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On the Dynamic Vibrations of an Elastic Beam in Frictional Contact with a Rigid Obstacle

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Abstract. Existence and uniqueness results are established for weak formulations of initial-boundary value problems which model the dynamic behavior of an Euler–Bernoulli beam that may come into frictional contact with a stationary obstacle. The beam is assumed to be situated horizontally and may move both horizontally and vertically, as a result of applied loads. One end of the beam is clamped, while the other end is free. However, the horizontal motion of the free end is restricted by the presence of a stationary obstacle and when this end contacts the obstacle, the vertical motion of the end is assumed to be affected by friction. The contact and friction at this end is modelled in two different ways. The first involves the classic Signorini unilateral or nonpenetration conditions and Coulomb's law of dry friction; the second uses a normal compliance contact condition and a corresponding generalization of Coulomb's law. In both cases existence and uniqueness are established when the beam is subject to Kelvin–Voigt damping. In the absence of damping, existence of a solution is established for a problem in which the normal contact stress is regularized.

Key words: frictional contact, Euler–Bernoulli beam, Signorini unilateral condition, dynamic vibrations, Coulomb's law of dry friction, normal compliance, Kelvin–Voigt viscoelastic law.

Mathematics Subject Classifications (1991): primary 35L85; secondary 73F15, 73K05, 73T05.

1. Introduction

Problems involving contact and friction phenomena have received a great deal of attention in recent years and by now there is a considerable body of engineering literature devoted to this subject. In contrast, there are relatively few general mathematical results available in this area, due to the substantial difficulties encountered in establishing existence results for initial-boundary value problems that model these phenomena. Moreover, in both cases, most of the existing literature deals with static situations, or, occasionally with a sequence of static problems, which arise from the time discretization of an evolution problem. Modelling and mathematical analysis of such problems can be found in Duvaut and Lions [5], Moreau et al. [22], Kikuchi and Oden [11], and Telega [32], and the references therein (see also Curnier [4]). There are, however, some recent results on quasistatic and dynamic behavior in Andersson [1], Telega [33], Klarbring et al. [15], and Oden and Martins [24].

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The classic approach to modelling the frictional contact of elastic bodies employs Signorini's nonpenetration condition and Coulomb's law of dry friction. But, as has been pointed out in [5, 14, 15, 24, 32, 33] and elsewhere, there are both physical and mathematical difficulties associated with the presence of these conditions in models of dynamic contact phenomena. From the physical point of view, unilateral contact conditions seem unrealistic except for very smooth surfaces, as they assert that there is no mutual penetration of the contacting bodies. Mathematically, the Signorini–Coulomb conditions lead to initial-boundary value problems for which the existence of solutions has only been shown in some special cases, e.g., ([8], [9], [23] and the references in [21]). The work of Duvuat and Lions [5, Chapter III] especially illustrates the mathematical difficulties that may be encountered in handling dynamic problems in this area, even in situations where the friction bound is treated as a prescribed function.

In an effort to overcome these difficulties, which are often reflected in the behavior of numerical solutions obtained from algorithms that are based on these two conditions (see, e.g., Raous et al. [27]), some investigators have introduced alternative models for the contact interface. Oden and Martins [24], for example, have proposed a model where the contact interface has a normal compliance characterized by a power-law relationship between the normal pressure and the penetration. A similar generalization of Coulomb's law was also proposed. The computational, theoretical and experimental justification for these conditions has been developed in a series of papers (see, e.g., [10, 14, 15, 16, 21, 25, 34]) and there is by now a considerable body of work devoted to this topic.

In the present paper we incorporate these normal compliance conditions into a model for the dynamic vibrations of an Euler–Bernoulli beam that is in frictional contact with an obstacle. The model takes into account both horizontal and vertical displacements of locations along the beam. Well-posedness results are obtained for a weak formulation of an initial-boundary value problem containing a constitutive relation that includes Kelvin–Voigt damping. More surprisingly, we can also show the existence and uniqueness of a problem that employs the Signorini–Coulomb conditions and Kelvin–Voigt damping. In both cases it possible to separate the problem of finding the horizontal displacement from the problem of finding the vertical displacement. The horizontal displacement problem is solved first and then the normal contact stress so obtained enters the friction functional in the vertical displacement problem. It is this latter fact that presents the greatest mathematical difficulties, particularly in the problem involving the Signorini–Coulomb conditions, where the normal contact possesses minimal regularity. Nevertheless, we can show the existence of a weak solution to the vertical displacement problem, using a crucial fractional Sobolev space estimate derived from the Fourier transform and interpolation. In the absence of damping we can also show the existence of a solution to an Euler–Bernoulli beam problem that employs the Signorini–Coulomb conditions and a regularized normal contact stress.

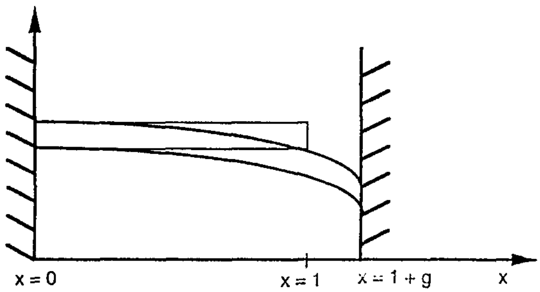

Figure 1. The reference configuration and the deflected beam.

We can now describe the remaining sections of this paper. Section 2 contains a description of the models and statements of the main results. Section 3 is devoted to establishing the well-posedness of a vertical displacement problem for a viscoelastic beam with a prescribed distributional friction bound. The proof uses such classic techniques as Galerkin approximation, convex regularization, interpolation, and compactness. By using this result the proofs of the main results are completed in Section 4.

2. The Models and Statements of Results

In this section, we present models for the dynamic evolution of a viscoelastic beam in frictional contact with a rigid obstacle. The beam is attached to a wall as its left end, but its right end is free to come into frictional contact with a rigid obstacle situated some distance to the right. The physical setting is depicited in Figure 1.

We assume that the area-center of gravity of the beam in its (stress free) reference configuration coincides with the interval $0 \leqslant x \leqslant 1$. We let $g > 0$ denote the initial gap between the end $x = 1$ and the obstacle. For $T > 0$, we set $\Omega_T = (0,1)\times(0,T)$, and let $u = u(x,t)$ and $v = v(x,t), (x,t) \in \Omega_T$ represent the horizontal and vertical displacements of the beam at location x and time t.

Let $\sigma_N = \sigma_N(x,t)$ be the contact pressure at (x,t) and let $\sigma_T = \sigma_T(x,t)$ be the shear stress at (x,t). Then the equations of motion in nondimensional units take the form

$$u_{tt} - (\sigma_N)_x(x,t) = k, \tag{2.1}$$

$$v_{tt} - (\sigma_T)_x(x,t) = f, \tag{2.2}$$

where f and k denote the vertical and horizontal applied forces, respectively. Here we have normalized the equations so that the coefficients of the acceleration terms are 1. We take for our constitutive relationships the Kelvin–Voigt viscoelasticity laws

$$\sigma_N(x,t) = au_x(x,t) + c_1 u_{xt}(x,t), \quad (x,t) \in \Omega_T, \tag{2.3}$$

and

$$\sigma_T(x,t) = -(v_{xxx}(x,t) + c_2 v_{xxxt}(x,t)), \quad (x,t) \in \Omega_T. \tag{2.4}$$

Here a is the coefficient of elasticity and c_1 and c_2 represent the viscosities in the horizontal and vertical directions, respectively. To complete the model we must include appropriate initial and boundary conditions. The initial conditions take the form

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad 0 \leqslant x \leqslant 1, \tag{2.5}$$

$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad 0 \leqslant x \leqslant 1, \tag{2.6}$$

where u_0, u_1, v_0 and v_1 are prescribed functions. Since we suppose that the beam is rigidly attached at its left end, we set

$$u(0,t) = 0 \tag{2.7}$$

$$v(0,t) = 0, \qquad v_x(0,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{2.8}$$

For the boundary conditions at the free end we consider several alternative conditions. For the horizontal displacement the first of these is the classic Signorini nonpenetration conditions

$$u(1,t) \leqslant g, \qquad \sigma_N(1,t) \leqslant 0, \quad \text{and}$$

$$\sigma_N(1,t)(g - u(1,t)) = 0, \quad 0 \leqslant t \leqslant T. \tag{2.9}$$

As was previously mentioned there are both physical and mathematical difficulties associated with the inclusion of this condition in mathematical models of frictional contact. Consequently, following [14] and [21], we shall also consider the alternative normal compliance condition

$$\sigma_N(1,t) = -c_N(u(1,t) - g)_+^{m_N}, \quad 0 \leqslant t \leqslant T, \tag{2.10}$$

where c_N and m_N are two positive constants, and $(u(1,t) - g)_+ = \max\{u(1,t) - g, 0\}$ represents the positive part of the function $u(1,t) - g$. We note that in both (2.9) and (2.10) that when there is no contact, i.e., when $u(1,t) < g$, then $\sigma_N(1,t) = 0$. However, (2.10) permits the contacting end to penetrate the obstacle, i.e., it permits $u(1,t) > g$. (2.9) may be thought of as a limiting case of (2.10) as c_N tends to infinity and for this reason it is typical to choose c_N large (see, e.g., [16]).

For the vertical displacement problem we assume that the sum of the moments acting on the free end is zero, i.e.,

$$v_{xx}(1,t) + c_2 v_{xxt}(1,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{2.11}$$

and we also include a friction law of the form

$$|\sigma_T(1,t)| \leqslant h(t), \tag{2.12}$$

$$\text{if } |\sigma_T(1,t)| = h(t) \quad \text{then} \quad v_t(1,t) = -\lambda\sigma_T(1,t), \text{ for some } \lambda \geqslant 0, \tag{2.13}$$

$$\text{if } |\sigma_T(1,t)| < h(t) \quad \text{then} \quad v_t(1,t) = 0, \tag{2.14}$$

for $0 \leqslant t \leqslant T$. Physically, the function $h(t)$ may be thought of as a friction bound and the conditions (2.12)–(2.14) interpreted in the following way: when the shear stress σ_T equals $\pm h$, then the shear will be in the direction opposite to the slip, and when the shear is strictly less (in absolute value) than the friction bound, then the end sticks to the obstacle.

The function h can be chosen in at least three different ways. One way is to treat h as a prescribed function. With this choice, equations (2.2), (2.4), (2.6), (2.8), (2.11)–(2.14) become an independent model for the transverse vibrations of a viscoelastic beam in frictional contact with a rigid obstacle. In Section 3 of this paper will show that such problems have weak solutions provided h is in $H^{-\varepsilon}(0,T)$, for some $0 < \varepsilon < \frac{1}{8}$. A second way of treating h is to let

$$h(t) = -\mu\sigma_N(1,t), \quad 0 \leqslant t \leqslant T, \tag{2.15}$$

where μ is a positive constant, called the coefficient of friction. With this choice, the conditions (2.12)–(2.15) constitute the classic Coulomb law of dry friction. Of course this law may be used in combination with either the Signorini condition (2.9) or the normal compliance condition (2.10). However, in the latter case, we may also allow for a more general h of the form

$$h(t) = c_T(u(1,t) - g)_+^{m_T}, \quad 0 \leqslant t \leqslant T, \tag{2.16}$$

where c_T and m_T are two positive constants which may be chosen independently of c_N and m_N. This completes our description of the various boundary conditions we will consider.

The alert reader will have noticed that equations (2.1), (2.3), (2.5), (2.7), and either (2.9) or (2.10) constitute an initial-boundary value problem for the horizontal displacement u that can be solved independently of the vertical displacement v. Indeed, problems of this type have already been considered in [12, 13, 17, 20, 21, 28] and we will use those results in Section 4 of this paper. The main interest in this paper thus lies in the vertical displacement problem which will be treated in Section 3.

Now, as is well known, a friction law of the form (2.12)–(2.14) imposes a regularity ceiling which, generally, precludes the existence of classical solutions to problems containing this boundary condition. Thus, it is natural to consider weak, or variational inequality, formulations of the above equations. We will give two such formulations in this section. Toward that end we introduce the following spaces and notation. (For definitions of any unexplained notation we refer the reader to [18] or [19]).

Let $H = L^2(0,1), E = \{w \in H^1(0,1) : w(0) = 0\}$ and $V = \{w \in H^2(0,1) : w(0) = w'(0) = 0\}$. Clearly we have

$$V \subseteq E \subseteq H = H' \subseteq E' \subseteq V',$$

where E' and V' are, respectively the topological duals of E and V.

Our first formulation incorporates the normal compliance condition (2.10) and the generalized Coulomb condition (2.15) and so it is convenient to introduce

$$j_N(u,w) = c_N \int_0^T (u(1,t) - g)_+^{m_N} w(1,t)\,\mathrm{d}t, \tag{2.17}$$

the '*normal compliance*' functional, and

$$j_T(u,w) = c_T \int_0^T (u(1,t) - g)_+^{m_T} |w(1,t)|\,\mathrm{d}t, \tag{2.18}$$

the '*friction*' functional. Note that if $m_N \geqslant 1$ and $m_T \geqslant 1$, then both functionals are defined and convex on $L^2(0,T;E) \times L^2(0,T;E)$ but that, unlike $j_N(u,\cdot)$, $j_T(u,\cdot)$ is not Gâteaux differentiable.

We can now give our first weak formulation, which is obtained in the usual way by multiplying (2.1)–(2.2) by suitable test functions and integrating by parts.

DEFINITION 2.1. A pair of functions $(u,v) \in L^2(0,T;E) \times L^2(0,T;V)$ is said to be a weak solution to (2.1)–(2.8), (2.10), provided that

$$u_t \in L^2(0,T;E), \qquad u_{tt} \in L^2(0,T;E'), \quad u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \tag{2.19}$$

$$v_t \in L^2(0,T;V), \qquad v_{tt} \in L^2(0,T;V'), \quad v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1, \tag{2.20}$$

and for each $\varphi \in L^2(0,T;E)$ and $w \in L^2(0,T;V)$

$$\begin{aligned} &\int_0^T \langle u_{tt}, \varphi\rangle\,\mathrm{d}t + a\int_0^T (u_x, \varphi_x)\,\mathrm{d}t + \\ &\quad + c_1 \int_0^T (u_{xt}, \varphi_x)\,\mathrm{d}t + j_N(u,\varphi) = \int_0^T \langle k, \varphi\rangle\,\mathrm{d}t, \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\int_0^T \langle v_{tt}, w - v_t\rangle\,\mathrm{d}t + \int_0^T (v_{xx}, w_{xx} - v_{xxt})\,\mathrm{d}t + \\ &\quad + c_2 \int_0^T (v_{xxt}, w_{xx} - v_{xxt})\,\mathrm{d}t + \end{aligned}$$

$$+j_T(u,w) - j_T(u,v_t)$$

$$\geqslant \int_0^T (f, w - v_t)\,\mathrm{d}t. \tag{2.22}$$

In the integrands contained in (2.21) and (2.22), $\langle\cdot,\cdot\rangle$ denotes respectively the duality pairing between E and E' and between V and V' while in both cases $(\cdot,\cdot)$ denotes the inner product in H.

We have the following existence and uniqueness result for the above problem.

THEOREM 2.2. *Let $c_1 \geqslant 0, a > 0$, and $c_2 > 0$. Let $k, k_t \in L^2(0,T;E'), f \in L^2(\Omega_T), u_0 \in E, v_0 \in V, u_1, v_1 \in H$, and $m_N \geqslant 1, m_T \geqslant 1$. Then there exists a unique solution to problem* (2.19)–(2.22), *provided that in the case when $c_1 = 0$, the requirement in* (2.21) *that $u_t \in L^2(0,T;E)$ is replaced by $u_t \in L^2(0,T;H)$.*

As previously indicated, the unique solvability of (2.19) and (2.21) when $c_1 = 0$ is established in [21, Theorem 4.1]. However, the proof given there can be modified in obvious ways to obtain the result for $c_1 > 0$. The unique solvability of (2.20) and (2.22), is proved in Section 4 by using the result of Section 3.

We have, in addition, the following stability result.

THEOREM 2.3. *Let $(y_i, z_i), i = 1, 2$, be two solutions to* (2.19)–(2.22) *corresponding to initial data $(y_{i0}, y_{i1}) \in E \times H$ and $(z_{i0}, z_{i1}) \in V \times H$ and applied forces $(f_i, k_i) \in L^2(\Omega_T) \times L^2(\Omega_T)$. Then there is an absolute constant $C > 0$, and a constant $C_1 > 0$ which only depends boundedly on the norms of the data for the horizontal problem, such that*

$$\begin{aligned}
\|y_1 - y_2\|_{H^1(\Omega_T)} &\leqslant C_1(\|k_1 - k_2\|_{L^2(\Omega_T)} + \|y_{10}' - y_{20}'\|_H + \|y_{11} - y_{21}\|_H),\\
\|z_1 - z_2\|_{W^{2,1}(\Omega_T)} &\leqslant C_1(\|k_1 - k_2\|_{L^2(\Omega_T)} + \|y_{10}' - y_{20}'\|_H + \|y_{11} - y_{21}\|_H)\\
&\quad + C(\|f_1 - f_2\|_{L^2(\Omega_T)} + \|z_{10}'' - z_{20}''\|_H + \|z_{11} - z_{21}\|_H).
\end{aligned}$$

The proof of this theorem is given in Section 4.

Our second formulation incorporates the Signorini condition (2.9) and the friction law (2.12)–(2.14). For this purpose it is convenient to set

$$K = \{f \in E : f(1) \leqslant g\}.$$

We also recall the scale of Hilbert spaces of distributions $H^\varepsilon(0,T), \varepsilon \in (-\infty, \infty)$, as defined in [19, Vol. I] and introduce, for $\varepsilon \geqslant 0$, the functional

$$J(z) = \langle h, |z(1,\cdot)|\rangle_{-\varepsilon,\varepsilon},$$

where $h \in H^{-\varepsilon}(0,T)$ and $\langle\cdot,\cdot\rangle_{-\varepsilon,\varepsilon}$ denotes the duality pairing between $H^\varepsilon(0,T)$ and $H^{-\varepsilon}(0,T)$.

The variational formulation of the problem with Signorini and Coulomb boundary conditions is as follows.

DEFINITION 2.4. Let $\varepsilon \geqslant 0$, and suppose $h \in H^{-\varepsilon}(0,T)$. A pair of functions $(u,v) \in L^2(0,T;K) \times L^2(0,T;V)$ is said to be a weak solution to (2.1)–(2.9), (2.11)–(2.14) provided that

$$u_t \in L^2(0,T;E), \quad u_{tt} \in L^2(0,T;E'), \quad u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \tag{2.23}$$

$$v_t \in L^2(0,T;V), \quad v_{tt} \in H^{-\varepsilon}(0,T;V'), \quad v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1, \tag{2.24}$$

and for each $\varphi \in L^2(0,T;K)$ and $w \in H^{\varepsilon}(0,T;V)$,

$$\begin{aligned}&\int_0^T \langle u_{tt}, \varphi - u\rangle \, dt + a\int_0^T (u_x, \varphi_x - u_x)\, dt\\ &\quad + c_1 \int_0^T (u_{xt}, \varphi_x - u_x)\, dt \geqslant \int_0^T (k, \varphi - u)\, dt,\end{aligned} \tag{2.25}$$

$$\begin{aligned}&\int_0^T \langle v_{tt}, w - v_t\rangle \, dt + \int_0^T (v_{xx}, w_{xx} - v_{xxt})\, dt\\ &\quad + c_2 \int_0^T (v_{xxt}, w_{xx} - v_{xxt})\, dt\\ &\quad + J(w) - J(v_t)\\ &\geqslant \int_0^T (f, w - v_t)\, dt.\end{aligned} \tag{2.26}$$

We have the following existence and uniqueness result.

THEOREM 2.5. *Let* $c_1 \geqslant 0, a > 0$, *and* $c_2 > 0$. *Let* $k \in L^2(\Omega_T), f \in L^2(\Omega_T), u_0 \in K, v_0 \in V$ *and* $u_1, v_1 \in H$. *If* $h = -\mu\sigma_N(1,\cdot)$, *where* μ *is a positive constant and* σ_N *is given by* (2.3), *then there exists a unique solution to problem* (2.23)–(2.26), *provided that in the case when* $c_1 = 0$ *the requirement in* (2.23) *that* $u_t \in L^2(0,T;E)$ *is replaced by* $u_t \in L^2(0,T;H)$.

Once again the solution to the horizontal problem (2.23) and (2.25) can be found in [12], [13], or [28] for the case when $c_1 = 0$ and in [17] for the case when $c_1 > 0$. Hence, the main interest lies in establishing the solution to the vertical displacement problem (2.24) and (2.26). This will be done in Section 4.

Finally, we consider a problem which causes the standard Signorini–Coulomb conditions without Kelvin–Voigt damping. This is the kind of model for frictional contact most frequently employed in engineering applications, despite the difficulties discussed in Section 1. In this setting we find it necessary to introduce a positive regularization operator, i.e., an operator $R : L^2(0,T) \to H^2_L(0,T)$ such that $R(\nu) \geqslant 0$ if $\nu \geqslant 0$ and such that there exists an absolute constant $C > 0$ so that

$$\|R(\nu)\|_{H^2_L(0,T)} \leqslant C\|\nu\|_{L^2(0,T)}$$

for each $\nu \in L^2(0,T)$. Here $H^2_L(0,T) = \{w \in H^2(0,T) : w(0) = 0\}$. One way to construct such an operator is by using the convolution with a positive C^∞ kernel (see, e.g., [2] or [7]).

We now have the following result.

THEOREM 2.6. *Let* $c_1 = c_2 = 0$ *and* $a > 0$. *Let* $k \in L^2(\Omega_T)$, $f \in H^1(0,T;H)$, $u_0 \in K$, $u_1 \in H$, $v_0 \in H^4(0,1)$, *and* $v_1 \in V$ *with* $v_0(0) = v_0'(0) = v_0''(1) = v_0'''(1) = 0$. *If* $h = R(-\mu\sigma_N(1,\cdot))$, *where* μ *is a positive constant and* σ_N *is given by* (2.3), *then there exists a solution to problem* (2.23)–(2.26). *Moreover,* v *can be chosen so that* $v \in W^{1,\infty}(0,T;V)$ *and* $v_{tt} \in L^\infty(0,T;H)$.

The proof of this theorem will also be given in Section 4.

We conclude this section by sketching what results can be obtained for a quasistatic formulation of the problem with Signorini–Coulomb conditions and no damping. In this formulation we omit the acceleration and damping terms from (2.1)–(2.4). Consequently, the problem for u becomes a pure boundary-value problem and by formally integrating (2.1) twice and using the boundary conditions (2.7) and (2.9) we arrive at the following explicit representation for u:

$$\begin{aligned} u(x,t) &= x \min\left\{g, -\int_0^1 yk(y,t)\,\mathrm{d}y\right\} \\ &\quad -x\int_0^1 (1-y)k(y,t)\,\mathrm{d}y + \int_0^x (x-y)k(y,t)\,\mathrm{d}y. \end{aligned} \tag{2.27}$$

Thus the calculation

$$\begin{aligned} h(t) &= -\mu\sigma_N(1,t) = -\mu u_x(1,t) \\ &= -\mu\left(\min\left\{g, -\int_0^1 yk(y,t)\,\mathrm{d}y\right\} + \int_0^1 yk(y,t)\,\mathrm{d}y\right) \end{aligned}$$

shows how h inherits its regularity properties from k. Similarly, by formally integrating (2.2) four times and using the boundary conditions (2.8) and (2.11) we obtain the following expression for v:

$$\begin{aligned} v(x,t) &= (1/6)(x^3 - 3x^2)v_{xxx}(1,t) \\ &\quad + \int_0^x \int_0^{x_4} \int_{x_3}^1 \int_{x_2}^1 f(x_1,t)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4. \end{aligned} \tag{2.28}$$

This expression implies that v lies in $H^1(0,T;H^4(0,1))$ provided $f \in H^1(0,T;H)$ and $v_{xxx}(1,t) \in H^1(0,T)$. It also implies that to achieve this degree of regularity the single initial condition v_0 must satisfy the compatability condition

$$v_0 = (1/6)(x^3 - 3x^2)v_0'''(1) + \int_0^x \int_0^{x_4} \int_{x_3}^1 \int_{x_2}^1 f(x_1, 0)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4.$$

Now differenting (2.28) we obtain

$$v_t = (1,t) = (-1/3)v_{xxxt}(1,t) + \int_0^1 \int_0^{x_4} \int_{x_3}^1 \int_{x_2}^1 f_t(x_1,t)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4.$$

By using this equation in combination with the friction conditions (2.12)–(2.14) and the initial value $v_0'''(1)$ it is possible to obtain $v_{xxx}(1,t)$ as the solution of a first order quasivariational inequality. Once $v_{xxx}(1,t)$ is obtained then (2.28) gives v. For an exact statement of the results and the hypotheses required we refer the reader to [29] and [30].

3. Solution of the Vertical Displacement Problem

The purpose of this section is to establish the following theorem.

THEOREM 3.1. *Let* $c_2 > 0$. *Let* $f \in L^2(\Omega_T), v_0 \in V, v_1 \in H$, *let* $0 < \varepsilon < 1/8$, *and let h be a nonnegative distribution in* $H^{-\varepsilon}(0,T)$. *Then there exists a unique* $v \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V) \cap [\cap_{\delta>0} W^{2,(3/2)-\delta}(\Omega_T)]$ *which satisfies*

$$v(\cdot,0) = v_0, v_t(\cdot,0) = v_1, \tag{3.1}$$

$$v_{xxt} \in L^2(\Omega_T), \tag{3.2}$$

$$v_{tt} \in H^{-\varepsilon}(0,T;V'), \tag{3.3}$$

and such that for all $w \in H^\varepsilon(0,T;V)$,

$$\begin{aligned}&\int_0^T \langle v_{tt}, w - v_t\rangle\,\mathrm{d}t + \int_0^T (v_{xx}, w_{xx} - v_{xxt})\,\mathrm{d}t\\ &\quad + c_2 \int_0^T (v_{xxt}, w_{xx} - v_{xxt})\,\mathrm{d}t + J(w) - J(v_t)\\ &\geqslant \int_0^T (f, w - v_t)\,\mathrm{d}t,\end{aligned} \tag{3.4}$$

where $J(z) = \langle h, |z(1,\cdot)|\rangle_{-\varepsilon,\varepsilon}$, *and* $\langle\cdot,\cdot\rangle_{-\varepsilon\varepsilon}$ *denotes the duality pairing between* $H^{-\varepsilon}(0,T)$ *and* $H^\varepsilon(0,T)$. *Moreover, the following stability result holds: if* $z_i, i = 1,2$ *are solutions in* $W^{2,1+\varepsilon}(\Omega_T)$ *of* (3.2) *and* (3.4) *corresponding to initial data*

$(z_{i0}, z_{i1}) \in V \times H$ *and applied forces* $f_i \in L^2(\Omega_T)$, *then there is an absolute constant* $C > 0$ *such that*

$$\|z_1 - z_2\|_{W^{2,1}(\Omega_T)} \leqslant C(\|f_1 - f_2\|_{L^2(\Omega_T)} + \|z_{10}'' - z_{20}''\|_H + \|z_{11} - z_{21}\|_H). \tag{3.5}$$

The proof of Theorem 3.1 will proceed in several stages, but the general idea is to use Galerkin's method to establish solutions to finite-dimensional versions of (3.4) in a setting where h is a nonnegative element of $L^1(0, T)$ and the functional J has undergone a convex regularization. We then establish apriori estimates which give us sufficient compactness to be able to pass to the limit and establish (3.4).

So we begin by assuming that h is a nonnegative element of $L^1(0, T)$. In this case, the functional J is given by

$$J(z) = \int_0^T h(t)|z(1, t)|\, \mathrm{d}t.$$

As was previously indicated, the functional J is not Gâteaux differentiable and so we introduce the convex resularization J_m of J given by

$$J_m(z) = \int_0^T h(t)\psi_m[z(1, t)]\, \mathrm{d}t,$$

where, for each positive integer m, the function ψ_m is defined by

$$\psi_m(s) = \begin{cases} |s| - (1/2m), & |s| \geqslant 1/m, \\ ms^2/2, & |s| \leqslant 1/m. \end{cases}$$

Note that the Gâteaux derivative of the functional J_m is given by

$$\langle J_m'(z), w\rangle = \int_0^T h(t)\psi_m'[z(1, t)]w(1, t)\, \mathrm{d}t,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on the boundary. We now proceed to create a sequence of finite-dimensional versions of (3.4) using J_m'. To this end, we select a sequence $\{z_n\}$ in $C^\infty(0, 1)$ with $z_j(0) = z_j'(0) = 0$ such that $\{z_n\}$ is an orthonormal basis for H whose linear span is strongly dense in V. Consequently, we may find $a_{m,j}$ and $b_{m,j}$ such that if

$$v_{0m} = \sum_{j=1}^m a_{m,j} z_j \quad \text{and} \quad v_{1m} = \sum_{j=1}^m b_{m,j} z_j,$$

then

$$v_{0m} \to v_0 \quad \text{strongly in } V, \tag{3.6}$$

$$v_{1m} \to v_1 \quad \text{strongly in } H. \tag{3.7}$$

Define the linear subspace $W_{T,m}$ of $H^{\varepsilon}(0,T;V)$ as the set of all functions of the form $\sum_{j=1}^{m} \phi_j z_j$, where $(\phi_1,\ldots,\phi_m)$ varies over all elements of $H^{\varepsilon}(0,T)^m$. We note that $W_{T,m} \subseteq W_{T,m+1}$, $m = 1,2,3,\ldots$. Consider now the problem of finding a function v_m of the form

$$v_m(x,t) = \sum_{j=1}^{m} c_j(t) z_j(x), \quad (x,t) \in \Omega_T,$$

for which

$$\begin{aligned}&(v_{mtt}, w) + (v_{mxx}, w_{xx}) + c_2(v_{mxxt}, w_{xx})\\ &\quad + h\psi_m'[v_{mt}(1,\cdot)]w(1,\cdot) = (f, w),\end{aligned} \tag{3.8}$$

for all $w \in W_{T,m}$, and which also satisfies the initial conditions

$$v_m(\cdot,0) = v_{0m}, \qquad v_{mt}(\cdot,0) = v_{1m}. \tag{3.9}$$

Problem (3.8)–(3.9) is equivalent to a system of second-order ordinary differential equations for the coefficients $c_1,\ldots,c_m$, which, as a consequence of Theorem 2.1.1 of [3], has a solution that exists on some interval $[0,t_m]$. The estimate (3.11) that we will obtain below together with Theorem 2.1.3 of [3] will show that these solutions in fact exist on $[0,T]$.

Now it follows from (3.8) that $v_{mt} \in W_{t_m,m}$, and so we may set $w = v_{mt}$ in (3.8) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v_{m\tau}(\cdot,t)\|_H^2 + \|v_{mxx}(\cdot,t)\|_H^2) + c_2\|v_{mxx\tau}(\cdot,t)\|_H^2\\ &\quad + h(t)\psi_m'[v_{m\tau}(1,t)]v_m(1,t)\\ &= (f(\cdot,t), v_{m\tau}(\cdot,t)).\end{aligned}$$

Integrating this equation from 0 to $t > 0$ yields, for all $t \in [0,t_m]$,

$$\begin{aligned}&\tfrac{1}{2}(\|v_{m\tau}(\cdot,t)\|_H^2 + \|v_{mxx}(\cdot,t)\|_H^2) + c_2\int_0^t\int_0^1 v_{mxx\tau}^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad + \int_0^t h\psi_m'[v_{m\tau}(1,\cdot)]v_{m\tau}(1,\cdot)\,\mathrm{d}\tau\\ &= \int_0^t\int_0^1 f v_{m\tau}\,\mathrm{d}x\,\mathrm{d}\tau + \tfrac{1}{2}(\|v_{1m}\|_H^2 + \|v_{0m}''\|_H^2)\end{aligned}$$

$$\leqslant C_m + \frac{1}{2}\int_0^t \int_0^1 v_{m\tau}^2 \, dx \, d\tau,$$

where

$$C_m = \tfrac{1}{2}(\|f\|^2_{L^2(\Omega_T)} + \|v_{1m}\|^2_H + \|v''_{0m}\|^2_H).$$

Observing that $\{C_m\}$ is a bounded sequence and that $s\psi'_m(s) \geqslant 0$, for all $s \in (-\infty, \infty)$, we conclude from (3.10) and Cauchy's and Gronwall's inequalities that there exists a constant $C > 0$, depending only on f, v_0, v_1 and T, such that

$$\|v_{mt}(\cdot, t)\|^2_H + \|v_{mxx}(\cdot, t)\|^2_H + 2c_2\|v_{mxxt}\|^2_{L^2(\Omega_T)} \leqslant C, \tag{3.11}$$

$\forall t \in [0, T], \forall m = 1, 2, 3, \ldots$. Therefore, the sequence $\{v_m\}$ is uniformly bounded in $W^{1,\infty}(0, T; H) \cap H^1(0, T; V)$. However, this elementary estimate does not provide sufficient compactness to allow us to pass to the limit in m and obtain (3.4). In order to do this we need to establish further estimates. The next lemma will assist us in this endeavor.

LEMMA 3.2. *Let $\phi \in L^2(-\infty, \infty) \cap L^\infty(-\infty, \infty)$. For each $\varepsilon \in [0, 1]$, ϕ defines a bounded operator $M_\phi : H^{-\varepsilon}(-\infty, \infty) \to H^{-\varepsilon}(-\infty, \infty)$, with operator norm not exceeding*

$$C_\varepsilon \max\{\|\phi\|_{L^\infty(-\infty,\infty)}, \|\phi\|_{L^2(-\infty,\infty)}\},$$

and which agrees with pointwise multiplication by ϕ on $L^2(-\infty, \infty)$ and $L^1(-\infty, \infty) \cap H^{-1}(-\infty, \infty)$.

Proof. We suppose first that $\varepsilon = 1$ and $\phi \in C_0^\infty(-\infty, \infty)$. Then, by [6, Theorem 9.3.5], multiplication by ϕ in S' defines a bounded operator M_ϕ of $H^{-1}(-\infty, \infty)$ into $H^{-1}(-\infty, \infty)$, with operator norm not exceeding

$$C\int_{-\infty}^{\infty} (1 + |s|)^{-1} |\tilde{\phi}(s)| \, ds \leqslant C\|\phi\|_{L^2(-\infty,\infty)}, \tag{3.12}$$

where $\tilde{\phi}$ denotes the Fourier transform of ϕ. For a general ϕ, we find a sequence $\{\phi_n\} \subseteq C_0^\infty(-\infty, \infty)$ with $\{\phi_n\}$ bounded in $L^\infty(-\infty, \infty)$, and for which $\phi_n \to \phi$ strongly in $L^2(-\infty, \infty)$, as well as pointwise almost everywhere on $(-\infty, \infty)$. We may thus define M_ϕ as the uniform limit of M_{ϕ_n} and conclude, by (3.12), that the operator norm of M_ϕ does not exceed a constant multiple of $\|\phi\|_{L^2(-\infty,\infty)}$. We also deduce from the strong and pointwise convergence that M_ϕ agrees with pointwise multiplication by ϕ on $L^2(-\infty, \infty)$ and on $L^1(-\infty, \infty) \cap H^{-1}(-\infty, \infty)$, and hence maps $H^0(-\infty, \infty) = L^2(-\infty, \infty)$ into $L^2(-\infty, \infty)$ with operator norm $\|\phi\|_{L^\infty(-\infty,\infty)}$. That M_ϕ maps $H^{-\varepsilon}(-\infty, \infty)$ into $H^{-\varepsilon}(-\infty, \infty)$ with the desired operator norm estimate for $\varepsilon \in (0, 1)$ now follows by interpolation [19, vol. I, Theorem 1.5.1 and Remark 1.7.5]. This completes the proof of Lemma 3.2.

Before proceeding further we point out the following consequence of Lemma 3.2. Let h be a function integrable on $(0,T)$ and denote by h_0 the function on $(-\infty,\infty)$ which agrees with h on $(0,T)$ and which vanishes outside of $(0,T)$. Then for each $\varepsilon \in [0,1]$, the $H^{-\varepsilon}(0,T)$-norm of h is equivalent to $\|h_0\|_{-\varepsilon}$. Hence, in what follows, we identify the $H^{-\varepsilon}(0,T)$-norm of h with $\|h_0\|_{-\varepsilon}$ for $\varepsilon \in [0,1]$.

LEMMA 3.3. *Let $c_2 > 0$. For each $\delta > 0, \{v_m\}$ is a bounded sequence in $W^{2,(3/2)-\delta}(\Omega_T)$.*

Proof. Extend h and f to $(0,+\infty)$ and $(0,1)\times(0,+\infty)$, respectively, by defining them to be 0 for $t > T$. Extend v_m to $(0,+\infty)$ by observing that the system of ordinary differential equations (3.8)–(3.9) can be solved on $(0,+\infty)$ so as to agree with v_m on $[0,T]$. Let v_m^* denote the even extension of v_m in t, i.e.

$$v_m^*(x,t) = \begin{cases} v_m(x,t), & t \geqslant 0, \\ v_m(x,-t), & t < 0. \end{cases}$$

Then

$$v_{mt}^* = \text{odd extension of } v_{mt} \text{ in } t, \tag{3.13}$$

$$v_{mtt}^* = 2v_{1m}\delta_0 + \text{even extension of } v_{mtt} \text{ in } t, \tag{3.14}$$

where δ_0 is the Dirac functional concentrated at $t=0$.

Let ϕ be a fixed element in the span of $\{z_1 \ldots, z_m\}$. Then by (3.8), (3.13), and (3.14),

$$\begin{aligned}(v_{mtt}^*,\phi) &+ 2\delta_0(v_{1m},\phi) + (v_{mxx}^*,\phi'') \\ &+ c_2(\operatorname{sgn}(\cdot)v_{mxxt}^*,\phi'') + h^*\psi_m'[\operatorname{sgn}(\cdot)v_{mt}^*(1,\cdot)]\phi(1) \\ &= (f^*,\phi),\end{aligned} \tag{3.15}$$

for $t \in (-\infty,\infty)$, where h^* and f^* are the even extensions in t of h and f, respectively.

Set

$$g_m = h^*\psi_m'[\operatorname{sgn}(\cdot)v_{mt}^*(1,\cdot)],$$

and let $\omega \in C_0^\infty(-\infty,\infty)$ satisfy $0 \leqslant \omega \leqslant 1, \omega \equiv 1$ on $(-1,T), \omega \equiv 0$ on $(-\infty,-2)\cup(T+1,+\infty)$. Then by (3.15),

$$\begin{aligned}((\omega v_{mt}^*)_t,\phi) &= (\omega' v_{mt}^*,\phi) + (\omega v_{mtt}^*,\phi) \\ &= (w' v_{mt}^*,\phi) + \omega[(f^*,\phi) - 2\delta_0(v_{1m},\phi) \\ &\quad -g_m\phi(1) - (v_{mxx}^*,\phi'') - c_2(\operatorname{sgn}(\cdot)v_{mxxt}^*,\phi'')].\end{aligned} \tag{3.16}$$

For a function g, we let $\tilde{g}$ denote its Fourier transform in t, i.e.,

$$\tilde{g}(x,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x,t)\, \mathrm{e}^{ist}\, \mathrm{d}t, \quad s \in (-\infty, \infty).$$

We take the Fourier transform of (3.16) to obtain

$$\begin{aligned} -is([\omega v_{mt}^*]^{\sim}, \phi) &= ([\omega f^*]^{\sim}, \phi) + ([\omega' v_{mt}^*]^{\sim}, \phi) - ([\omega v_{mxx}^*]^{\sim}, \phi'') \\ &\quad - c_2([\omega \operatorname{sgn}(\cdot) v_{mxxt}^*]^{\sim}, \phi'') \\ &\quad -2(v_{1m}, \phi) - (\omega g_m)^{\sim} \phi(1), \end{aligned} \tag{3.17}$$

where we have used the fact that $\omega \delta_0(t) = \delta_0(t)$ since $\omega(0) = 1$.

Next, let $\delta > 0$ and, for each $s \in (-\infty, \infty)$, choose

$$\phi = i \operatorname{sgn}(s)(1 - \chi_{(-1,1)}(s))|s|^{-2\delta} \overline{(\omega v_{mt}^*)^{\sim}}(\cdot, s)$$

in (3.17), where $\chi_{(-1,1)}$ denotes the characteristic function of the interval $(-1, 1)$ and the bar over $(\omega v_{mt}^*)(\cdot, s)$ denotes the complex conjugate. We then integrate the equation which results over $-\infty < s < \infty$ to obtain

$$\begin{aligned} &\int_{|s| \geqslant 1} \int_0^1 |s|^{1-2\delta} |(w v_{mt}^*)^{\sim}|^2 \, \mathrm{d}x \, \mathrm{d}s \\ &= i \int_{|s| \geqslant 1} \int_0^1 \operatorname{sgn}(s) |s|^{-2\delta} (\omega f^*)^{\sim} \overline{(\omega v_{mt}^*)^{\sim}} \, \mathrm{d}x \, \mathrm{d}s \\ &\quad + i \int_{|s| \geqslant 1} \int_0^1 \operatorname{sgn}(s) |s|^{-2\delta} (\omega' v_{mt}^*)^{\sim} \overline{(\omega v_{mt}^*)^{\sim}} \, \mathrm{d}x \, \mathrm{d}s \\ &\quad - i \int_{|s| \geqslant 1} \int_0^1 \operatorname{sgn}(s) |s|^{-2\delta} (\omega v_{mxx}^*)^{\sim} \overline{(\omega v_{mxxt}^*)^{\sim}} \, \mathrm{d}x \, \mathrm{d}s \\ &\quad - c_2 i \int_{|s| \geqslant 1} \int_0^1 \operatorname{sgn}(s) |s|^{-2\delta} (\omega \operatorname{sgn}(\cdot) v_{mxxt}^*)^{\sim} \overline{(\omega v_{mxxt}^*)^{\sim}} \, \mathrm{d}x \, \mathrm{d}s \\ &\quad - 2i \int_{|s| \geqslant 1} \int_0^1 \operatorname{sgn}(s) |s|^{-2\delta} v_{1m} \overline{(\omega v_{mt}^*)^{\sim}} \, \mathrm{d}x \, \mathrm{d}s \\ &\quad - i \int_{|s| \geqslant 1} \operatorname{sgn}(s) |s|^{-2\delta} (\omega g_m)^{\sim} \overline{[\omega v_{mt}^*(1, \cdot)]^{\sim}} \, \mathrm{d}s \\ &= I_{m1} + \cdot + I_{m6}. \end{aligned} \tag{3.18}$$

We now proceed to estimate the first five terms on the right-hand side of (3.18). In what follows C will denote a constant whose value may change from line to line but which is always independent of m. It follows from (3.11) for T sufficiently large, the fact that the Fourier transform is an isometry of $L^2(-\infty,\infty)$, and the Schwartz inequality, that $\{I_{m1}\},\ldots,\{I_{m4}\}$ are all bounded with respect to m. (Note that in estimating $\{I_{m3}\}$ and $\{I_{m4}\}$ we have made crucial use of the presence of the viscosity term in (3.11)).

In the modulus of the integrand in I_{m5}, we factor $|s|^{-2\delta}$ as $|s|^{-(1/2)-\delta}|s|^{(1/2)-\delta}$, group the first and second factors with v_{1m} and $(\omega v^*_{mt})^\sim$, respectively, and then use the H-boundedness of $\{v_{1m}\}$ to obtain, via the Cauchy inequality, that

$$|I_{m5}| \leqslant C + \frac{1}{2}\int_{|s|\geqslant 1}\int_0^1 |s|^{1-2\delta}|(\omega v^*_{mt})^\sim|^2\,\mathrm{d}x\,\mathrm{d}s.$$

It is this estimate which requires the inclusion of the decay factor $|s|^{-2\delta}$ in the choice of ϕ above. Now since $\{\omega v_{mt}\}$ is bounded in $L^2((0,1)\times(-\infty,\infty))$, there is a constant C such that

$$\begin{aligned}&\int_{|s|\leqslant 1}\int_0^1 |s|^{1-2\delta}|(\omega v^*_{mt})^\sim|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\leqslant \int_{-\infty}^{\infty}\int_0^1 |(\omega v^*_{mt})^\sim|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C.\end{aligned}$$

Using this fact and the estimates obtained thus far in (3.18) we arrive at

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_0^1 |s|^{1-2\delta}|(\omega v^*_{mt})^\sim|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\leqslant C + 2\int_{-\infty}^{\infty} |(\omega g_m)^\sim|\,|[\omega v^*_{mt}(1,\cdot)]^\sim|\,\mathrm{d}s. \qquad (3.19)\end{aligned}$$

We now wish to estimate the integral on the right-hand side of (3.19). We begin by observing that it follows from the Schwartz inequality that it does not exceed

$$\|\omega g_m\|_{-\alpha}\|\omega v^*_{mt}(1,\cdot)\|_\alpha, \qquad (3.20)$$

with $\alpha = \frac{1}{8} - (\frac{3}{4})\delta > 0$. Now we apply Lemma 3.2 with $\phi = \phi_m = \omega\psi'_m [\operatorname{sgn}(\cdot)v^*_{mt}(1,\cdot)]$ and use the fact that $|\psi'_m| \leqslant 1$ to conclude that, for all δ sufficiently small,

$$\begin{aligned}\|\omega g_m\|_{-\alpha} &\leqslant C\|h^*\|_{-\alpha}\max\{\|\phi_m\|_{L^\infty(-\infty,\infty)}, \|\phi_m\|_{L^2(-\infty,\infty)}\}\\ &\leqslant C\|h\|_{H^{-\epsilon}(0,T)}, \quad \forall m = 1,2,3,\ldots, \qquad (3.21)\end{aligned}$$

where the second inequlity in (3.21) follows from the remark after the proof of Lemma 3.2 and the fact that $(h^*)^\sim$ is twice the real part of $\tilde{h}_0$. Using (3.19)–(3.21) and Cauchy's inequality with ε we have that, for each $\eta > 0$, there exists a constant $C_\eta > 0$ for which

$$\int_{-\infty}^{\infty}\int_0^1 |s|^{1-2\delta}|(\omega v_{mt}^*)^\sim|^2 \,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant C_\eta + \eta\|\omega v_{mt}^*(1,\cdot)\|_\alpha^2, \quad \forall m = 1,2,3,\ldots. \tag{3.22}$$

Now let $Q = (0,1)\times(-\infty,\infty)$ and note by (3.11) that $\{\omega v_m^*\}$ is bounded in $W^{2,1}(Q)$. We can thus add the square of the $W^{2,1}(Q)$-norm of ωv_m^* to both sides of (3.22) and conclude that, for each $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$\|\omega v_m^*\|_{W^{2,(3/2)-\delta}(Q)}^2 \leqslant C_\eta + \eta\|\omega v_{mt}^*(1,\cdot)\|_\alpha^2. \tag{3.23}$$

Now we apply Theorem 4.2.1 of [19, vol. II] to find $C > 0$ such that

$$\|(\omega v_{mt}^*)(1,\cdot)\|_\alpha \leqslant C\|\omega v_{mt}^*\|_{W^{2,(3/2)-\delta}(Q)}. \tag{3.24}$$

Hence by choosing η sufficiently small, we conclude from (3.23) and (3.24) that $\{\omega v_m^*\}$ is bounded in $W^{2,(3/2)-\delta}(Q)$, and so $\{v_m\}$ is bounded in $W^{2,(3/2)-\delta}(\Omega_T)$. This concludes the proof of the Lemma.

Returning to the proof of Theorem 3.1, we can now conclude from (3.11) and Lemma 3.3 that there exists

$$v \in W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap\left[\bigcap_{\delta>0} W^{2,(3/2)-\delta}(\Omega_T)\right]$$

such that, after passing to a subsequence,

$$v_m \to v, \quad \text{weak}^* \text{ in } W^{1,\infty}(0,T;H)\cap H^1(0,T;V), \tag{3.25}$$

$$v_{mt}(1,\cdot) \to v_t(1,\cdot), \quad \text{weakly in } H^\alpha(0,T), \tag{3.26}$$

provided $\alpha = \frac{1}{8} - (\frac{3}{4})\delta > 0$. Note that for such α we have that $H^\alpha(0,T)$ embeds compactly in $L^2(0,T)$, [19, vol. I, Theorem 1.16.1]. Thus we may suppose that

$$v_{mt}(1,\cdot) \to v_t(1,\cdot), \quad \text{pointwise a.e. on } (0,T). \tag{3.27}$$

Next, note that it follows from (3.8) that

$$(v_{mtt}, w - v_{mt}) + (v_{mxx}, w_{xx} - v_{mxxt}) + c_2(v_{mxxt}, w_{xx} - v_{mxxt})$$

$$+h\psi_m'[v_{mt}(1,\cdot)](w-v_{mt})(1,\cdot)$$
$$=(f,w-v_{mt}),\quad \forall w\in W_{T,m}. \tag{3.28}$$

Now it follows from the convexity of J_m that for all z,w in the domain of J_m,

$$J_m(w)-J_m(z)\geqslant \langle J_m'(z),w-z\rangle. \tag{3.29}$$

If we now integrate (3.28) over $(0,T)$ and use (3.29) we obtain

$$\begin{aligned}
&-\int_0^T(v_{mt},w_t)\,\mathrm{d}t+\int_0^1[v_{mt}(x,T)w(x,T)-v_{1m}(x)w(x,0)]\,\mathrm{d}x\\
&\quad-\tfrac12(\|v_{mt}(\cdot,T)\|_H^2-\|v_{1m}\|_H^2)+\int_0^T(v_{mxx},w_{xx})\,\mathrm{d}t\\
&\quad-\tfrac12(\|v_{mxx}(\cdot,T)\|_H^2-\|v_{0m}''\|_H^2)\\
&\quad+c_2\int_0^T(v_{mxxt},w_{xx}-v_{mxxt})\,\mathrm{d}t+J_m(w)-J_m(v_{mt})\\
&\geqslant\int_0^T(f,w-v_{mt})\,\mathrm{d}t,\quad \forall w\in W_{T,m}\cap H^1(\Omega_T).
\end{aligned}\tag{3.30}$$

Now let p be a fixed positive integer and take $w\in W_{T,p}\cap H^1(\Omega_T)$ in (3.30). We now pass to the limit in m in the various terms in (3.30) so as to establish (3.4). By (3.25) we may pass to the limit in the first and fourth terms on the left of (3.30), and in the term on the right-hand side. (3.25) and the weak lower-semicontinuity of the norm in $L^2(\Omega_T)$ also yield

$$\int_0^T(v_{xxt},w_{xx}-v_{xxt})\,\mathrm{d}t\geqslant\overline{\lim_m}\int_0^T(v_{mxxt},w_{xx}-v_{mxxt})\,\mathrm{d}t,$$

and so we may pass to the limit in the sixth term on the left of (3.30). We observe next that $|\psi_m(s)-|s||\leqslant 1/m$, whence by (3.27), $\psi_m[v_{mt}(1,\cdot)]\to|v_t(1,\cdot)|$, pointwise a.e. on $(0,T)$. Hence, by Fatou's lemma,

$$\begin{aligned}
J(v_t)&=\int_0^T h|v_t(1,\cdot)|\,\mathrm{d}t\\
&\leqslant\underline{\lim_m}\int_0^T h\psi_m[v_{mt}(1,\cdot)]\,\mathrm{d}t\\
&=\underline{\lim_m}\,J_m(v_{mt}),
\end{aligned}\tag{3.31}$$

and so we may pass to the limit in the seventh and eighth terms on the left of (3.30).

We next prove that $v_{xx}(\cdot,T)$ and $v_t(\cdot,T)$ exist as elements of H, and that

$$v_{mxx}(\cdot,T) \to v_{xx}(\cdot,T), \quad \text{and} \tag{3.32}$$

$$v_{mt}(\cdot,T) \to v_t(\cdot,T), \quad \text{weakly in } H. \tag{3.33}$$

Once (3.32) and (3.33) have been established, (3.6), (3.7), and the weak lower-semicontinuity of the norm will allow us to pass to the limit in the second, third and fifth terms on the left of (3.30). This will complete passage to the limit in all the terms in (3.30).

To establish (3.32) we note that it follows from (3.11) that there exists $r \in H$ such that

$$v_{mxx}(\cdot,T) \to r, \quad \text{weakly in } H, \tag{3.34}$$

and (3.25) implies that

$$v_m(\cdot,T) \to v(\cdot,T) \quad \text{strongly in } H. \tag{3.35}$$

Then a standard argument shows that $r = v_{xx}(\cdot,T)$, and hence (3.32) holds.

Establishing (3.33) is more involved. We begin by observing that for each measurable subset E of $(0,T)$,

$$\int_E h|\psi_m'[v_{mt}(1,t)]\,\mathrm{d}t \leqslant \int_E h\,\mathrm{d}t, \quad \forall m = 1,2,3,\ldots,$$

and hence $\{h\psi_m'[v_{mt}(1,\cdot)]\}$ is relatively weakly compact in $L^1(0,T)$. Since Lemma 3.2 implies that this sequence is bounded in $H^{-\varepsilon}(0,T)$, it follows that there exists $\ell \in H^{-\varepsilon}(0,T) \cap L^1(0,T)$ for which

$$h\psi_m'[v_{mt}(1,\cdot)] \to \ell, \quad \text{weakly in } L^1(0,T).$$

Integrating (3.8) and $(0,T)$ and passing to the limit in the equation which results, we conclude that, for all $w \in \{z \in \cup_p W_{T,p}\colon\ z(\cdot,T) = z(\cdot,0) = 0\} \cap H^1(\Omega_T)$,

$$-\int_0^T (v_t, w_t)\,\mathrm{d}t + \int_0^T (v_{xx}, w_{xx})\,\mathrm{d}x + c_2\int_0^T (v_{xxt}, w_{xx})\,\mathrm{d}t$$

$$+\int_0^T \ell w(1,t)\,\mathrm{d}t = \int_0^T (f,w)\,\mathrm{d}t,$$

from which it follows that

$$v_{tt} + v_{xxxx} + c_2 v_{xxxxt} = f \quad \text{in } \mathcal{D}'(\Omega_T).$$

Thus, by (3.11), $v_{tt} \in L^2(0,T;H^{-2}(0,1))$, and so

$$v_t \in C([0,T];H^{-2}(0,1)). \tag{3.36}$$

But it follows from (3.11) and (3.25) that, for all $\phi \in C_0^\infty(0,1)$,

$$\operatorname*{ess\,sup}_{t\in[0,T]} |\langle v_t(\cdot,t),\phi\rangle| \leqslant C\|\phi\|_H,$$

and so, by (3.36), $|\langle v_t(\cdot,T),\phi\rangle| \leqslant C\|\phi\|_H$, from which it follows that $v_t(\cdot,T)$ defines on element of H. By (3.25), we have

$$\lim_m \int_0^T \int_0^1 w(x)\phi(t)v_{mt}(x,t)\,\mathrm{d}s\,\mathrm{d}t$$

$$= \int_0^T \int_0^1 w(x)\phi(t)v_t(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

for all $\phi \in L^1(0,T)$, $w \in H$. In particular, setting $\phi = (1/\eta)\chi_{[T-\eta,T]}$ for $0 < \eta < T$, it follows that

$$\lim_m \frac{1}{\eta}\int_{T-\eta}^T \int_0^1 v_{mt}(x,t)w(x)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\eta}\int_{T-\eta}^T \int_0^1 v_t(x,t)w(x)\,\mathrm{d}x\,\mathrm{d}t,$$

and hence

$$\lim_{\eta\to 0}\lim_m \frac{1}{\eta}\int_{T-\eta}^T \int_0^1 v_{mt}(x,t)w(x)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \lim_{\eta\to 0} \frac{1}{\eta}\int_{T-\eta}^T \int_0^1 v_t(x,t)w(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.37}$$

We now wish to interchange limits on the left-hand side of (3.37) and evaluate the limit on the right hand side. For this purpose we need the following lemma.

LEMMA 3.4 *Let w be in the span of $\{z_1,\ldots,z_p\}$. Then the set*

$$\left\{\int_0^1 v_{mt}(x,\cdot)w(x)\,\mathrm{d}x:\ m \geqslant p\right\}$$

is uniformly equicontinuous and hence relatively norm compact in $C[0,T]$.

Proof. Choose s and t in $[0,T]$ with $s < t$. It follows from (3.8) that

$$\begin{aligned}\int_0^1 [v_{m\tau}(x,t) - v_{m\tau}(x,s)]w(x)\,\mathrm{d}x &= \int_0^1 \int_s^t v_{m\tau\tau}(x,\tau)w(x)\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int_s^t \int_0^1 f(x,\tau)w(x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad - \int_s^t \int_0^1 v_{mxx}(s,\tau)w''(x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad - c_2 \int_s^t \int_0^1 v_{mxx\tau}(x,\tau)w''(x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad - w(1)\int_s^t h\psi_m'[v_{m\tau}(1,\cdot)]\,\mathrm{d}\tau.\end{aligned}$$

Since $f \in L^2(\Omega_T)$, the first term on the right tends to zero as $s - t \to 0$, independently of m. It follows from (3.11) and the Schwartz inequality that the second and third terms are dominated by $C\|w''\|_H\sqrt{t-s}$. Since $|\psi_m'| \leqslant 1$, the last term is dominated by $|w(1)| \int_s^t h\,\mathrm{d}\tau$. Therefore

$$\int_0^1 [v_{m\tau}(x,t) - v_{m\tau}(x,s)]w(x)\,\mathrm{d}x \to 0$$

as $t - s \to 0$, independently of m. This completes the proof of the Lemma.

Returning to (3.37), we find upon passing to a subsequence that for each w in the linear span of $\{z_n\}$, there exists $q \in C[0,T]$ for which

$$\int_0^1 w(x)v_{mt}(x,\cdot)\,\mathrm{d}x \to q, \quad \text{strongly in } C[0,T].$$

Since we already have that

$$\int_0^1 w(x)v_{mt}(x,\cdot)\,\mathrm{d}x \to \int_0^1 w(x)v_t(x,\cdot)\,\mathrm{d}x, \quad \text{weak* in } L^\infty(0,T),$$

it follows that

$$\int_0^1 w(x)v_t(x,\cdot)\,\mathrm{d}x = q \in C[0,T].$$

This is sufficient to conclude that

$$\lim_{\eta\to 0} \frac{1}{\eta} \int_{T-\eta}^T \int_0^1 w(x)v_t(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^1 w(x)v_t(x,T)\,\mathrm{d}x.$$

The uniform convergence also allows us to interchange the limits on the left-hand side of (3.37), and so we conclude that

$$\lim_m \int_0^1 v_{mt}(x,T)w(x)\,\mathrm{d}x = \int_0^1 v_t(x,T)w(x)\,\mathrm{d}x.$$

Since the linear span of $\{z_n\}$ is dense in H, (3.33) follows. We have now shown that we can pass to the limit in all terms in (3.30) for a fixed $w \in W_{T,p} \cap H^1(\Omega_T)$.

Thus (3.4) holds for all $w \in (\cup_p W_{T,p}) \cap H^1(\Omega_T)$. The density of $(\cup_p W_{T,p}) \cap H^1(\Omega_T)$ in $H^\varepsilon(0,T;V)$ will yield (3.4), for all $w \in H^\varepsilon(0,T;V)$, provided we show that

$$v_{tt} \in H^{-\varepsilon}(0,T;V'). \tag{3.38}$$

In order to establish (3.38), we integrate by parts and pass to the limit in (3.8) to conclude, with the aid of (3.33), that

$$\begin{aligned}\int_0^T \langle v_{tt}, w\rangle\,\mathrm{d}t = \int_0^T (f,w)\,\mathrm{d}t - \int_0^T (v_{xx}, w_{xx})\,\mathrm{d}t \\ -c_2\int_0^T (v_{xxt}, w_{xx})\,\mathrm{d}t - \int_0^T \ell w(1,\cdot)\,\mathrm{d}t,\end{aligned} \tag{3.39}$$

$\forall w \in \cup_p W_{T,p}$. Therefore, for each ϕ in the linear span of $\{z_n\}$,

$$\begin{aligned}\langle v_{tt}(\cdot,t),\phi\rangle &= (f(\cdot,t),\phi) - (v_{xx}(\cdot,t),\phi'') \\ &\quad -c_2(v_{xxt}(\cdot,t),\phi'') - \ell(t)\phi(1),\end{aligned} \tag{3.40}$$

for a.e. $t \in [0,T]$. It follows from (3.40), the fact that $H^2(0,1)$ embeds continuously into $C[0,1]$, and the strong density of the linear span of $\{z_n\}$ in V, that the function

$$t \to \langle v_{tt}(\cdot,t),\phi\rangle, \quad t \in (0,T),$$

is an element of $H^{-\varepsilon}(0,T)$ for all $\phi \in V$. This implies that (3.38) holds and hence (3.4) holds. We also note that the $H^{-\varepsilon}(0,T;V')$-norm of v_{tt} does not exceed $C(1+\|h\|_{H^{-\varepsilon}(0,T)})$ for some $C > 0$.

We now check that the initial conditions (3.1) hold for v. That $v(\cdot,0) = v_0$ is immediate from (3.6), (3.9), (3.25), and the boundedness of the trace operator from $H^1(\Omega_T)$ into H. To see that $v_t(\cdot,0) = v_1$, we observe that a simple modification of the argument used to verify (3.33) shows that $v_t(\cdot,0) \in H$ and

$$v_{1m} = v_{mt}(\cdot,0) \to v_t(\cdot,0), \quad \text{weakly in } H,$$

whence by (3.7), $v_t(\cdot,0) = v_1$. This completes the proof of the existence assertion in Theorem 3.1 in the case where h is assumed to be in $L^1(0,T)$.

Turning to the general case, we now assume that h is a nonnegative distribution in $H^{-\varepsilon}(0,T)$, for some $\varepsilon \in [0, 1/8)$. Then by ([19, Vol. I, Remarks 1.7.4 and 1.12.5.]) there exists a sequence of nonnegative functions $\{h_n\}$ in $C^\infty[0,T]$ for which

$$h_n \to h \quad \text{strongly in } H^{-\varepsilon}(0,T). \tag{3.41}$$

By the version of Theorem 3.1 which has just been established, for each n, there exists a function v_n such that for all $w \in H^\varepsilon(0,T;V)$,

$$\begin{aligned}
&\int_0^T \langle v_{ntt}, w - v_{nt}\rangle \,\mathrm{d}t + \int_0^T (v_{nxx}, w_{xx} - v_{nxxt})\,\mathrm{d}t \\
&\quad + c_2 \int_0^T (v_{nxxt}, w_{xx} - v_{nxxt})\,\mathrm{d}t \\
&\quad + \int_0^T h_n |w(1,t)|\,\mathrm{d}t - \int_0^T h_n |v_{nt}(1,t)|\,\mathrm{d}t \\
&\geqslant \int_0^T (f, w - v_{nt})\,\mathrm{d}t,
\end{aligned} \tag{3.42}$$

and for which

$$v_n(\cdot,0) = v_0, \quad v_{nt}(\cdot,0) = v_1,$$

$$\{v_{nxx}(\cdot,T)\} \text{ and } \{v_{nt}(\cdot,T)\} \text{ are bounded in } H,$$

$$\{v_n\} \text{ is bounded in } W^{1,\infty}(0,T;L^2(0,1)) \cap H^1(0,T;H^2(0,1)) \cap \left[\bigcap_{\delta>0} W^{2,(3/2)-\delta}(\Omega_T)\right],$$

$$\{v_{ntt}\} \text{ is bounded in } H^{-\varepsilon}(0,T;V').$$

We may thus find

$$v \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap \left[\bigcap_{\delta>0} W^{2,(3/2)-\delta}(\Omega_T)\right]$$

which satisfies (3.1)–(3.3) and for which

$$v_n \to v \text{ weak}^* \quad \text{in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V)$$

and

$$v_{ntt} \to v_{tt} \text{ weakly in } H^{-\varepsilon}(0,T;V').$$

Consequently, we may pass to the limit in the second and third terms on the left-hand side of (3.42) and in the term on the right.

To pass to the limit in the first term on the left-hand side of (3.42), we integrate it by parts and observe that our previous reasoning will allow passage to the limit in the expression which results, provided that an appropriate version of Lemma 3.4 holds for the sequence $\{v_{nt}\}$. But if we choose $w \in C_0^\infty(0,1)$, then our previous argument with (3.40) used in place of (3.8) shows that the set

$$\left\{\int_0^1 v_{nt}(\cdot,x)w(x)\,\mathrm{d}x \colon n = 1,2,3,\ldots\right\}$$

is uniformly equicontinuous on $[0,T]$, and this is sufficient for our purposes.

It only remains to pass to the limit in the fourth and fifth terms on the left side of (3.42). We first deduce from [19, vol. I, Theorem 1.10.2] that, for each $\alpha \in [0,\frac{1}{2})$, the $H^\alpha(0,T)$-norm of $|v_{nt}(1,\cdot)|$ does not exceed the $H^\alpha(0,T)$-norm of $v_{nt}(1,\cdot)$. Using the boundedness of the sequence $\{v_{nt}(1,\cdot)\}$ in $H^\alpha(0,T)$ for $\alpha \in (0,\frac{1}{8})$ and the compactness of the embedding of $H^\alpha(0,T)$ into $H^\varepsilon(0,T)$ for $\alpha > \varepsilon$, we conclude, after passing to a subsequence, that

$$|v_{nt}(1,\cdot)| \to |v_t(1,\cdot)|, \quad \text{strongly in } H^\varepsilon(0,T).$$

This, together with (3.41), suffices to pass to the limit in the fourth and fifth terms on the left of (3.42). Thus v satisfies (3.4). This concludes the proof of the existence assertion contained in Theorem 3.1.

Finally, we establish the stability assertion of Theorem 3.1, which will also establish the uniqueness. We begin by observing that if $v \in W^{2,1+\varepsilon}(\Omega_T)$ satisfies (3.2), then $v_t \in H^\varepsilon(0,T;V)$. Now fix $t \in [0,T]$, let $\chi_{(0,t)}$ denote the characteristic function of the interval $(0,t)$, and recall that $\chi_{(0,t)}$ is a multiplier of $H^\alpha(0,T)$ for all $\alpha \in [0,\frac{1}{2})$ ([31, Th 6.11(b), p. 162]). Then, for each solution v of (3.2) and (3.4) in $W^{2,1+\varepsilon}(\Omega_T)$ and for each $w \in H^\varepsilon(0,T;V)$, we have that $(w - v_t)\chi_{(0,t)} + v_t \in H^\varepsilon(0,T;V)$. We may thus replace w in (3.4) by this function to conclude that

$$\int_0^t \langle v_{\tau\tau}, w - v_\tau\rangle\,\mathrm{d}\tau + \int_0^t (v_{xx}, w_{xx} - v_{xx\tau})\,\mathrm{d}\tau$$

$$+c_2\int_0^t (v_{xx\tau}, w_{xx} - v_{xx\tau})\,\mathrm{d}t$$

$$+J[(w - v_\tau)\chi_{(0,t)} + v_\tau] - J(v_\tau)$$

$$\geqslant \int_0^t (f, w - v_\tau)\,\mathrm{d}t, \quad \forall w \in H^\varepsilon(0,T;V). \tag{3.43}$$

Now let z_i, $i = 1, 2$ be solutions in $W^{2,1+\varepsilon}(\Omega_T)$ of (3.2) and (3.4) corresponding to applied forces $f_i \in L^2(\Omega_T)$ and with initial data $(z_{i0}, z_{i1}) \in V \times H$. Letting $v = z_1, w = z_{2\tau}, f = f_1$ in (3.43), and then $v = z_2, w = z_{1\tau}, f = f_2$ in (3.43), adding the resulting inequalities, and setting $z = z_1 - z_2$, we obtain

$$\begin{aligned}
&-\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}(\|z_\tau(\cdot,\tau)\|_H^2 + \|z_{xx}(\cdot,\tau)\|_H^2)\,\mathrm{d}\tau - c_2\int_0^t\int_0^1 z_{xx\tau}^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad + J(-z_\tau\chi_{(0,t)} + z_{1\tau}) + J(z_\tau\chi_{(0,t)} + z_{2\tau}) - J(z_{1\tau}) - J(z_{2\tau}) \\
&\geqslant \int_0^t (f_2 - f_1, z_\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{3.44}$$

But the sum of the last four terms on the left-hand side of (3.44) is the limit, as $n \to +\infty$, of

$$\begin{aligned}
&\int_0^T h_n|(z_{2\tau} - z_{1\tau})(1,\tau)\chi_{(0,t)} + z_{1\tau}(1,\tau)|\,\mathrm{d}\tau - \int_0^T h_n|z_{1\tau}(1,\tau)|\,\mathrm{d}\tau \\
&\qquad + \int_0^T h_n|(z_{1\tau} - z_{2\tau})(1,\tau)\chi_{(0,t)} + z_{2\tau}(1,\tau)|\,\mathrm{d}\tau - \int_0^T h_n|z_{2\tau}(1,\tau)|\,\mathrm{d}\tau \\
&= \int_0^t h_n|z_{2\tau}(1,\tau)|\,\mathrm{d}\tau - \int_0^t h_n|z_{1\tau}(1,\tau)|\,\mathrm{d}\tau \\
&\qquad + \int_0^t h_n|z_{1\tau}(1,\tau)|\,\mathrm{d}\tau - \int_0^t h_n|z_{2\tau}(1,\tau)|\,\mathrm{d}\tau \\
&= 0,
\end{aligned}$$

and hence is also zero. It follows that for each $\eta > 0$ and for a.e. $t \in [0,T]$,

$$\begin{aligned}
\|z_\tau(\cdot,t)\|_H^2 + \|z_{xx}(\cdot,t)\|_H^2 \leqslant{}& \|z_{10}'' - z_{20}''\|_H^2 + \|z_{11} - z_{21}\|_H^2 \\
&+ \frac{\eta}{2}\|z_t\|_{L^2(\Omega_T)}^2 + \frac{1}{2\eta}\|f_1 - f_2\|_{L^2(\Omega_T)}^2,
\end{aligned}$$

which implies (3.5), for $\eta > 0$ sufficiently small. The proof of Theorem 3.1 is now complete.

We remark in concluding this section that the proof we give of Theorem 3.1 shows that in the case when the distribution h is a function in $L^1(0,T)$, the full sequence $\{v_m\}$ of Galerkin approximations obtained directly from the regularized problems converges weak* in $W^{1,\infty}(0,T;H) \cap H^1(0,T;V)$ to the solution of (3.1)–(3.4). Consequently, these approximations may be used in developing numerical methods for calculating the vertical displacement of the beam in specific examples.

4. Proofs of the Main Results

In this section we use the results of Section 3 to prove Theorems 2.2, 2.3, 2.5 and 2.6.

Proof of Theorem 2.2. As has been mentioned above, it follows from [21, Theorem 4.1] that problem (2.19) and (2.21) has a unique solution u. (Although the proof as stated there only covers the case $c_1 = 0$, it can be modified in obvious ways to obtain the results for $c_1 > 0$.) Consequently, we need only establish the existence of a unique solution to (2.20) and (2.22). To this end we note that when $c_1 > 0$ we have that $u \in L^2(0,T;E)$ and $u_t \in L^2(0,T;E)$ and therefore $u(1,\cdot)$, the trace of u on $x = 1$, belongs to $H^1(0,T)$. Consequently, if we now put $h = c_T(u(1,\cdot) - g)_+^{m_T}$ we have that $h \in L^\infty(0,T)$. In the case when $c_1 = 0$ we have that $u(1,\cdot) \in H^{1/2}(0,T)$ and hence by [26] $u(1,\cdot) \in L^q(0,T)$ for every finite q. It follows that h is in $L^2(0,T)$. In either case we can apply Theorem 3.1 with this choice of h to conclude that problem (2.20) and (2.22) has a unique solution v. This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. In this section we let C_1 denote a constant whose value may change from line to line but which only depends on the norms of the data for the horizontal problem. Let y_1 and y_2 be as in the hypothesis of the theorem. Let $t \in [0,T]$ and let $\chi_{(0,t)}$ be the characteristic function of $(0,t)$. Since we have that $y_{i\tau} \in L^2(0,T;E), i = 1,2$, we may set $\varphi = u_\tau\chi_{(0,t)}$ in (2.21) for $u = y_1$ and $u = y_2$. We subtract one of the equations so obtained from the other, and integrate the first two terms in the resulting equation. We thus obtain, for all $\varepsilon > 0$ and all $t \in [0,T]$,

$$
\begin{aligned}
&\|y_\tau(\cdot,t)\|_H^2 + a\|y_x(\cdot,t)\|_H^2 + c_1\int_0^t \|y_{x\tau}\|_H^2\,\mathrm{d}\tau \\
&\quad + c_N\int_0^t [(y_1(1,\cdot) - g)_+^{m_N} \\
&\quad - (y_2(1,\cdot) - g)_+^{m_N} y_\tau(1,\cdot)\,\mathrm{d}\tau \\
&\leqslant \|y_{10}' - y_{20}'\|_H^2 + \|y_{11} - y_{21}\|_H^2 \\
&\quad + \frac{1}{2\varepsilon}\int_0^t\int_0^1 (k_1 - k_2)^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\varepsilon}{2}\int_0^t\int_0^1 y_\tau^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{4.1}
$$

where $y = y_1 - y_2$. Using the fact that $y_\tau \in L^2(0,T;E)$, we have that

$$c_N \left| \int_0^t [(y_1(1,\tau) - g)_+^{m_N} - (y_2(1,\tau) - g)_+^{m_N}] y_\tau(1,\tau)\, \mathrm{d}\tau \right|$$

$$\leqslant \varepsilon \int_0^t \|y_{x\tau}\|_H^2 \, \mathrm{d}\tau + \frac{c_N^2}{4\varepsilon} \int_0^t |(y_1(1,\tau) - g)_+^{m_N}$$

$$-(y_2(1,\tau) - g)_+^{m_N}|^2 \, \mathrm{d}\tau. \tag{4.2}$$

Assuming that $m_N > 1$ we use the Mean Value Theorem and Hölder's inequality for $q > \max\{1, 1/(2(m_N - 1))\}$ to obtain

$$\int_0^t |(y_1(1,\tau) - g)_+^{m_N} - (y_2(1,\tau) - g)_+^{m_N}|^2 \, \mathrm{d}\tau$$

$$\leqslant C \|y(1,\cdot)\|_{L^2(0,t)}^2$$

$$+ C \max\{\|y_1(1,\cdot)\|_{L^{2(m_N-1)q}(0,T)}^{2(m_N-1)},$$

$$\|y_2(1,\cdot)\|_{L^{2(m_N-1)q}(0,T)}^{2(m_N-1)}\} \|y(1,\cdot)\|_{L^{2p}(0,t)}^2, \tag{4.3}$$

where p is such that $p^{-1} + q^{-1} = 1$. If $m_N = 1$ then the estimate (4.3) holds with the second term on the right deleted.

Now recall that the trace operator from $H^1(\Omega_T)$ into $L^r(0,T)$, for $1 \leqslant r < +\infty$, is bounded, and so by rescaling the inequality, we can find $C > 0$, independent of t, for which

$$\|y(1,\cdot)\|_{L^r(0,t)}^2 \leqslant C \max\{t^{(2/r)+1}, t^{(2/r)-1}\} \|y\|_{H^1(\Omega_t)}^2, \quad \forall t \in (0,T), \tag{4.4}$$

where we have set $\Omega_t = (0,1) \times (0,t)$. We also note that it follows from the proof of [21, Theorem 4.1] that the $H^1(\Omega_T)$-norms of y_1 and y_2 do not exceed a constant which depends boundedly on the appropriate norms of the data. If we thus set

$$m(t) = \max\{1, t^2\} + \max\{t^{(1/p)+1}, t^{(1/p)-1}\},$$

it follows from (4.3)–(4.4) that there exists a constant $C_1 > 0$, depending boundedly on the norms of the data, for which

$$\int_0^t |(y_1(1,\tau) - g)_+^{m_N} - (y_2(1,\tau) - g)_+^{m_N}|^2 \, \mathrm{d}\tau$$

$$\leqslant C_1 m(t) \|y\|_{H^1(\Omega_t)}^2, \quad \forall t \in (0,T). \tag{4.5}$$

By using the fact that $y \in L^2(0,T;E)$, it now follows from (4.1)–(4.5), that, for ε sufficiently small and for all $t \in [0,T]$,

$$\|y(\cdot,t)\|_H^2 + \|y_x(\cdot,t)\|_H^2 + \|y_\tau(\cdot,t)\|_H^2$$
$$\leqslant C_1(\|y'_{10} - y'_{20}\|_H^2 + \|y_{11} - y_{21}\|_H^2 + \|k_1 - k_2\|_{L^2(\Omega_T)}^2)$$
$$+C_1 m(t)\|y\|_{H^1(\Omega_t)}^2.$$

The stability result for y is now a consequence of Gronwall's inequality, since $m(t)$ is integrable on $(0,T)$.

We next let z_1 and z_2 be as in the hypothesis of Theorem 2.3 and put $z = z_1 - z_2$. By an argument similar to the one used in the proof of Theorem 3.1, we have, for all $\varepsilon > 0$ and for all $t \in [0,T]$, that

$$\|z_\tau(\cdot,t)\|_H^2 + \|z_{xx}(\cdot,t)\|_H^2 + c_2\int_0^t\int_0^1 z_{xx\tau}^2\,dx\,d\tau$$
$$\leqslant \|z''_{10} - z''_{20}\|_H^2 + \|z_{11} - z_{21}\|_H^2 + \varepsilon\|z_t\|_{L^2(\Omega_T)}^2$$
$$+\varepsilon\int_0^t |z_\tau(1,\tau)|^2 d\tau + 1/4\varepsilon\|f_1 - f_2\|_{L^2(\Omega_T)}^2$$
$$+1/4\varepsilon\int_0^T |(y_1(1,\tau) - g)_+^{m_T} - (y_2(1,\tau) - g)_+^{m_T}|^2 d\tau. \tag{4.6}$$

Since $z_\tau \in L^2(0,T;V)$, we have that

$$\int_0^t |z_\tau(1,\tau)|^2 d\tau \leqslant \int_0^t\int_0^1 z_{xx\tau}^2\,dx\,d\tau.$$

From (4.5) (with m_N replaced by m_T), we have that the last term on the right-hand side of (4.6) does not exceed a constant multiple of $m(T)\|y\|_{H^1(\Omega_T)}^2$. Using these facts in (4.6), taking ε sufficiently small and combining this with the stability result just obtained for y, we obtain the stability result for z. This concludes the proof of Theorem 2.3.

Proof of Theorem 2.5. In this setting the existence of a unique solution to the horizontal problem (2.23) and (2.25) follows from [28, Theorem 2.1] for the case when $c_1 = 0$ and [KS] for the case when $c_1 > 0$. The proof in Section 2.2 of [28] can also be modified to show that if $u \in L^2(0,T;E), u_t \in L^2(0,T;E)$ and $u_{tt} - au_{xx} - c_1u_{xxt}$ (defined in the sense of distributions) is in $L^2(\Omega_T)$ then the trace $\sigma_N(1,\cdot) = au_x(1,\cdot) + c_1u_{xt}(1,\cdot)$ exists as an element of $H^{-1/2}(0,T)$. (In the case when $c_1 = 0$ this conclusion follows if the only assumption made on u_t is that $u_t \in L^2(0,T;H)$). Moreover, it is shown there that if u satisfies (2.25) then $-\sigma_N(1,\cdot)$ is nonnegative. In [13] it is shown that in fact $\sigma_N(1,\cdot)$ is in $L^2(0,T)$. Consequently, if we put $h = -\mu\sigma_N(1,\cdot)$ in Theorem 3.1 we can again conclude that there exists a unique v satisfying (2.24) and (2.26).

Proof of Theorem 2.6. As in the proof of Theorem 2.5, we have a unique solution to the horizontal problem (2.23) and (2.25) such that the trace $-\sigma_N(1,\cdot) = -cu_x(1,\cdot)$ is a nonnegative function in $L^2(0,T)$. However, in this case, we cannot rely on Theorem 3.1, as stated, to produce the solution to the vertical problem since when $c_2 = 0$ the critical Lemma 3.3 is no longer valid. However, we can begin the proof in the same way, choosing z_j to satisfy the additional conditions $z_j''(1) = z_j'''(1) = 0$ and then taking the convergence in (3.6) and (3.7) in $H^4(0,1)$ and V, respectively. We thus obtain the elementary estimate (3.11), with the term $||v_{mxxt}||_{L^2(\Omega_T)}$ deleted. This alone, of course, is not sufficient to pass to the limit in (3.30). But if we take a C^2-smoothing of ψ and let $h = R(-\mu\sigma_N(1,\cdot)) \in H^2(0,1)$ then it is possible to differentiate

$$(v_{mtt}, z_j) + (v_{mxx}, z_{jxx}) + h\psi_m'[v_{mt}(1,\cdot)]z_j(1) = (f, z_j),$$

with respect to t, multiply by a_j'' and sum over j to obtain a second round of estimates of the form

$$||v_{mtt}(\cdot,t)||_H^2 + ||v_{mxxt}(\cdot,t)||_H^2 \leqslant C,$$

where C depends only on the data. In this process the additional assumptions on the initial data are used. These estimates are sufficient to pass to the limit in (3.30) and establish (3.4). Since this approach is by now well-known (see, e.g., [5, Chapter III, Section 5.5.4]), we omit the details. We can thus conclude that there exists a solution to the vertical problem (2.24) and (2.26). This concludes the proof of Theorem 2.6.

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