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On the existence, uniqueness and completeness of displacements and stress functions in linear elasticity

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1. Introduction

In the absence of incompatibilities (non-elastic strains as, e.g., caused by thermal stresses, plastic deformation, electrical and magnetic processes), the strain equations of compatibility for a continuous elastic body take the form

$$\nabla \times \boldsymbol{E} \times \nabla = 0 \tag{I}$$

(strain tensor $\boldsymbol{E}$ symmetric), and in the absence of body-forces the stress equations of equilibrium are

$$\nabla \cdot \boldsymbol{S} = 0 \tag{II}$$

(stress tensor $\boldsymbol{S}$ symmetric). It is well known that both pairs of equations can be satisfied *a priori*, namely (I) by deriving the strain $\boldsymbol{E}$ from the displacement vector $\boldsymbol{u}$ according to

$$\boldsymbol{E} = (\nabla \boldsymbol{u} + \boldsymbol{u}\nabla)/2, \tag{I*}$$

and (II) by deriving the stress $\boldsymbol{S}$ from a symmetric stress function tensor $\boldsymbol{A}$ according to

$$\boldsymbol{S} = \nabla \times \boldsymbol{A} \times \nabla. \tag{II*}$$

The second representation (II*) dates back to Beltrami (1892), the first (I*) comes from Cesaro (1906).

It was also Cesaro who showed that, for *simply*-connected bodies, the eqs. (I) are necessary and sufficient conditions for the existence of a single-valued displacement field $\boldsymbol{u}$. For *multiply*-connected bodies, however, this is no longer true: here discontinuities of the displacement may occur, caused by dislocations and disclinations respectively, "hidden" in the elastic continuum (Volterra's "distorsioni" (1907); cp. Zastrow (1985)).

Similarly, every solution of (II) admits the representation (II*), if and only if the elastic body under consideration is *simply*-bordered (Rieder (1960)). In the general case of a *multiply*-bordered continuum, (II*) can at most represent *totally self-equilibrated* stress fields (Gurtin (1963)), i.e., stress fields not only obeying (II), but also with the resultant force and moment vanishing on every closed surface contained in the body.

Thus in both approaches, the topology of the region Ω covered by the elastic body plays an important role. This truism was already stressed, even if in a more general context, by Maxwell (1873, art. 18–22) in whose terminology multiply-*connected* regions (in modern topology characterized by the *first* Betti number b_1) and multiply-*bordered* regions (denoted by the *second* Betti number b_2) were called "*cyclic*" and "*periphractic*" respectively.

The present paper makes use of the well-known duality between the displacement and the stress function method. But we will already note here that the formal analogy applies only up to a certain point. This is due to a basic difference (apart from the different tensor rank and level of integration) between the two approaches: whereas the displacement field $\boldsymbol{u}$, defined by (I*), is determined by *three* start functions, the stress function approach (II*) works with *six* independent components. Since stress function tensors of the form $A = \nabla \boldsymbol{v} + \boldsymbol{v}\nabla$ yield no stresses ("*zero*-stress function tensors"), one may impose certain gauges on A and thus adapt its form to the problem at hand. Gauges in Cartesian coordinates were already established by Maxwell (1870) and Morera (1892) (and later by Blokh (1964)), coordinate-*invariant* gauges by Schaefer (1953), Kroener & Marguerre (1954, 1955) and Blokh (1964) (for an extensive bibliography see Truesdell (1959), Gurtin (1972)).

The advantage common to the cited *invariant* formulations is the fact that the stress function tensor A becomes biharmonic ($\Delta\Delta A = 0$). The corresponding displacement fields $\boldsymbol{u}(A)$ are closely connected with the displacement functions of Galerkin (1930) and Papkovich (1932) (dating back, essentially, to Boussinesq (1885); for details see Stippes (1966, 1967) and Gurtin (1972)). The completeness of the latter has been shown, e.g., by Gurtin (1962), Pecknold (1971) and Gurtin (1972), usually by means of the Stokes-Helmholtz decomposition $\boldsymbol{u} = c_1 \operatorname{grad}\phi + c_2 \operatorname{curl}\boldsymbol{v}$. Complcteness proofs for the stress function method (II*) were given for *simply*-bordered regions by Guenther (1954) and Truesdell (1959). For *multiply*-bordered regions, completeness proofs were put forward by Rieder (1960) and Gurtin (1963), based, as mentioned above, on the concept of *totally self-equilibrated* stress fields (cp. also Carlson (1967)).

The present paper aims at completing the afore-mentioned works in two respects: (a) Our results enable us to make not only a qualitative analysis, but also a quantitative statement (in terms of an equation) about the relation between the topology of the elastic body under consideration and the number of those strain and stress fields which are *not* representable as derivatives of displacements and stress functions. (b) Algebraic topology, i.e., applying algebraic methods to study certain questions about topological spaces and mappings, has become a standard branch of modern mathematics (see, e.g., Goldberg (1962), Spivak (1974)). For the purpose of this paper, de Rham's *cohomology* turns out to be the most appropriate mathematical tool. Besides its distinct definition of the Betti numbers, it offers the advantage of being applicable to arbitrary open regions of the $\mathbb{R}^3$; therefore our results are generally valid, without any restrictions on the regularity of the boundary or on the boundedness of the elastic body.

2. Mathematical preliminaries: some remarks on tensor analysis and algebraic topology

We follow Gurtin's (1972) scheme of notation and use lightface letters for scalars (ϕ, λ). Boldface letters stand for pth-order tensors, namely minuscules for vectors ($\boldsymbol{v}$, $\boldsymbol{w}$) and majuscules for second-order tensors ($\boldsymbol{V}$, $\boldsymbol{W}$). Both the symbolic notation and the indicial notation are used: we write v_i and V_{ij} resp. (p subscripts) for the components of $\boldsymbol{v}$ and $\boldsymbol{V}$ resp. in the underlying Cartesian coordinate frame r_1, r_2, r_3. $\boldsymbol{I}$ is the second-order identity tensor, corresponding to Kronecker's delta δ_{ij} in indicial notation, and $\boldsymbol{\varepsilon}$ is the Levi-Civita-tensor, in indicial notation ε_{ijk}. Summation and differentiation conventions are employed as usual: summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate ($v_{j,i} = \partial_i v_j$). All subscripts range over the integers 1, 2, 3.

We write $\boldsymbol{V}^T$ for the transpose of the second-order tensor $\boldsymbol{V}$, sym$\{\boldsymbol{V}\}$ and skw$\{\boldsymbol{V}\}$ denote the symmetric and the skew part of $\boldsymbol{V}$ resp., and tr$\{\boldsymbol{V}\}$ is the trace of $\boldsymbol{V}$. Analogously to the vector differential operators div$\boldsymbol{v}$, curl$\boldsymbol{v}$ and gradϕ, the corresponding tensor differential operators are defined by

$$\mathrm{Div}\boldsymbol{V} = \nabla \cdot \boldsymbol{V} = \boldsymbol{V}^T \cdot \nabla = \boldsymbol{h} \cong h_j = V_{ij,i}, \tag{2.1}$$

$$\mathrm{Curl}\boldsymbol{V} = \nabla \times \boldsymbol{V} = -(\boldsymbol{V}^T \times \nabla)^T = \boldsymbol{W} \cong W_{ij} = \varepsilon_{irs} V_{sj,r}, \tag{2.2}$$

$$\mathrm{Grad}\boldsymbol{v} = \nabla \boldsymbol{v} = (\boldsymbol{v}\nabla)^T = \boldsymbol{G} \cong G_{ij} = v_{j,i}. \tag{2.3}$$

Following Kroener's notation (1954, 1960) (and differing from Gurtin's (1972)), we write for the latter Div, Curl and Grad in order to indicate that they range one tensor-order higher than the operators div, curl and grad. We will also adopt Kroener's abbreviation Def$\boldsymbol{v}$ (in Gurtin's notation $\hat{\nabla}\boldsymbol{v}$) for the symmetric part of Grad$\boldsymbol{v}$,

$$\mathrm{Def}\boldsymbol{v} = \mathrm{sym}\,\{\mathrm{Grad}\boldsymbol{v}\} = (\nabla\boldsymbol{v} + \boldsymbol{v}\nabla)/2 = \boldsymbol{U} \cong (v_{j,i} + v_{i,j})/2 = U_{ij} \tag{2.4}$$

(read: "deformator of $\boldsymbol{v}$"), and Kroener's operator Inc$\boldsymbol{V}$ (read: "incompatibility of $\boldsymbol{V}$"),

$$\mathrm{Inc}\boldsymbol{V} = \nabla \times \boldsymbol{V} \times \nabla = \boldsymbol{W} \cong W_{ij} = -\varepsilon_{irs}\varepsilon_{jtu} V_{su,rt}\,. \tag{2.5}$$

The definitions (2.1)–(2.3) yield the well-known identities

$$\mathrm{curl}\;\mathrm{grad}\phi = 0, \quad \mathrm{Curl}\;\mathrm{Grad}\boldsymbol{v} = 0; \qquad (2.6),\ (2.6^*)$$

$$\mathrm{div}\;\mathrm{curl}\boldsymbol{v} = 0, \quad \mathrm{Div}\;\mathrm{Curl}\boldsymbol{V} = 0; \qquad (2.7),\ (2.7^*)$$

and for the operator Inc the following two identities hold:

$$\mathrm{Inc}\;\mathrm{Def}\boldsymbol{v} = 0, \tag{2.8}$$

$$\mathrm{Div}\;\mathrm{Inc}\boldsymbol{V} = 0. \tag{2.9}$$

For our later considerations, we will summarize some basic definitions of algebraic topology (in particular de Rham cohomology; for details, we refer to the standard works of Goldberg (1960), Spivak (1974)). Of special importance for the following are the concept of the *quotient space* and the definition of the first and second *Betti numbers.*

Let $\mathscr{V}$, $\mathscr{W}$ denote two vector spaces, and Φ a linear mapping $\Phi: \mathscr{V} \to \mathscr{W}$. The *image* of Φ (Im$\{\Phi\}$) is the set of all vectors $\boldsymbol{w} \in \mathscr{W}$ which can be represented as $\boldsymbol{w} = \Phi(\boldsymbol{v})$ for some $\boldsymbol{v} \in \mathscr{V}$. The *kernel* of Φ (Ker$\{\Phi\}$) is defined as the set of all $\boldsymbol{v} \in \mathscr{V}$ with $\Phi(\boldsymbol{v}) = 0$. Obviously, Φ is onto, if and only if Im$\{\Phi\} = \mathscr{W}$, and Φ is one to one, if and only if Ker$\{\Phi\} = \{0\}$. – As an example, let us consider a system of linear equations $\boldsymbol{v} \to \boldsymbol{w} = \boldsymbol{L} \cdot \boldsymbol{v}$; then Im$\{\Phi\}$ is the set of all vectors $\boldsymbol{w} \in \mathscr{W}$ for which the inhomogeneous system $\boldsymbol{w} = \boldsymbol{L} \cdot \boldsymbol{v}$ admits a solution, and Ker$\{\Phi\}$ is the set of all vectors $\boldsymbol{v}$ with $\boldsymbol{L} \cdot \boldsymbol{v} = 0$, i.e., the homogeneous solutions of the system. – Now, the identities (2.8), (2.9) (and analogously (2.6)–(2.7*)) can be written in the form

$$\mathrm{Im}\,\{\mathrm{Def}\} \subset \mathrm{Ker}\,\{\mathrm{Inc}\}, \tag{2.10}$$

$$\mathrm{Im}\,\{\mathrm{Inc}\} \subset \mathrm{Ker}\,\{\mathrm{Div}\}. \tag{2.11}$$

For two vector spaces $\mathscr{V}$, $\mathscr{W}$ with $\mathscr{W} \subset \mathscr{V}$, we call two vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \mathscr{V}$ *equivalent*, if $(\boldsymbol{v}_1 - \boldsymbol{v}_2) \in \mathscr{W}$. By $[\boldsymbol{v}]$ we denote the class of all vectors in $\mathscr{V}$ which are equivalent to $\boldsymbol{v}$. Obviously, the set of all classes $[\boldsymbol{v}]$ forms again a vector space, the *quotient space* of $\mathscr{V}$ with respect to $\mathscr{W}$, denoted by $\mathscr{V}/\mathscr{W}$. Practically speaking, the construction of a quotient space serves the purpose of excluding all inessential properties of the objects under consideration. E.g., we will discuss the quotient space Ker{Inc}/Im{Def}, since we are interested in those strain tensor fields $\boldsymbol{E}$ which, while fulfilling the compatibility equation $\text{Inc}\boldsymbol{E} = 0$, are *not* representable as $\boldsymbol{E} = \text{Def}\boldsymbol{u}$. – With the concept of the quotient space, the linear mapping $\Phi: \mathscr{V} \to \mathscr{W}$ gives rise to a subsequent linear mapping

$$\tilde{\Phi}: \mathscr{V}/\text{Ker}\{\Phi\} \to \mathscr{W}, \tag{2.12}$$

defined by $\tilde{\Phi}([\boldsymbol{v}]) := \Phi(\boldsymbol{v})$. It is easily proved that then $\text{Ker}\{\tilde{\Phi}\} = \{0\}$, i.e., $\tilde{\Phi}$ is one to one.

Let ω now be a differential form of degree k on a manifold Ω. By $d_k\omega$ we denote the outer derivative of ω; $d_k\omega$ is a differential form of degree $k + 1$. Poincare's lemma for differential forms then states that $d_{k+1}d_k\omega = 0$ for any ω and k (Spivak (1974), Goldberg (1962)). In the language of linear mapping, this identity takes the form

$$\text{Im}\{d_k\} \subset \text{Ker}\{d_{k+1}\}. \tag{2.13}$$

The k-th *de Rham cohomology* vector space of Ω is defined as the quotient space†

$$\mathscr{H}^k(\Omega) = \text{Ker}\{d_{k+1}\}/\text{Im}\{d_k\}. \tag{2.14}$$

The dimension of $\mathscr{H}^k(\Omega)$ is called the k-th *Betti number* $b_k(\Omega)$ and is a topological invariant of Ω (see Goldberg (1962)). For an open region Ω of $\mathbb{R}^3$, we have the special cases

$$\mathscr{H}^1(\Omega) = \text{Ker}\{\text{curl}\}/\text{Im}\{\text{grad}\}, \tag{2.15}$$

$$\mathscr{H}^2(\Omega) = \text{Ker}\{\text{div}\}/\text{Im}\{\text{curl}\}; \tag{2.16}$$

only these two cases will be of interest in the following.

† One might suspect that $\mathscr{H}^k(\Omega)$ depends on the smoothness assumptions made for the differential forms under consideration. But this is not the case (see Goldberg (1962)); therefore minimal differentiability properties, i.e., the existence of all occurring differentials, are sufficient in the present context.

3. On the integrability of the strain tensor

The following considerations are divided into three groups: we first consider the relation between displacement $\boldsymbol{u}$ and strain $\boldsymbol{E}$ for simply-connected regions Ω and the integral of Cesaro; the structure of its kernel then leads us to a linear mapping which carries $\boldsymbol{E}$ to two tensor fields $\boldsymbol{E} \times \nabla$ and $\boldsymbol{E} + (\boldsymbol{E} \times \nabla) \times \boldsymbol{r}$ the discussion of which forms the main part. This finally reveals the relation between the compatibility and integrability of the strain tensor field $\boldsymbol{E}$ and the topology of Ω.

3.1. Integrable and non-integrable strain tensor fields

An elastic body is an open region $\Omega \in \mathbb{R}^3$ (not necessarily bounded). Given a displacement vector field $\boldsymbol{u}$ on Ω, the corresponding (linear) strain tensor field $\boldsymbol{E} = \boldsymbol{E}^T$ is defined as

$$\boldsymbol{E} = \mathrm{Def}\boldsymbol{u}. \tag{3.1}$$

From (2.8) it follows that the homogeneous equations of compatibility

$$\mathrm{Inc}\boldsymbol{E} = 0 \tag{3.2}$$

are then identically fulfilled. It was Cesaro (1906) who proved that, for *simply*-connected regions Ω, the converse is also true: given a symmetric tensor field $\boldsymbol{E}$ with $\mathrm{Inc}\boldsymbol{E} = 0$, there is a vector field $\boldsymbol{u}$ with $\boldsymbol{E} = \mathrm{Def}\boldsymbol{u}$, i.e., $\boldsymbol{E}$ is integrable (for a modern proof see Tran-Cong (1985)).

For a *multiply*-connected region Ω, however, there will generally exist fields $\boldsymbol{E}$ with $\mathrm{Inc}\boldsymbol{E} = 0$, which are *not* representable in such a way, i.e., which are *non*-integrable. In order to determine the variety of these non-integrable strain fields, it is first necessary to define under what relation two non-integrable strain fields $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ are said to be "of the same type". We will call them *equivalent*, if a vector field $\boldsymbol{u}$ exists from which their difference can be derived according to $(\boldsymbol{E}_1 - \boldsymbol{E}_2) = \mathrm{Def}\boldsymbol{u}$; in other words: we will consider two non-integrable strain fields $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ as "essentially identical", if we can obtain $\boldsymbol{E}_2$ by adding an integrable strain field to $\boldsymbol{E}_1$. With $\mathscr{E}(\Omega)$ denoting the space of all strain fields under this equivalence relation, it then holds that (cp. (2.10))

$$\mathscr{E}(\Omega) = \mathrm{Ker}\{\mathrm{Inc}\}/\mathrm{Im}\{\mathrm{Def}\}. \tag{3.3}$$

We have thus reduced the investigation of all possible non-integrable strain fields $\boldsymbol{E}$ to the determination of the *quotient space* (3.3) and its dimension.

3.2. The integral of Cesaro

According to Cesaro's theorem, for any strain tensor field $\boldsymbol{E} = \mathrm{Def}\boldsymbol{u}$ on Ω, the corresponding displacement vector field $\boldsymbol{u}$ can be represented (up to a rigid body displacement) in the form

$$\boldsymbol{u}(\boldsymbol{r}) = \int_{\mathring{\boldsymbol{r}}}^{\boldsymbol{r}} \mathrm{d}\bar{\boldsymbol{r}} \cdot \{\boldsymbol{E}(\bar{\boldsymbol{r}}) + (\boldsymbol{E}(\bar{\boldsymbol{r}}) \times \bar{\nabla}) \times (\bar{\boldsymbol{r}} - \boldsymbol{r})\}$$

where $\mathring{\boldsymbol{r}}$ is a fixed reference point. It follows directly that the vanishing of this integral for any closed path $\mathscr{C}$ in Ω is a necessary condition for the integrability of $\boldsymbol{E}$. Therefore, a *non*-integrable strain field $\boldsymbol{E}$ will generally yield a non-vanishing Burgers vector

$$\boldsymbol{b}(\boldsymbol{r}) = \oint_{\mathscr{C}} (\partial \boldsymbol{u}/\partial s)\, \mathrm{d}s$$

for closed paths $\mathscr{C}$ in Ω. The integral $\boldsymbol{b}(\boldsymbol{r})$ can be decomposed into

$$\boldsymbol{b}(\boldsymbol{r}) = \mathring{\boldsymbol{b}} + \boldsymbol{d} \times (\boldsymbol{r} - \mathring{\boldsymbol{r}}) \tag{3.4}$$

with

$$\mathring{\boldsymbol{b}} = + \oint_{\mathscr{C}} \mathrm{d}\bar{\boldsymbol{r}} \cdot \{(\boldsymbol{E}(\bar{\boldsymbol{r}}) + \boldsymbol{E}(\bar{\boldsymbol{r}}) \times \bar{\nabla}) \times (\bar{\boldsymbol{r}} - \mathring{\boldsymbol{r}})\}$$

and

$$\boldsymbol{d} = - \oint_{\mathscr{C}} \mathrm{d}\bar{\boldsymbol{r}} \cdot \{\boldsymbol{E}(\bar{\boldsymbol{r}}) \times \bar{\nabla}\}.$$

$\mathring{\boldsymbol{b}}$ is called the *dislocation*, $\boldsymbol{d}$ the *disclination* of $\boldsymbol{E}$ around the path $\mathscr{C}$ (for a closer discussion see Zastrow (1985)).

The kernels of the two integrals $\boldsymbol{d}$ and $\mathring{\boldsymbol{b}}$ respectively are given by the tensor fields

$$\boldsymbol{D}^E(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) \times \nabla, \quad \boldsymbol{B}^E(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) + (\boldsymbol{E}(\boldsymbol{r}) \times \nabla) \times \boldsymbol{r}. \tag{3.5}$$

From the assumption (3.2) $\mathrm{Inc}\boldsymbol{E} = 0$, we obtain

$$\mathrm{Curl}\boldsymbol{D}^E = 0, \quad \mathrm{Curl}\boldsymbol{B}^E = 0 \qquad \text{(3.6a), (3.6b)}$$

(for (3.6b), use eq. (A1) of the Appendix and $\mathrm{tr}\{\boldsymbol{E} \times \nabla\} = 0$ because

$\boldsymbol{E} = \boldsymbol{E}^T$). This enables us to define a linear mapping

$$\Phi\colon \mathrm{Ker}\{\mathrm{Inc}\} \rightarrow \mathscr{H}^1(\Omega)^3 \times \mathscr{H}^1(\Omega)^3, \tag{3.7}$$

$$\boldsymbol{E} \mapsto ([\boldsymbol{D}^E], [\boldsymbol{B}^E]) \tag{3.7*}$$

(the symbol "x" means the Cartesian product of two vector spaces, and a tensor field is conceived as a collection of three vector fields). By discussing the properties of this mapping, the next section will prove that $\dim\{\mathscr{E}(\Omega)\} = 6b_1(\Omega)$.

3.3. Compatibility and integrability of E

We will first prove that the tensors $\boldsymbol{D}^E$ and $\boldsymbol{B}^E$ respectively are *gradient* tensors if $\boldsymbol{E}$ is representable as $\boldsymbol{E} = \mathrm{Def}\boldsymbol{u}$ and vice versa, i.e., that for the mapping Φ (3.7), we obtain the relation

LEMMA I:

$$\mathrm{Ker}\{\Phi\} = \mathrm{Im}\{\mathrm{Def}\}. \tag{3.8}$$

Proof: With $\boldsymbol{E} = \mathrm{Def}\boldsymbol{u}$ the definition (3.5) gives

$$\begin{aligned} \boldsymbol{D}^E &= (\nabla\boldsymbol{u} + \boldsymbol{u}\nabla) \times \nabla/2 = (\nabla\boldsymbol{u}) \times \nabla/2 \\ &= \mathrm{Grad}(\boldsymbol{u} \times \nabla/2), \end{aligned} \tag{3.9a}$$

$$\begin{aligned} \boldsymbol{B}^E &= (\nabla\boldsymbol{u} + \boldsymbol{u}\nabla)/2 + ((\nabla\boldsymbol{u} + \boldsymbol{u}\nabla) \times \nabla) \times \boldsymbol{r}/2 \\ &= \mathrm{Grad}(\boldsymbol{u} - (\nabla\boldsymbol{u} - \boldsymbol{u}\nabla)\cdot\boldsymbol{r}/2) \end{aligned} \tag{3.9b}$$

(for (3.9a), use the identity (2.6*), and for (3.9b) see eq. (A2) of the Appendix); hence (according to (2.15))

$$[\boldsymbol{D}^E] = 0, \quad [\boldsymbol{B}^E] = 0. \qquad (3.9a^*), (3.9b^*)$$

Now let us assume, on the contrary,

$$\boldsymbol{D}^E = \mathrm{Grad}\boldsymbol{v}, \quad \boldsymbol{B}^E = \mathrm{Grad}\boldsymbol{w}. \tag{3.10}$$

The corresponding tensor $\boldsymbol{E}$ is, according to (3.5),

$$\boldsymbol{E} = \boldsymbol{B}^E - \boldsymbol{D}^E \times \boldsymbol{r} = \nabla \boldsymbol{w} - (\nabla \boldsymbol{v}) \times \boldsymbol{r}$$

$$= \nabla(\boldsymbol{w} - \boldsymbol{v} \times \boldsymbol{r}) - \boldsymbol{v} \times \boldsymbol{I},$$

and thus, because $\boldsymbol{E} = \boldsymbol{E}^T$ and $\boldsymbol{v} \times \boldsymbol{I}$ is skew symmetric,

$$\boldsymbol{E} = \mathrm{sym}\{\nabla(\boldsymbol{w} - \boldsymbol{v} \times \boldsymbol{r})\}$$

$$= \mathrm{Def}(\boldsymbol{w} - \boldsymbol{v} \times \boldsymbol{r}). \tag{3.11}$$

This enables us, according to (2.12), to define a subsequent mapping $\tilde{\Phi}$ of the quotient space Ker{Inc}/Im{Def},

$$\tilde{\Phi}: \mathscr{E}(\Omega) \rightarrow \mathscr{H}^1(\Omega)^3 \times \mathscr{H}^1(\Omega)^3, \tag{3.12}$$

$$[\boldsymbol{E}] \mapsto ([\boldsymbol{D}^E], [\boldsymbol{B}^E]). \tag{3.12*}$$

$\tilde{\Phi}$ is one to one. The following lemma II will complete our proof.

LEMMA II: *$\tilde{\Phi}$ is also onto.*

Proof: Given two tensor fields $\boldsymbol{D}$, $\boldsymbol{B}$ with $\mathrm{Curl}\boldsymbol{D} = \mathrm{Curl}\boldsymbol{B} = 0$ (cp. (2.15)), we define

$$\boldsymbol{E} = \mathrm{sym}\,\{\boldsymbol{B} - \boldsymbol{D} \times \boldsymbol{r}\}, \tag{3.13}$$

according to (3.11). It then holds that

$$\boldsymbol{D}^E = \boldsymbol{E} \times \nabla = \boldsymbol{D} + \mathrm{Grad}\boldsymbol{v}, \tag{3.14a}$$

$$\boldsymbol{B}^E = \boldsymbol{E} + (\boldsymbol{E} \times \nabla) \times \boldsymbol{r} = \boldsymbol{B} + \mathrm{Grad}\boldsymbol{w} \tag{3.14b}$$

(for the proof and the expressions $\boldsymbol{v}$, $\boldsymbol{w}$ see section A3 of the Appendix); hence

$$[\boldsymbol{D}^E] = [\boldsymbol{D}], \quad [\boldsymbol{B}^E] = [\boldsymbol{B}]. \qquad (3.14a^*), (3.14b^*)$$

It has finally to be shown that the strain tensor $\boldsymbol{E}$ according to (3.13) is compatible: indeed (3.14a) yields $\mathrm{Curl}\boldsymbol{D}^E = \mathrm{Curl}(\boldsymbol{D} + \mathrm{Grad}\boldsymbol{v}) = 0$ and thus $\mathrm{Curl}(\boldsymbol{E} \times \nabla) = \mathrm{Inc}\boldsymbol{E} = 0$. Therefore (3.12) changes into

$$\tilde{\Phi}([\boldsymbol{E}]) = \Phi(\boldsymbol{E}) = ([\boldsymbol{D}^E], [\boldsymbol{B}^E]) = ([\boldsymbol{D}], [\boldsymbol{B}]). \tag{3.15}$$

as required. We can now summarize our results and establish

THEOREM I: *The vector space* $\mathscr{E}(\Omega)$ *is* isomorphic *to* $\mathscr{H}^1(\Omega)^3 \times \mathscr{H}^1(\Omega)^3$, *i.e. its dimension is*

$$\dim\{\mathscr{E}(\Omega)\} = 6b_1(\Omega). \tag{3.16}$$

4. On the integrability of the stress tensor

As this chapter is structured in a similar way to the previous one, we will denote corresponding equations with corresponding equation numbers (e.g., eq. (4.3) is formally analogous to eq. (3.3)).

4.1. Integrable and non-integrable stress tensor fields

We call a stress tensor field $\boldsymbol{S}$ (on the open region $\Omega \in \mathbb{R}^3$) integrable, if it can be derived from a second-order tensor $\boldsymbol{A} = \boldsymbol{A}^T$ according to

$$\boldsymbol{S} = \mathrm{Inc}\boldsymbol{A}. \tag{4.1}$$

$\boldsymbol{A}$ is called the stress function tensor (Beltrami (1892); cp Truesdell (1959)). According to (2.9) the homogeneous equations of equilibrium

$$\mathrm{Div}\boldsymbol{S} = 0 \tag{4.2}$$

are identically fulfilled. For *simply*-bordered regions Ω, the stress tensor $\boldsymbol{S}$ (with $\mathrm{Div}\boldsymbol{S} = 0$) is always integrable (cp. Trans-Cong (1985)). For *multiply*-bordered regions Ω, however, this is no longer true (Rieder (1960), cp. Gurtin (1963)). In order to get a survey of all possible *non*-integrable states of stress on Ω, we will, as in the analogous geometrical case (cp. (3.3)), investigate the vector space (cp. (2.11))

$$\mathscr{S}(\Omega) = \mathrm{Ker}\{\mathrm{Div}\}/\mathrm{Im}\{\mathrm{Inc}\}. \tag{4.3}$$

It will be shown that its dimension equals the sixfold of the second Betti number $b_2(\Omega)$.

4.2. The integral of Peretti–Guenther

Given the stress field $\boldsymbol{S}$, the resulting force $\boldsymbol{f}$ acting on a surface $\mathcal{S}$ in Ω is

$$\boldsymbol{f} \;=\; +\int_{\mathcal{S}} \bar{\boldsymbol{n}}\, \mathrm{d}\bar{a} \cdot \boldsymbol{S}(\bar{\boldsymbol{r}}).$$

($\boldsymbol{n}$ is the outward normal to $\mathcal{S}$, and $\mathrm{d}\boldsymbol{a}$ the element of area); and the resulting moment $\mathring{\boldsymbol{m}}$ is calculated from $\boldsymbol{S}$ according to

$$\mathring{\boldsymbol{m}} \;=\; -\int_{\mathcal{S}} \bar{\boldsymbol{n}}\, \mathrm{d}\bar{a} \cdot \{\boldsymbol{S}(\bar{\boldsymbol{r}}) \times (\bar{\boldsymbol{r}} - \mathring{\boldsymbol{r}})\}$$

where $\mathring{\boldsymbol{r}}$ is a fixed reference point (Peretti (1949), Guenther (1954)). The integral $\mathring{\boldsymbol{m}}$ can be decomposed into

$$\mathring{\boldsymbol{m}} \;=\; \boldsymbol{m}(\boldsymbol{r}) - \boldsymbol{f} \times (\boldsymbol{r} - \mathring{\boldsymbol{r}}) \qquad (4.4)$$

where

$$\boldsymbol{m}(\boldsymbol{r}) \;=\; -\int_{\mathcal{S}} \bar{\boldsymbol{n}}\, \mathrm{d}\bar{a} \cdot \{\boldsymbol{S}(\bar{\boldsymbol{r}}) \times (\bar{\boldsymbol{r}} - \boldsymbol{r})\}.$$

The analogy of (4.4) and (3.4) is obvious: the resulting moment vector $\mathring{\boldsymbol{m}}$ corresponds to the dislocation vector $\mathring{\boldsymbol{b}}$, and the resulting force vector $\boldsymbol{f}$ to the disclination vector $\boldsymbol{d}$.

It was Rieder (1960; cp. Gurtin (1963), (1972)) who observed that a necessary condition for the integrability of $\boldsymbol{S}$ is the vanishing of $\boldsymbol{m}(\boldsymbol{r})$ for any closed surface $\mathcal{S}$ in Ω (in Gurtin's notation (1963), $\boldsymbol{S}$ is then a *totally self-equilibrated* stress field). Therefore, a *non*-integrable stress field $\boldsymbol{S}$ will in general yield a non-vanishing moment vector $\boldsymbol{m}(\boldsymbol{r})$.

Now the integrals (4.4) vanish identically, if the stress $\boldsymbol{S} = \mathrm{Inc}\boldsymbol{A}$ itself becomes zero; for a simply-connected region, this means that the stress function tensor $\boldsymbol{A}$ is representable as $\boldsymbol{A} = \mathrm{Def}\boldsymbol{v}$:

$$\boldsymbol{S} \;=\; \mathrm{Inc}\boldsymbol{A} \;=\; \mathrm{Inc}\,\mathrm{Def}\boldsymbol{v} \;=\; 0.$$

By applying Stokes' theorem, the surface integral $\boldsymbol{m}(\boldsymbol{r})$ can be transformed into the contour integral

$$\boldsymbol{m}(\boldsymbol{r}) = -\oint_{\mathscr{C}} \mathrm{d}\bar{\boldsymbol{r}} \cdot \{\boldsymbol{A}(\bar{\boldsymbol{r}}) + (\boldsymbol{A}(\bar{\boldsymbol{r}}) \times \bar{\nabla}) \times (\bar{\boldsymbol{r}} \times \boldsymbol{r})\}$$

$$= -\oint_{\mathscr{C}} (\partial \boldsymbol{v}/\partial s)\, \mathrm{d}s,$$

where the "generating" vector $\boldsymbol{v}(\boldsymbol{r})$ is obtained from $\boldsymbol{A}$ according to

$$\boldsymbol{v}(\boldsymbol{r}) = \int_{\hat{r}}^{r} \mathrm{d}\bar{\boldsymbol{r}} \cdot \{\boldsymbol{A}(\bar{\boldsymbol{r}}) + (\boldsymbol{A}(\bar{\boldsymbol{r}}) \times \bar{\nabla}) \times (\bar{\boldsymbol{r}} - \boldsymbol{r})\}.$$

This is, of course, again Cesaro's integral (the "*zero*-stress function tensor" with $\boldsymbol{S} = \mathrm{Inc}\boldsymbol{A} = \mathrm{Inc\,Def}\boldsymbol{v} = 0$ corresponds to the compatibility condition $\mathrm{Inc}\boldsymbol{E} = \mathrm{Inc\,Def}\boldsymbol{u} = 0$), and the analogy to Section 3.2 is complete.

Now we proceed in the same way as in the geometrical case and consider the kernels of the two integrals $\mathring{\boldsymbol{m}}$ and $\boldsymbol{f}$ respectively from (4.4):

$$\boldsymbol{F}^S(\boldsymbol{r}) = \boldsymbol{S}(\boldsymbol{r}), \quad \boldsymbol{M}^S(\boldsymbol{r}) = \boldsymbol{S}(\boldsymbol{r}) \times \boldsymbol{r}. \tag{4.5}$$

Since we assumed equilibrium ((4.2): $\mathrm{Div}\boldsymbol{S} = 0$), the tensor fields $\boldsymbol{F}^S$ and $\boldsymbol{M}^S$ possess the property

$$\mathrm{Div}\boldsymbol{F}^S = 0, \quad \mathrm{Div}\boldsymbol{M}^S = 0 \qquad \text{(4.6a), (4.6b)}$$

(of course, $\boldsymbol{F}^S$ was only introduced to lay stress on the analogy to the geometrical case; for (4.6b), see eq. (A4) of the Appendix). We define the linear mapping

$$\Psi: \mathrm{Ker}\{\mathrm{Div}\} \to \mathscr{H}^2(\Omega)^3 \times \mathscr{H}^2(\Omega)^3, \tag{4.7}$$

$$\boldsymbol{S} \mapsto ([\boldsymbol{F}^S], [\boldsymbol{M}^S]) \tag{4.7*}$$

and will prove in the following that $\dim\{\mathscr{S}(\Omega)\} = 6b_2(\Omega)$.

4.3. *Equilibrium and integrability of* S

The tensors $\boldsymbol{F}^S = \boldsymbol{S}$ and $\boldsymbol{M}^S = \boldsymbol{S} \times \boldsymbol{r}$ are *solenoidal*, if and only if $\boldsymbol{S}$ is representable as $\boldsymbol{S} = \mathrm{Inc}\boldsymbol{A}$:

LEMMA III†:

$$\text{Ker}\,\{\Psi\} = \text{Im}\{\text{Inc}\}. \tag{4.8}$$

Proof: From $\boldsymbol{S} = \text{Inc}\boldsymbol{A}$ we obtain

$$\boldsymbol{F}^S = \boldsymbol{S} = \nabla \times \boldsymbol{A} \times \nabla = \text{Curl}(\boldsymbol{A} \times \nabla), \tag{4.9a}$$

$$\boldsymbol{M}^S = (\nabla \times \boldsymbol{A} \times \nabla) \times \boldsymbol{r} = \text{Curl}(\boldsymbol{A} + (\boldsymbol{A} \times \nabla) \times \boldsymbol{r}) \tag{4.9b}$$

(for (4.9b), use eq. (A1) of the Appendix and $\text{tr}\,\{\boldsymbol{A} \times \nabla\} = 0$ because $\boldsymbol{A} = \boldsymbol{A}^T$). Hence

$$[\boldsymbol{F}^S] = 0, \quad [\boldsymbol{M}^S] = 0. \qquad \text{(4.9a*), (4.9b*)}$$

If we assume, on the other hand

$$\boldsymbol{F}^S = \boldsymbol{S} = \text{Curl}\boldsymbol{V}, \quad \boldsymbol{M}^S = \text{Curl}\boldsymbol{W}, \tag{4.10}$$

the corresponding stress tensor field $\boldsymbol{S}$ is

$$\boldsymbol{S} = \nabla \times (\boldsymbol{W} - \boldsymbol{V} \times \boldsymbol{r}) \times \nabla - \text{tr}\,\{\boldsymbol{V}\}\boldsymbol{I} \times \nabla$$

(see eq. (A5) of the Appendix); since $\boldsymbol{S} = \boldsymbol{S}^T$ and $\text{tr}\,\{\boldsymbol{V}\}\boldsymbol{I} \times \nabla$ is skew symmetric, it follows

$$\boldsymbol{S} = \text{Inc}(\text{sym}\{\boldsymbol{W} - \boldsymbol{V} \times \boldsymbol{r}\}). \tag{4.11}$$

Therefore the mapping $\tilde{\Psi}$ of the quotient space Ker{Div}/Im{Inc}

$$\tilde{\Psi}: \mathscr{S}(\Omega) \to \mathscr{H}^2(\Omega)^3 \times \mathscr{H}^2(\Omega)^3, \tag{4.12}$$

$$[\boldsymbol{S}] \mapsto ([\boldsymbol{F}^S], [\boldsymbol{M}^S]) \qquad \text{(4.12*)}$$

can be defined, and $\tilde{\Psi}$ is one to one.

† Lemma III is equivalent to Gurtin's result (1963): a theorem on solenoidal fields (see Goldberg (1962), Gurtin (1972)) states that for a vector field $\boldsymbol{v}$ on Ω, $\text{div}\boldsymbol{v} = 0$ implies $\boldsymbol{v} = \text{curl}\boldsymbol{w}$, if and only if $\int \boldsymbol{n}\, \text{d}a \cdot \boldsymbol{v} = 0$ for every closed surface $\mathscr{S}$ in Ω. Analogously, $\boldsymbol{S} \in \text{Ker}\{\Psi\}$, if and only if $\int \boldsymbol{n}\, \text{d}a \cdot \boldsymbol{S} = 0$ and $\int \boldsymbol{n}\, \text{d}a \cdot (\boldsymbol{S} \times \boldsymbol{r}) = 0$ for every closed surface $\mathscr{S}$ in Ω, i.e., if the stress tensor field $\boldsymbol{S}$ is *totally self-equilibrated* (cp. Section 4.2).

LEMMA IV: $\tilde{\Psi}$ *is also onto.*

Given two tensor fields $\boldsymbol{F}$, $\boldsymbol{M}$ with Div$\boldsymbol{F}$ = Div$\boldsymbol{M}$ = 0 (cp.(2.16)), it can indeed be shown that there always exists a stress field $\boldsymbol{S}$ with $[\boldsymbol{F}^S] = [\boldsymbol{F}]$, $[\boldsymbol{M}^S] = [\boldsymbol{M}]$. But at this point, the formal analogy between the geometrical and the statical case breaks down, and we have to use a different approach:

(a) In order to obtain the equivalence $[\boldsymbol{S}] = [\boldsymbol{F}]$, let us first consider a tensor field $\boldsymbol{S}^a = \boldsymbol{F} + \nabla \times \boldsymbol{G}$ with $\boldsymbol{G} = \boldsymbol{R}^T$. The symmetry of $\boldsymbol{S}^a$ is secured, if $\boldsymbol{R}$ is chosen in such a way (see Section A6 of the Appendix) that tr$\{\boldsymbol{R}\} = 0$ and $\nabla \cdot \boldsymbol{R} = -\boldsymbol{\varepsilon} \cdot\cdot\, \text{skw}\{\boldsymbol{F}\}$.

(b) But when thus defined, the stress tensor $\boldsymbol{S}^a$ can generally *not* yield the desired second equivalence $[\boldsymbol{M}^S] = [\boldsymbol{S}^a \times \boldsymbol{r}] = [\boldsymbol{M}]$. Therefore $\boldsymbol{S}^a$ has to be completed by a solenoidal field $\boldsymbol{S}^b = \nabla \times \boldsymbol{U}$, and hence $\boldsymbol{M}^S = \boldsymbol{S}^a \times \boldsymbol{r} + (\nabla \times \boldsymbol{U}) \times \boldsymbol{r}$. Comparison with eq. (A1) of the Appendix prompts the choice of $\boldsymbol{U} = (\boldsymbol{S}^a \times \boldsymbol{r} - \boldsymbol{M})^T$. Indeed $\boldsymbol{S}^b = \nabla \times \boldsymbol{U}$ can, like $\boldsymbol{S}^a$, always be symmetrized by a suitable choice of $(\boldsymbol{S}^a \times \boldsymbol{r} - \boldsymbol{M})$, (note that $\boldsymbol{M}$ is only determined up to an arbitrary solenoidal field), and we now obtain $\boldsymbol{M}^S = \boldsymbol{M} + \nabla \times \boldsymbol{W}$, hence the desired result $[\boldsymbol{M}^S] = [\boldsymbol{M}]$. Let us therefore define

$$\boldsymbol{S} = \boldsymbol{S}^a + \boldsymbol{S}^b$$

with

$$\boldsymbol{S}^a = \boldsymbol{F} + \nabla \times \boldsymbol{R}^T,$$

and

$$\boldsymbol{S}^b = -\nabla \times (\boldsymbol{M} - \boldsymbol{S}^a \times \boldsymbol{r})^T. \tag{4.13}$$

First, we have to show:

COROLLARY: ***R*** *and* ***M*** *can always be chosen in such a way that* $\boldsymbol{S}^a$ *and* $\boldsymbol{S}^b$ *are symmetric.*

According to (4.13) the skew parts of $\boldsymbol{S}^a$ and $\boldsymbol{S}^b$ respectively are

$$\text{skw}\{\boldsymbol{S}^a\} = +\text{skw}\{\nabla \times \boldsymbol{R}^T\} + \text{skw}\{\boldsymbol{F}\},$$

$$\text{skw}\{\boldsymbol{S}^b\} = -\text{skw}\{\nabla \times (\boldsymbol{M} - \boldsymbol{S}^a \times \boldsymbol{r})^T\}.$$

As shown in Section A6 of the Appendix, S^a and S^b become symmetric, if

$$\operatorname{tr}\{R\} = 0, \qquad \nabla \cdot R = -\varepsilon \cdot\cdot \operatorname{skw}\{F\};$$

$$\operatorname{tr}\{M - S^a \times r\} = 0, \quad \nabla \cdot (M - S^a \times r) = 0.$$

In contrast to the other three conditions, the last one is identically fulfilled: $\nabla \cdot M = 0$ and $\nabla \cdot (S^a \times r) = 0$ (according to eq. (A4) of the Appendix). The first three can always, without loss of generality, be satisfied (see A6).

Proof of Lemma IV: It now holds that

$$\text{(a)} \quad F^S = S^a + S^b = F + \nabla \times \{R^T - (M - S^a \times r)^T\}$$

$$= F + \nabla \times V, \tag{4.14a}$$

and hence

$$[F^S] = [F]. \tag{4.14a*}$$

$$\text{(b)} \quad M^S = S \times r = S^a \times r - (\nabla \times (M - S^a \times r)^T) \times r$$

can be transformed according to eq. (A1) of the Appendix into

$$M^S = S^a \times r - \nabla \times ((M - S^a \times r)^T \times r)$$

$$- \operatorname{tr}\{M - S^a \times r\}I + (M - S^a \times r);$$

if we choose $\operatorname{tr}\{M - S^a \times r\} = 0$ (as above), we obtain

$$M^S = M - \nabla \times ((M - S^a \times r)^T \times r)$$

$$= M + \nabla \times W, \tag{4.14b}$$

hence

$$[M^S] = [M]. \tag{4.14b*}$$

With (4.14a*) and (4.14b*), the mapping (4.12) changes into

$$\tilde{\Psi}([S]) = \Psi(S) = ([F^S], [M^S])$$

$$= ([F], [M]); \tag{4.15}$$

and we can summarize our results in

THEOREM II: *The vector space* $\mathcal{S}(\Omega)$ *is* isomorphic *to* $\mathcal{H}^2(\Omega)^3 \times \mathcal{H}^2(\Omega)^3$, *i.e. its dimension is*

$$\dim\{\mathcal{S}(\Omega)\} = 6b_2(\Omega). \tag{4.16}$$

5. Conclusions and examples

Theorems I and II respectively can be interpreted in the following way:

I. For any elastic body Ω, there exist $6b_1(\Omega)$ *non*-integrable strain fields $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_{6b_1}$ on Ω such that any strain field $\boldsymbol{E}$ with Inc$\boldsymbol{E} = 0$ can be represented in the form

$$\boldsymbol{E} = \text{Def}\boldsymbol{u} + \sum_\lambda e_\lambda \boldsymbol{E}_\lambda \quad (\lambda = 1, 2, \ldots, 6b_1)$$

where the coefficients e_λ are unique, and $\boldsymbol{u}$ is unique up to a rigid body displacement.

II. Analogously, for any elastic body Ω, there exist $6b_2(\Omega)$ *non*-integrable stress fields $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_{6b_2}$ on Ω such that any stress field $\boldsymbol{S}$ with Div$\boldsymbol{S} = 0$ can be represented in the form

$$\boldsymbol{S} = \text{Inc}\boldsymbol{A} + \sum_\lambda s_\lambda \boldsymbol{S}_\lambda \quad (\lambda = 1, 2, \ldots, 6b_2)$$

where the coefficients s_λ are unique, and $\boldsymbol{A}$ is unique up to an arbitrary "*zero*-stress function tensor".

The geometrical meaning of the Betti numbers $b_1(\Omega)$ and $b_2(\Omega)$ can be illustrated as follows: the second Betti number $b_2(\Omega)$ characterizes multiply-bordered regions Ω ("periphractic" in Maxwell's terminology (1873)) and denotes the number of connected surfaces $\mathcal{S}$ of Ω, diminished by one.† For a *simply*-bordered body, we have $b_2(\Omega) = 0$, i.e., there are zero non-integrable stress fields $\boldsymbol{S}$; for a ball with 1, 2, . . . holes, there are in in general 6, 12, . . . independent non-integrable stress fields.

The first Betti number $b_1(\Omega)$ describes the connectedness of the region Ω ("cyclic" in Maxwell's terminology (1873)) and denotes the number of topologically independent closed loops $\mathcal{C}$ in Ω which are not shrinkable to a single point. For a *simply*-connected body Ω ($b_1 = 0$), there are no non-integrable strain fields $\boldsymbol{E}$, and one obtains Cesaro's theorem.

† In particular cases, considerable mathematical difficulties may arise in counting the number $b_2(\Omega)$ of connected surfaces $\mathcal{S}$ (e.g., if some surfaces are degenerate), or in deciding (for $b_1(\Omega)$) whether two closed loops are "topologically independent" or not (cp. the loops $\mathcal{C}_5$, $\mathcal{C}_6$ of Fig. 3). Such problems are avoided by using the distinct definitions of de Rham cohomology.

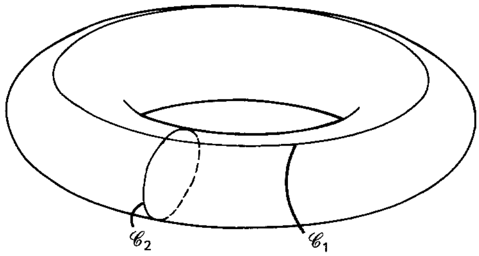

Fig. 1. Hollow torus ($b_1 = 2$) with the two independent closed loops $\mathscr{C}_1$ and $\mathscr{C}_2$.

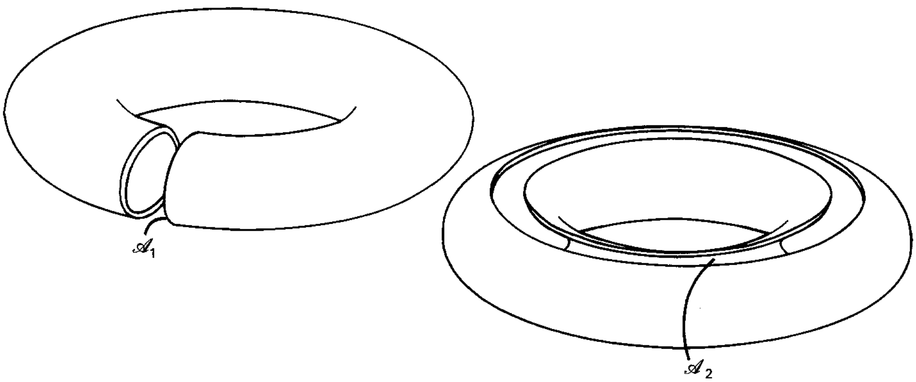

Fig. 2. Hollow torus ($b_1 = 2$) with the two cut surfaces $\mathscr{A}_1$ and $\mathscr{A}_2$.

For a full torus ($b_1 = 1$), there is one such closed loop ($\mathscr{C}_1$: cp. Fig. 1 and 2), for a hollow torus ($b_1 = 2$) two, as shown in Fig. 1.

The corresponding $6 \cdot 2 = 12$ non-integrable strain fields are generated by (1) cutting the torus open along a surface (e.g., $\mathscr{A}_1$ or $\mathscr{A}_2$) perpendicular to the loop ($\mathscr{C}_1$ or $\mathscr{C}_2$), (2) by translating ("dislocation") or/and rotating ("disclination") the two lips of each cut relative to each other, and (3) by cementing the two lips together again (Volterra's "distorsioni" (1907); cp. Zastrow (1985)) (see Fig. 2).

Whereas the strain field $\boldsymbol{E}$ in the deformed torus is still single-valued, continuous and compatible (Inc$\boldsymbol{E} = 0$), the displacement vector $\boldsymbol{u}$ will now show a discontinuity across the surface with every closed circuit.

An example for a region Ω with $b_1(\Omega) = 4$ is the connected sum of two hollow tori. Figure 3 shows the 4 independent closed loops $\mathscr{C}_1 - \mathscr{C}_4$.

Here we have $6 \cdot 4 = 24$ independent non-integrable strain fields, generated by the operations (1)–(3) described above. The two loops $\mathscr{C}_5$, $\mathscr{C}_6$ drawn in Fig. 3 are examples of topologically *non*-independent closed loops: $\mathscr{C}_5$ is topologically equivalent to the sum of $\mathscr{C}_1$ and $\mathscr{C}_3$, and $\mathscr{C}_6$ is topologically equivalent to a loop which is shrinkable to a point.

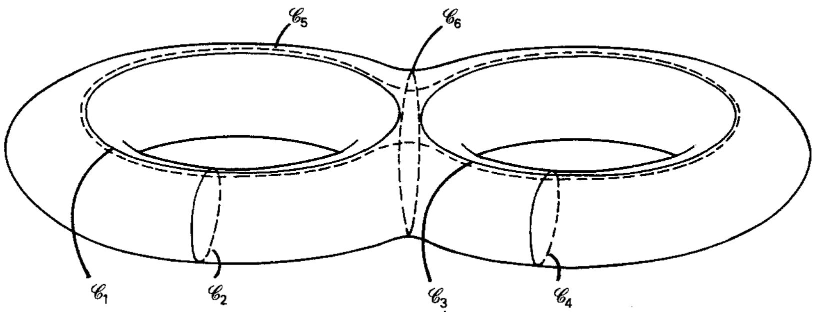

Fig. 3. Connected sum of two hollow tori ($b_1 = 4$) with the 4 independent closed loops $\mathscr{C}_1$–$\mathscr{C}_4$.

Appendix

For the calculation of the corresponding formulae of Sections 3 and 4, we will mainly use indicial notation. The following transformations are based on the identity

$$\varepsilon_{jkl}\varepsilon_{jmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$$

and on the expansion

$$(V_{ij}r_h)_{,t} = V_{ij,t}r_h + V_{ij}\delta_{ht}$$

for the product of an arbitrary second-order tensor V_{ij} and the position vector r_h.

A1. By means of these two formulae, it can easily be shown that

$$\begin{aligned} W_{hk} &= \varepsilon_{hij}\varepsilon_{kmn}(V_{jm}r_n)_{,i} \\ &= \varepsilon_{hij}\varepsilon_{kmn}V_{jm,i} - (\delta_{hk}\delta_{jm} - \delta_{hm}\delta_{jk})V_{jm} \\ &= \varepsilon_{hij}\varepsilon_{kmn}V_{jm,i} - \delta_{hk}V_{jj} + V_{kh}, \end{aligned} \tag{A1}$$

or in symbolic notation respectively

$$W = \nabla \times (V \times r) = (\nabla \times V) \times r - \mathrm{tr}\,\{V\}I + V^T. \tag{A1*}$$

A2. Proof of eq. (3.9b): $((\mathrm{Def}\boldsymbol{u}) \times \nabla) \times \boldsymbol{r} = (\nabla(\nabla \boldsymbol{u} - \boldsymbol{u}\nabla)) \cdot \boldsymbol{r}/2$:

$$\begin{aligned} H_{ij} &= \varepsilon_{jkl}\varepsilon_{kmn}(u_{m,i} + u_{i,m})_{,n} r_l/2 \\ &= ((u_{l,i} + u_{i,l})_{,j} - (u_{j,i} + u_{i,j})_{,l}) r_l/2 \\ &= (u_{l,j} - u_{j,l})_{,i} r_l/2 \end{aligned} \tag{A2}$$

(the tensor $\boldsymbol{W} = (\nabla \boldsymbol{u} - \boldsymbol{u}\nabla)/2 = \mathrm{skw}\{\mathrm{Grad}\boldsymbol{u}\}$ is the rotation tensor (see, e.g., Gurtin (1972)).

A3. Proof of eq. (3.14a): $\boldsymbol{D}^E = (\mathrm{sym}\{\boldsymbol{B} - \boldsymbol{D} \times \boldsymbol{r}\}) \times \nabla = \boldsymbol{D} + \nabla \boldsymbol{v}$:

$$2D^E_{ij} = \varepsilon_{jkl}(B_{ik} + B_{ki} - (\varepsilon_{kmn}D_{im} + \varepsilon_{imn}D_{km})r_n)_{,l};$$

with $\varepsilon_{irs}B_{sj,r} = 0$ and hence $B_{sj,r} = B_{rj,s}$ (and correspondingly for D_{sj}), it follows that

$$\begin{aligned} 2D^E_{ij} &= \varepsilon_{jkl}B_{ik,l} - (D_{is,s}r_j - D_{ij,s}r_s - 2D_{ij}) - (\delta_{ij}D_{ss} - D_{ij}) \\ &= \varepsilon_{jkl}B_{lk,i} - (D_{ss,i}r_j - D_{sj,i}r_s - 2D_{ij}) - (\delta_{ij}D_{ss} - D_{ij}) \\ &= \varepsilon_{jkl}B_{lk,i} + (D_{tj}r_t - D_{ss}r_j)_{,i} + 2D_{ij} \\ &= 2v_{j,i} + 2D_{ij}. \end{aligned} \tag{A3}$$

With this result, a similar subsequent calculation yields eq. (3.14b): $\boldsymbol{B}^E = \boldsymbol{E} + \boldsymbol{D}^E \times \boldsymbol{r} = \cdots = \boldsymbol{B} + \nabla(\boldsymbol{v} \times \boldsymbol{r})$.

A4. Proof of eq. (4.6b): $\nabla \cdot \boldsymbol{M}^S = \nabla \cdot (\boldsymbol{S} \times \boldsymbol{r}) = 0$:

$$\begin{aligned} M^S_{ij,i} &= \varepsilon_{jkl}(S_{ik}r_l)_{,i} \\ &= \varepsilon_{jkl}S_{ik,i}r_l + \varepsilon_{jkl}S_{lk} = 0 \end{aligned} \tag{A4}$$

because of $S_{ik,i} = 0$ and the symmetry $S_{lk} = S_{kl}$.

A5. Proof of eq. (4.11): $\boldsymbol{S} = \mathrm{Inc}(\mathrm{sym}\{\boldsymbol{W} - \boldsymbol{V} \times \boldsymbol{r}\})$:
With $\boldsymbol{S} = \nabla \times \boldsymbol{V} = -\boldsymbol{V}^T \times \nabla$, it follows that

$$\begin{aligned} \boldsymbol{M}^S &= \nabla \times \boldsymbol{W} = \boldsymbol{S} \times \boldsymbol{r} = (\nabla \times \boldsymbol{V}) \times \boldsymbol{r} \\ &= \nabla \times (\boldsymbol{V} \times \boldsymbol{r}) - \boldsymbol{V}^T + \mathrm{tr}\,\{\mathrm{V}\}\boldsymbol{I} \end{aligned}$$

(see eq. (A1)); hence

$$S = -V^T \times \nabla = (\nabla \times W - \nabla \times (V \times r) - \operatorname{tr}\{V\}I) \times \nabla$$
$$= \operatorname{Inc}(W - V \times r) - \operatorname{tr}\{V\}I \times \nabla. \tag{A5}$$

A6. Proof that S^a and S^b respectively according to eq. (4.13) can always be symmetrized: The skew part of $S^a = F + \nabla \times R^T$ has the general form

$$\operatorname{skw}\{S^a\} = \varepsilon_{ijt} v_t \cong -I \times v.$$

The double inner product with $\varepsilon \cong \varepsilon_{hij}$ yields

$$2v_h = \varepsilon_{hij}\varepsilon_{ijt} v_t \cong \varepsilon \cdot\cdot \operatorname{skw}\{S^a\}.$$

On the other hand, the calculation of $\varepsilon \cdot\cdot \operatorname{skw}\{\nabla \times R^T\}$ yields

$$\varepsilon_{hij}(\varepsilon_{irs} R_{js,r} - \varepsilon_{jrs} R_{is,r})/2 = R_{ih,i} - R_{ss,h}.$$

Thus the tensor S^a becomes symmetric, if

$$(1)\ \nabla \cdot R = -\varepsilon \cdot\cdot \operatorname{skw}\{F\},$$

$$(2)\ \operatorname{tr}\{R\} = 0.$$

Now R can always, without loss of generality, be chosen in such a way that both conditions are fulfilled: indeed it is always possible (see, e.g., Spivak (1974)), for a given vector field v, to find a tensor field V satisfying (1) $v = \operatorname{Div} V$ (one tensor rank lower, this corresponds to the determination of a vector field w with prescribed divergence $\phi = \operatorname{div} w$). – For (2), consider R^* with $\operatorname{tr}\{R^*\} \neq 0$. We can choose a vector field v with $\nabla \cdot v = \operatorname{tr}\{R^*\}$ and take $R = R^* + \nabla \times (I \times v)/2$. Then computing of $\operatorname{tr}\{R\}$ yields

$$R_{ss} = R^*_{ss} - \varepsilon_{sjk}\varepsilon_{kst} v_{t,j}/2$$
$$= R^*_{ss} + (v_{k,k} - \delta_{kk} v_{t,t})/2$$
$$\cong \operatorname{tr}\{R^*\} - \nabla \cdot v = 0.$$

The corresponding proof for the symmetry of the second stress tensor S^b follows the same lines.

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