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Determination of $C^{1/2}$, $C^{-1/2}$ and more general isotropic tensor functions of C

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1. Introduction

In the polar decomposition

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}, \quad \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I}, \tag{1.1}$$

where $\boldsymbol{F}$ is the (invertible) deformation gradient tensor, $\boldsymbol{U}$ and $\boldsymbol{V}$ are, respectively, the (symmetric) right and left stretch tensors, $\boldsymbol{R}$ is an orthogonal tensor and $\boldsymbol{I}$ is the identity tensor, one faces the problem of determining $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{R}$ in terms of $\boldsymbol{F}$. The superscript T in Eq. (1.1) stands for the transpose. Using the right and left Cauchy - Green tensors $\boldsymbol{C}$ and $\boldsymbol{B}$ defined by

$$\boldsymbol{U}^2 = \boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}, \quad \boldsymbol{V}^2 = \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}, \tag{1.2}$$

we have

$$\left.\begin{aligned} &\boldsymbol{U} = \boldsymbol{C}^{1/2}, \\ &\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}, \\ &\boldsymbol{V} = \boldsymbol{B}^{1/2} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{\mathrm{T}}. \end{aligned}\right\} \tag{1.3}$$

The problem reduces to the determination of $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ in terms of $\boldsymbol{C}$.

There are several ways of determining $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ in terms of $\boldsymbol{C}$, (see [1–3], for example). Notice that $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ are special examples of isotropic tensor functions of $\boldsymbol{C}$. Using the representation theorem for isotropic tensor functions [4] together with the diagonalization of a symmetric tensor, we present an alternate way of determining $\boldsymbol{U}$, $\boldsymbol{U}^{-1}$ and other more general isotropic tensor functions in terms of $\boldsymbol{C}$. Since $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ are very simple functions of $\boldsymbol{C}$, one can determine them easily without recourse to the representation theorem and the diagonalization. This is shown in Sections 2 and 3. In Section 4 we treat $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ as isotropic tensor functions of $\boldsymbol{C}$. In doing so, we show that $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ can be represented by lower order powers of $\boldsymbol{C}$ when $\boldsymbol{U}$ has a repeated eigenvalue.

2. Determination of U

Let λ_i, $(i = 1, 2, \ldots, n)$ be the eigenvalues of $\boldsymbol{U}$ in the n-dimensional space. The eigenvalues of $\boldsymbol{C}$ are λ_i^2, $(i = 1, 2, \ldots, n)$.

2a. Two-dimensional space

Using the Cayley-Hamilton theorem, we have

$$\boldsymbol{U}^2 - I_U \boldsymbol{U} + II_U \boldsymbol{I} = \boldsymbol{0}, \tag{2.1}$$

where

$$I_U = \lambda_1 + \lambda_2, \quad II_U = \lambda_1 \lambda_2 \tag{2.2}$$

are the principal invariants of $\boldsymbol{U}$. Since $\boldsymbol{U}^2 = \boldsymbol{C}$, we have

$$\boldsymbol{U} = I_U^{-1}(II_U \boldsymbol{I} + \boldsymbol{C}). \tag{2.3}$$

This agrees with the $\boldsymbol{U}$ obtained in [2] * and (3.3) of [3].

2b. Three-dimensional space

Again, from the Cayley-Hamilton theorem we obtain

$$\boldsymbol{U}^3 - I_U \boldsymbol{U}^2 + II_U \boldsymbol{U} - III_U \boldsymbol{I} = \boldsymbol{0}, \tag{2.4}$$

where the invariants are given by

$$\left.\begin{aligned} I_U &= \lambda_1 + \lambda_2 + \lambda_3, \\ II_U &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\ III_U &= \lambda_1\lambda_2\lambda_3. \end{aligned}\right\} \tag{2.5}$$

If we multiply (2.4) by $\boldsymbol{U}$,

$$\boldsymbol{U}^4 - I_U \boldsymbol{U}^3 + II_U \boldsymbol{U}^2 - III_U \boldsymbol{U} = \boldsymbol{0}, \tag{2.6}$$

and eliminate $\boldsymbol{U}^3$ between (2.4) and (2.6), we have, after replacing $\boldsymbol{U}^4$ and $\boldsymbol{U}^2$ by $\boldsymbol{C}^2$ and $\boldsymbol{C}$, respectively,

$$\boldsymbol{U} = (I_U II_U - III_U)^{-1}\{I_U III_U \boldsymbol{I} + (I_U^2 - II_U)\boldsymbol{C} - \boldsymbol{C}^2\}. \tag{2.7}$$

It can be shown that (2.7) is identical to (3.7) in [3].

2c. Four- and higher-dimensional space

In the four-dimensional space, we have

$$\boldsymbol{U}^4 - I_U \boldsymbol{U}^3 + II_U \boldsymbol{U}^2 - III_U \boldsymbol{U} + IIII_U \boldsymbol{I} = \boldsymbol{0}, \tag{2.8}$$

where I_U, $II_U, \ldots$ are the principal invariants of $\boldsymbol{U}$. Two more equations are obtained by multiplying Eq. (2.8) by $\boldsymbol{U}$ and $\boldsymbol{U}^2$. By eliminating $\boldsymbol{U}^3$ and $\boldsymbol{U}^5$ between the three equations and replacing $\boldsymbol{U}^2$, $\boldsymbol{U}^4$, $\boldsymbol{U}^6$ by $\boldsymbol{C}$, $\boldsymbol{C}^2$, $\boldsymbol{C}^3$, one obtains $\boldsymbol{U}$ in terms of $\boldsymbol{I}$, $\boldsymbol{C}$, $\boldsymbol{C}^2$ and $\boldsymbol{C}^3$.

A similar procedure can be used to determine $\boldsymbol{U}$ in a higher-dimensional space.

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* $\boldsymbol{U}$ on page 55 of [2] has a different expression from (2.3) presented here because I_U and II_U are replaced in terms of I_C and II_C. The minus sign inside the parentheses containing $\boldsymbol{C}$ on page 55 of [2] should be a plus sign.

2d. Repeated eigenvalues

Let $\boldsymbol{U}$ be in a three-dimensional space and let

$$\lambda_1 \neq \lambda_2 = \lambda_3. \tag{2.9}$$

Then, the representation given by Eq. (2.7) is not unique. In fact Eq. (2.3), which was derived for two-dimensional $\boldsymbol{U}$, applies to three-dimensional $\boldsymbol{U}$ when (2.9) holds. We will return to this point in Section 4.

If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, say, it is not difficult to show that $\boldsymbol{U} = \lambda \boldsymbol{I}$, $\boldsymbol{C} = \lambda^2 \boldsymbol{I}$.

A similar reduction can be made for $\boldsymbol{U}$ in a higher-dimensional space with repeated eigenvalues.

3. Determination of U^{-1}.

3a. Two-dimensional space

If we multiply Eq. (2.1) by $\boldsymbol{U}^{-1}$ and substitute $\boldsymbol{U}$ from (2.3), we obtain

$$\boldsymbol{U}^{-1} = (I_U II_U)^{-1}\{(I_U^2 - II_U)\boldsymbol{I} - \boldsymbol{C}\}. \tag{3.1}$$

3b. Three-dimensional space

We multiply Eq. (2.4) by $\boldsymbol{U}^{-1}$, replace $\boldsymbol{U}^2$ by $\boldsymbol{C}$ and $\boldsymbol{U}$ by (2.7). We then have

$$\begin{aligned}\boldsymbol{U}^{-1} = III_U^{-1}(I_U II_U - III_U)^{-1}\{&[I_U II_U^2 - III_U(I_U^2 + II_U)]\boldsymbol{I}\\ &-[III_U + I_U(I_U^2 - 2II_U)]\boldsymbol{C} + I_U\boldsymbol{C}^2\}.\end{aligned} \tag{3.2}$$

3c. Remarks

It can be shown that Eqs. (3.1) and (3.2) are identical to (4.1) and (4.2) of [3], respectively. For $\boldsymbol{U}^{-1}$ in four- and higher-dimensional spaces, a procedure similar to the one in deriving (3.2) can be employed. For the cases of repeated eigenvalues, the discussion in Section 2d applies here also.

4. Isotropic tensor functions

A tensor-valued tensor function $\boldsymbol{G}(\boldsymbol{C})$ is isotropic if the relation [4]

$$\boldsymbol{Q}\boldsymbol{G}(\boldsymbol{C})\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{G}(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^{\mathrm{T}}), \tag{4.1}$$

holds for all orthogonal tensors $\boldsymbol{Q}$. $\boldsymbol{Q}$ may be, but need not be, the orthogonal tensor $\boldsymbol{R}$ defined in (1.1). It is not difficult to show that $\boldsymbol{C}^m$, where m is an integer, is isotropic. Consequently, tensor power series in $\boldsymbol{C}$ are isotropic tensor functions. In particular, if we define

$$\left.\begin{aligned} e^{\boldsymbol{C}} &= \boldsymbol{I} + \boldsymbol{C} + \frac{1}{2!}\boldsymbol{C}^2 + \dots,\\ \sin \boldsymbol{C} &= \boldsymbol{C} - \frac{1}{3!}\boldsymbol{C}^3 + \dots,\\ \ln(\boldsymbol{I} + \boldsymbol{C}) &= \boldsymbol{C} - \tfrac{1}{2}\boldsymbol{C}^2 + \tfrac{1}{3}\boldsymbol{C}^3 \dots, \end{aligned}\right\} \tag{4.2}$$

$e^{\boldsymbol{C}}$, $\sin \boldsymbol{C}$ and $\ln(\boldsymbol{I} + \boldsymbol{C})$ are isotropic tensor functions.

To show that $C^{1/2}$ is isotropic, we use the relation:

$$(QC^{1/m}Q^T)^m = (QC^{1/m}Q^T)(QC^{1/m}Q^T)\dots(QC^{1/m}Q^T)$$
$$= Q(C^{1/m})^m Q^T = QCQ^T, \tag{4.3}$$

or

$$QC^{1/m}Q^T = (QCQ^T)^{1/m}. \tag{4.4}$$

Hence $C^{1/m}$ where m is an integer is an isotropic tensor function. Notice that if $G(C)$ is isotropic, so is $G^{-1}(C)$. Therefore, $C^{-1/m}$ is isotropic. Hence, $U = C^{1/2}$ and $U^{-1} = C^{-1/2}$ discussed in Sections 2 and 3 are isotropic tensor functions of C.

The representation theorem [4] states that $G(C)$ is isotropic if and only if it has a representation (we consider three-dimensional space here)

$$G(C) = \phi_0 I + \phi_1 C + \phi_2 C^2, \tag{4.5}$$

where ϕ_0, ϕ_1, ϕ_2 are functions of the invariants of C. Since the invariants of C and U are related [3], ϕ_0, ϕ_1 and ϕ_2 can be expressed as functions of λ_1, λ_2, λ_3 or I_U, II_U and III_U.

The fact that C is symmetric and positive definite suggests that we may diagonalize C by

$$P^T C P = \Lambda^2, \tag{4.6}$$

where P is an orthogonal tensor formed by the eigenvectors of C while the components of Λ form a diagonal matrix whose diagonal elements are λ_1, λ_2 and λ_3. If we pre-multiply Eq. (4.5) by P^T, post-multiply by P and make use of Eqs. (4.1) and (4.6), we obtain

$$G(\Lambda^2) = \phi_0 I + \phi_1 \Lambda^2 + \phi_2 \Lambda^4. \tag{4.7}$$

For $G(C) = C^{1/2}$, (4.7) can be written in full as

$$\left.\begin{aligned} g(\lambda_1^2) &= \phi_0 + \phi_1\lambda_1^2 + \phi_2\lambda_1^4, \\ g(\lambda_2^2) &= \phi_0 + \phi_1\lambda_2^2 + \phi_2\lambda_2^4, \\ g(\lambda_3^2) &= \phi_0 + \phi_1\lambda_3^2 + \phi_2\lambda_3^4, \end{aligned}\right\} \tag{4.8}$$

where $g(\lambda^2) = \lambda$. For $G(C) = C^{-1/2}$, e^C, $\sin C$ and $\ln(I + C)$, we have $g(\lambda^2) = \lambda^{-1}$, e^{λ^2}, $\sin \lambda^2$ and $\ln(1 + \lambda^2)$, respectively.

Equation (4.8) has a unique solution for ϕ_0, ϕ_1, ϕ_2 when λ_i are distinct. For $G(C) = C^{1/2}$, $g(\lambda^2) = \lambda$ and we have,

$$\begin{aligned} \phi_0 &= \lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)[(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)]^{-1} \\ &= I_U III_U (I_U II_U - III_U)^{-1}, \\ \phi_1 &= (\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)[(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)]^{-1} \\ &= (I_U^2 - II_U)(I_U II_U - III_U)^{-1}, \\ \phi_2 &= -[(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)]^{-1} = -(I_U II_U - III_U)^{-1}. \end{aligned} \tag{4.9}$$

Equation (4.5) then reproduces Eq. (2.7). If $\lambda_1 \neq \lambda_2 = \lambda_3$, one may consider Eqs. $(4.8)_1$ and $(4.8)_2$ only since $(4.8)_3$ is identical to $(4.8)_2$. We then have a one-parameter family of solutions for ϕ_0, ϕ_1, ϕ_2 and the representation given by (4.5) is not unique. A particular representation in which C^2 is not present can be obtained by setting $\phi_2 = 0$ in $(4.8)_1$ and $(4.8)_2$ to obtain

$$\left.\begin{aligned} \phi_0 &= \lambda_1\lambda_2(\lambda_1+\lambda_2)^{-1}, \\ \phi_1 &= (\lambda_1+\lambda_2)^{-1}, \\ \phi_2 &= 0. \end{aligned}\right\} \tag{4.10}$$

Equation (4.5) now reproduces (2.3) for U in three-dimensional space when $\lambda_1 \neq \lambda_2 = \lambda_3$.

One may reproduce Eqs. (3.1) and (3.2) for U^{-1} by the present approach. It should be pointed out that in using the approach of this section the eigenvectors of C are not needed. Only the eigenvalues of C or U are needed.

Finally, consider the function $G(C) = (C + cI)^{-1}$ where c is a function of the invariants of C. This function is isotropic. Setting $g(\lambda^2) = (\lambda^2 + c)^{-1}$ in Eq. (4.8) we have

$$\left.\begin{aligned} \phi_0 &= \left[\left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2\right) + \left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)c + c^2\right]K^{-1} \\ &= \left(II_C + I_C c + c^2\right)K^{-1}, \\ \phi_1 &= -\left[\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)+c\right]K^{-1} \\ &= -(I_C + c)K^{-1}, \\ \phi_2 &= K^{-1}, \end{aligned}\right\} \tag{4.11}$$

where

$$\begin{aligned} K &= \left(\lambda_1^2+c\right)\left(\lambda_2^2+c\right)\left(\lambda_3^2+c\right) \\ &= III_C + II_C c + I_C c^2 + c^3. \end{aligned} \tag{4.12}$$

With (4.11) and (4.12), Eq. (4.5) reproduces (2.2) in [3].

References

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