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On the polynomial invariants of the elasticity tensor

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Abstract. We consider the old problem of finding a basis of polynomial invariants of the fourth rank tensor C of elastic moduli of an anisotropic material. Decomposing C into its irreducible components we reduce this problem to finding joint invariants of a triplet (a, b, D), where a and b are traceless symmetric second rank tensors, and D is completely symmetric and traceless fourth rank tensor ($D \in T_4^{ss}$). We obtain by reinterpreting the results of classical invariant theory a polynomial basis of invariants for D which consists of 9 invariants of degrees 2 to 10 in components of D. Finally we use this result together with a well-known description of joint invariants of a number of second-rank symmetric tensors to obtain joint invariants of the triplet (a, b, D) for a *generic D*.

0. Introduction. Motivation for the work

The main purpose of the work is to construct a classification of linear anisotropic elastic materials. In colloquial terms we ask the question: How do we give distinct names to distinct anisotropic elastic materials? Clearly a designation based on the 21 components of the tensor C of elastic moduli in a fixed reference frame is not good for this purpose because it provides, generically, different names for different orientations of a given material. As the material is rotated the tensor C moves on its *orbit* in the space T_4^e of elasticity tensors. Thus what is needed is a parametrization of distinct orbits. The set O_c of the distinct orbits of elasticity moduli is a manifold of (21-3) = 18 dimensions that has a fairly complicated boundary. The problem of naming the distinct orbits would be solved if O_c, the manifold of distinct orbits, can be mapped in a one-to-one and continuous manner into the linear space $\mathbf{R}^n$; the coordinates of an image point would then serve as the name of the associated orbit. It is of interest to know what minimal dimension n is needed for this purpose. The following examples may be of help in thinking about this idea:

If the manifold of interest is a circle which is one dimensional then the minimal dimension $n = 2$. If the manifold of interest is the group $SO(3)$ of rigid body rotations (which is three dimensional) then $n = 5$. One needs at most a 37 dimensional space to map a smooth manifold of dimension 18 in the above described manner.

In the present paper we show that 39 polynomial invariants of C can be used to designate a certain generic set of materials.

1. Decomposition of the elasticity tensor

Let C be the tensor of elastic moduli of a homogeneous material at some reference temperature θ_0. Fix some orthonormal basis in $\mathbf{R}^3$ and let C_{ijkl} stand for the components of C in this basis. Then for infinitesimal isothermal deformations and rotations Hooke's law for the material has the form:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \sigma = C\varepsilon.$$

Thus, C can be regarded as a linear map $C\colon T_2^s \to T_2^s$, where T_2^s is the space of symmetric rank two tensors. This map is self-adjoint. From this follows that C has the following symmetries: $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$. We will denote the space of all rank 4 tensors satisfying these conditions by T_4^e; it has dimension 21.

If we rotate the material by $g \in SO(3)$ then the elasticity moduli of the rotated material are given by the tensor which we denote gC; its components are given by

$$(gC)_{ijkl} = g_{ip}g_{jq}g_{kr}g_{ls}C_{pqrs}.$$

This defines the action of $SO(3)$ on the space T_4^e and renders it a finite-dimensional representation of $SO(3)$. For the calculation of invariants of C it will be convenient to decompose T_4^e into simpler pieces. Indeed, it can be shown (cf. [2]) that under the action of $SO(3)$ the space T_4^e decomposes into direct sum of spaces of dimensions 1, 1, 5, 5, 9:

$$T_4^e \simeq \mathbf{R} \oplus \mathbf{R} \oplus T_2^{ss} \oplus T_2^{ss} \oplus T_4^{ss}$$

where T_n^{ss} stands for the space of completely symmetric traceless tensors of rank n, which has the dimension $2n + 1$; in other words, we have the following lemma:

LEMMA. *Each $C_{ijkl} \in T_4^e$ can be represented uniquely in the form:*

$$\begin{aligned} C_{ijkl} = {} & \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ & + \delta_{ij}a_{kl} + \delta_{kl}a_{ij} \\ & + \delta_{ik}b_{jl} + \delta_{jl}b_{ik} + \delta_{il}b_{jk} + \delta_{jk}b_{il} \\ & + D_{ijkl} \end{aligned}$$

where λ, $\mu \in \mathbf{R}$, a, $b \in T_2^{ss}$ and $D \in T_4^{ss}$ are completely symmetric traceless tensors (thus, for instance, for every permutation p, $D_{p(ijkl)} = D_{ijkl}$, and $D_{iikl} = 0$), and this decomposition is conserved by the action of $SO(3)$. Furthermore, λ, μ and a_{kl}, b_{kl}, D_{ijkl} are linear combinations of components C_{ijkl}; for example, $\lambda = \frac{1}{15}(2C_{iipp} - C_{ippi})$.

REMARK. The tensors a, b, D inherit the symmetries of C. For example, for isotropic materials we have $gC = C$ for all $g \in SO(3)$, which implies $a = b = 0$, $D = 0$ and we have only λ, μ which are exactly the Lamé constants.

Now we want to formulate some general results about finite-dimensional representations of $SO(3)$; later we will apply them to T_4^e.

Let V be a finite-dimensional representation of $SO(3)$. A *polynomial invariant* of V is a polynomial function p on V (i.e., if we choose a basis in V then $p(v)$ can be written as a polynomial in coordinates of v) which is invariant under the action of $SO(3)$, namely

$$p(gv) = p(v) \text{ for any } g \in SO(3).$$

REMARK. We can omit the condition of polynomiality and obtain a definition of an invariant function on V; for example, one may consider continuous invariants. However, in this paper we only consider polynomial invariants.

The important property of polynomial invariants is that they "separate the orbits". This means that if we know the values $p(v)$ for all polynomial invariants p we can recover v uniquely up to the action of $SO(3)$ (cf. [1]).

This property is not very useful in itself, since there exists infinite number of polynomial invariants. Our main goal is to find a finite set of polynomial invariants which would separate the orbits. For this purpose we introduce the following definition:

DEFINITION. A finite set $p_1, \dots, p_k$ of polynomial invariants of V is called an *integrity basis* if every polynomial invariant of V can be written as a polynomial in $p_1, \dots, p_k$.

This is one of the basic notions of the classical invariant theory. One of the main results of this theory claims existence of such a set (cf. [1], [3]):

THEOREM. *For any finite-dimensional representation V of $SO(3)$ there exists a (finite) integrity basis.*

Now we can apply this result to the problem of separating the orbits by invariants. Let us introduce one more definition:

DEFINITION. A finite set $p_1, \ldots, p_n$ of invariants of V is called a *functional basis* if

$$p_i(v_1) = p_i(v_2) \text{ for all } i = 1, \ldots, n$$

implies $v_1 = gv_2$ for some $g \in SO(3)$.

Note that in this definition we do not require that all these invariants be polynomial; nevertheless all functional bases considered here will consist of polynomial invariants. Finally we mention the following simple theorems:

THEOREM. *Any integrity basis is a functional basis.*

This theorem follows immediately from the definitions and the fact that the set of all polynomial invariants separates orbits.

Note that the converse is false: for example, for the representation of $SO(3)$ in the space of symmetric traceless matrices T_2^{ss} it is known that the integrity basis is given by the invariants $I_2(a) = \operatorname{tr} a^2$, $I_3(a) = \operatorname{tr} a^3$. Since $\operatorname{tr} a^2 \geqslant 0$, one easily sees that the invariants I_2^3, I_3 form a functional basis which is not an integrity basis.

So we have proved the following theorem:

THEOREM. *For any finite-dimensional representation V of $SO(3)$ there exists a (finite) functional basis consisting of polynomial invariants.*

Now let us return to our problem of finding a functional basis of polynomial invariants of T_4^e. Obviously, λ and μ are invariants and the functional basis of invariants of the elasticity tensor can be obtained by adding λ and μ to the invariants of the triple (a, b, D). We can construct the functional basis of invariants of this triplet by first finding the invariants of each element of the triplet and then adding to this list the joint invariants. It is well known that the functional basis of invariants of $a \in T_2^{ss}$ is given by $\operatorname{tr} a^2$, $\operatorname{tr} a^3$ (similar result for b); what is not so well known is the basis of invariants of D, which we discuss in Section 2. In Section 3 we use this result together with a well known description of joint invariants of a number of symmetric second rank tensors to obtain the joint invariants of the triplet (a, b, D) for a generic D.

2. Invariants of symmetric traceless tensor of rank 4

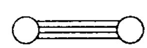
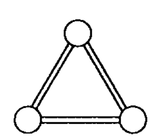
In this section we obtain the integrity basis (and therefore, the functional basis) of polynomial invariants of $D \in T_4^{ss}$ using the results of classical invariant theory. For this purpose we first show (cf. Appendix 1) that this problem reduces to finding invariants of homogeneous polynomials of degree 8 in x, y under the action of $SL(2, \mathbf{C})$. This last problem was considered in the 1880's by Sylvester and von Gall ([4], [5]). Their result when interpreted in the language of tensors states that the minimal integrity basis of invariants of $D \in T_4^{ss}$ consists of 9 invariants of degrees $2, 3, \ldots, 10$ in D_{ijkl}.[1]

To give the explicit expressions for these invariants we adopt the following conventions: Small letters $a, b, \ldots$ are used for second rank tensors, while capital letters denote fourth rank tensors. Given two tensors $a = a_{ij}$, $b = b_{kl}$ we define their product to be the second rank tensor given by $(ab)_{il} = a_{ij} b_{jl}$ (we assume summation over repeating indices). Similarly, we define the product of two fourth rank tensors to be given by $(AB)_{ijkl} = A_{ijpq} B_{pqkl}$, and the product of two tensors of ranks two and four as $(Ab)_{ij} = A_{ijkl} b_{kl}$. We introduce the following traces: $\operatorname{tr} a = a_{ii}$, $\operatorname{tr} A = A_{ijij}$. We also need the second rank tensor d_2 defined as

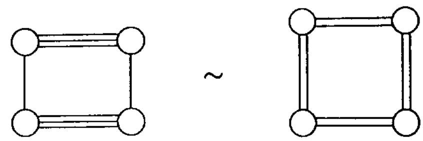
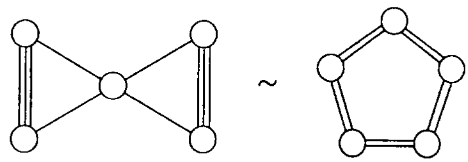
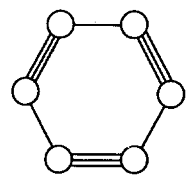
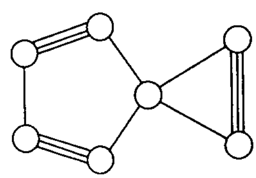
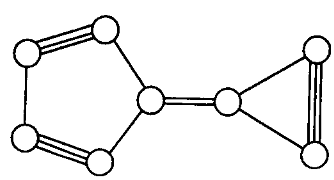
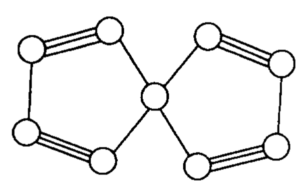
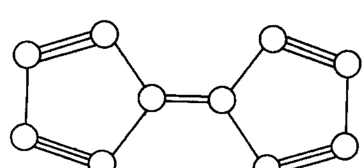
$$(d_2)_{ij} = D_{ipqr} D_{jpqr}.$$

Now we can formulate our main result:

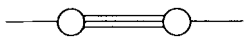
THEOREM 1. *The nine invariants $J_2, \ldots, J_{10}$ listed below, which are of degrees $2, \ldots, 10$ respectively form an integrity (and, therefore functional) basis of polynomial invariants of $D \in T_4^{ss}$.*

Invariant	degree	diagram
$J_2 = \operatorname{tr} d_2$	2	
$J_3 = \operatorname{tr} D^3$	3	

[1]These 9 invariants are not independent (one easily checks that $\dim(T_n^{ss}/SO(3)) = 6$, and hence only six of them are independent); in other words, there are polynomial relations between them; however, these relations do not enable one to express one of these invariants as a polynomial of the remaining ones. The complete list of relations among these invariants can be found in the dissertation [6].

Invariant	degree	diagram
$J_4 = \operatorname{tr} d_2^2$	4	
$J_5 = \operatorname{tr} d_2 D d_2$	5	
$J_6 = \operatorname{tr} d_2^3$	6	
$J_7 = \operatorname{tr} d_2^2 D d_2$	7	
$J_8 = \operatorname{tr} d_2^2 D^2 d^2$	8	
$J_9 = \operatorname{tr} d_2^2 D(d_2^2)$	9	
$J_{10} = \operatorname{tr} d_2^2 D^2(d_2^2)$	10	

Note that the third column of the list contains a diagrammatical description of the invariants J_i, where each circle of a diagram represents D and a line connecting two circles corresponds to a summation over one pair of indices.[2] For example, $(d_2)_{ij} = D_{ipqr}D_{jpqr}$ is represented by the following diagram:



It should be pointed out immediately that this set is different from the integrity basis of invariants given in [5], but these sets are equivalent in the following sense: We call two invariants X and Y of the same degree equivalent (i.e. $X \sim Y$) if $X - Y$ can be expressed as a polynomial of invariants of lower degrees. It should be also clear that if we substitute each invariant in a functional basis by an equivalent one, we still get a functional basis. The above list contains two examples of equivalent invariants. We also note that $J_6 \nsim \operatorname{tr} D^6$.

Thus, to prove the theorem it suffices to show the equivalence of our invariants with those given in [5]. We start the proof by showing such equivalence for degree 8. Except for degree 6 (to which we shall return below), equivalence of remaining invariants in the two sets can be proved by using similar arguments.

The invariant of degree 8 given in [5], when rewritten in terms of tensors (cf. Appendix 1) is defined as

$$i_{\Delta k} = \{\Delta\}_{ij}\{D^2\}_{ijkl}\{d_2\}_{kl},$$

where curly brackets stand for the symmetric and traceless part of the tensor they bracket, and $\Delta = \{d_2\}^2$. Now we claim that $J_8 = i_{\Delta k}$. Since $\{D^2\}$ is completely symmetric and traceless, it is obvious that $i_{\Delta k} = \operatorname{tr}\{d_2\}^2\{D^2\}d_2$. Next, since for a second rank symmetric tensor $\{a\}_{ij} = a_{ij} - \frac{1}{3}\delta_{ij}\operatorname{tr} a$, one easily sees that

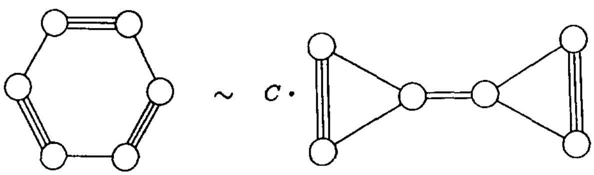
$$i_{\Delta k} = \operatorname{tr} d_2^2\{D^2\}d_2 + \operatorname{tr} d_2(\text{some invariant of degree 6}) \sim \operatorname{tr} d_2^2\{D^2\}d_2.$$

Now let us return to $J_8 = \operatorname{tr} d_2^2 D^2 d_2$. It is easy to see that $D^2 \in T_4^e$ and hence the decomposition of Section 1 holds. When we substitute this decomposition for D^2 in the expression for J_8, we see that $J_8 \sim \operatorname{tr} d_2^2\{D^2\}d_2 + c \cdot \operatorname{tr} d_2^4$. But it follows from Hamilton-Cayley theorem applied to d_2 that $\operatorname{tr} d_2^4$ can be expressed via invariants of lower degrees, namely $\operatorname{tr} d_2$, $\operatorname{tr} d_2^2$, $\operatorname{tr} d_2^3$. Thus

$$J_8 \sim \operatorname{tr} d_2^2\{D^2\}d_2 \sim i_{\Delta k}.$$

[2] One notices that these diagrams resemble the structural formulas in chemistry; but as was noted a long time ago (cf. [3, Appendix 1]), this resemblance is fortuitous.

For degree 6 we need the following non-trivial relation



which can be deduced from the Gordon's identities for invariants (cf. [3], [6]). This completes the proof of the theorem. □

3. Invariants of the elasticity tensor C

As we have seen in the Section 1, the problem of finding the invariants of the elasticity tensor C reduces to the problem of finding the invariants of the triple $(a, b, D) \in T_2^{ss} \oplus T_4^{ss} \oplus T_4^{ss}$. It is known from the classical invariant theory (cf. [3], ch. VIII) that a set composed of a number of tensors possesses a finite integrity basis of polynomial invariants and there exists an algorithm for finding such a set. However this algorithm involves consideration of certain diagrams of increasing complexity. Moreover, as the number and the ranks of the tensors forming the set increase the number of diagrams and relations between them to be considered grows very fast. We can show that even in our case of the triplet (a, b, D) which is relatively simple the execution of the algorithm by hand is prohibitively long. This may be the reason why an integrity basis of polynomial invariants of (a, b, D) seems not to be available in the literature. On the other hand we are convinced that the problem can now be solved with the help of a computer.

Nevertheless an integrity basis for invariants (a, b, D) is likely to be large because of the presence of exceptional cases where a, b and D have various kinds of symmetries. Therefore it is reasonable to look for a set of polynomial invariants which would determine "non-exceptional" or "generic" triplets (a, b, D) uniquely up to rotations. Indeed we shall show below that this more modest problem can be solved without any calculations and the resulting set consists of 39 invariants.

To define rigorously the notion of "genericity" we need the second rank symmetric traceless tensor d_s given by $(d_s)_{ij} = D_{ijkl}D_{kpqr}D_{lpqr}$; recall also that $(d_2)_{ij} = D_{ipqr}D_{jpqr}$.

DEFINITION. *A tensor $D \in T_4^{ss}$ is called generic if d_2, d_s do not have a common principal axis.*

To justify the use of the word "generic" in the above definition we should show that the set of all non-generic D's is "small". For this purpose we prove first (see Appendix 2) that this set is defined by a system of 6 polynomial equations each of degree 15 in components of D. A well known result in algebraic geometry implies that the set of solutions of this system can be either the whole space T_4^{ss} (this means that the system holds for any $D \in T_4^{ss}$) or some algebraic geometry implies that the set of solutions of this system can be either computer) an example D which does not satisfy this system (and hence is generic). This in turn implies that the non-generic set is an algebraic subvariety of dimension at most 8 (in fact, we believe the dimension is 7) in the 9-dimensional space T_4^{ss}. From this it follows that the set of non-generic D's has measure zero. Informally speaking, this means that the probability of a randomly chosen $D \in T_4^{ss}$ being generic is 1.

We hasten to note that many interesting examples are non-generic in our sense: for example, if D has any non-trivial symmetries (i.e., $gD = D$ for some $g \in SO(3)$, $g \neq I$) then d_2 and d_s inherit these symmetries and this means that they have a common principal axis; therefore D is not generic.

We will also call $C \in T_4^e$ generic if the corresponding D is generic.

Now we can give the following definition:

DEFINITION. A finite set $p_1, \ldots, p_n$ of invariants of T_4^e is called a *weak functional basis* if

$$p_i(C_1) = p_i(C_2) \text{ for all } i = 1, \ldots, n$$

implies $C_1 = gC_2$ for some $g \in SO(3)$ provided that C_1, C_2 are generic.

Once more, we note that a weak functional basis of invariants of T_4^e can be obtained by adding λ, μ to a weak functional basis of invariants of the triplet (a, b, D). In constructing such a basis for (a, b, D) we prove first the following theorem:

THEOREM 2. *A generic D is uniquely determined by d_2, d_s, J_3, J_8, J_9, J_{10}.*

Proof. First observe that having d_2 and d_s we can calculate the invariants J_2, J_4, J_5, J_6, J_7. This implies that the knowledge of d_2, d_s, J_3, J_8, J_9, J_{10} provides us with all the invariants of D and therefore determines D uniquely up to rotations; in other words, the data set of the theorem gives us the orbit $O(D)(= \{gD, g \in SO(3)\})$ of D.

Now we prove that two distinct points on the orbit $D, D' = gD$ give rise to distinct pairs (d_2, d_s). Indeed suppose that $d_2(D) = d_2(D')$. Hence $d_2(D) = d_2(gD) = gd_2(D)$. But one easily sees that $d_2 = gd_2$ is possible only if g is a rotation about a principal axis of $d_2(D)$ (of 180 degrees if eigenvalues of d_2 are distinct) or $g = I$. The same argument applies to d_s if $d_s(D') = d_s(D)$. But by definition D is generic implies that d_2 and d_s do not have a common principal axis, therefore $d_2(D') = d_2(D)$, $d_s(D') = d_s(D)$ is possible only when $g = I$, or equivalently when $D' = D$. Therefore there is a one-to-one correspondence between the orbit of D and (d_2, d_s) for generic D. (Incidentally, for generic D both of these orbits are isomorphic to $SO(3)$).

Thus we see that the pair d_2, d_s determines the position of a (generic) D on its orbit. □

It may be useful to give an example when the data set of the theorem does not determine D uniquely. Such an example is given by any tensor $D \in T_4^{ss}$ which has the symmetries of a cube (i.e., $gD = D$ if and only if $g \in S_4$, the group of symmetries of a cube). It is easily seen from the symmetry considerations that in this example $d_s = 0$ and $d_2 = \lambda I$, where λ is an invariant of D. We see that the orbit of (d_2, d_s) is a single point and hence (d_2, d_s) gives no information about the position of D on its orbit. Of course in this example D is not generic, so there is no contradiction with our theorem.

Now recalling the decomposition of T_4^e given in Section 1, we can formulate the main result of this paper:

THEOREM 3. *A weak functional basis of invariants of C is given by the following list:*

λ, μ

$J_2, \ldots, J_{10}$

$tr\, a^2$, $tr\, a^3$, $tr\, b^2$, $tr\, b^3$

$tr\, ab$, $tr\, a^2 b$, $tr\, ab^2$, $tr\, a^2 b^2$ *and similar invariants*

for each of the pairs (a, d_2), (b, d_2), (a, d_s), $(b\, d_s)$,

$tr\, abd_2$, $tr\, abd_s$, $tr\, ad_2 d_s$, $tr\, bd_2 d_s$

Proof. First, a weak functional basis of invariants of C is obtained by adding λ, μ to a weak functional basis of invariants of (a, b, D). Next from the previous theorem it follows that a weak functional basis of invariants of (a, b, D) can be obtained by adding J_3, J_8, J_9, J_{10} to the functional basis of invariants of the quadruple (a, b, d_2, d_s). For the latter quadruple the functional basis is known (cf. [7]): it is composed by the traces

$$tr\, x,\ tr\, x^2,\ tr\, x^3;\quad tr\, xy,\ tr\, x^2 y,\ tr\, xy^2,\ tr\, x^2 y^2;\quad tr\, xyz$$

where x, y, z run over the set $\{a, b, d_2, d_s\}$.

The functional basis of invariants of C thus obtained contains J_3, J_8, J_9, J_{10} plus 9 invariants of the pair (d_2, d_s). But these 13 invariants can be expressed as polynomials of the generating invariants $J_2, \ldots, J_{10}$ by Theorem 1. With this observation we arrive at the set of 39 invariants given in the statement of the theorem. □

Note that this set does not contain any invariant of degree higher than 10.

REMARK. The set of invariants given in the theorem is not a functional basis but only a weak functional basis. For example, as remarked before, in the case where D has symmetries of a cube the pair (d_2, d_s) does not give any information about the orientation of D and therefore the invariants of the theorem which were constructed for the set (a, b, d_2, d_s) do not allow us to recover (a, b, D) if a, b are non-zero tensors.

Appendix 1: Reduction to the Classical Case

We now show that the problem of finding polynomial invariants of irreducible representation T_4^{ss} of the group $SO(3)$ is equivalent to a similar problem for the group $SL(2, \mathbf{C})$. Indeed, we will show below that the algebra of polynomial invariants of T_4^{ss} is isomorphic to the algebra of polynomial invariants of the space V_8 of homogeneous polynomials of degree 8 under the action of $SL(2, \mathbf{C})$. On the other hand, structure of the latter algebra was studied thoroughly by Sylvester and von Gall ([4], [5]). They found that

(i) The algebra of invariants is generated by 9 invariants of degrees $2, 3, \ldots, 10$ respectively.
(ii) They gave explicit expressions for these invariants.

Thus in view of the above mentioned isomorphism a minimal functional basis of polynomial invariants of $D \in T_4^{ss}$ consists of 9 invariants of degrees $2, 3, \ldots, 10$.

To give the explicit formulas for these invariants we need one more notion from the classical invariant theory. Namely if we are given two vectors $a \in V_{2n}$, $b \in V_{2m}$ where V_{2k} stands for the representation of $SL(2, \mathbf{C})$ in the space of homogeneous polynomials of degree $2k$ (see below) then there is a canonical way to construct a new vector $c = (a, b)^i \in V_{2(m+n-i)}$. These operations are usually called *transvectants* (German *Überschiebung*). In modern terms they can be defined with the help of the following theorem:

THEOREM. (*Clebsch-Gordon*) *For each triplet of non-negative integers m, n, i such that $0 \leqslant i \leqslant m + n$ there exists a unique up to a constant factor linear map*

$$V_{2n} \otimes V_{2m} \to V_{2(n+m-i)}$$

commuting with the action of $SO(3)$.

Now we can easily give the expressions for all the generating invariants of Sylvester and von Gall. For example, the invariant of degree 8 looks as follows:

$$i_{\Delta k}(f) = (((f, f)^4, k)^4, \Delta)^4,$$

where $f \in V_8$, $k = (f, f)^6$, $\Delta = (k, k)^2$. One can easily check that $i_{\Delta,k} \in V_0 = \mathbf{R}$; from the definition of the operations $(a, b)^i$ it follows that $i_{\Delta,k}$ is an invariant.

To find the formulas for corresponding invariants of $D \in T_4^{ss}$ we note that for even i we have an analog of the operations $(a, b)^i$ for the tensors: given $a = a_{i_1 \ldots i_m} \in T_m^{ss}$, $b = b_{j_1 \ldots j_n} \in T_n^{ss}$ there is a canonical way to construct a tensor $c = (a, b)^{2l} \in T_{n+m-2l}^{ss}$; in terms of components c can be written as follows: $c_{i_1 \ldots i_{m-l} j_1 \ldots j_{n-l}} = \{a_{i_1 \ldots i_{m-l} p_1 \ldots p_l} b_{j_1 \ldots j_{n-l} p_1 \ldots p_l}\}$ (here curly brackets stand for the symmetric and traceless part of the tensor they bracket); in particular, $(D, D)^4 = \{D^2\}$. Thus it is clear that the generating invariant of degree 8 of a tensor $D \in T_4^{ss}$ is given by

$$i_{\Delta k} = \{\Delta\}_{ij} \{D^2\}_{ijkl} \{d_2\}_{kl},$$

where $\Delta = \{d_2\}^2$. We have used this expression in Section 2.

We do not give here the expressions for the invariants of other degrees: they can be found in [5].

We now return, as promised above, to the relationship between the representations of the groups $SO(3)$ and $SL(2, \mathbf{C})$. We first note that there exists an embedding $SO(3) \to SL(2, \mathbf{C})/\pm 1$, given by the well-known isomorphism $SO(3) \simeq SU(2)/\pm 1$. In some sense, $SL(2, \mathbf{C})/\pm 1$ can be considered as the complexification of $SO(3)$. This can be defined rigorously using the notion of Lie algebras. In particular, we have the following lemma (cf. [1]):

LEMMA. *Any action of $SO(3)$ in a real space T can be uniquely extended by complex analyticity to the action of $SL(2, \mathbf{C})/\pm 1$ (and hence, $SL(2, \mathbf{C})$) in the space $T \oplus iT$. If T is irreducible, so is $T \oplus iT$.*

In particular, we can apply this lemma to our representation T_n^{ss}. It is known (cf. [1]) that this representation is irreducible. Then we obtain an irreducible representation $T_n^{ss} \oplus iT_n^{ss}$ of $SL(2, \mathbf{C})$ of dimension $2n + 1$ over $\mathbf{C}$. But we know that there is a unique (up to isomorphism) irreducible representation of $SL(2, \mathbf{C})$ of dimension $2n + 1$. This representation can be realized as the action of $SL(2, \mathbf{C})$ in the space V_{2n} of all homogeneous polynomials with complex coefficients of degree $2n$ in two variables. The action of $SL(2, \mathbf{C})$ in this space is given by the formula $gf(x, y) = f(x', y')$ where $x' = g_{11}x + g_{12}y$, $y' = g_{21}x + g_{22}y$. Thus, as representations of $SL(2, \mathbf{C})$, $T_n^{ss} \oplus iT_n^{ss} \simeq V_{2n}$.

We next consider the algebras of all polynomial complex-valued functions on T_n^{ss} and on V_{2n}, which we denote by $S(T_n^{ss})$ and $S(V_{2n})$ respectively. One easily sees that since $V_{2n} \simeq T_n^{ss} \oplus iT_n^{ss}$ the algebras of polynomial functions are isomorphic: $S(T_n^{ss}) \simeq S(V_{2n})$.

Now the following theorem appears naturally (once more, we refer to [1] for the proof of this theorem)

THEOREM. *Let us denote by* $S(V_{2n})^{SL(2,\mathbf{C})}$ *the algebra of those polynomials from* $S(V_{2n})$ *which are invariant under the action of* $SL(2,\mathbf{C})$, *and by* $S(T_n^{ss})^{SO(3)}$ *the algebra of those polynomials from* $S(T_n^{ss})$ *which are invariant under the action of* $SO(3)$. *Then these algebras are isomorphic:*

$$S(V_{2n})^{SL(2,\mathbf{C})} \simeq S(T_n^{ss})^{SO(3)}.$$

This theorem allowed us to borrow results from the classical invariant theory.

Appendix 2: Genericity

In this appendix we show that the set of all non-generic D's can be described by a system of polynomial equations. To do this we first need a criterion for two symmetric 3×3 matrices a, b to have a common principal axis which could be expressed in terms of polynomials in components of a, b. We produce such a criterion as follows: first we take the commutator $ab-ba$. Next we can associate to this matrix a vector v in a standard way: $v_i = \frac{1}{2}\varepsilon_{ijk}(ab-ba)_{jk} = \varepsilon_{ijk}a_{jl}b_{lk}$. Now we have the following lemma:

LEMMA. *Let a, b be symmetric* 3×3 *matrices with real entries. Then a, b have a common principal axis if and only if the vector v defined above is a common eigenvector for a and b or zero (in other words, $av = \lambda v$, $bv = \mu v$ for some λ, $\mu \in \mathbf{R}$).*

Proof of this lemma is simple. The only subtlety arises when a or b have multiple eigenvalues but one easily discovers that the lemma remains valid.

Next one observes that the condition $av = \lambda v$ for some $\lambda \in \mathbf{R}$ is equivalent to the following system:

$$(av)_1 v_2 = v_1 (av)_2,$$

$$(av)_1 v_3 = v_1 (av)_3,$$

$$(av)_2 v_3 = v_2 (av)_3,$$

where $(av)_i$ stands for the i-th component of the vector av. This can be rewritten as $\varepsilon_{ijk}(av)_j v_k = 0$. Similarly, $bv = \mu v \Rightarrow \varepsilon_{ijk}(bv)_j v_k = 0$. Substituting $\varepsilon_{kpr}a_{ps}b_{sr}$ for v_k, we obtain that a and b have a common principal axis if and only if the following system of polynomial equations holds:

$$\begin{aligned} &\varepsilon_{ijk}\varepsilon_{mnl}\varepsilon_{kpr}a_{jm}a_{nt}a_{ps}b_{tl}b_{sr} = 0, \\ &\varepsilon_{ijk}\varepsilon_{mnl}\varepsilon_{kpr}b_{jm}b_{tl}b_{sr}a_{nt}a_{ps} = 0 \end{aligned} \qquad i = 1, 2, 3.$$

This is a system of 6 polynomial equations of degree 5 in the components of a and b.

Now substituting d_2 and d_s for a and b we obtain that d_2 and d_s have a common principal axis if and only if a system of 6 polynomial equations of degree 15 in D_{ijkl} holds.

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