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An admissibility condition for equilibrium shocks in finite elasticity *

ROHAN ABEYARATNE

College of Engineering, Michigan State University, East Lansing, Michigan 48824, U.S.A.

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Abstract

The equations governing the equilibrium of a finitely deformed elastic solid are derived from the Principle of Minimum Potential Energy. The possibility of the deformation gradient and the stresses being discontinuous across certain surfaces in the body – "equilibrium shocks" – is allowed for. In addition to the equilibrium equations, natural boundary conditions and traction continuity condition, a *supplementary jump condition* which is to hold across the surface of discontinuity is derived. This condition is shown to imply that a stable equilibrium shock must necessarily be dissipation-free.

1. Introduction

The differential equations governing the equilibrium of certain finitely deformed elastic solids may lose their ellipticity in the presence of sufficiently severe strains, and as a result, there may arise deformation fields which are not "smooth"; in particular, the deformation gradient and the stresses could fail to be continuous across one or more surfaces – *equilibrium shocks* – in the body [1]–[7]. The study of such problems is motivated, in part, by the fairly common observation that in certain highly deformed ductile solids a smooth deformation field more or less abruptly gives way to one which is less smooth and involves narrow bands of highly localized shear deformation, [8].

Certain features of the mathematical problem here are similar to those associated with the theory of steady, irrotational flow of an inviscid, compressible gas [9]. There, the lack of ellipticity accompanying supersonic flow often leads to the presence of shocks – surfaces of discontinuity of certain fluid properties. In gas dynamics, certain restrictions arising from the second law of thermodynamics are imposed on the change of entropy across a shock, these conditions being related to the dissipative character of the shocks in the inviscid fluid. An analogous condition – a "*dissipativity inequality*" – in the context of elastostatics was derived by Knowles and Sternberg [3] and Knowles [10] on energetic grounds. A thermodynamic motivation for this condition was given in [10]. They proposed that physically acceptable equilibrium fields should necessarily conform to this inequality.

Enlarging the class of admissible elastostatic fields by allowing for equilibrium

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shocks can often lead to a multiplicity of solutions to the relevant boundary-value problem. This is illustrated by the examples in [5] and [11] and is analogous to the well-known nonuniqueness occurring in the theory of quasi-linear hyperbolic partial differential equations [12]. In such circumstances, it is essential to introduce additional criteria which rule out physically inadmissible solutions from among the many solutions admitted by the differential equations. The dissipativity inequality alluded to previously plays such a role in finite elastostatics. However, while it does serve to disallow certain solutions as being unacceptable on energetic grounds, this condition does not in general rule out solutions which are unstable (see example in [5]). It is therefore of some interest to examine the possibility of imposing restrictions which are, in fact, stronger than the dissipativity inequality.

Ericksen [11], in examining the equilibrium of a bar, successfully enforced a stability condition based on the classical energy criterion to single out a physically acceptable solution. Subsequently Abeyaratne [5] utilized this criterion in a similar manner in the study of the finite twisting of a tube, which in turn suggests the possible importance of such a criterion in general. Certainly solutions admitted by such a criterion would necessarily be stable (in the sense of minimizing the corresponding potential energy.)

In this paper we examine the general issue of *minimizing the potential energy* associated with a *weak solution* – an elastostatic field involving equilibrium shocks. We derive, as necessary conditions, two jump conditions which are to hold across the shock. One of these is a statement of the requirement that the tractions are to be continuous across the shock-surface. The second turns out to be precisely the dissipativity inequality with the inequality sign replaced by equality. It follows that a "stable" shock is required to be *dissipation-free*.

There are several examples of problems in the calculus of variations which do not possess solutions in the class of "smooth" functions but which do in an extended class of "piecewise smooth" functions. (For example, see Section 15 of [13].) In such an event certain jump conditions, *Weierstrass-Erdmann corner conditions*, must necessarily hold at the point of lack of smoothness (the "corners"). The dissipation-free condition derived here corresponds to one of these corner conditions.

2. Basic equations. Equilibrium shocks

Let $\mathfrak{R}$ be the three-dimensional open region occupied by the interior of a body in an undeformed configuration. A deformation of the body is described by a sufficiently smooth and invertible transformation

$$\mathbf{y} = \mathbf{x} + \mathbf{u}(\mathbf{x}) \quad \text{on} \quad \mathfrak{R} \tag{2.1}$$

which maps the domain $\mathfrak{R}$ onto a domain $\mathfrak{R}_*$. Here $\mathbf{y}$ is the position vector after deformation of the particle which in the undeformed configuration was located at $\mathbf{x}$, while $\mathbf{u}(\mathbf{x})$ denotes the displacement vector field. For the moment, $\mathbf{u}$ is assumed to be twice continuously differentiable on $\mathfrak{R}$. The deformation gradient tensor $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{1} + \nabla\mathbf{u}, \qquad \det\mathbf{F} > 0 \quad \text{on} \quad \mathfrak{R}. \tag{2.2}$$ *

* Notation: $\nabla\mathbf{u}$ is the tensor field with components $\partial u_i/\partial x_j$. div $\boldsymbol{\sigma}$ is the vector field with components $\partial\sigma_{ij}/\partial x_j$. $W_{\mathbf{F}}(\mathbf{F})$ is the tensor with components $\partial W/\partial F_{ij}$. Here we have used standard indicial notation and Cartesian coordinates.

Let $\boldsymbol{\sigma}(\mathbf{x})$ be the nominal stress tensor field accompanying the deformation at hand. In the absence of body forces, the local conditions for force and moment equilibrium are

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \qquad \boldsymbol{\sigma}\mathbf{F}^T = \mathbf{F}\boldsymbol{\sigma}^T \quad \text{on} \quad \mathcal{R} \tag{2.3}$$

provided that $\boldsymbol{\sigma}$ is continuously differentiable on $\mathcal{R}$. The nominal traction vector $\mathbf{s}$ on the boundary of a sub-domain of the body is given by

$$\mathbf{s} = \boldsymbol{\sigma}\mathbf{n} \tag{2.4}$$

where $\mathbf{n}$ is the unit outward normal vector.

Suppose now that the body under consideration is homogeneous and elastic and possesses an elastic potential * $W = W(\mathbf{F})$: W represents the strain-energy density per unit undeformed volume. The nominal stress tensor is now given by

$$\boldsymbol{\sigma} = W_{\mathbf{F}}(\mathbf{F}). \tag{2.5}$$

Finally, we suppose that the displacement field $\mathbf{u}$ and the nominal traction field $\mathbf{s}$ are prescribed on the portions $\partial\mathcal{R}_1$, $\partial\mathcal{R}_2$ of the boundary, where $\partial\mathcal{R} = \partial\mathcal{R}_1 \cup \partial\mathcal{R}_2$ denotes the entire boundary of the region $\mathcal{R}$;

$$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) \quad \text{on} \quad \partial\mathcal{R}_1, \qquad \mathbf{s}(\mathbf{x}) = \bar{\mathbf{s}}(\mathbf{x}) \quad \text{on} \quad \partial\mathcal{R}_2, \tag{2.6}$$

$\bar{\mathbf{u}}$ and $\bar{\mathbf{s}}$ being suitably smooth functions.

Given $\bar{\mathbf{u}}$, $\bar{\mathbf{s}}$ and W, we are to determine the displacement field $\mathbf{u}(\mathbf{x})$ and the stress field $\boldsymbol{\sigma}(\mathbf{x})$ in accordance with (2.2)–(2.6).

The particular examples studied in [5] and [11] were shown to possess *no* solutions (of the requisite smoothness) to the problem formulated above. However, upon relaxing the degree of smoothness demanded of the field quantities it was demonstrated that solutions did exist. This need to relax the differentiability requirements was in turn related to a loss of ellipticity of the governing displacement equations of equilibrium, $\operatorname{div} W_{\mathbf{F}}(\mathbf{F}) = \mathbf{0}$.

We wish to focus attention on such "weak solutions" and will now require the field quantities to possess the classical degree of smoothness everywhere in $\mathcal{R}$ *except* on one or more regular surfaces within the body. Consequently, we allow for the possibility that there is a surface $\mathbb{S}$ in $\mathcal{R}$ such that $\boldsymbol{\sigma}$ and $\mathbf{F}$ are continuously differentiable everywhere in $\mathcal{R}$ except on $\mathbb{S}$ and such that $\boldsymbol{\sigma}$ and $\mathbf{F}$ suffer finite jump discontinuities across it. The displacement field $\mathbf{u}$ is presumed to be continuous everywhere in $\mathcal{R}$. Such a surface has been called an *equilibrium shock* [10].

On going through the usual arguments **, one finds from the global conditions for equilibrium that

$$\begin{aligned} &\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \qquad \boldsymbol{\sigma}\mathbf{F}^T = \mathbf{F}\boldsymbol{\sigma}^T \quad \text{on} \quad \mathcal{R} - \mathbb{S}, \\ &[\boldsymbol{\sigma}]_-^+ \mathbf{N} = \mathbf{0} \quad \text{on} \quad \mathbb{S}. \end{aligned} \tag{2.7}$$

The jump condition in (2.7) states that the nominal tractions are to be continuous across $\mathbb{S}$. Here $[f]_-^+ = \overset{+}{f} - \bar{f}$ where $\overset{+}{f}$ and $\bar{f}$ are the limiting values of the quantity f

* The dependence of W on $\mathbf{F}$ is, of course, restricted by the principle of material frame indifference, $W(\mathbf{F}) = W((\mathbf{F}^T\mathbf{F})^{1/2})$. Further, W is assumed to be infinitely differentiable for every nonsingular tensor $\mathbf{F}$.

** See Chadwick [15], p. 114.

(presumed to exist) as a point on $\mathbb{S}$ is approached from each side, and $\mathbf{N}$ is a unit normal to $\mathbb{S}$.

Given the prescribed vector fields $\bar{\mathbf{u}}$, $\bar{\mathbf{s}}$, one is now to determine the displacement field $\mathbf{u}(\mathbf{x})$ and the stress field $\boldsymbol{\sigma}(\mathbf{x})$ (having the required degree of relaxed smoothness) in accordance with (2.2), (2.4)–(2.7). Weak solutions of this form to problems in finite elastostatics have been determined in a number of specific examples [5]–[7], [11].

As illustrated by [5] and [11], enlarging the class of admissible stress and displacement fields in this way admits the possibility of multiple solutions to a given problem. Some of these solutions, though conforming to the appropriate differential equations and boundary conditions, may be unacceptable on physical grounds (such as instability). In such a situation it is essential to introduce additional criteria which rule out physically inadmissible solutions from among the many nominal solutions. One such criterion, a "*dissipativity inequality*" analogous to the entropy condition in fluid mechanics, was proposed by Knowles and Sternberg [3] and Knowles [10] on energetic grounds. They considered a quasi-static time dependent family of equilibrium states and showed that the rate at which elastic energy is being stored (in every subdomain of the body) does not exceed the corresponding rate at which work is being done if and only if

$$[W - \mathbf{FN}\cdot\boldsymbol{\sigma}\mathbf{N}]_-^+ \geqslant 0 \quad \text{on} \quad \mathbb{S}. \tag{2.8}$$

Here the positive side of $\mathbb{S}$ is taken as the side toward which $\mathbb{S}$ moves in this quasi-static motion and $\mathbf{N}$ is the unit normal directed into the positive side.

3. Variational formulation. An admissibility condition

We now re-examine the formulation of the problem described in the previous section by making use of the Principle of Minimum Potential Energy. By admitting the possibility of equilibrium fields possessing the relaxed degree of smoothness described previously, we derive the equilibrium equations $(2.7)_1$, the natural boundary condition $(2.6)_2$, the traction continuity condition $(2.7)_3$ as well as an *additional jump condition* which is to hold across an equilibrium shock.

Given the vector fields $\bar{\mathbf{u}}(\mathbf{x})$, $\bar{\mathbf{s}}(\mathbf{x})$ and the strain energy density W as described previously, the potential energy functional $V\{\mathbf{w}\}$ is defined by

$$V\{\mathbf{w}\} = \int_{\mathfrak{R}} W(\mathbf{1} + \nabla\mathbf{w})\,dv_x - \int_{\partial\mathfrak{R}_2} \bar{\mathbf{s}}\cdot\mathbf{w}\,dA_x \tag{3.1}$$

for all functions $\mathbf{w}$ in some set $\mathcal{Q}$. Here we are assuming that the loading on $\partial\mathfrak{R}_2$ is dead. We suppose that the set of "admissible virtual displacements", $\mathcal{Q}$, is the set of all vector-valued functions defined on $\mathfrak{R}$ which obey the following requirements:

(i) $\mathbf{w}(\mathbf{x})$ is continuous on $\mathfrak{R}$,

(ii) $\mathbf{w}(\mathbf{x})$ has continuous first and second derivatives in $\mathfrak{R}$ except possibly along one or more regular surfaces,

(iii) $\mathbf{w}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})$ on $\partial\mathfrak{R}_1$.

Since this limited degree of smoothness is all that is required of an equilibrium displacement field, it seems reasonable *not* to impose more severe smoothness requirements on the virtual displacement $\mathbf{w}$. Then from among all functions $\mathbf{w}$ in $\mathcal{Q}$ we wish to determine a function for which the functional $V\{\mathbf{w}\}$ has a weak extremum.

We assume the existence of a function $\mathbf{u}(\mathbf{x})$ in $\mathcal{Q}$ which extremizes V. Suppose that there is a regular surface $\mathbb{S}$ in $\mathcal{R}$ across which the first derivatives of $\mathbf{u}$ suffer a finite jump discontinuity, while $\mathbf{u}$ is twice continuously differentiable at every other point. Let $\mathbf{N}$ be a unit normal to $\mathbb{S}$.

Now consider first the one parameter family of virtual displacements $\mathbf{w}$

$$\mathbf{w}(\mathbf{x};\epsilon)=\mathbf{u}(\mathbf{x})+\epsilon\boldsymbol{\delta}\mathbf{u}(\mathbf{x}),\qquad -\epsilon_1<\epsilon<\epsilon_1, \tag{3.2}$$

where ϵ_1 is a sufficiently small positive constant. Here the variation $\boldsymbol{\delta}\mathbf{u}(\mathbf{x})$ is assumed to be a twice continuously differentiable vector field on $\mathcal{R}$ with $\boldsymbol{\delta}\mathbf{u}(\mathbf{x})=\mathbf{0}$ on the boundary $\partial\mathcal{R}_1$. Thus, for each ϵ, $\mathbf{w}(\mathbf{x};\epsilon)$ is in $\mathcal{Q}$. By assumption the associated family of potential energies $\hat{V}(\epsilon)=V\{\mathbf{u}+\epsilon\boldsymbol{\delta}\mathbf{u}\}$ has a minimum at $\epsilon=0$ so that necessarily

$$\left.\frac{d\hat{V}}{d\epsilon}\right|_{\epsilon=0}=0. \tag{3.3}$$

On going through the usual arguments (see for example [15]) one finds from (3.1)–(3.3) that

$$\begin{aligned}
&\operatorname{div} W_{\mathbf{F}}(\mathbf{1}+\nabla\mathbf{u})=\mathbf{0} && \text{on}\quad \mathcal{R}-\mathbb{S},\\
&W_{\mathbf{F}}(\mathbf{1}+\nabla\mathbf{u})\mathbf{n}=\bar{\mathbf{s}} && \text{on}\quad \partial\mathcal{R}_2,\\
&[W_{\mathbf{F}}(\mathbf{1}+\nabla\mathbf{u})]_-^+\mathbf{N}=\mathbf{0} && \text{on}\quad \mathbb{S}.
\end{aligned} \tag{3.4}$$ *

In view of (2.4) and (2.5), these are precisely the equilibrium equations $(2.7)_1$, traction boundary condition $(2.6)_2$ and traction continuity condition $(2.7)_3$ respectively.

The family of virtual displacements $\mathbf{w}(\mathbf{x};\epsilon)$ defined by (3.2) possesses a discontinuity in its gradient $\nabla\mathbf{w}$ across the *fixed* surface $\mathbb{S}$, i.e. $\mathbb{S}$ is independent of the parameter ϵ. It is, of course, possible to choose other families of admissible virtual displacements $\mathbf{w}(\mathbf{x};\epsilon)$ which do not possess this feature, and it is natural to wonder whether such a choice would yield any further necessary conditions, in addition to those obtained above. Since, in any given boundary-value problem, the location of an equilibrium shock $\mathbb{S}$ in the body is not known *a priori*, one might expect such a "variation of the location" of $\mathbb{S}$ to yield information of interest. We now proceed to choose such a class of functions $\mathbf{w}(\mathbf{x};\epsilon)$ with $\mathbf{w}(\mathbf{x};0)=\mathbf{u}(\mathbf{x})$ and such that its gradient $\nabla\mathbf{w}$ is discontinuous across a variable surface $\mathbb{S}(\epsilon)$, $\mathbb{S}(0)=\mathbb{S}$. This will lead to a field equation in $\mathcal{R}-\mathbb{S}$ and a jump condition on $\mathbb{S}$, the former being essentially equivalent to $(3.4)_1$ while the latter is an entirely independent condition.

The displacement field $\mathbf{u}(\mathbf{x})$ which minimizes the potential energy V may be described parametrically by

$$\mathbf{u}=\mathbf{u}(\boldsymbol{\xi}),\qquad \mathbf{x}=\boldsymbol{\xi}\quad\text{for}\quad\boldsymbol{\xi}\text{ in }\mathcal{R}. \tag{3.5}$$

Now consider the one parameter family of virtual displacements $\mathbf{w}(\mathbf{x};\epsilon)$ defined by

$$\mathbf{w}=\mathbf{u}(\boldsymbol{\xi}),\qquad \mathbf{x}=\boldsymbol{\xi}+\epsilon\boldsymbol{\delta}\mathbf{x}(\boldsymbol{\xi})\quad\text{for}\quad\boldsymbol{\xi}\text{ in }\mathcal{R},\qquad -\epsilon_1<\epsilon<\epsilon_1. \tag{3.6}$$ **

Here the variation $\boldsymbol{\delta}\mathbf{x}(\boldsymbol{\xi})$ is a twice continuously differentiable vector field on $\mathcal{R}$.

* Here $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{R}$ while $\mathbf{N}$ is a unit normal to $\mathbb{S}$.

** The analysis here is no more than a generalization of Section 2.6 of Pars [16]. Observe that the virtual displacement field chosen previously, (3.2), may be equivalently described by $\mathbf{w}=\mathbf{u}(\boldsymbol{\xi})+\epsilon\boldsymbol{\delta}\mathbf{u}(\boldsymbol{\xi})$, $\mathbf{x}=\boldsymbol{\xi}$.

Furthermore, for each value of ϵ in $(-\epsilon_1, \epsilon_1)$ we require the mapping $\boldsymbol{\xi} \to \mathbf{x}$, $(3.6)_2$, to be one to one. Finally suppose that

$$\delta\mathbf{x}(\boldsymbol{\xi}) = \mathbf{0} \qquad \text{for} \qquad \boldsymbol{\xi} \text{ in } \partial\mathcal{R},$$
$$\det \mathbf{H} > 0, \quad H_{ij} = \frac{\partial x_i}{\partial \xi_j} = \delta_{ij} + \epsilon \frac{\partial \delta x_i}{\partial \xi_j}(\boldsymbol{\xi}) \qquad \text{for} \qquad \boldsymbol{\xi} \text{ in } \mathcal{R}, \qquad -\epsilon_1 < \epsilon < \epsilon_1. \tag{3.7}$$

Consequently (3.6) defines a family of admissible virtual displacements. Moreover, by virtue of the properties of $\mathbf{u}$ (see paragraph preceding (3.2)), the gradient $\nabla_x \mathbf{w}(\mathbf{x}; \epsilon)$ is continuous in $\mathcal{R}$ except across the surface $\mathbb{S}(\epsilon)$ which is the image of $\mathbb{S}$ under the mapping $\boldsymbol{\xi} \to \mathbf{x}$. Note that $\mathbf{w}(\mathbf{x}; 0) = \mathbf{u}(\mathbf{x})$ so that $\mathbb{S}(0) = \mathbb{S}$.

In view of (3.1) and (3.7) the potential energy $\hat{V}(\epsilon)$ associated with (3.6) may be written as

$$\hat{V}(\epsilon) = V\{\mathbf{w}(\mathbf{x}; \epsilon)\} = \int_{\mathcal{R}} W\Big(\mathbf{1} + \underset{\xi}{\nabla} \mathbf{u}\mathbf{H}^{-1}\Big) \det \mathbf{H} \mathrm{d}v_{\xi} - \int_{\partial\mathcal{R}_2} \bar{\mathbf{s}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}) \mathrm{d}A_x. \tag{3.8}$$

We note that the volume integral in (3.8) is taken over the region $\mathcal{R}$ since by virtue of $(3.7)_1$ the mapping $\mathbf{x} \to \boldsymbol{\xi}$, the inverse of $(3.6)_2$, takes the region $\mathcal{R}$ onto itself. In writing the area integral in (3.8) we have made use of (3.6), $(3.7)_1$ which implies that $\mathbf{w}(\mathbf{x}; \epsilon) = \mathbf{u}(\mathbf{x})$ on $\partial\mathcal{R}$. Keeping in mind that $\nabla_{\xi}\mathbf{u}(\boldsymbol{\xi})$, and hence the integrand of (3.8), is discontinuous across the fixed surface $\mathbb{S}(=\mathbb{S}(0)$, we may differentiate (3.8) with respect to ϵ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\hat{V}(\epsilon) = \int_{\mathcal{R}} \left\{ \frac{\partial W}{\partial F_{ij}} \frac{\partial u_i}{\partial \xi_k} \frac{\partial}{\partial \epsilon}\big(H_{kj}^{-1}\big) \det \mathbf{H} + W \frac{\partial}{\partial \epsilon} (\det \mathbf{H}) \right\} \mathrm{d}v_{\xi}. \tag{3.9}$$

On using (3.7) and a standard formula to calculate $\partial\mathbf{H}^{-1}/\partial\epsilon$, $\partial(\det \mathbf{H})/\partial\epsilon$, (for example see Chapter 1 of [14]), (3.9) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\hat{V}(\epsilon) = \int_{\mathcal{R}} \left\{ \frac{\partial W}{\partial F_{ij}} \frac{\partial u_i}{\partial \xi_k} \left[\frac{1}{\det \mathbf{H}} \epsilon_{kqm}\epsilon_{jpn} H_{nm} - H_{kj}^{-1} H_{qp}^{-1} \right] \frac{\partial \delta x_p}{\partial \xi_q} + W H_{qp}^{-1} \frac{\partial \delta x_p}{\partial \xi_q} \right\} \det \mathbf{H} \mathrm{d}v_{\xi}, \tag{3.10}$$

where ϵ_{ijk} is the permutation symbol. Setting $\epsilon = 0$ in (3.10) gives

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\hat{V}\bigg|_{\epsilon=0} = \int_{\mathcal{R}} \left[W\delta_{pq} - \frac{\partial W}{\partial F_{iq}} \frac{\partial u_i(\mathbf{x})}{\partial x_p} \right] \frac{\partial \delta x_p(\mathbf{x})}{\partial x_q} \mathrm{d}v_x, \tag{3.11}$$

since we now have $\mathbf{H} = \mathbf{1}$ and $\boldsymbol{\xi} = \mathbf{x}$. The argument of W in (3.11) is $\mathbf{1} + \nabla\mathbf{u}(\mathbf{x})$. The divergence theorem, $(3.7)_1$ and (3.11) yield

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\hat{V}\bigg|_{\epsilon=0} = \int_{\mathbb{S}} \left[W\delta_{pj} - \frac{\partial W}{\partial F_{ij}} \frac{\partial u_i}{\partial x_p} \right]_+^- N_j \delta x_p \mathrm{d}A_x - \int_{\mathcal{R}} \frac{\partial}{\partial x_j} \left[W\delta_{pj} - \frac{\partial W}{\partial F_{ij}} \frac{\partial u_i}{\partial x_p} \right] \delta x_p \mathrm{d}v_x,$$

which by assumption vanishes for all admissible variations $\delta\mathbf{x}$. Here $\mathbf{N}$ is the unit normal to $\mathbb{S}$ which is directed into its positive side. Consequently it follows that

$$\left[W\delta_{pj} - \frac{\partial W}{\partial F_{ij}} \frac{\partial u_i}{\partial x_p} \right]_+^- N_j = 0 \qquad \text{on} \qquad \mathbb{S},$$

$$\frac{\partial}{\partial x_j} \left[W\delta_{pj} - \frac{\partial W}{\partial F_{ij}} \frac{\partial u_i}{\partial x_p} \right] = 0 \qquad \text{on} \qquad \mathfrak{R} - \mathbb{S}. \tag{3.12}$$

The equilibrium equations $(3.4)_2$ ensure that the field equations $(3.12)_2$ hold automatically, whereas the continuity of tractions $(3.4)_3$ allow the first of (3.12) to be written as

$$[W\mathbf{1} - \mathbf{F}^T\boldsymbol{\sigma}]_-^+ \mathbf{N} = \mathbf{0} \qquad \text{on} \qquad \mathbb{S}. \tag{3.13}$$

Here we have set $\boldsymbol{\sigma} = W_{\mathbf{F}}$ and $\mathbf{F} = \mathbf{1} + \nabla \mathbf{u}$. Equation (3.13) is a *jump condition which must necessarily be satisfied* if the displacement field $\mathbf{u}$ is to minimize the potential energy over the assumed class of virtual displacements. In the terminology of the calculus of variations, the jump conditions $(3.4)_3$, (3.13) are the *Weierstrass-Erdmann corner conditions* (see * Section 15 of [13]).

4. Discussion

An elastostatic field which involves equilibrium shocks must necessarily satisfy the appropriate field equations, jump conditions and boundary conditions described in Section 2. If this field is also to minimize the associated Potential Energy of the body it must, in addition, conform to the Weierstrass-Erdmann corner condition (3.13). In view of the known (and conjectured) relation between the stability of an equilibrium configuration and its role as a minimizer of the Potential Energy, see pages 195–206 of [19], one might require that all physically admissible equilibrium shocks should necessarily conform to the additional jump condition (3.13).

We note that $\mathbf{P} = W\mathbf{1} - \mathbf{F}^T\boldsymbol{\sigma}$ is the so-called energy-momentum tensor and (3.13) written as $[\mathbf{P}]_-^+\mathbf{N} = \mathbf{0}$, may be interpreted as requing that the "force per unit area" (in the sense of Eshelby [17]) on the shock-surface $\mathbb{S}$ be zero.

Next, taking the dot product of (3.13) with the unit normal $\mathbf{N}$ leads to

$$[W - \mathbf{FN}\cdot\boldsymbol{\sigma}\mathbf{N}]_-^+ = 0 \text{ on } \mathbb{S}. \tag{4.1}$$

From this and (2.8) we conclude that a "stable" equilibrium shock is necessarily *dissipation-free*. ** It is not difficult to show that in view of displacement and traction continuity across $\mathbb{S}$, (3.13) is in fact equivalent to (4.1). In order to show this, we note that the vector condition (3.13) is equivalent to the three scalar conditions obtained by

* Nemat-Nasser [18] has considered a variational approach to nonlinear problems with discontinuous fields. The primary interest in this study was in composite materials and consequently the surfaces of strain discontinuity (corresponding to the interfaces between the different material) were located at *fixed* positions with respect to the undeformed body. As a result he had $\mathbb{S}(\epsilon) = \mathbb{S}$ throughout his analysis and does not arrive at the jump condition (3.13).

** The analysis and results here are, of course, based on the assumption that the system here can be characterized by a potential energy functional and that this is given by (3.1). Additional effects, such as "surface energy" associated with the shock surface, would undoubtedly change the results.

taking its dot product with any three linearly independent vectors. For this purpose choose $\mathbf{N}$, $\mathbf{L}_1$ and $\mathbf{L}_2$ where $\mathbf{L}_\alpha$ ($\alpha = 1, 2$) are two vectors tangent to $\mathbb{S}$. Then, (3.13) leads to (4.1) and $[\mathbf{F}\mathbf{L}_\alpha \cdot \boldsymbol{\sigma}\mathbf{N}]_-^+ = 0$. Since the continuity of displacements imply that $[\mathbf{F}\mathbf{L}_\alpha]_-^+ = \mathbf{0}$, it follows from $(2.7)_3$ that the latter holds automatically. This establishes our claim.

Calculations similar to those performed here may be carried out in the case of *incompressible* elastic materials. These result in the identical jump condition (4.1).

In the special case of *anit-plane shear* the displacement vector only has a component in the x_3-direction, $u_3 = u(x_1, x_2)$. If the material is incompressible and has an elastic potential W depending solely on the first invariant I_1 of the deformation, $W = W(I_1)$, $I_1 = \mathrm{tr}\mathbf{F}\mathbf{F}^{\mathrm{T}}$, one finds [20] that

$$\begin{aligned}
&F_{\alpha\beta} = \delta_{\alpha\beta}, \quad F_{3\alpha} = u,_{\alpha} \quad F_{\alpha 3} = 0, \quad F_{33} = 1,\\
&I_1 = 3 + |\nabla u|^2,\\
&\sigma_{\alpha\beta} = \left[2W'(I_1) - p\right]\delta_{\alpha\beta}, \quad \sigma_{3\alpha} = 2W'(I_1)u,_{\alpha}, \quad \sigma_{\alpha 3} = pu,_{\alpha},\\
&\sigma_{33} = 2W'(I_1) - p, \quad p = 2W'(I_1) + d_0 u + d_0 x_3 + d_1.
\end{aligned} \tag{4.2 *}$$

Consequently, the "dissipation-free condition" (4.1) reads

$$\left[W(3 + |\nabla u|^2) - 2W'(3 + |\nabla u|^2)\left(\frac{\partial u}{\partial n}\right)^2\right]_-^+ = 0 \quad \text{on} \quad \mathbb{S}, \tag{4.3}$$

where $\partial u/\partial n$ denotes the derivative of u in a direction normal to the cylindrical shock-surface $\mathbb{S}$. Continuity of traction and displacements require

$$\left[2W'(3 + |\nabla u|^2)\frac{\partial u}{\partial n}\right]_-^+ = \left[\frac{\partial u}{\partial s}\right]_-^+ = 0, \tag{4.4}$$

respectively, with $\partial u/\partial s$ being a tangential derivative of u along $\mathbb{S}$. Consider a particular point on the shock-surface $\mathbb{S}$ and define the function $\hat{\tau}(k)$ at that point by

$$\hat{\tau}(k) = 2kW'\left(3 + k^2 + \left(\frac{\partial u}{\partial s}\right)^2\right), \qquad -\infty < k < \infty. \tag{4.5}$$

This function is apparently a "stress response function" which in the special case when $\partial u/\partial s = 0$ reduces to the shear stress response function in simple shear. The jump condition (4.3) can now be written in the form

$$\int_{\partial\bar{u}/\partial n}^{\partial\overset{+}{u}/\partial n} \hat{\tau}(k)\mathrm{d}k = \hat{\tau}\left(\frac{\partial\overset{+}{u}}{\partial n}\right)\left(\frac{\partial\overset{+}{u}}{\partial n} - \frac{\partial\bar{u}}{\partial n}\right). \tag{4.6}$$

resulting in the particularly simple interpretation that the area under the graph of $\hat{\tau}$(k) between k $= \partial\overset{+}{u}/\partial n$ and $k = \partial\bar{u}/\partial n$ must equal the area of the rectangle which lies on the same base and has height $\hat{\tau}(\partial\overset{+}{u}/\partial n)$ $(= \hat{\tau}(\partial\bar{u}/\partial n))$. If $\partial u/\partial s$ happens to vanish at this point (as is for example the case in one dimensional or purely axi-symmetric problems) the graph referred to here is simply the "shear stress-amount of shear" response curve of the material. We *emphasize* however that in general $\hat{\tau}(k)$ is not a function of the material alone but rather depends on the local orientation of the shock-surface as well.

Alternatively, one may interpret the dissipation-free condition (4.3) in terms of the

* d_0, d_1 are constants.

energy function $\hat{W}(k)$

$$\hat{W}(k) = W\left(3 + k^2 + \left(\frac{\partial u}{\partial s}\right)^2\right), \qquad -\infty < k < \infty \tag{4.7}$$

defined at the particular point on $\mathbb{S}$ under consideration. The first of (4.4) now states that the tangents to the curve $\hat{W}(k)$ vs. k at $k = \partial \vec{u}/\partial n$ and $k = \partial \bar{u}/\partial n$ are parallel, while (4.3) goes on to require that they, in fact, coincide.

Knowles and Sternberg [6] and Abeyaratne [7] have examined the anti-plane shear deformation field near the tip of a crack for certain classes of elastic materials which suffer a loss of ellipticity at sufficiently severe deformation levels. In these studies, they found deformation fields which involved equilibrium shocks issuing from the crack-tips [6] or points on the crack-faces [7] and terminating in the interior of the body. It is not difficult to show that these equilibrium fields do *not* conform to the dissipation-free condition (4.1) and therefore are presumably unstable.

Finally, it is interesting to note that as a consequence of (3.13) the **J**-integral defined by

$$\mathbf{J} = \int_{\Sigma} (W\mathbf{1} - \mathbf{F}^T\boldsymbol{\sigma})\mathbf{N}\,\mathrm{d}A \tag{4.8}$$

on some arbitrary closed surface Σ (containing points in $\mathcal{R}$ only) will vanish *even if* Σ intersects a stable equilibrium shock. This would, of course, not be the case if the jump condition (3.13) were not imposed. In fracture mechanics, a common technique of determining the stress-field near the tip of a crack is to perform an asymptotic calculation in the vicinity of the crack-tip and to subsequently utilize the **J**-integral to determine the parameter left undetermined by the asymptotic analysis (Rice [21]). One would, of course, not be able to use this technique in the presence of equilibrium shocks since, in general, the **J**-integral no longer vanishes on a surface intersecting a shock. However, we see here that if one were to admit only equilibrium shocks which conform to the dissipation-free condition (3.13), one *could* in fact use such a technique.

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