbf{T}}_{-1}$. As in Example 1, $\mathbf{T}$ as given by (5.2) is an explicit function of $\overset{o}{\mathbf{T}}_{-1}$ (through $\overset{o}{\mathbf{B}}_{-1}$) and contains only the Mooney–Rivlin

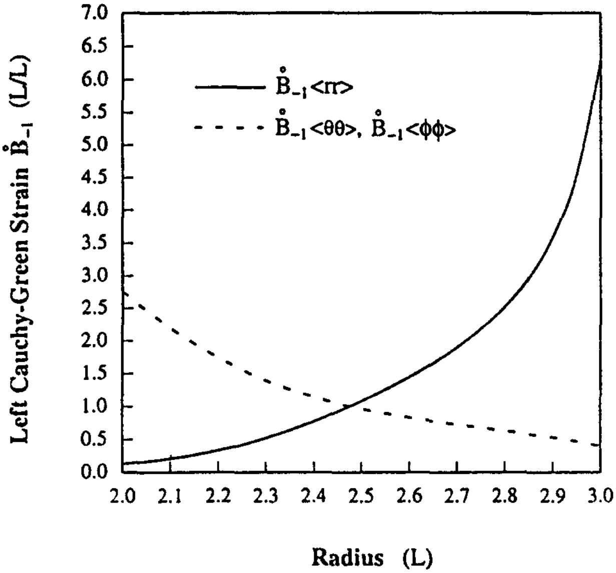

Fig. 8. Physical components of the left Cauchy–Green strain tensor in the spherical shell of Example 2, for which $\alpha = -1$.

constants c_1 and c_2. With this example we have demonstrated the formulation of a constitutive equation for a body that has a virtual configuration with an infinite number of parts with zero volume. From a comparison of Examples 1 and 2, it is clear that the process of constructing a constitutive equation for a specific residually stressed body is independent of the classification of the body.

6. The Virtual Configuration

The concept of the virtual configuration was introduced in Section 3.2 as a tool with which to develop an intuitive understanding of the derivation of the constitutive equation. However, it is of practical value in modeling destructive experiments that are used to determine residual stress. We begin with a brief discussion of these experiments.

In the past, destructive experiments were used primarily to determine residual stresses in products of heavy industry, such as railway wheels and rails, tractor wheels, and welded pipe. However, the principles of destructive experiments can be applied in the elastic regime of any residually stressed body. An interesting example is described in [25], wherein a destructive experiment is used to determine the residual stress in a multi-layer thin film. The thin film in this case was produced by sputtering a thin layer of chromium on both sides of a Kapton substrate sheet, followed by a thin layer of copper on both sides. (The Kapton substrate was 50 μm thick, and single layers of chromium and copper were 535Å and 8.8 kÅ, respectively.) Residual stresses generated in the film during fabrication were deter-

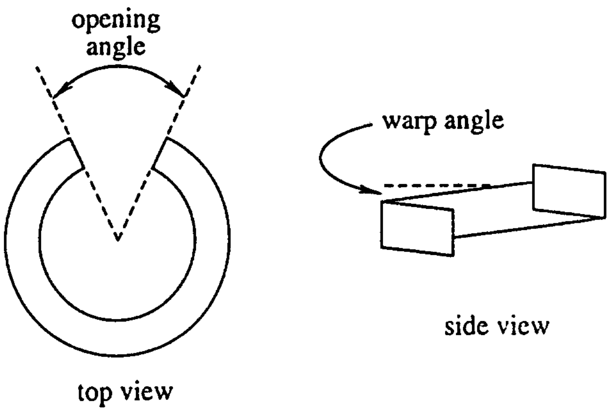

Fig. 9. The configuration of an equatorial slice of a tight-skin mouse left ventricle after a single radial cut is made in the original ring of tissue (adapted from Fig. 1 of [27]).

mined by chemically etching a single layer of metal from one side of the film. This produced curvature in the remaining film that was measured and used to calculate the original residual stress. Similar methods have been used to obtain information about the residual stress in very thin (e.g. 2 μm thick) silicon cantilever beams that simulate structures in micromechanical devices [26].

Destructive experiments have also been used in the context of residually stressed biological tissues. A recent example is the determination of the residual stress in the left ventricle of the tight-skin mouse [27]. In this experiment, a thin equatorial slice (1–2 mm thick) was removed from the left ventricle and placed in a fluid to minimize effects of friction and gravity. In this residually stressed configuration, the slice is a continuous ring of tissue. The ring was then cut radially at one location, which caused it to open into a 'c' shape and also to warp out of plane, as shown in Fig. 9. The cut configuration is assumed to be stress free, and the residual stress originally supported by the ring is assumed to be the stress that results from deforming the ring back into the uncut configuration.

The specific examples just described give an idea of how some features of a residual stress field can be determined with destructive experiments. It is worthwhile, however, to discuss certain aspects more formally. Destructive experiments in general proceed by one of the following methods: the body is sectioned into separate pieces, a portion of the surface is removed (by grinding or etching, for example), or a discrete piece is removed. In all cases, the goal is to measure the surface displacements that result from the relief of the residual stress, and to use that data to deduce properties of the residual stress field.

Ideally, the cutting process can be continued until any further removal of material produces negligible strain by the standards of the particular experiment under consideration. It is assumed that each of the pieces that result from the destructive experiment is free of residual stress. Of course, there exists the possibility of undiscovered, non-negligible residual stresses in this material. One must take care to verify that any remaining residual stresses can actually be neglected.

An estimate of the residual stress originally supported by the intact configuration of this material can then be obtained from the solution of the associated displacement boundary value problem. Displacement boundary conditions are specified for the (assumed) stress free configuration such that the boundary is returned to the original residually stressed configuration, according to the total displacement measured from all of the cuts. The stress that results from this deformation can be calculated using the constitutive equation for the natural material, and is assumed to be the residual stress in the intact body. If the constitutive equation for the natural material is not known, it may be possible to measure the mechanical properties of a piece of material in the approximate stress free configuration and use this data to formulate a constitutive equation.

The derivation of the constitutive equation presented in Section 3 and illustrated in Sections 4 and 5 does not require that a region $\mathcal{R}_\varepsilon$ (or $\mathcal{R}$ in the special case of Section 3.1) be determined. The role of $\mathcal{R}_\varepsilon$ thus far has been to provide a physical motivation for the mathematics used to derive the constitutive equation. However, since the objective of the destructive experiments described above is to relieve the residual stress from a portion of the body, it would be useful to know what the approximate stress free configuration (e.g., $\mathcal{R}_\varepsilon$ for small ε) of a part of the body would be for a given residual stress field. Of course, in the experimental situation the actual residual stress field is not known. But suppose, for example, that we have a hypothesis of what the residual stress might be in the body. In this case, the hypothetical residual stress field could be used to determine the approximate stress free configuration of a part, which in turn could be used to determine how best to cut the body in order to test the hypothesis.

Recall from Section 3.2 that we have defined the virtual configuration of a part of the residually stressed body to be the stress free configuration of that part, and the virtual configuration of the body to be the closure of the union of virtual configurations of the individual parts. We have also established that the virtual configuration of a part $\mathcal{P}_{0_\varepsilon}$ of a typical residually stressed body is attained in the limit as ε approaches zero, and therefore has zero volume. In order to apply the concept of the virtual configuration to destructive experiments, such a virtual configuration must be approximated as a region with finite volume. In this section the approximation of a virtual configuration is demonstrated in the context of the residually stressed body described in Section 5. That example is then used to illustrate that the virtual configuration of a residually stressed body may not be unique. We close this section with a discussion of the relationship between approximate virtual configurations and destructive experiments.

6.1. Approximation of the Virtual Configuration: An Example

In this section we determine an approximate virtual configuration for the thick-walled spherical shell used in the example of Section 5. First we establish that a stress free configuration with finite volume does not exist for any part of this body. Therefore, in a destructive experiment the virtual configuration can only be approximated. To show this we verify that the strain associated with the residual stress is locally incompatible at all points in the spherical shell. Second, we construct a mathematical approximate virtual configuration of the residually stressed body, and use the constitutive equation formulated in Section 5 to calculate the stresses in this approximate virtual configuration. Since the actual virtual configuration is stress free, the magnitude of these stresses is a measure of the accuracy of the approximation.

Compatibility

If a virtual configuration with non-zero volume exists for a part of a residually stressed body, then there is a compatible strain tensor associated with the deformation that maps the virtual configuration into the residually stressed configuration. That is, if $\mathbf{y}$ is the deformation from the virtual configuration into the residually stressed configuration, then there exists a compatible left Cauchy–Green strain field $\overset{\circ}{\mathbf{B}}$ given by

$$\overset{\circ}{\mathbf{B}} = (\nabla\mathbf{y})(\nabla\mathbf{y})^T. \tag{6.1}$$

The inverse of $\mathbf{y}$ maps the residually stressed part into a stress free configuration with finite volume that can be achieved experimentally without approximation. This is clearly the case in Example 1, where a stress free spherical shell is everted to produce a residually stressed spherical shell, and where the deformation $\mathbf{y}$ is the eversion. The eversion of a hollow elastic tube is another deformation that satisfies (6.1). A discussion of compatibility as it relates to residual stress can be found in [28], and a more comprehensive discussion of compatibility is given in [29] in the context of finite deformations.

We now turn to the specific example presented in Section 5, in which the residually stressed configuration $\mathcal{B}_0$ (the hollow spherical shell) supports the residual stress $\overset{\circ}{\mathbf{T}}_{-1}$ shown in Fig. 7. To establish that there are no virtual configurations with finite volume for this body, we show that the strain tensor $\overset{\circ}{\mathbf{B}}_{-1}$ associated with $\overset{\circ}{\mathbf{T}}_{-1}$ is locally incompatible in $\mathcal{B}_0$, i.e., it does not satisfy (6.1). Therefore, there is no deformation that will completely relieve the residual stress $\overset{\circ}{\mathbf{T}}_{-1}$ in any finite part of $\mathcal{B}_0$.

By (4.5), the strain $\overset{\circ}{\mathbf{B}}_{-1}$ (Fig. 8) is of the form

$$[\overset{\circ}{\mathbf{B}}_{-1}] = \begin{bmatrix} \overset{\circ}{\mathrm{B}}_{-1}\langle rr\rangle(r) & 0 & 0 \\ 0 & \overset{\circ}{\mathrm{B}}_{-1}\langle\theta\theta\rangle(r) & 0 \\ 0 & 0 & \overset{\circ}{\mathrm{B}}_{-1}\langle\theta\theta\rangle(r) \end{bmatrix} \tag{6.2}$$

with respect to the orthonormal basis $\mathbf{e}\langle i\rangle$, where i ranges over the spherical coordinates r, θ, and ϕ. Due to the simplicity of (6.2), the incompatibility of $\overset{\circ}{\mathbf{B}}_{-1}$ can be shown by direct computation of possible deformations $\mathbf{y}$ that satisfy (6.1) for $\overset{\circ}{\mathbf{B}}_{-1}$. We will see that no such deformations exist.

Since $\overset{\circ}{\mathbf{B}}_{-1}$ does not depend on θ and ϕ, the strain $\overset{\circ}{\mathbf{B}}_{-1}$ is constant on any surface of constant r in $\mathcal{B}_0$. Therefore, the virtual configuration must also be a spherical shell, because any other geometry would require that $\overset{\circ}{\mathbf{B}}_{-1}$ have both θ and ϕ dependence on a surface of constant r. A criterion for the deformation of a spherical shell into another spherical shell, to within a rigid rotation, is that the rotation $\overset{\circ}{\mathbf{R}}_{-1}$ from the polar decomposition $(\nabla\mathbf{y}) = \overset{\circ}{\mathbf{V}}_{-1}\overset{\circ}{\mathbf{R}}_{-1}$ have physical components such that

$$|\overset{\circ}{\mathrm{R}}_{-1}\langle ij\rangle| = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{6.3}$$

Since $[\overset{\circ}{\mathbf{V}}_{-1}]$ and $[\overset{\circ}{\mathbf{R}}_{-1}]$ are diagonal with respect to the basis $\mathbf{e}\langle i\rangle, [\nabla\mathbf{y}]$ is also diagonal. We can therefore write $\mathbf{y} = \psi(r)\mathbf{e}\langle r\rangle$, and (6.1) becomes

$$\begin{bmatrix} \dfrac{\mathrm{d}\psi(r)}{\mathrm{d}r} & 0 & 0 \\ 0 & \dfrac{\psi(r)}{r} & 0 \\ 0 & 0 & \dfrac{\psi(r)}{r} \end{bmatrix} = \begin{bmatrix} \sqrt{\overset{\circ}{\mathrm{B}}_{-1}\langle rr\rangle(r)} & 0 & 0 \\ 0 & \sqrt{\overset{\circ}{\mathrm{B}}_{-1}\langle\theta\theta\rangle(r)} & 0 \\ 0 & 0 & \sqrt{\overset{\circ}{\mathrm{B}}_{-1}\langle\theta\theta\rangle(r)} \end{bmatrix} \tag{6.4}$$

in physical components. The equation for the $\langle\theta\theta\rangle$ components in (6.4) gives

$$\psi(r) = r\sqrt{\overset{\circ}{\mathrm{B}}_{-1}\langle\theta\theta\rangle(r)},$$

but

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sqrt{\overset{\circ}{\mathrm{B}}_{-1}\langle\theta\theta\rangle(r)}\right) \neq \sqrt{\overset{\circ}{\mathrm{B}}_{-1}\langle rr\rangle(r)},$$

so the equation for the $\langle rr\rangle$ components in (6.4) is not satisfied. Therefore, no deformation $\mathbf{y}$ exists that satisfies (6.4), so $\overset{\circ}{\mathbf{B}}_{-1}(r)$ is a locally incompatible strain field. We have shown that no portion of $\mathcal{B}_0$ has a virtual configuration with finite volume.

The Approximate Virtual Configuration

We now proceed to construct a discontinuous mapping from the residually stressed configuration $\mathcal{B}_0$ of the thick-walled spherical shell into an approximate virtual configuration. We will confirm that this approximate virtual configuration is a set of individual parts that cannot be assembled into a stress free, continuous whole.

Due to the absence of θ and ϕ dependence in this problem, the set of all points at a given radius in $\mathcal{B}_0$ can be mapped by a single function into the virtual configuration of that set, which is a continuous spherical surface. To see this, consider the outermost, infinitely thin layer of $\mathcal{B}_0$, in which the tangential stress is compressive. If this infinitely thin spherical subshell were removed from $\mathcal{B}_0$ and unloaded, it would expand radially until the compressive tangential stress was relieved.

An infinitely thin shell is clearly unattainable in a destructive experiment, so we are motivated to consider an experiment in which this virtual configuration is approximated. Suppose that a spherical subshell with a small but finite thickness could be removed from $\mathcal{B}_0$ and then allowed to assume a traction free configuration. This configuration of the subshell is essentially composed of regions $\mathcal{R}_\varepsilon$ (defined in Section 3.2) and supports the residual stress field $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$. However, from the theorem in the appendix we know that $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ vanishes in this subshell as the thickness approaches zero. Therefore, we refer to the unloaded subshell with finite thickness as the 'experimental approximation' of the virtual configuration, and note that this approximation improves as the thickness of the subshell is decreased.

Our objective is to construct a mapping from a given subshell of $\mathcal{B}_0$ into an approximate virtual configuration of that subshell. The experimental approximation obtained through a destructive experiment seems at first to be the obvious configuration for which to construct this mapping. However, the residual stress field $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ in the experimental approximation is unknown, so it is unclear how the available information can be used to determine the deformation from $\mathcal{B}_0$ into this approximate virtual configuration. As an alternative, we can formulate a mapping from a subshell of $\mathcal{B}_0$ into an approximate virtual configuration that differs from the experimental approximation, but becomes sufficiently similar to it as the thickness of the subshell is decreased. This new approximate virtual configuration will be referred to as the 'mathematical approximation', and will serve our purpose of illustrating the properties of an approximate virtual configuration of a general residually stressed body.

First we consider the subdivision of $\mathcal{B}_0$ into a finite number of thin spherical shells. For n subshells, the thickness δr of each is $\delta r = (r_e - r_i)/n$, where r_i and r_e are the internal and external radii of $\mathcal{B}_0$, respectively. Let r_k be the radial coordinate

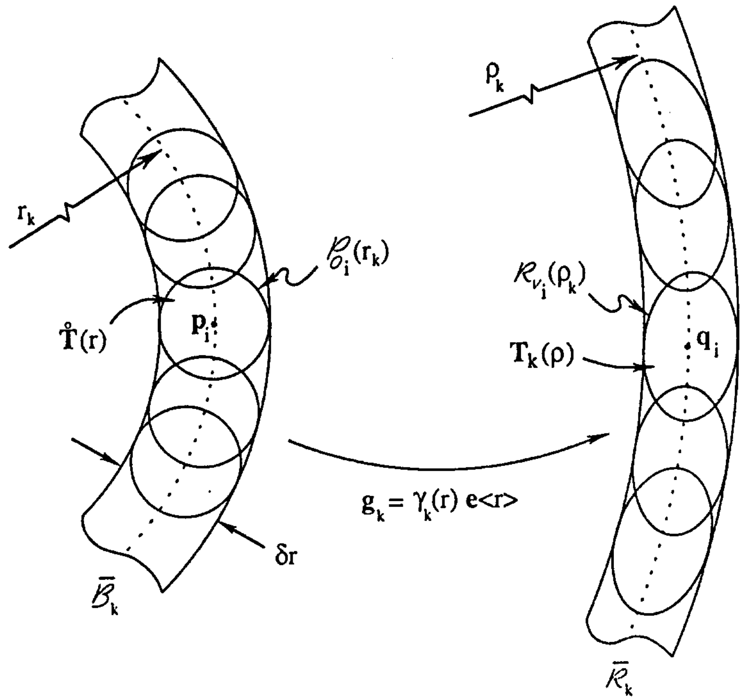

Fig. 10. $\overline{\mathcal{B}}_k$ is the kth subshell of the hollow spherical shell in Example 2, and is defined as the closure of the union $\bigcup_{i=1}^{\infty} \mathcal{P}_{0_i}(r_k)$. A few representative spherical neighborhoods $\mathcal{P}_0(r_k)$ are shown in $\overline{\mathcal{B}}_k$. The approximate virtual configuration $\overline{\mathcal{R}}_k$ is defined by the deformation $\mathbf{g}_k$, which is constructed so that the stress $\mathbf{T}_k$ is zero at ρ_k. Because $\overline{\mathcal{R}}_k$ is approximate, $\mathbf{T}_k$ is non-zero at other radii in $\overline{\mathcal{R}}_k$, but approaches zero as δr becomes small.

of the points that lie midway through the kth shell wall, shown in Fig. 10. The kth shell in $\mathcal{B}_0$ can be defined as the closure of the union of an infinite number of parts $\mathcal{P}_{0_i}(r_k)$, where $\mathcal{P}_{0_i}(r_k)$ is the neighborhood with diameter δr of the ith point at the radius r_k (Fig. 10). We denote this set by $\overline{\mathcal{B}}_k \equiv \bigcup_{i=1}^{\infty} \mathcal{P}_{0_i}(r_k)$. A deformation $\mathbf{g}_k = \gamma_k(r)\mathbf{e}\langle r\rangle$ maps all parts $\mathcal{P}_{0_i}(r_k)$ into approximate virtual configurations $\mathcal{R}_{v_i}(\rho_k)$, where $\rho_k = \gamma_k(r_k)$. In general, $\rho = \gamma_k(r)$ for $(r_k - (\delta r/2)) \leqslant r \leqslant (r_k + (\delta r/2))$. The image of the kth shell under the deformation $\mathbf{g}_k$ is the mathematical approximate virtual configuration, and is defined as $\overline{\mathcal{R}}_k \equiv \bigcup_{i=1}^{\infty} \mathcal{R}_{v_i}(\rho_k)$.

We construct a deformation $\mathbf{g}_k$ for the kth shell according to the following two criteria. First, the deformation $\mathbf{g}_k$ must be isochoric because the natural material associated with $\mathcal{B}_0$ is incompressible. Therefore,

$$\det(\nabla \mathbf{g}_k) = 1. \tag{6.5}$$

The second criterion is that $\overline{\mathcal{R}}_k$ approach the true virtual configuration of the material at r_k in $\overline{\mathcal{B}}_k$, in the limit as δr approaches zero. Although these criteria can be met in a number of ways, we choose to construct $\mathbf{g}_k$, and therefore $\overline{\mathcal{R}}_k$, as follows. Let $\mathbf{T}_k(\rho)$ be the stress at any radius ρ within $\overline{\mathcal{R}}_k$. We require that the material at $\rho_k = \gamma_k(r_k)$ in $\overline{\mathcal{R}}_k$ be stress free, i.e.,

$$\mathbf{T}_k(\rho_k) = \mathbf{0}. \tag{6.6}$$

Surface tractions* would be required on $\partial\overline{\mathcal{R}}_k$ to physically achieve an approximate virtual configuration that satisfies (6.6). However, we will see that the stress $\mathbf{T}_k(\rho)$ approaches zero as the thickness of $\overline{\mathcal{B}}_k$ approaches zero, and therefore the surface tractions approach zero as well.

We choose to construct $\overline{\mathcal{R}}_k$ according to condition (6.6) because this condition allows us to use the strain $\overset{\circ}{\mathbf{B}}_{-1}$ from Fig. 8 to determine the stress free configuration of the material at r_k in $\overline{\mathcal{B}}_k$. Recall from Example 2 that $\overset{\circ}{\mathbf{B}}_{-1}$ is the strain associated with the deformation from the stress free configuration into the residually stressed configuration. Hence, we will use the strain $\overset{\circ}{\mathbf{B}}_{-1}\ (r_k)$ from Fig. 8 to determine the radius ρ_k of the points originally at r_k. The mapping $\mathbf{g}_k(r)$ for other radii in $\overline{\mathcal{B}}_k$ is determined by the incompressibility of the material, i.e., condition (6.5).

Note the contrast between the mathematical approximate virtual configuration that satisfies (6.6), and the experimental approximate virtual configuration obtained from the destructive experiment discussed previously. The experimental approximation, which supports the residual stress $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$, is free of surface tractions and in general contains no points that are stress free. The mathematical approximation requires small surface tractions in order to maintain a state of zero stress at the radius ρ_k. However, as the thickness of $\overline{\mathcal{B}}_k$ is decreased, both approximate virtual configurations approach the true virtual configuration of the points originally at r_k.

We now construct the deformation $\mathbf{g}_k(r)$. From (6.6) and the definition of $\overset{\circ}{\mathbf{B}}_{-1}$, we know that $(\nabla\mathbf{g}_k^{-1})(\nabla\mathbf{g}_k^{-1})^T = \overset{\circ}{\mathbf{B}}_{-1}$ only at r_k (since $\overline{\mathcal{R}}_k$ is stress free only at ρ_k). Therefore, we also have $(\nabla\mathbf{g}_k)^T(\nabla\mathbf{g}_k) = (\overset{\circ}{\mathbf{B}}_{-1})^{-1}$ only at r_k. With (6.2)–(6.4), it is easily shown that

$$\nabla\mathbf{g}_k(r_k) = \sqrt{(\overset{\circ}{\mathbf{B}}_{-1})^{-1}(r_k)}, \tag{6.7}$$

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* For this example, the tractions on $\partial\overline{\mathcal{R}}_k$ consist of an inflation (or deflation) pressure superposed on a balanced internal and external pressure. The inflation (deflation) pressure adjusts $\mathrm{T}_k\langle\theta\theta\rangle(\rho_k)$ to zero, and the balanced pressure (either tensile or compressive) adjusts $\mathrm{T}_k\langle rr\rangle(\rho_k)$ to zero.

and from (6.5) and (6.7),

$$\gamma_k(r) = \left[r_k^3 \left(\frac{1}{(\overset{\circ}{\mathbf{B}}_{-1}\langle\theta\theta\rangle(r_k))^{3/2}} - 1 \right) + r^3 \right]^{1/3}, \qquad (6.8)^*$$

where

$$\left(r_k - \frac{\delta r}{2}\right) \leqslant r \leqslant \left(r_k + \frac{\delta r}{2}\right).$$

To determine the accuracy of the approximation given by (6.8), the constitutive equation (5.2) for $\mathcal{B}_0$ can be used with $\mathbf{F}_k = \nabla \mathbf{g}_{\underline{k}}$ to calculate the stress $\mathbf{T}_k$ in each $\overline{\mathcal{R}}_k$. The tensor $\nabla \mathbf{g}_k$ exists at all radii in each $\overline{\mathcal{B}}_k$ since the deformation $\mathbf{g}_k$, as defined by (6.8), is differentiable at all radii contained in $\overline{\mathcal{B}}_k$. We present a numerical example: $\mathcal{B}_0$ is subdivided into ten shells of equal thickness, numbered one through ten from the interior to the exterior of $\mathcal{B}_0$. The largest magnitude of $\mathrm{T}_k\langle\theta\theta\rangle$ in the outermost shell of the approximate virtual configuration is $\mathrm{T}_{10}\langle\theta\theta\rangle = -2.37F/L^2$. This is an order of magnitude smaller than the residual stress at the corresponding point in $\mathcal{B}_0$, which is $\overset{\circ}{\mathrm{T}}_{-1}\langle\theta\theta\rangle = -21.18F/L^2$. To see that the approximation improves as the number of subshells increases, $\mathcal{B}_0$ is subdivided into 50 and then into 100 shells. The maximum stress in the outermost shell decreases to $\mathrm{T}_{50}\langle\theta\theta\rangle = -0.51F/L^2$ for 50 subshells, and to $\mathrm{T}_{100}\langle\theta\theta\rangle = -0.26F/L^2$ for 100 subshells.

For the numerical example in which $\mathcal{B}_0$ is subdivided into 100 shells, the radius ρ_k of the innermost approximate virtual configuration is $\rho_1 = 1.21L$, and the radius of the outermost is $\rho_{100} = 4.71L$. Recall that the internal and external radii of $\mathcal{B}_0$ are $2.0L$ and $3.0L$, respectively. Clearly, the incompressible material of $\mathcal{B}_0$ cannot occupy all of the space between ρ_1 and ρ_{100}. Therefore, as the number of subshells in $\mathcal{B}_0$ approaches infinity, the approximate virtual configuration of $\mathcal{B}_0$ approaches an infinite number of isolated spherical shells.

This example demonstrates how the virtual configuration of a residually stressed body can be approximated using the strain (e.g., $\overset{\circ}{\mathbf{B}}_{-1}$) associated with the residual stress. The example also demonstrates that the local compatibility (or incompatibility) of this strain can be used to determine whether or not an exact virtual configuration with finite volume can be obtained for a specific part of the body. This is valuable information in the context of destructive experiments, where the ideal (but rarely attainable) situation is to work with a relatively large portion of stress free material. We have also shown that once a mapping from $\mathcal{B}_0$ into an approximate virtual configuration has been constructed, the constitutive equation for $\mathcal{B}_0$ can be used to evaluate the accuracy of the approximation.

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* Note that $\gamma_k(r)$ is of the form $\gamma_k(r) = (A + r^3)^{1/3}$, where A is a constant. This deformation is the universal solution for inflation of an incompressible spherical shell [23].

6.2. Non-uniqueness of the Virtual Configuration

Every residually stressed body has a virtual configuration composed of an infinite number of discrete points. For some residually stressed bodies, however, the virtual configuration can be more structured, as in the example of the hollow sphere discussed in Section 6.1. Recall that for that example, the virtual configurations of points at a fixed radius in $\mathcal{B}_0$ could be assembled into a continuous spherical surface of points that serves as an equally valid virtual configuration. The everted spherical shell of Example 1 (Section 4) also has multiple virtual configurations. That spherical shell can be partitioned into two equal parts, each of which can be 'uneverted' to produce a stress free hemispherical shell with finite thickness. These hemispherical shells are virtual configurations with finite volume, and each can be subdivided in an infinite number of ways to create new virtual configurations with finite volume. Multiple virtual configurations can be obtained in this way for any residually stressed body in the special class discussed in Section 3.1.

This idea can be stated more formally as follows. If the closure of the union of virtual configurations $\mathcal{P}_{v_i}$, $i = 1$ to n, of a set of n parts of $\mathcal{B}_0$ can be assembled into a continuous configuration, denoted by $\overline{\mathcal{R}}_s \equiv \overline{\bigcup_{i=1}^{n} \mathcal{P}_{v_i}}$, by rigid body translations and rotations, and if $\overline{\mathcal{R}}_s$ can be mapped into a part of $\mathcal{B}_0$ by a single continuous function, then $\overline{\mathcal{R}}_s$ is also a virtual configuration.

The virtual configuration consisting of an infinite number of discrete points is adequate for the derivation of the constitutive equation. For experimental applications, however, it is advantageous to determine the largest possible continuous virtual configuration, since larger portions of material are usually easier to excise from the body and measure.

6.3. Destructive Experiments and the Virtual Configuration

The concept of the virtual configuration combines with the mathematics used in the derivation to provide a sound theoretical framework for destructive experiments. In this section we briefly examine the relationship between our theory and the physical aspects of these experiments. We first address the use of a destructive experiment to obtain the data required to formulate a constitutive equation for the residually stressed material. This is followed by a discussion of the role of the virtual configuration in the design of destructive experiments.

Characterization of a Residually Stressed Material

As we discussed briefly in the introduction, the properties of the material in a residually stressed body cannot be obtained by the same experimental procedure used for stress free bodies. Specifically, mechanical testing of a discrete portion of a residually stressed body does not provide the properties of that material in the intact residually stressed configuration. The reason for this is that the effective elastic properties of the residually stressed material depend on the residual stress, and removal of the test specimen will typically relieve some or all of the residual stress. Clearly, if all of the residual stress is relieved, mechanical tests performed on this specimen provide the properties of the natural material. These ideas are central to the following discussion.

The constitutive equation for a residually stressed material can be constructed using the derivation presented in Section 3 and the data from a destructive experiment. Recall that in the typical destructive experiment, an approximate virtual configuration is obtained and the displacement from this virtual configuration to the residually stressed configuration is measured. In the notation of Section 3, the measured displacements are used to calculate an approximation of the deformation gradient $\overset{\circ}{\mathbf{F}}$. Mechanical testing of a piece of approximately stress free material provides the data needed to formulate a constitutive equation for the natural material (e.g., equation (2.6)). This equation is then evaluated with the approximate deformation gradient to give an estimate of the residual stress $\overset{\circ}{\mathbf{T}}$ originally supported by the material in the intact body; that is, $\overset{\circ}{\mathbf{T}} = \tilde{\mathfrak{T}}(\overset{\circ}{\mathbf{F}})$. Of course, the accuracy of $\overset{\circ}{\mathbf{T}}$ depends on the accuracy of $\overset{\circ}{\mathbf{F}}$.

In Section 3 we sought a constitutive equation for the residually stressed material that depends explicitly on the residual stress and the properties of the natural material. In Sections 4 and 5 we demonstrated the formulation of such constitutive equations in the context of a given residual stress field and an invertible response function for the natural material. Destructive experiments, on the other hand, provide a different data set; namely, they provide an approximation of both the deformation gradient $\overset{\circ}{\mathbf{F}}$ and the properties of the natural material. If the explicit dependence on residual stress is not important in a particular application, the constitutive equation for the residually stressed material can be formulated in terms of $\overset{\circ}{\mathbf{F}}$. That is, rather than inverting $\tilde{\mathfrak{T}}$ to obtain an expression for $\overset{\circ}{\mathbf{F}}$ as a function of $\overset{\circ}{\mathbf{T}}$, as in (3.15), the $\overset{\circ}{\mathbf{F}}$ calculated from the measured displacements can be substituted directly into (3.16) to give a constitutive equation of the form $\mathbf{T} = \tilde{\mathfrak{T}}(\mathbf{F}\overset{\circ}{\mathbf{F}})$ for the residually stressed material. We emphasize that construction of the constitutive equation from such experimental data requires only that the excised portion of the body be elastic and nearly stress free.

Design of Destructive Experiments

From the previous discussion it is clear that destructive experiments, the concept of the virtual configuration, and the theory underlying the derivation of the constitutive equation are all closely related. To illustrate this relationship further, we briefly discuss a situation in which the calculation of an approximate virtual configuration can be of use.

The first issue to address in a destructive experiment is how the residually stressed body should be cut in order to determine the residual stress most efficiently and accurately. Of course, as we have already mentioned, the actual residual stress field is not known a-priori. In some cases, however, it may be possible to use the equilibrium equation $(2.4)_1$, the zero surface traction condition (2.5), and the body geometry to postulate the form of the residual stress field in the body [e.g., 30]. The choice of candidate residual stress fields could be further narrowed by information about the microstructure of the material or by a model of the mechanism suspected to cause the residual stress. An approximate virtual configuration that is constructed using one of these hypothetical residual stress fields offers a suggestion for how to begin the destructive experiment.

7. Summary

In this paper we have presented a method for the derivation of constitutive equations for a broad class of elastic residually stressed materials. These constitutive equations depend explicitly on the residual stress and include only the material properties required to describe the natural material. In addition to a constitutive equation, our derivation provides an intuitive mathematical model for standard destructive experiments commonly performed on residually stressed bodies. The direct parallels between these destructive experiments and the theory used in the derivation are made physically meaningful by the concept of the virtual configuration.

Appendix

In the theorem and proof that follow, we rely on a certain familiarity with the behavior of elastic residually stressed bodies. We present a brief discussion of this behavior here in order to simplify the presentation of the proof.

If an elastic residually stressed body is composed of a natural material for which the material properties are smooth functions of position in the body, and if the body supports a residual stress field that is also a smooth function of position, then the mechanical response of the residually stressed body is expected to be smooth throughout the body. By smooth mechanical response we mean that if the body is subjected to a smooth surface traction field, the resulting deformation will also be smooth. It is also physically well motivated to assume that if a portion of the body is excised and unloaded, as discussed in Section 3.2, the resulting deformation is smooth and the unloaded part will support a new residual stress field that is also smooth.

In the absence of an explicit constitutive equation for the residually stressed material, we cannot substantiate the above ideas more concretely. However, we cannot assume a specific constitutive equation for use in the following proof and still obtain the general result that we seek. Therefore, the above assumptions must rest on our intuitive understanding of the influence of the residual stress on the effective elastic properties of the material in the residually stressed configuration.

From this point of view we state mathematical requirements for the existence of a stress free configuration of the material at a point in a residually stressed body. Of course, these conditions do not hold for every residually stressed body. At the end of this appendix we discuss how the derivation can be handled when these conditions hold only in finite subregions of a residually stressed body.

THEOREM. *Let $\mathcal{P}_{0_\varepsilon}$ be part of an elastic residually stressed body, as defined in Section 3.2, such that for all $\varepsilon < \overline{\varepsilon}$, where $\overline{\varepsilon} > 0$, the stress $\overset{\circ}{\mathbf{T}}$ and the properties of the natural material in $\mathcal{P}_{0_\varepsilon}$ are both C^1 functions of position $\mathbf{p}$ in $\mathcal{P}_{0_\varepsilon}$. In addition, let the deformation $\overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$ from $\mathcal{P}_{0_\varepsilon}$ to $\mathcal{R}_\varepsilon$ be a C^2 function of position $\mathbf{p}$ in $\mathcal{P}_{0_\varepsilon}$, and let $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ in $\mathcal{R}_\varepsilon$ be given by a constitutive equation of the form**

$$\overset{\circ}{\mathbf{T}}{}'_\varepsilon = \mathfrak{F}(\overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1}), \tag{A.1}$$

where $\overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1} \equiv \nabla \overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$, and where the response function $\mathfrak{F}$ is once-differentiable with respect to both $\overset{\circ}{\mathbf{T}}$ and $\overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1}$. Then

$$\lim_{\varepsilon \to 0} \overset{\circ}{\mathbf{T}}{}'_\varepsilon(\overline{\mathbf{q}}) = \mathbf{0}. \tag{A.2}$$

Proof. Each region $\mathcal{R}_\varepsilon$ is a residually stressed body by definition, and therefore $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ satisfies equations (2.4) and (2.5):

$$\begin{aligned} \operatorname{div} \overset{\circ}{\mathbf{T}}{}'_\varepsilon &= \mathbf{0}, \\ \overset{\circ}{\mathbf{T}}{}'_\varepsilon &= (\overset{\circ}{\mathbf{T}}{}'_\varepsilon)^T, \\ \overset{\circ}{\mathbf{T}}{}'_\varepsilon \mathbf{n}' &= \mathbf{0}, \end{aligned} \tag{A.3}$$

where $\mathbf{n}'$ is the outward unit normal to $\partial\mathcal{R}_\varepsilon$. By Signorini's mean stress theorem [31], the volume average of $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$, denoted by $\overset{\circ}{\mathbf{T}}{}'_M(\mathcal{R}_\varepsilon)$, is given by

$$\begin{aligned} \overset{\circ}{\mathbf{T}}{}'_M(\mathcal{R}_\varepsilon) &= \frac{1}{V(\mathcal{R}_\varepsilon)} \int_{\mathcal{R}_\varepsilon} \overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})\, \mathrm{d}V \\ &= \frac{1}{V(\mathcal{R}_\varepsilon)} \left[\int_{\partial\mathcal{R}_\varepsilon} (\mathbf{q} - \overline{\mathbf{q}}) \otimes (\overset{\circ}{\mathbf{T}}{}'_\varepsilon \mathbf{n}')\, \mathrm{d}A - \int_{\mathcal{R}_\varepsilon} (\mathbf{q} - \overline{\mathbf{q}}) \otimes (\operatorname{div} \overset{\circ}{\mathbf{T}}{}'_\varepsilon)\, \mathrm{d}V \right], \end{aligned}$$

* This assumption is motivated by constitutive equations previously derived for residually stressed and initially stressed bodies [11,13–17], under assumptions more specific than those made here. Note that since the surface tractions $\overset{\circ}{\mathbf{T}}\overset{\circ}{\mathbf{n}}$ on $\partial\mathcal{P}_{0_\varepsilon}$ are non-zero, $\mathcal{P}_{0_\varepsilon}$ is an *initially* stressed body. Indeed, it is reasonable to assume that the residual stress $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ at a point in $\mathcal{R}_\varepsilon$ depends in a smooth way on the initial stress $\overset{\circ}{\mathbf{T}}$ at the corresponding point in $\mathcal{P}_{0_\varepsilon}$, and on the deformation at that point.

where $V(\mathcal{R}_\varepsilon)$ is the volume of $\mathcal{R}_\varepsilon$ for a given ε. From (A.3), $\overset{\circ}{\mathbf{T}}{}'_M(\mathcal{R}_\varepsilon) = \mathbf{0}$ for all $\varepsilon < \bar{\varepsilon}$, and in rectangular Cartesian coordinates, the components of the mean stress (or equivalently, the mean values of the components of $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$) are given by

$$[\overset{\circ}{\mathbf{T}}{}'_M(\mathcal{R}_\varepsilon)]_{ij} = \{[\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}\}_M = \frac{1}{V(\mathcal{R}_\varepsilon)} \int_{\mathcal{R}_\varepsilon} [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij} \, \mathrm{d}V = 0 \tag{A.4}$$

for all ij. By the mean value theorem there exists at least one point $\mathbf{q}_0$ in $\mathcal{R}_\varepsilon$ such that

$$[\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q}_0)]_{ij} = \{[\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}\}_M = 0. \tag{A.5}$$

Based on the physical arguments discussed above (i.e., $\overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$, and the properties of the natural material are all smooth functions of position in $\mathcal{P}_{0_\varepsilon}$), it is well motivated to assume that $\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})$ is a smooth function of position in $\mathcal{R}_\varepsilon$. Therefore, $[\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}$ is bounded in each $\mathcal{R}_\varepsilon$:

$$|[\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}| < \mathrm{M}, \quad \mathrm{M} \in \text{Real},$$

for all ij. In addition, since the Cartesian components $[\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}$ must have both positive and negative values in $\mathcal{R}_\varepsilon$ in order that (A.4) hold for each ε, we know that

$$\max\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij} \geqslant 0$$

and (A.6)

$$\min\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij} \leqslant 0.$$

We establish with the following arguments that $\max\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}$ and $\min\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}$ approach the same value as ε approaches zero. From (A.1),

$$\begin{aligned} \nabla_q \overset{\circ}{\mathbf{T}}{}'_\varepsilon &= D_{\overset{\circ}{\mathbf{T}}} \mathfrak{F}(\overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1})[(\nabla_p \overset{\circ}{\mathbf{T}})\, \overset{\circ}{\mathbf{F}}_\varepsilon] + \\ &\quad + D_{\overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1}} \mathfrak{F}(\overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1})[(\nabla_p \overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1}) \overset{\circ}{\mathbf{F}}_\varepsilon], \end{aligned} \tag{A.7}$$

where $D_{\overset{\circ}{\mathbf{T}}} \mathfrak{F}(\overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1})$ and $D_{\overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1}} \mathfrak{F}(\overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1})$ are derivatives of the tensor-valued function $\mathfrak{F}$ with respect to $\overset{\circ}{\mathbf{T}}$ and $\overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1}$, respectively. These derivatives exist by

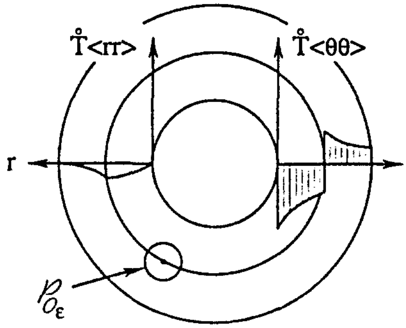

Fig. 11. Physical components of the residual stress as a function of radius in a hollow elastic cylinder that is composed of two press-fit hollow elastic cylinders (adapted from Fig. 3.6.1 of [32]).

assumption (see Footnote p. 35), and therefore are bounded. As ε approaches zero, both $\overset{\circ}{\mathbf{T}}$ and the properties of the natural material in $\mathcal{P}_{0_\varepsilon}$ become increasingly uniform functions of position in $\mathcal{P}_{0_\varepsilon}$ (see Section 3.2). Hence, the deformations $\overset{\circ}{\mathbf{f}}_\varepsilon$ and $\overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}$ become increasingly homogeneous as ε approaches zero. Therefore, $\overset{\circ}{\mathbf{F}}_\varepsilon$ approaches a constant as ε approaches zero, and $\lim_{\varepsilon\to 0} \nabla\overset{\circ}{\mathbf{F}}{}_\varepsilon^{-1} = \mathbf{0}$. From these arguments and equation (A.7) it is clear that

$$|\nabla_q \overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})| < \mathrm{G}, \quad \mathrm{G} \in \text{Real},$$

and therefore that

$$\max\, |\nabla_q[\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}| \leqslant \mathrm{G}, \tag{A.8}$$

for all $\varepsilon < \overline{\varepsilon}$, and for all ij. Hence, if d_ε is the diameter of a sphere that encloses $\mathcal{R}_\varepsilon$ for a given ε,

$$(\max\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij} - \min\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}) \leqslant \mathrm{d}_\varepsilon \mathrm{G} \tag{A.9}$$

for each ε. Then, since $\lim_{\varepsilon\to 0} V(\mathcal{R}_\varepsilon) = 0$, and since the dimensions of R_ε must remain finite, $\lim_{\varepsilon\to 0} \mathrm{d}_\varepsilon = 0$, and from (A.9), $\lim_{\varepsilon\to 0}(\max\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij} - \min\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}) = 0$ as well. That is, $\max\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}$ and $\min\, [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\mathbf{q})]_{ij}$ approach the same value as ε approaches zero, and that value is zero by (A.5). Finally, since $\mathcal{R}_\varepsilon$ reduces to a point at $\overline{\mathbf{q}}$ as ε approaches zero, $\lim_{\varepsilon\to 0} [\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\overline{\mathbf{q}})]_{ij} = 0$ for all ij, and therefore $\lim_{\varepsilon\to 0}\overset{\circ}{\mathbf{T}}{}'_\varepsilon(\overline{\mathbf{q}}) = \mathbf{0}$.

Discussion

Recall that our initial assumptions were that $\overset{\circ}{\mathbf{T}}$ and the properties of the natural material are smooth functions of position throughout $\mathcal{B}_0$, and therefore in any part

$\mathcal{P}_{0_\varepsilon}$ of $\mathcal{B}_0$ as well. In some residually stressed bodies, however, these assumptions hold only in subregions of the body. We explore this situation with the following qualitative example.

Consider the residually stressed body composed of two concentric, thick-walled hollow cylinders that are both composed of the same elastic natural material. Suppose that the outer cylinder is press-fit around the inner cylinder to produce the residual stress, and that the cylinders are fused together at the interface to form a continuous body. The tangential residual stress is discontinuous at the interface, as shown in Fig. 11 [32, pp. 97–98]. Consider the neighborhood $\mathcal{P}_{0_\varepsilon}$ of a point $\overline{\mathbf{p}}$ located on the interface between the two cylinders (Fig. 11). If $\mathcal{P}_{0_\varepsilon}$ were removed and unloaded, the resulting region $\mathcal{R}_\varepsilon$ (Fig. 3) would contain a surface through the point $\overline{\mathbf{q}} = \overset{\circ}{\mathbf{f}}{}_\varepsilon^{-1}(\overline{\mathbf{p}})$ across which the stress $\overset{\circ}{\mathbf{T}}{}'_\varepsilon$ is discontinuous. This is true for all neighborhoods $\mathcal{P}_{0_\varepsilon}$ of this point $\overline{\mathbf{p}}$ as ε approaches zero. Hence, equation (A.2) does not hold for this part $\mathcal{P}_{0_\varepsilon}$, and the constitutive equation for the point $\overline{\mathbf{p}}$ in the residually stressed body cannot be derived as in Section 3. However, the interface can be approached from both sides with infinitesimal neighborhoods $\mathcal{P}_{0_\varepsilon}$ to derive the constitutive equation for points within each subregion.

In this example, the properties of the natural material are smooth throughout the residually stressed body, but the residual stress is not. The results are very similar for the case where the constituent cylinders are composed of different natural materials.

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