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Barnett-Lothe tensors and their associated tensors for monoclinic materials with the symmetry plane at $x_3 = 0$

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Abstract. The three Barnett-Lothe tensors **S**, **H**, **L** and the three associated tensors **S**(θ), **H**(θ), **L**(θ) appear frequently in the real form solutions to two-dimensional anisotropic elasticity problems. Explicit expressions of the components of these tensors are derived and presented for monoclinic materials whose plane of material symmetry is at $x_3 = 0$. We use the algebraic formalism for these tensors but the results are derived not by the straight-forward substitution of the complex matrices **A** and **B** into the formulae. Instead, we find the product $-\mathbf{AB}^{-1}$, whose real and imaginary parts are $\mathbf{SL}^{-1}$ and $\mathbf{L}^{-1}$, respectively. The tensors **S**, **H**, **L** are then determined from $\mathbf{SL}^{-1}$ and $\mathbf{L}^{-1}$. For **S**(θ), **H**(θ), **L**(θ) we again avoid the direct substitution by employing an alternate approach. The new approaches require minimal algebra and, at the same time, provide simple and concise expressions for the components of these tensors. Although the new approaches can be extended, in principle, to monoclinic materials whose plane of symmetry is not at $x_3 = 0$ and to materials of general anisotropy, the explicit expressions for these materials are too complicated. More studies are needed for these materials.

1. Introduction

In a fixed rectangular coordinate system x_i, $i = 1, 2, 3$, let u_i and σ_{ij} be the displacement and stress, respectively. The stress-strain laws and the equations of equilibrium are

$$\sigma_{ij} = C_{ijks} u_{k,s}, \tag{1.1}$$

$$C_{ijks} u_{k,sj} = 0, \tag{1.2}$$

in which C_{ijks} are the elastic stiffnesses, repeated indices imply summation and a comma stands for differentiation. We assume that C_{ijks} are fully symmetric and positive definite such that the strain energy is positive. Let

$$\begin{aligned} Q_{ik}(\theta) &= C_{ijks} n_j(\theta) n_s(\theta), \\ R_{ik}(\theta) &= C_{ijks} n_j(\theta) m_s(\theta), \\ T_{ik}(\theta) &= C_{ijks} m_j(\theta) m_s(\theta), \end{aligned} \tag{1.3}$$

$$n_i(\theta) = (\cos\theta, \sin\theta, 0), \quad m_i(\theta) = (-\sin\theta, \cos\theta, 0),$$

where θ is a real parameter and, in matrix notation,

$$\left.\begin{aligned} \mathbf{N}_1(\theta) &= -\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta), \quad \mathbf{N}_2(\theta) = \mathbf{T}^{-1}(\theta), \\ \mathbf{N}_3(\theta) &= \mathbf{R}(\theta)\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta) - \mathbf{Q}(\theta). \end{aligned}\right\} \tag{1.4}$$

The superscript T stands for the transpose. Define the incomplete integrals

$$\left.\begin{aligned} \mathbf{S}(\theta) &= \frac{1}{\pi}\int_0^\theta \mathbf{N}_1(\theta')\,d\theta', \\ \mathbf{H}(\theta) &= \frac{1}{\pi}\int_0^\theta \mathbf{N}_2(\theta')\,d\theta', \\ \mathbf{L}(\theta) &= -\frac{1}{\pi}\int_0^\theta \mathbf{N}_3(\theta')\,d\theta', \end{aligned}\right\} \tag{1.5}$$

and the complete integrals

$$\mathbf{S} = \mathbf{S}(\pi), \quad \mathbf{H} = \mathbf{H}(\pi), \quad \mathbf{L} = \mathbf{L}(\pi). \tag{1.6}$$

The three complete integrals **S**, **H**, **L** are the Barnett-Lothe tensors [1]. The other three incomplete integrals $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$ are the associated tensors. The dependence of $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$ on θ will be indicated explicitly except when $\theta = \pi$. They appear often in the real form solutions of anisotropic elasticity problems. For instance, in the problem of an infinite anisotropic material subject to a line force **f** and a line dislocation with the Burgers vector $\mathbf{b}_0$, the real form solution is [2, 3, 4]

$$\left.\begin{aligned} \mathbf{u} &= \frac{-1}{\pi}(\ln r)\mathbf{h} - \mathbf{S}(\theta)\mathbf{h} + \mathbf{H}(\theta)\mathbf{g}, \\ \boldsymbol{\phi} &= \frac{1}{\pi}(\ln r)\mathbf{g} + \mathbf{S}^T(\theta)\mathbf{g} + \mathbf{L}(\theta)\mathbf{h}, \\ x_1 &= r\cos\theta, \quad x_2 = r\sin\theta, \end{aligned}\right\} \tag{1.7}$$

where **g**, **h** are real constants given by

$$2\mathbf{g} = \mathbf{L}\mathbf{b}_0 - \mathbf{S}^T\mathbf{f},$$

$$2\mathbf{h} = \mathbf{S}\mathbf{b}_0 + \mathbf{H}\mathbf{f},$$

and $\boldsymbol{\phi}$ is the stress function from which one obtains

$$\sigma_{i2} = \phi_{i,1} \quad \sigma_{i1} = -\phi_{i,2}. \tag{1.8}$$

It is interesting to note that the form of solution (1.7) remains valid if the line force and the line dislocation are applied at the origin of a composite space which consists of an arbitrary number of wedges of different wedge angles and materials [4]. In fact, (1.7) also applies to inhomogeneous anisotropic materials whose elastic stiffnesses depend on the polar angle θ [5, 6]. Solutions to other anisotropic elasticity problems in which the Barnett-Lothe tensors and their associated tensors appeared can be found in [7–12].

The integrations in (1.5) require a numerical approximation except for certain anisotropic materials with special material symmetry. An alternate to the integral formalism is the algebraic formalism which we present briefly below.

If we assume that [13, 14, 15]

$$\mathbf{u} = \mathbf{a}f(z), \quad z = x_1 + px_2,$$

where p and $\mathbf{a}$ are constant and f is an arbitrary function of z, the equations of equilibrium (1.2) are satisfied if

$$[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}]\mathbf{a} = \mathbf{0}, \tag{1.9}$$

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{1.10}$$

We see that $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are $\mathbf{Q}(\theta)$, $\mathbf{R}(\theta)$, $\mathbf{T}(\theta)$ of (1.3) with $\theta = 0$. By introducing the new vector

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}, \tag{1.11}$$

in which the second equality follows from (1.9), the stresses σ_{ij} obtained from (1.1) can be written in the form of (1.8) where $\boldsymbol{\phi}$ is the stress function

$$\boldsymbol{\phi} = \mathbf{b}f(z). \tag{1.12}$$

The two equations in (1.11) can be recast in the standard eigenrelation

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi},$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix},$$

where $\mathbf{N}_i$ are $\mathbf{N}_i(\theta)$ of (1.4) with $\theta = 0$. There are six eigenvalues p_α and six associated eigenvectors $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$, $\alpha = 1, 2, \ldots, 6$. Since p_α cannot be real if the strain energy is positive [13, 16], we let

$$p_{\alpha+3} = \bar{p}_\alpha, \quad \operatorname{Im} p_\alpha > 0, \ \alpha = 1, 2, 3,$$

where the overbar denotes the complex conjugate and Im stands for the imaginary part. If we introduce the 3×3 matrices

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3], \tag{1.13}$$

where $\mathbf{b}_\alpha$ are related to $\mathbf{a}_\alpha$ through (1.11), it can be shown that [1, 15]

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = 2i\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^T, \tag{1.14}$$

$\mathbf{I}$ being the unit matrix, provided $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ are normalized such that

$$2\mathbf{a}_\alpha, \mathbf{b}_\alpha = 1, \quad \alpha \text{ not summed.} \tag{1.15}$$

Equations (1.14) are the algebraic equivalents of the complete integrals $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ of (1.6). The algebraic equivalence of the incomplete integrals $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$ will be presented later on.

It should be pointed out that $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ are related by

$$\mathbf{HL} - \mathbf{SS} = \mathbf{I}. \tag{1.16}$$

The matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric and positive definite while $\mathbf{SL}^{-1}$ and $\mathbf{H}^{-1}\mathbf{S}$ are anti-symmetric. More studies on the relationships between $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ and their structure and invariance properties can be found in [17, 18].

Other related tensors $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$, where v is the real wave speed, appear in the problem of steady state moving line dislocation and Rayleigh surface waves [15, 19–24]. The integral formalism for $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ is again the complete integrals of (1.6) if, in leading to the integrals in (1.5), we replace $\mathbf{Q}(\theta)$, $\mathbf{R}(\theta)$, $\mathbf{T}(\theta)$ of (1.3) by

$$\mathbf{Q}(\theta, v) = \mathbf{Q}(\theta) - \rho v^2 \cos^2 \theta \mathbf{I},$$

$$\mathbf{R}(\theta, v) = \mathbf{R}(\theta) + \rho v^2 \sin \theta \cos \theta \, \mathbf{I},$$

$$\mathbf{T}(\theta, v) = \mathbf{T}(\theta) - \rho v^2 \sin^2 \theta \mathbf{I},$$

where ρ is the mass density. The algebraic formalism for $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ is given by (1.14) if we replace $\mathbf{Q}$ of $(1.10)_1$ by

$$\mathbf{Q}(v) = \mathbf{Q} - \rho v^2 \mathbf{I}.$$

One drawback with the integral formalism is that, except for special anisotropic materials, it is difficult to obtain the integrations explicitly in closed form. This is particularly so for the incomplete integrals. For the complete integrals, Chadwick and his co-workers have obtained $\mathbf{S}(v)$ for cubic materials [21] and for transversely isotropic materials [23] in which the axis of symmetry is in the (x_1, x_2) plane or the (x_1, x_3) plane. Recently, Chadwick and Wilson [24] obtained explicit expressions of $\mathbf{S}(v)$ for monoclinic materials for which the plane of symmetry is at $x_3 = 0$.

The algebraic formalism avoids the integration and is particularly suitable for the incomplete integrals. The expressions are explicit, but the formalism is not without drawbacks. Firstly, the expressions (1.14) tacitly assume that the 6×6 matrix $\mathbf{N}$ is simple or semisimple, i.e., the eigenvalues p_α are distinct or, if there is a multiple eigenvalue, the eigenvectors span the six-dimensional space. Modifications required when $\mathbf{N}$ is non-semisimple can be found in [25]. This drawback however does not present much problem because one can always assume that the p_α are distinct during the algebraic calculation. If the final results do not contain p_α explicitly, the problem of repeated eigenvalue disappears and the results apply to non-semisimple $\mathbf{N}$ also. If the final results contain p_α explicitly and if $p_1 = p_2$, say, we can take the proper limit in the final results. Secondly, although the algebraic expressions are explicit, they require calculation of the eigenvalues p_α, the eigenvectors $\mathbf{a}_\alpha, \mathbf{b}_\alpha$, and the normalization factors of the eigenvectors. The hidden algebraic calculation in general demands an inordinate effort as manifested in the recent work for orthotropic materials [26].

In this paper we present explicit expressions for $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$ for monoclinic materials whose plane of symmetry is at $x_3 = 0$. We adopt the algebraic formalism, but we have found a way to circumvent most of the complicated algebraic calculations. As it turns out, the algebraic calculations are minimal and the final results are rather concise.

2. The eigenvalues and eigenvectors

On using the contracted notation C_{ab} for C_{ijks}, (1.9) for monoclinic materials with plane of symmetry at $x_3 = 0$ becomes [19]

$$\begin{bmatrix} C_{11} + 2pC_{16} + p^2C_{66} & C_{16} + p(C_{12} + C_{66}) + p^2C_{26} & 0 \\ C_{16} + p(C_{12} + C_{66}) + p^2C_{26} & C_{66} + 2pC_{26} + p^2C_{22} & 0 \\ 0 & 0 & C_{55} + 2pC_{45} + p^2C_{44} \end{bmatrix} \mathbf{a} = \mathbf{0}. \tag{2.1}$$

The eigenvalues p_1, p_2, p_3 are the roots of the determinant of the 3×3 matrix on the left. Noticing that the imaginary part of p_α is positive, we let

$$p_3 = (-C_{45} + iq)C_{44}^{-1}, \quad q = (C_{44}C_{55} - C_{45}^2)^{1/2}. \tag{2.2}$$

Then p_1, p_2 are the two roots with positive imaginary part of the quartic equation which can be written as, using the identities (A12) derived in Appendix A,

$$s'_{11}p^4 - 2s'_{16}p^3 + (s'_{66} + 2s'_{12})p^2 - 2s'_{26}p + s'_{22} = 0. \tag{2.3}$$

In (2.3),

$$s'_{ij} = s_{ij} - (s_{i3}s_{3j})s_{33}^{-1}, \tag{2.4}$$

where s_{ij} are the elastic compliances which are the inverse of the elastic stiffnesses C_{ab}.

For the matrices **A** and **B** of (1.13), we obtain $\mathbf{a}_\alpha$ from (2.1) and $\mathbf{b}_\alpha$ from $(1.11)_1$. Following [14], however, we determine $\mathbf{b}_\alpha$ first and then $\mathbf{a}_\alpha$ as follows. From the 3×3 matrix shown in (2.1), it is readily seen that **A** has the structure

$$\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}, \tag{2.5}$$

where the $*$ denotes a possibly non-zero element. In fact **B** also has the same structure. From (1.8), (1.12) and the fact that $\sigma_{12} = \sigma_{21}$, we obtain [14]

$$b_1 = -pb_2,$$

which holds for matrials of general anisotropy. Therefore we can write **B** as (see Eq. (80) of [14])

$$\mathbf{B} = \begin{bmatrix} -k_1p_1 & -k_2p_2 & 0 \\ k_1 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \tag{2.6}$$

in which k_1, k_2, k_3 are arbitrary complex constants to be determined by the normalization condition (1.15).

We next turn to the matrix **A**. For $\mathbf{a}_3$, we let

$$\mathbf{a}_3 = k_3 \begin{bmatrix} 0 \\ 0 \\ -iq^{-1} \end{bmatrix}.$$

This satisfies (2.1) for $p = p_3$ and, when substituted in $(1.11)_1$ and on using (2.2), leads to $\mathbf{b}_3$ in (2.6). For a_1 and a_2, we let

$$\mathbf{a} = \frac{k}{\gamma(p)} \begin{bmatrix} -(C_{66} + 2pC_{26} + p^2C_{22}) \\ C_{16} + p(C_{12} + C_{66}) + pC_{26}^2 \\ 0 \end{bmatrix}, \tag{2.7}$$

where $\gamma(p)$ is a function of p to be determined. This satisfies (2.1) for $p = p_1$ or p_2. Substituting into $(1.11)_1$, we obtain $\mathbf{b}_1$ or $\mathbf{b}_2$ shown in (2.6) if we choose

$$\gamma(p) = (C_{16}C_{26} - C_{12}C_{66}) - p(C_{12}C_{26} - C_{16}C_{22}) + p^2(C_{22}C_{66} - C_{26}^2). \tag{2.8}$$

It is shown in Appendix A that the components of $\mathbf{a}$ in (2.7) can be rewritten as

$$\left.\begin{aligned} -(C_{66} + 2pC_{26} + p^2C_{22})[\gamma(p)]^{-1} &= s'_{12} - s'_{16}p + s'_{11}p^2 = \xi(p), \\ [C_{16} + p(C_{12} + C_{66}) + p^2C_{26}][\gamma(p)]^{-1} &= s'_{22}p^{-1} - s'_{26} + s'_{21}p = \eta(p), \end{aligned}\right\} \tag{2.9}$$

say. Therefore $\mathbf{A}$ has the expression

$$\mathbf{A} = \begin{bmatrix} k_1\xi(p_1) & k_2\xi(p_2) & 0 \\ k_1\eta(p_1) & k_2\eta(p_2) & 0 \\ 0 & 0 & -ik_3q^{-1} \end{bmatrix}. \tag{2.10}$$

The complex constants k_1, k_2, k_3 are, using (2.6), (2.10) and the normalization condition (1.15),

$$\left.\begin{aligned} 2k_1^2[\eta(p_1) - p_1\xi(p_1)] &= 1, \\ 2k_2^2[\eta(p_2) - p_2\xi(p_2)] &= 1, \\ -2ik_3^2q^{-1} &= 1. \end{aligned}\right\} \tag{2.11}$$

We have thus obtained the matrices $\mathbf{A}$ and $\mathbf{B}$ shown in (2.10) and (2.6) in which k_1, k_2, k_3 are given in (2.11). As we will see in the next two Sections, the new approaches to be employed do not require employment of k_1 and k_2.

The formulation presented here follows that of Stroh [14, 19]. Another formulation due to Lekhnitskii [16] would have produced the matrices $\mathbf{A}$ and $\mathbf{B}$ more directly [27]. However, Lekhnitskii's formulation does not produce the identities (1.14). It is interesting to point out that Stroh [14] was the first to obtain $\mathbf{B}$ in (2.6) and $\mathbf{L}^{-1}$ in the next section using an alternate formulation which, unknown to him, was identical to Lekhnitskii's formulation.

3. Explicit expressions for S, H, L

With **A** and **B** given by (2.10) and (2.6) which are in the form of (2.5), it is easily seen that **S**, **H**, **L** of (1.14) also have the same structure (2.5). By substituting **A** and **B** into (1.14), each term contains the factors k_1^2, k_2^2, k_3^2 which can be eliminated by using (2.11). This straightforward approach was employed in [26]. The algebra is unwieldy for orthotropic materials studied in [26] and is forbidding for the monoclinic materials considered here except for the S_{33}, H_{33}, L_{33} components.

We outline the alternate approach below. First we observe the simplicity of the expression for **B** which has the simple inverse

$$\mathbf{B}^{-1} = \frac{1}{p_1 - p_2}\begin{bmatrix} -k_1^{-1} & -p_2 k_1^{-1} & 0 \\ k_2^{-1} & p_1 k_2^{-1} & 0 \\ 0 & 0 & (p_1 - p_2)k_3^{-1} \end{bmatrix}, \tag{3.1}$$

and we have

$$-\mathbf{A}\mathbf{B}^{-1} = \begin{bmatrix} is'_{11}\,\mathrm{Im}(p_1 + p_2) & s'_{11}p_1\bar{p}_1 - s'_{12} & 0 \\ s'_{12} - s'_{11}\bar{p}_1\bar{p}_2 & is'_{11}\,\mathrm{Im}[p_1 p_2(\bar{p}_1 + \bar{p}_2)] & 0 \\ 0 & 0 & iq^{-1} \end{bmatrix}, \tag{3.2}$$

where use has been made of the relations between the roots p_1, p_2 and the coefficients of the quartic equation in (2.3). The normalization factors k_1, k_2, k_3 drop out during multiplications. Hence the presence of k_1, k_2, k_3 in **A** and **B** is redundant in computing $-\mathbf{AB}^{-1}$. Equation (3.2), with a different expression for $(\mathbf{AB}^{-1})_{22}$, was obtained by Suo [27]. The imaginary part of Suo's result was in Stroh's first paper [14]. Next, since

$$-\mathbf{AB}^{-1} = -(\mathbf{AB}^T)(\mathbf{BB}^T)^{-1}$$

and using (1.14), we have

$$-\mathbf{AB}^{-1} = \mathbf{SL}^{-1} + i\mathbf{L}^{-1}. \tag{3.3}$$

Therefore the real and imaginary parts of (3.2) are $\mathbf{SL}^{-1}$ and $\mathbf{L}^{-1}$, respectively. With $\mathbf{SL}^{-1}$ and $\mathbf{L}^{-1}$ so determined, we find **L** from $\mathbf{L}^{-1}$ and **S** from

$$\mathbf{S} = (\mathbf{SL}^{-1})\mathbf{L}.$$

Finally, **H** is determined from (1.16) as

$$\mathbf{H} = \mathbf{L}^{-1} + \mathbf{S}(\mathbf{SL}^{-1}).$$

We now list below explicit expressions of **S**, **H**, **L**. We have $S_{33}=0$ and

$$L_{33}=H_{33}^{-1}=q. \tag{3.4}$$

For the remaining non-zero components, we use 2×2 matrices for $S_{\alpha\beta}$, $H_{\alpha\beta}$, $L_{\alpha\beta}$, ($\alpha, \beta = 1$ or 2). Therefore the matrices below are the "reduced" matrices which are obtained by deleting the 3rd row and 3rd column of the original 3×3 matrices. We have, in the order the results are obtained,

$$\left.\begin{aligned} &\mathbf{SL}^{-1}=s'_{11}g\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \quad \mathbf{L}^{-1}=s'_{11}\begin{bmatrix}b & d\\ d & e\end{bmatrix},\\ &\mathbf{L}=\frac{h}{s'_{11}}\begin{bmatrix}e & -d\\ -d & b\end{bmatrix}, \quad \mathbf{S}=gh\begin{bmatrix}d & -b\\ e & -d\end{bmatrix},\\ &\mathbf{H}=s'_{11}(1-g^2h)\begin{bmatrix}b & d\\ d & e\end{bmatrix},\end{aligned}\right\} \tag{3.5}$$

where

$$\left.\begin{aligned} &p_1+p_2=a+bi, \quad p_1p_2=c+di,\\ &e=ad-bc>0, \quad g=(s'_{12}/s'_{11})-c>0,\\ &h=(be-d^2)^{-1}>0, \quad b>0, \quad g^2h<1.\end{aligned}\right\} \tag{3.6}$$

In the above, $b>0$ because the imaginary part of p_1 and p_2 are positive. Alternately, since $\mathbf{L}^{-1}$ is positive definite [15, 28, 29], from the expression for $\mathbf{L}^{-1}$ in (3.5) we conclude that b as well as e and h are positive as indicated in (3.6). The inequality $g^2h<1$ follows from the positive definiteness of **H**. A proof that $g>0$, which means $S_{21}>0$ and $S_{12}<0$, is given in Appendix B.

It should be pointed out that a, b, c, d, e, g, h are all non-dimensional real constants. The relations between the roots p_1, p_2 and the coefficients of the quartic equation (2.3) provide the following identities.

$$\left.\begin{aligned} &a=s'_{16}/s'_{11}, \quad 2c+a^2+b^2=(s'_{66}+2s'_{12})/s'_{11},\\ &ac+bd=s'_{26}/s'_{11}, \quad c^2+d^2=s'_{22}/s'_{11}.\end{aligned}\right\} \tag{3.7}$$

It is shown in [30] that the component $(\mathbf{SL}^{-1})_{21}$ is an invariant under rotation about the x_3-axis. Hence $s'_{11}g$ is an invariant. It can also be shown that the traces of $\mathbf{L}^{-1}$ and **H** in (3.5) are invariants. Therefore $s'_{11}(b+e)$ and g^2h are invariants.

4. Explicit expressions for $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$

In view of the fact that $\mathbf{N}_i(\theta)$ in (1.5) are periodic in θ with periodicity π, we have

$$\mathbf{S}(\pi+\theta) = \mathbf{S} + \mathbf{S}(\theta),$$

$$\mathbf{H}(\pi+\theta) = \mathbf{H} + \mathbf{H}(\theta),$$

$$\mathbf{L}(\pi+\theta) = \mathbf{L} + \mathbf{L}(\theta).$$

It suffices therefore to consider $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$ for $0 \leqslant \theta \leqslant \pi$.

The algebraic equivalence of the incomplete integrals (1.5) can be shown to be [31]

$$\left.\begin{aligned} \mathbf{S}(\theta) &= \frac{2}{\pi}\,\mathrm{Re}[\mathbf{A}\boldsymbol{\Psi}(\theta)\mathbf{B}^T], \\ \mathbf{H}(\theta) &= \frac{2}{\pi}\,\mathrm{Re}[\mathbf{A}\boldsymbol{\Psi}(\theta)\mathbf{A}^T], \\ \mathbf{L}(\theta) &= -\frac{2}{\pi}\,\mathrm{Re}[\mathbf{B}\boldsymbol{\Psi}(\theta)\mathbf{B}^T], \end{aligned}\right\} \tag{4.1}$$

where Re stands for the real part and $\Psi(\theta)$ is the diagonal matrix

$$\begin{aligned} \boldsymbol{\Psi}(\theta) &= \mathrm{diag}[\ln\zeta_1(\theta), \ln\zeta_2(\theta), \ln\zeta_3(\theta)], \\ \zeta_\alpha(\theta) &= \cos\theta + p_\alpha \sin\theta. \end{aligned} \tag{4.2}$$

It is readily shown that $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$ $\mathbf{L}(\theta)$ also have the structure (2.5). For the $S_{33}(\theta)$, $H_{33}(\theta)$, $L_{33}(\theta)$ components, if we substitute A_{33} and B_{33} from (2.10) and (2.6) into (4.1) and eliminate k_3^2 from $(2.11)_3$, we obtain

$$\left.\begin{aligned} S_{33}(\theta) &= \frac{1}{2\pi}\ln\left[\cos^2\theta + \frac{C_{55}}{C_{44}}\sin^2\theta - \frac{C_{45}}{C_{44}}\sin 2\theta\right], \\ L_{33}(\theta) &= q^2 H_{33}(\theta) = \frac{q}{\pi}\tan^{-1}\left[\frac{q\sin\theta}{C_{44}\cos\theta - C_{45}\sin\theta}\right]. \end{aligned}\right\} \tag{4.3}$$

The remaining non-zero components are $S_{\alpha\beta}(\theta)$, $H_{\alpha\beta}(\theta)$, $L_{\alpha\beta}(\theta)$, ($\alpha, \beta = 1$ or 2). For these components we will not employ the straight-forward approach of substituting **A** and **B** into (4.1) and eliminating k_1^2, k_2^2 from (2.11). We will present the results below first and then outline an alternate derivation of the results. It is understood that the matrices appearing below are the reduced 2×2 matrices which are obtained by deleting the 3rd row and 3rd column of the original 3×3 matrices. We have

$$\left.\begin{aligned} \mathbf{S}(\theta) &= \Lambda(\theta)\mathbf{S} + \Gamma_+(\theta)\mathbf{S}_+ + \Gamma_-(\theta)\mathbf{S}_- + \Phi(\theta)\mathbf{I}, \\ \mathbf{H}(\theta) &= \Lambda(\theta)\mathbf{H} + \Gamma_+(\theta)\mathbf{H}_+ + \Gamma_-(\theta)\mathbf{H}_-, \\ \mathbf{L}(\theta) &= \Lambda(\theta)\mathbf{L} + \Gamma_+(\theta)\mathbf{L}_+ + \Gamma_-(\theta)\mathbf{L}_-, \end{aligned}\right\} \tag{4.4}$$

in which $\mathbf{L}_\pm$, $\mathbf{S}_\pm$, $\mathbf{H}_\pm$ are the 2×2 real constant matrices listed below.

$$\mathbf{L}_+ = \frac{bh}{s'_{11}}\begin{bmatrix} t_2 & d \\ d & -b \end{bmatrix}, \quad \mathbf{L}_- = \frac{h}{s'_{11}}\begin{bmatrix} t_4 & t_1 \\ t_1 & -t_3 \end{bmatrix},$$

$$\mathbf{S}_+ = ghb\begin{bmatrix} -d & b \\ -e & d \end{bmatrix} + \begin{bmatrix} a & -2 \\ 2(c+g) & -a \end{bmatrix},$$

$$\mathbf{S}_- = gh\begin{bmatrix} -t_1 & t_3 \\ t_4 & t_1 \end{bmatrix} + \begin{bmatrix} b & 0 \\ 2d & -b \end{bmatrix},$$

$$\mathbf{H}_+ = s'_{11}(1+g^2h)b\begin{bmatrix} b & d \\ d & -t_2 \end{bmatrix} - 2s'_{11}g\begin{bmatrix} 2 & a \\ a & 2c \end{bmatrix},$$

$$\mathbf{H}_- = s'_{11}(1+g^2h)\begin{bmatrix} t_3 & t_1 \\ t_1 & -t_4 \end{bmatrix} - 2s'_{11}g\begin{bmatrix} 0 & b \\ b & 2d \end{bmatrix},$$

$$t_1 = 2e - ad, \quad t_2 = e - 2d^2b^{-1},$$

$$t_3 = 2d - ab, \quad t_4 = 2cd - ae.$$

The functions Φ, Λ, Γ_+ and Γ_- in (4.4) depend on θ and are given by

$$\left.\begin{aligned} 2\pi\Phi(\theta) &= \ln(X^2+Y^2)^{1/2}, \\ 2\pi\Lambda(\theta) &= \tan^{-1}(Y/X), \end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned} 4\pi\Gamma_+(\theta) &= (\chi\cos v_- + \Delta\sin v_-)/w_-, \\ 4\pi\Gamma_-(\theta) &= (\chi\sin v_- - \Delta\cos v_-)/w_-, \end{aligned}\right\} \tag{4.6}$$

where

$$
\left.\begin{aligned}
X &= \cos^2\theta + (w_+^2 \cos 2v_+ - w_-^2 \cos 2v_-)\sin^2\theta + a\sin\theta\cos\theta,\\
Y &= \sin\theta[b\cos\theta + (w_+^2 \sin 2v_+ - w_-^2 \sin 2v_-)\sin\theta],\\
\chi &= \tanh^{-1}\left\{\frac{w_-[\cos v_- \sin 2\theta + (a\cos v_- + b\sin v_-)\sin^2\theta]}{\cos^2\theta + (w_+^2 + w_-^2)\sin^2\theta + a\sin\theta\cos\theta}\right\},\\
\Delta &= \tan^{-1}\left\{\frac{w_-[\sin v_- \sin 2\theta + (a\sin v_- - b\cos v_-)\sin^2\theta]}{\cos^2\theta + (w_+^2 - w_-^2)\sin^2\theta + a\sin\theta\cos\theta}\right\}.
\end{aligned}\right. \tag{4.7}
$$

The new notations $w_\pm$ and $v_\pm$ are real constants which are related to p_1 and p_2 by

$$
\left.\begin{aligned}
p_1 + p_2 &= 2w_+(\cos v_+ + i\sin v_+) = a + bi,\\
p_1 - p_2 &= 2w_-(\cos v_- + i\sin v_-), \quad w_\pm > 0.
\end{aligned}\right\} \tag{4.8}
$$

We now outline the derivation of the results presented above. Noticing that the matrices on the right of (4.1) are of the form shown in (2.5), (4.1) remain valid if all matrices are replaced by the reduced 2×2 matrices, we rewrite the reduced diagonal matrix $\boldsymbol{\Psi}(\theta)$ as

$$
\boldsymbol{\Psi}(\theta) = \tfrac{1}{2}\ln(\zeta_1\zeta_2)\mathbf{I} + \tfrac{1}{2}\ln(\zeta_1/\zeta_2)\mathbf{K}, \tag{4.9}
$$

$$
\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

$\mathbf{I}$ in (4.9) is the 2×2 identity matrix. By setting

$$
\ln(\zeta_1\zeta_2) = 2\pi(\Phi + i\Lambda), \tag{4.10}
$$

in which Φ and Λ are real and using (4.9) and (1.14), (4.1) can be written as

$$
\left.\begin{aligned}
\mathbf{S}(\theta) &= \Lambda(\theta)\mathbf{S} + \frac{1}{\pi}\operatorname{Re}[\ln(\zeta_1/\zeta_2)\mathbf{AKB}^T] + \Phi(\theta)\mathbf{I},\\
\mathbf{H}(\theta) &= \Lambda(\theta)\mathbf{H} + \frac{1}{\pi}\operatorname{Re}[\ln(\zeta_1/\zeta_2)\mathbf{AKA}^T],\\
\mathbf{L}(\theta) &= \Lambda(\theta)\mathbf{L} - \frac{1}{\pi}\operatorname{Re}[\ln(\zeta_1/\zeta_2)\mathbf{BKB}^T].
\end{aligned}\right\} \tag{4.11}
$$

We first determine $\ln(\zeta_1\zeta_2)$ and $\ln(\zeta_1/\zeta_2)$. With ζ_α defined in (4.2), it is easily shown that $\ln(\zeta_1\zeta_2)$ is given by (4.10) with Φ and Λ defined in (4.5). As to $\ln(\zeta_1/\zeta_2)$, we first write

$$p_1 = w_+(\cos v_+ + i\sin v_+) + w_-(\cos v_- + i\sin v_-),$$

$$p_2 = w_+(\cos v_+ + i\sin v_+) - w_-(\cos v_- + i\sin v_-),$$

using the notation of (4.8). We then have

$$\begin{aligned}\ln\zeta &= \tfrac{1}{2}\ln\{\cos^2\theta + (w_+^2 + w_-^2)\sin^2\theta + a\sin\theta\cos\theta \\ &\qquad \pm w_-[\cos v_- \sin 2\theta + (a\cos v_- + b\sin v_-)\sin^2\theta]\} \\ &\qquad + i\tan^{-1}\left\{\frac{(w_+\sin v_+ \pm w_-\sin v_-)\sin\theta}{\cos\theta + (w_+\cos v_+ \pm w_-\cos v_-)\sin\theta}\right\},\end{aligned}$$

in which the upper sign is for ζ_1 and the lower sign is for ζ_2. Making use of the identities

$$\tfrac{1}{2}\ln\frac{1+y}{1-y} = \tanh^{-1} y,$$

$$\tan^{-1} y_1 - \tan^{-1} y_2 = \tan^{-1}\frac{y_1 - y_2}{1 + y_1 y_2},$$

one can verify that

$$\ln(\zeta_1/\zeta_2) = \chi + i\Delta,$$

where χ and Δ are defined in (4.7).

We next consider the matrix products $\mathbf{AKB}^T$, $\mathbf{AKA}^T$ and $\mathbf{BKB}^T$ in (4.11). A straight-forward substitution of $\mathbf{A}$ and $\mathbf{B}$ into the products would lead to an extremely complicated algebra. This is because the normalization factors k_1^2 and k_2^2 appear in the process which have to be eliminated using (2.11). An alternate approach is to observe that

$$\left.\begin{aligned}\mathbf{BKB}^T &= (\mathbf{BKB}^{-1})(\mathbf{BB}^T), \\ \mathbf{AKB}^T &= (\mathbf{AB}^{-1})(\mathbf{BKB}^T), \\ \mathbf{AKA}^T &= (\mathbf{AKB}^T)(\mathbf{AB}^{-1})^T.\end{aligned}\right\} \tag{4.12}$$

We see that the only unknown is $\mathbf{BKB}^{-1}$ which can be shown to be

$$\mathbf{BKB}^{-1} = \frac{1}{p_1 - p_2}\begin{bmatrix} p_1 + p_2 & 2p_1p_2 \\ -2 & -(p_1+p_2) \end{bmatrix}$$

$$= \frac{\cos v_- - i \sin v_-}{2w_-}\begin{bmatrix} a + bi & 2(c + di) \\ -2 & -(a + bi) \end{bmatrix}.$$

In this calculation, the normalization factors k_1^2, k_2^2 drop out during multiplications. With $\mathbf{BB}^T$ given in terms of $\mathbf{L}$ in $(1.14)_3$, $\mathbf{BKB}^T$ is readily calculated from $(4.12)_1$. Equations $(4.12)_2$, $(4.12)_3$, in that order, produce $\mathbf{AKB}^T$ and $\mathbf{AKA}^T$ when $\mathbf{AB}^{-1}$ of (3.3) is employed. This completes the description of the derivation of $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$.

For a given value of argument, the arctangent is not unique. The value of arctangent in $(4.3)_3$, $(4.5)_2$, $(4.7)_2$ is determined uniquely as follows. In the (x, y) plane, we let the denominator and numerator in the argument of arctangent represent, respectively, the x and y coordinates. If (x, y) is in the first, second, third or fourth quadrant, the value of arctangent is in the range $(0, \pi/2)$, $(\pi/2, \pi)$, $(\pi, 3\pi/2)$ or $(3\pi/2, 2\pi)$, respectively.

Three special cases of the results obtained here should be noted. In the first, when $\theta = \pi$, $\Gamma_\pm(\theta)$ and $\Phi(\theta)$ vanish while $\Lambda(\theta) = 1$. Equations (4.4) reduce to (1.6).

The second special case is when $p_1 = p_2$ which means, by $(4.8)_3$, $w_- = 0$. If we take the limit $w_- = 0$ in (4.6), we have

$$\left.\begin{aligned} 4\pi\Gamma_+(\theta) &= \frac{\sin 2\theta + a \sin^2\theta}{\cos^2\theta + w_+^2 \sin^2\theta + a \sin\theta \cos\theta}, \\ 4\pi\Gamma_-(\theta) &= \frac{b \sin^2\theta}{\cos^2\theta + w_+^2 \sin^2\theta + a \sin\theta\cos\theta}. \end{aligned}\right\} \tag{4.13}$$

Alternately, it can be shown that $\Gamma_\pm$ of (4.6) are related to p_1, p_2 by

$$\frac{\ln(\zeta_1/\zeta_2)}{p_1 - p_2} = 2\pi(\Gamma_+ - i\Gamma_-). \tag{4.14}$$

Taking the limit $p_2 = p_1$ on the left and equating the real and imaginary parts on both sides of the equation also leads to (4.13). Equation (4.14) is more convenient in evaluating Γ_+ and Γ_- when p_1 and p_2 are nearly equal.

The third special case is when the material is orthotropic with its planes of symmetry at the coordinate planes. We then have $s'_{16} = s'_{26} = 0$. From (3.7),

$a = d = 0$ and [26]

$$c = -(s'_{22}/s'_{11})^{1/2} = -(C_{11}/C_{22})^{1/2},$$

$$b^2 = [2(s'_{11}s'_{22})^{1/2} + 2s'_{12} + s'_{66}](s'_{11})^{-1}$$

$$= [(C_{11}C_{22})^{1/2} + C_{12} + 2C_{66}][(C_{11}C_{22})^{1/2} - C_{12}](C_{22}C_{66})^{-1},$$

in which the second equalities for c and b^2 follow from (A12). Explicit expressions for p_1, p_2 can be obtained and the results reduce to that presented in [26].

5. Discussions and concluding remarks

The alternate algebraic approach employed here for **S**, **H**, **L** is to find the matrix product $-\mathbf{AB}^{-1}$ whose real and imaginary parts are $\mathbf{SL}^{-1}$ and $\mathbf{L}^{-1}$, respectively. A different product, $i\mathbf{AB}^{-1}$, is the inverse of the impedance matrix which is a positive definite Hermitian [15, 20, 22]. As shown in the paper, $\mathbf{SL}^{-1}$ and $\mathbf{L}^{-1}$ are the bases for computing **S**, **H**, **L**. For the monoclinic materials studied here, $\mathbf{SL}^{-1}$ has only one independent component while $\mathbf{L}^{-1}$ has four. Therefore there are a total of five independent components for **S**, **H**, **L**. For general anisotropic materials, the same approach can be taken to find $\mathbf{SL}^{-1}$ and $\mathbf{L}^{-1}$. Since $\mathbf{SL}^{-1}$ is antisymmetric and hence contains three independent components while $\mathbf{L}^{-1}$ contains six, there are a total of nine independent components for **S**, **H**, **L** for general anisotropic materials. This conclusion agrees with that in [32] where a different argument was followed.

For the problem of an interface crack in an anisotropic bimaterial, the stress singularities at the crack tip depend on $\mathbf{L}^{-1}$ and $\mathbf{SL}^{-1}$ only [33]. In particular, whether the displacement at the crack surface is oscillatory or not depends on $\mathbf{SL}^{-1}$ in the two materials [30, 33, 34]. For the crack problem therefore all we need is $\mathbf{L}^{-1}$ and $\mathbf{SL}^{-1}$, not $\mathbf{S}, \mathbf{H}, \mathbf{L}$. For the monoclinic materials considered here, $(\mathbf{SL}^{-1})_{21}$ is the only independent non-zero component of $\mathbf{SL}^{-1}$. It is shown in [30] that $(\mathbf{SL}^{-1})_{21}$ is invariant with rotation about the x_3 axis and that whether the crack surface displacement is oscillatory or not depends on whether $(\mathbf{SL}^{-1})_{21}$ in the two materials are different or not. The oscillatory phenomenon does not depend on the individual orientation of the two materials.

When $d = 0$, the matrices **H** and **L** are diagonal while the diagonal components of **S** vanish. From $(3.6)_2$, the vanishing of d implies that p_1 and p_2 have the form

$$p_1 = \rho_1 \, e^{i\delta}, \quad p_2 = -\rho_2 \, e^{-i\delta}, \quad 0 < \delta < \pi,$$

where δ, ρ_1, ρ_2 are real and positive. Hence

$$p_1 p_1 = -\rho_1 \rho_2 = c,$$

and c is a negative quantity. Equations $(3.7)_{1,3,4}$ then provide the condition for the vanishing of d:

$$s'_{26}(s'_{11})^{1/2} = -s'_{16}(s'_{22})^{1/2}. \tag{5.1}$$

Therefore, when (5.1) holds, **H** and **L** are diagonal while the diagonal components of **S** vanish. It is shown in Appendix A that (5.1) is equivalent to

$$C_{26} C_{11}^{1/2} = -C_{16} C_{22}^{1/2}.$$

Although the expressions for **S**, **H**, **L** remain valid when $p_1 = p_2$, the expressions for $\mathbf{S}(\theta)$, $\mathbf{H}(\theta)$, $\mathbf{L}(\theta)$ require a limiting process when $p_1 = p_2$. One does not have to solve the quartic equation (2.3) to find out if $p_1 = p_2$. When $p_1 = p_2$, the quartic equation has the form

$$s'_{11}(p - p_1)^2(p - \bar{p}_1)^2 = 0.$$

If we expand the product on the left, compare the coefficients of each term with (2.3) and use $(3.6)_{1,2}$ and (3.7), we obtain the conditions for $p_1 = p_2$:

$$\left.\begin{aligned} &s'_{26}(s'_{11})^{1/2} = s'_{16}(s'_{22})^{1/2}, \\ &2s'_{11}[(s'_{11}s'_{22})^{1/2} - s'_{12}] = s'_{11}s'_{66} - (s'_{16})^2. \end{aligned}\right\} \tag{5.2}$$

Again, $(5.2)_1$ can be shown to be equivalent to

$$C_{26} C_{11}^{1/2} = C_{16} C_{22}^{1/2}.$$

We see that $(5.2)_1$ differs from (5.1) only in sign. This implies that $d = 0$ and $p_1 = p_2$ cannot co-exist unless C_{16} and C_{26} both vanish, which is the case for orthotropic materials with planes of symmetry at the coordinate planes. In the latter case, $(5.2)_2$ reduces to

$$2[(s'_{11}s'_{22})^{1/2} - s'_{12}] - s'_{66} = 0.$$

Employing (A12), one sees that this is equivalent to

$$(C_{11}C_{22})^{1/2} - C_{12} - 2C_{66} = 0,$$

which is the condition for $p_1 = p_2$ for orthotropic materials [26].

Finally, if $d \neq 0$, one can re-orient the coordinate axes x_1, x_2 such that $d = 0$. If x_i^* is the new coordinate system which is obtained by rotating the x_i coordinate system about the x_3-axis an angle θ_0, we have

$$\mathbf{x}^* = \mathbf{\Omega x},$$

$$\mathbf{\Omega} = \begin{bmatrix} \cos\theta_0 & \sin\theta_0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is shown in [35] that $\mathbf{S}^*$ referred to the x_i^* coordinate system is

$$\mathbf{S}^* = \mathbf{\Omega S \Omega}^T.$$

It we substitute $\mathbf{S}$ from (3.5), we find that the diagonal components of $\mathbf{S}^*$ vanish if

$$\tan 2\theta_0 = \frac{2d}{b-e}. \tag{5.3}$$

Consequently, at the orientation θ_0 given by (5.3), $\mathbf{H}^*$, $\mathbf{L}^*$ are diagonal while the diagonal components of $\mathbf{S}^*$ vanish. The orientation θ_0 has an interesting physical meaning in relation to line forces and line dislocations in the infinite anisotropic material [32].

Appendix A

Using the contracted notation

$$\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3,$$

$$\sigma_{23} = \sigma_4, \quad \sigma_{31} = \sigma_5, \quad \sigma_{12} = \sigma_6,$$

$$\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3,$$

$$2\varepsilon_{23} = \varepsilon_4, \quad 2\varepsilon_{31} = \varepsilon_5, \quad 2\varepsilon_{12} = \varepsilon_6,$$

the stress-strain laws for the anisotropic elastic material can be written as

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \tag{A1}$$

in which $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are 6×1 column matrices and $\mathbf{C}$ is the 6×6 symmetric matrix of elastic stiffnesses. The inverse of (A1) is

$$\boldsymbol{\varepsilon} = \mathbf{s}\boldsymbol{\sigma}, \tag{A2}$$

$$\mathbf{sC} = \mathbf{Cs} = \mathbf{I}, \tag{A3}$$

where $\mathbf{s}$ is the 6×6 symmetric matrix of elastic compliances.

For the two-dimensional deformation considered here, $\varepsilon_3 = 0$ and the third equation of (A2) can be solved for σ_3 in terms of σ_j, $j \neq 3$. Substituting σ_3 so obtained in the remaining five equations of (A2), we have

$$\boldsymbol{\varepsilon}^0 = \mathbf{s}'\boldsymbol{\sigma}^0, \tag{A4}$$

in which $\boldsymbol{\varepsilon}^0$, $\boldsymbol{\sigma}^0$ are 5×1 column matrices obtained from $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ by deleting the ε_3, σ_3 components while $\mathbf{s}'$ is the 5×5 matrix whose components s'_{ij} are defined in (2.4) with $i, j \neq 3$. Likewise, if we ignore the third equation in (A1) and noticing that $\varepsilon_3 = 0$, we have

$$\boldsymbol{\sigma}^0 = \mathbf{C}^0\boldsymbol{\varepsilon}^0, \tag{A5}$$

where $\mathbf{C}^0$ is the 5×5 matrix which is obtained from $\mathbf{C}$ by deleting the third row and third column. Equations (A4) and (A5) imply that

$$\mathbf{s}'\mathbf{C}^0 = \mathbf{C}^0\mathbf{s}' = \mathbf{I}. \tag{A6}$$

Thus $\mathbf{s}'$ and $\mathbf{C}^0$ are the inverses of each other. Since the strain energy is

$$\tfrac{1}{2}(\boldsymbol{\sigma}^0)^T\boldsymbol{\varepsilon}^0 = \tfrac{1}{2}(\boldsymbol{\sigma}^0)^T\mathbf{s}'\boldsymbol{\sigma}^0 > 0,$$

$\mathbf{s}'$ is positive definite [29]. From (A6), $\mathbf{C}^0$ is also positive definite.

Equation (A6) and the positive definiteness of $\mathbf{C}^0$ and $\mathbf{s}'$ are properties of the elastic constants. Their validity should be independent of whether ε_3 vanishes or not. An alternate proof without assuming $\varepsilon_3 = 0$ can be obtained as follows. If we replace the third column and third row of C_{ij} by zero elements, we have

$$C^0_{ij} = C_{ij} - \delta_{i3}C_{3j} - C_{i3}\delta_{3j} + \delta_{i3}C_{33}\delta_{3j},$$

where δ_{ij} is the Kronecker delta and $i, j = 1, 2, \ldots, 6$. Using (2.4) and (A3), it can be shown that

$$C^0_{ij}s'_{jk} = \delta_{ik} - \delta_{i3}\delta_{3k}.$$

This is equivalent to (A6) because the third column and third rows of C^0_{ij}, s'_{ij} and the product $C^0_{ij}s'_{jk}$ contain only zero elements. The positive definiteness of C^0 follows from the fact that it is a principal minor of $\mathbf{C}$ which is positive definite. The positive definiteness of $\mathbf{s}'$ follows from (A6).

For monoclinic materials with the plane of symmetry at $x_3 = 0$, C_{3j}, C_{4j}, $(j = 1, 2, 3)$ and C_{i6}, $(i = 4, 5)$ vanish. We rearrange the rows and columns of $\mathbf{C}^0$ in the form [36]

$$\mathbf{C}^0 = \left[\begin{array}{ccc|cc} C_{11} & C_{12} & C_{16} & 0 & 0 \\ C_{12} & C_{22} & C_{26} & 0 & 0 \\ C_{16} & C_{26} & C_{66} & 0 & 0 \\ \hline 0 & 0 & 0 & C_{44} & C_{45} \\ 0 & 0 & 0 & C_{45} & C_{55} \end{array}\right] = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix}, \tag{A7}$$

say. Similarly, the 5×5 matrix $\mathbf{s}'$ is rearranged as

$$\mathbf{s}' = \left[\begin{array}{ccc|cc} s'_{11} & s'_{12} & s'_{16} & 0 & 0 \\ s'_{12} & s'_{22} & s'_{26} & 0 & 0 \\ s'_{16} & s'_{26} & s'_{66} & 0 & 0 \\ \hline 0 & 0 & 0 & s'_{44} & s'_{45} \\ 0 & 0 & 0 & s'_{45} & s'_{55} \end{array}\right] = \begin{bmatrix} \mathbf{s}'_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{s}'_2 \end{bmatrix}. \tag{A8}$$

With the rearranged $\mathbf{C}^0$ and $\mathbf{s}'$, (A6) remains valid and hence

$$\left.\begin{aligned} \mathbf{C}_1\mathbf{s}'_1 &= \mathbf{s}'_1\mathbf{C}_1 = \mathbf{I}, \\ \mathbf{C}_2\mathbf{s}'_2 &= s'_2\mathbf{C}_2 = \mathbf{I}. \end{aligned}\right\} \tag{A9}$$

Moreover, since $\mathbf{C}^0$ and $\mathbf{s}'$ are positive definite, so are $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{s}'_1$, $\mathbf{s}'_2$.

Let $\tilde{\mathbf{C}}_1$ be the adjoint of $\mathbf{C}_1$ so that

$$\mathbf{C}_1\tilde{\mathbf{C}}_1 = J\mathbf{I}, \quad J = \det \mathbf{C}_1 = (\det \mathbf{s}'_1)^{-1}. \tag{A10}$$

It follows from $(A9)_1$ that

$$\tilde{\mathbf{C}}_1 = J\mathbf{s}'_1. \tag{A11}$$

From $(A10)_1$ and (A11), the explicit expressions of the components of $\tilde{\mathbf{C}}_1$ are

$$\left.\begin{aligned} \tilde{C}_{11} &= C_{22}C_{66} - C_{26}^2 = Js'_{11}, \\ \tilde{C}_{22} &= C_{11}C_{66} - C_{16}^2 = Js'_{22}, \\ \tilde{C}_{66} &= C_{11}C_{22} - C_{12}^2 = Js'_{66}, \\ \tilde{C}_{12} &= C_{16}C_{26} - C_{12}C_{66} = Js'_{12}, \\ \tilde{C}_{16} &= C_{12}C_{26} - C_{16}C_{22} = Js'_{16}, \\ \tilde{C}_{26} &= C_{12}C_{16} - C_{11}C_{26} = Js'_{26}. \end{aligned}\right\} \tag{A12}$$

Equations (A12) are needed in reaching (2.3). Similarly, if $\tilde{\mathbf{s}}_1'$ is the adjoint of $\mathbf{s}_1'$, we have

$$\mathbf{s}_1'\tilde{\mathbf{s}}_1' = J^{-1}\mathbf{I}$$

and, comparing with $(\text{A9})_2$,

$$\tilde{\mathbf{s}}_1' = J^{-1}\mathbf{C}_1. \tag{A13}$$

Equation (A11) provides $\mathbf{s}_1'$ in terms of the components of $\mathbf{C}_1$ while (A13) gives $\mathbf{C}_1$ in terms of the components of $\mathbf{s}_1'$.

With (A12), $\gamma(p)$ of (2.8) can be written as

$$\gamma(p) = J[s_{12}' - s_{16}'p + s_{11}'p^2].$$

One can then show that, making use of (A13) and (2.3),

$$(s_{12}' - s_{16}' + s_{11}'p^2)\gamma(p) = -(C_{66} + 2pC_{26} + p^2C_{22}),$$

$$(s_{22}'p^{-1} - s_{26}' + s_{21}'p)\gamma(p) = C_{16} + p(C_{12} + C_{66}) + p^2C_{26}.$$

This is (2.9).

We next show that the conditions

$$s_{26}'(s_{11}')^{1/2} = \pm s_{16}'(s_{22}')^{1/2}, \tag{A14}$$

which appear in $(5.2)_1$ and (5.1), are identical to

$$C_{26}C_{11}^{1/2} = \pm C_{16}C_{22}^{1/2}. \tag{A15}$$

First, it can be shown that, using (A12) and $(\text{A10})_2$,

$$(s_{26}')^2s_{11}' - (s_{16}')^2s_{22}' = J^{-2}(C_{26}^2C_{11} - C_{16}^2C_{22}).$$

Therefore (A14) and (A15) are equivalent if we can prove that $s_{16}'s_{26}'$ and $C_{16}C_{26}$ have the same sign. To this end, we obtain from (A12)

$$\begin{aligned} J^2s_{16}'s_{26}' &= C_{16}C_{26}[C_{11}C_{22} + C_{12}^2 - C_{12}(C_{22}C_{16}C_{26}^{-1} + C_{11}C_{26}C_{16}^{-1})] \\ &= C_{16}C_{26}[(C_{11}C_{22})^{1/2} \mp C_{12}]^2, \end{aligned}$$

where use has been made of (A15). Therefore $s'_{16}s'_{26}$ and $C_{16}C_{26}$ have the same sign. This completes the proof.

Finally, for orthotropic materials we have $C_{16} = C_{26} = 0$, and using (A12) and $(A10)_2$, it can be shown that

$$[(C_{11}C_{22})^{1/2} \pm C_{12}]^{-1} = (s'_{11}s'_{22})^{1/2} \pm s'_{12}. \tag{A16}$$

On choosing the positive sign, the component $(\mathbf{SL}^{-1})_{21}$ is given by the expression on the left [30] or on the right [27].

Appendix B

We will show that $S_{21} > 0$ and $S_{12} < 0$, i.e., g of $(3.6)_4$ is positive.

First, we show that $g \neq 0$. If $g = 0$, we have from $(3.6)_4$ that

$$c = s'_{12}/s'_{11}.$$

If we obtain a from $(3.7)_1$, d^2 from $(3.7)_4$, b^2 from $(3.7)_3$ and use (A13), equation $(3.7)_2$ leads to

$$C_{22}C_{66} - C_{26}^2 = 0,$$

which violates the positive definiteness of $\mathbf{C}$. Hence $g \neq 0$. It should be pointed out that if g were zero, the tensor $\mathbf{S}$ would vanish identically.

Next, suppose that there exists a monoclinic material for which $g < 0$. Let $\mathbf{C}^{(+)}$, $\mathbf{C}^{(-)}$ denote the elastic constants of monoclinic materials for which $g > 0$ and $g < 0$, respectively. We know that $\mathbf{C}^{(+)}$ exist because $g > 0$ for the particular case of orthotropic materials [26]. Consider the following one parameter family of monoclinic materials

$$\mathbf{C}(\lambda) = \lambda\mathbf{C}^{(+)} + (1 - \lambda)\mathbf{C}^{(-)},$$

where $0 \leqslant \lambda \leqslant 1$ is a real parameter. If $\mathbf{C}^{(+)}$, $\mathbf{C}^{(-)}$ are positive definite, so is $\mathbf{C}(\lambda)$ and hence $\mathbf{C}(\lambda)$ is admissible. Since g is continuous in $\mathbf{C}$ [37] and $\mathbf{C}(\lambda)$ in λ, $g(\lambda)$ is continuous in λ. If $g(0) < 0$ and $g(1) > 0$, there exists a λ in the interval (0, 1) for which $g(\lambda) = 0$. This means that there exists a monoclinic material with positive definite $\mathbf{C}$ for which $g = 0$. This is in contradiction with the earlier conclusion that $g \neq 0$. Hence g cannot be negative. By $(3.5)_4$, $S_{21} > 0$ and $S_{12} < 0$.

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