Journal of Elasticity, Vol. 7, No. 1, January 1977
Noordhoff International Publishing - Leyden
Printed in The Netherlands

The determination of stress and strain concentrations at an ellipsoidal inclusion in an anisotropic material

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(Received June 9, 1975)

ABSTRACT

The aim of this paper is to produce elementary yet explicit formulae for the evaluation of stress and strain concentration factors at an ellipsoidal inclusion, for arbitrary anisotropy, under uniform loading at infinity. The results are such that the required formulae do not involve the solution of any boundary value problems or the knowledge of any Green's functions. An important feature of the analysis is that the solution of the interface problem is intimately related to the solution of the inclusion problem.

Introduction

One of the most often discussed problems in stress analysis is the determination of stress and strain concentration factors at the surface of an ellipsoidal inclusion in a matrix which is uniformly loaded at infinity. The vast majority of this work is restricted to isotropic materials and makes use of stress functions to solve the appropriate boundary value problem. Extension of this approach to anisotropic materials is not easy and has received very little attention.

A second approach is due to Eshelby [1] and relies on explicit knowledge of the Green's function. The difficulty here is that the Green's function is simple for isotropic materials, dreadfully complicated for transversely isotropic materials and unknown in general. Eshelby [1] only worked out the results for isotropic materials. The method depends upon the fact that the stress and strain within the inclusion are uniform. Eshelby [1] goes on to analyse the conditions which apply at the interface and is then able to deduce the stress and strain in the matrix at the interface. We remark that Eshelby's analysis of the interface conditions only works for isotropic materials.

A third method has recently been given by Hill [2]. Hill's [2] technique is to combine a thorough analysis of the jump conditions at an arbitrary interface with the standard solution to the ellipsoidal inclusion problem. While most of Hill's [2] work is restricted to isotropic materials, Laws [3] has shown how it can provide a basis for a complete treatment of the anisotropic problem. Roughly, the method given in [3] combines the separate solutions of two sub-problems, viz. the interface problem and the inclusion problem. In the interface problem we can think of the stress and strain on one side of the interface as known ;[1] the problem is to determine the stress and strain at the other side of the interface. The general solution is given in [3] and depends upon four tensors each of which depends only on the stiffness tensor and the unit normal to the interface. Furthermore any one of these tensors uniquely determines the other three. When the solution of the interface problem is combined with the solution of the inclusion problem, one obtains explicit formulae for the evaluation of stress and strain concentration factors, c.f. [3].

At the time of writing of my earlier paper [3] I said that applications of the general theory were limited by the fact that solutions of the inclusion problem were known only in a few special cases. This statement is not correct. Indeed further work has shown that the stress and strain within the inclusion can be determined whatever the degree of anisotropy. The important work in this connection is due to Kinoshita and Mura [4].

The aim of the present paper is to bring together the solutions of the separate problems. In doing so, we give a simplified treatment of the anisotropic inclusion problem. Furthermore it transpires that there is a fundamental connection between the interface problem and the ellipsoidal inclusion problem.

Ultimately it is shown that one can obtain stress and strain concentration factors at ellipsoidal inclusions in arbitrarily anisotropic materials without solving any boundary value problems and without knowing the Green's functions.

The interface problem

We use the notation of Laws [3]. Fourth order tensors are denoted by upper case, light face Latin letters, symmetric second order tensors are denoted by lower case, bold face, Greek letters, and vectors are denoted by lower case, bold face Latin letters. In addition it is often appropriate to use the usual suffix notation.

The interface problem is conveniently specified by decomposing any *symmetric* second order tensor $\boldsymbol{\tau}$ into the sum of its exterior part $\boldsymbol{\tau}_e$ and its interior part $\boldsymbol{\tau}_i$, where

$$\boldsymbol{\tau}_e = \boldsymbol{n} \otimes \boldsymbol{\tau n} + \boldsymbol{\tau n} \otimes \boldsymbol{n} - (\boldsymbol{n} \cdot \boldsymbol{\tau n}) \boldsymbol{n} \otimes \boldsymbol{n},$$

$$\boldsymbol{\tau}_i = (\boldsymbol{1} - \boldsymbol{n} \otimes \boldsymbol{n}) \boldsymbol{\tau} (\boldsymbol{1} - \boldsymbol{n} \otimes \boldsymbol{n}).$$

Here, $\boldsymbol{1}$ is the unit second order tensor and $\otimes$ denotes the tensor product.

Consider an interface between two media. On one side of the interface there is a linearly elastic solid with stress $\boldsymbol{\sigma}$, strain $\boldsymbol{\varepsilon}$ and stiffness tensor L and compliance tensor M. On the other side the stress is $\boldsymbol{\sigma}^*$ and the strain is $\boldsymbol{\varepsilon}^*$. In the interface

[1] This information is not necessary. In the terminology of [2] and [3] we only need to know the interior part of the strain and the exterior part of the stress.

problem one regards $\boldsymbol{\sigma}_e^*$ and $\boldsymbol{\varepsilon}_i^*$ as given. It can then be shown that $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are uniquely determined by the formulae, c.f. Laws [3],

$$\boldsymbol{\varepsilon} = C\boldsymbol{\varepsilon}^* + D\boldsymbol{\sigma}^*,$$
$$\boldsymbol{\sigma} = F\boldsymbol{\varepsilon}^* + G\boldsymbol{\sigma}^*.$$

Furthermore one can show that D is symmetric, $D = D^T$, and that

$$C = I - DL, \qquad G = LD,$$
$$F = L - LDL = F^T. \tag{1}$$

It then follows that

$$\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^* = D(\boldsymbol{\sigma}^* - L\boldsymbol{\varepsilon}^*), \tag{2}$$
$$\boldsymbol{\sigma} - \boldsymbol{\sigma}^* = F(\boldsymbol{\varepsilon}^* - M\boldsymbol{\sigma}^*).$$

A quick derivation of the component form of D can be given as follows. First, Hadamard's lemma implies that

$$\varepsilon_{ij} - \varepsilon_{ij}^* = \tfrac{1}{2}(a_i n_j + a_j n_i). \tag{3}$$

Second, continuity of surface traction requires that

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{\sigma}^*\boldsymbol{n},$$

and hence that

$$L(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^*)\boldsymbol{n} = (\boldsymbol{\sigma}^* - L\boldsymbol{\varepsilon}^*)\boldsymbol{n}. \tag{4}$$

If we temporarily put $\boldsymbol{\gamma} = \boldsymbol{\sigma}^* - L\boldsymbol{\varepsilon}^*$, then substitution of (3) into (4) leads to

$$\gamma_{il} n_l = L_{ijkl} n_j n_l a_k. \tag{5}$$

Since L_{ijkl} is positive definite and symmetric, the tensor $L_{ijkl} n_j n_l$ is invertible. We denote the inverse[2] by $\hat{g}_{ik}$:

$$L_{ijkl} n_j n_l = \hat{g}_{ik}^{-1}. \tag{6}$$

We can now solve (5) for a_i in the form

$$a_i = \hat{g}_{ik} \gamma_{kl} n_l.$$

Thus

$$\varepsilon_{ij} - \varepsilon_{ij}^* = \tfrac{1}{2}(n_i \hat{g}_{jk} n_l + n_j \hat{g}_{ik} n_l)\gamma_{kl}$$

and so

$$D_{ijkl} = \tfrac{1}{4}(n_i \hat{g}_{jk} n_l + n_j \hat{g}_{ik} n_l + n_i \hat{g}_{jl} n_k + n_j \hat{g}_{il} n_k). \tag{7}$$

We may interpret this as the general solution of the interface problem. We observe that (7) clearly shows that D is positive definite and symmetric.

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[2] Our notation anticipates the fact that $\hat{g}_{ij}$ is just the Fourier transform of the Green's function g_{ij}.

The inclusion problem

We begin by giving a convenient representation for the Green's function $g_{ij}(\boldsymbol{x})$ for an infinite solid with arbitrary anisotropy. The representation is implied, but not specifically stated, by Kinoshita and Mura[4]. In fact the result is readily obtainable from Synge's [5] calculation. The following results are given by Synge [5]:

$$
\begin{aligned}
g_{ij}(\boldsymbol{x}) &= \frac{1}{(2\pi)^3}\int \hat{g}_{ij}(\boldsymbol{k})\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})\,dV(\boldsymbol{k}) \\
&= \frac{1}{(2\pi)^3}\lim_{R\to\infty}\int_{\Omega}\hat{g}_{ij}(\boldsymbol{w})\frac{\sin(R\boldsymbol{w}\cdot\boldsymbol{x})}{\boldsymbol{w}\cdot\boldsymbol{x}}\,dS(\boldsymbol{w}),
\end{aligned}
\tag{8}
$$

where Ω denotes the surface of the unit sphere. Furthermore since

$$
\lim_{R\to\infty}\frac{1}{\pi}\frac{\sin(R\boldsymbol{w}\cdot\boldsymbol{x})}{\boldsymbol{w}\cdot\boldsymbol{x}} = \delta(\boldsymbol{w}\cdot\boldsymbol{x}),
$$

it follows that

$$
g_{ij}(\boldsymbol{x}) = \frac{1}{8\pi^2}\int_{\Omega}\hat{g}_{ij}(\boldsymbol{w})\delta(\boldsymbol{x}\cdot\boldsymbol{w})\,dS(\boldsymbol{w}). \tag{9}
$$

Now Synge [5] does not write down equation (9) but goes directly from (8) to the formula

$$
g_{ij}(\boldsymbol{x}) = \frac{1}{8\pi^2|\boldsymbol{x}|}\int \hat{g}_{ij}(\boldsymbol{w})\,ds, \tag{10}
$$

the integral being taken around the unit circle which has its centre at the origin and lies in the plane perpendicular to $\boldsymbol{x}$. As far as I am aware, all subsequent work, including that of Kinoshita and Mura [4], is based on (10). My aim is to show that the representation (9) helps to produce a simple solution to the standard inclusion problem.

Consider a bounded inclusion of arbitrary shape and volume V. The uniform transformation stress is denoted by $\boldsymbol{s}$. It is known that the strain field in the inclusion and in the infinite matrix is given by

$$
\varepsilon_{ij}(\boldsymbol{x}) = P_{ijkl}(\boldsymbol{x})s_{kl},
$$

where

$$
\begin{aligned}
P_{ijkl}(\boldsymbol{x}) = -\frac{1}{4}\int_V \Bigg\{&\frac{\partial^2 g_{ik}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j\,\partial x_l} + \frac{\partial^2 g_{jk}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i\,\partial x_l} \\
&+ \frac{\partial^2 g_{il}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j\,\partial x_k} + \frac{\partial^2 g_{jl}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i\,\partial x_k}\Bigg\}\,dV(\boldsymbol{y}).
\end{aligned}
\tag{11}
$$

Now

$$\int_V \frac{\partial^2 g_{ik}(\boldsymbol{x}-\boldsymbol{y})}{\partial x_j\, \partial x_l}\, dV(\boldsymbol{y}) = \frac{1}{8\pi^2}\int_V \frac{\partial^2}{\partial x_j\, \partial x_l}\int_\Omega \hat{g}_{ij}(\boldsymbol{w})\delta((\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{w})\, dS(\boldsymbol{w})\, dV(\boldsymbol{y})$$

$$= \frac{1}{8\pi^2}\frac{\partial^2}{\partial x_j\, \partial x_l}\int_\Omega \hat{g}_{ij}(\boldsymbol{w})\psi(\boldsymbol{x}, \boldsymbol{w})\, dS(\boldsymbol{w}) \tag{12}$$

where

$$\psi(\boldsymbol{x}, \boldsymbol{w}) = \int_V \delta((\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{w})\, dV(\boldsymbol{y}). \tag{13}$$

Thus $\psi(\boldsymbol{x}, \boldsymbol{w})$ is just the area of the section of V cut off by the plane through $\boldsymbol{x}$ perpendicular to $\boldsymbol{w}$. We note that equation (12), albeit with a longer derivation, has been given previously by Kinoshita and Mura [4].

The simplifying feature of the ellipsoidal inclusion is that one can easily perform the integration in (13) when $\boldsymbol{x}$ lies within the inclusion. Let the ellipsoid be given by

$$\alpha_{ij} y_i y_j = 1,$$

then a simple change of variable

$$z_i = \beta_{ij} y_j, \qquad \boldsymbol{\beta}^2 = \boldsymbol{\alpha},$$

transforms the ellipsoid into a sphere. It is then easy to show that for $\boldsymbol{x} \in V$

$$\psi(\boldsymbol{x}, \boldsymbol{w}) = \frac{\pi}{t^3 \alpha^{1/2}}\{t^2 - (\boldsymbol{x}\cdot \boldsymbol{w})^2\}, \tag{14}$$

where

$$t^2 = \alpha_{ij}^{-1} w_i w_j, \qquad \alpha = \det \alpha_{ij}.$$

From (11), (12) and (14) we now have, for $\boldsymbol{x} \in V$,

$$P_{ijkl} = \frac{1}{16\pi\alpha^{1/2}}\int_\Omega \frac{1}{t^3}\{\hat{g}_{ik} w_j w_l + \hat{g}_{jk} w_i w_l + \hat{g}_{il} w_j w_k + \hat{g}_{jl} w_i w_k\}\, dS(\boldsymbol{w}). \tag{15}$$

Thus P is constant for $\boldsymbol{x} \in V$. In addition comparison of (7) and (15) shows that for $\boldsymbol{x} \in V$

$$P = \frac{1}{4\pi a^{1/2}}\int_\Omega \frac{1}{t^3} D\, dS(\boldsymbol{w}). \tag{16}$$

Since D is positive definite (and obviously symmetric) so is P. It is remarkable that the solution of the inclusion problem is related to the solution of the interface problem through (16). Even more surprising is the fact that the connection extends still further. To see this we refer to Hill's [6] exposition of the ellipsoidal inclusion problem which contains details of all the tensors currently used in the problem.

Foremost amongst these is Eshelby's S tensor which is given by

$$S = PL. \tag{17}$$

Trivially, from (1), (16) and (17) we obtain the additional connection

$$S^T = \frac{1}{4\pi a^{1/2}} \int \frac{1}{t^3} G \, dS(\boldsymbol{w}).$$

It is also appropriate to emphasise here that equation (15) shows that the strain, and hence the stress, within the inclusion is constant—a result first proved by Eshelby [1]. The advantage of the above solution is that it provides explicit formulae for the evaluation of P and S.

Stress and strain concentration factors

Consider now the problem of an ellipsoidal inclusion with stiffness L^* and compliance M^* embedded in an infinite matrix with stiffness L and compliance M. The *uniform* stress at infinity is $\boldsymbol{\sigma}(\infty)$ and the corresponding uniform strain is $\boldsymbol{\varepsilon}(\infty)$. It is quite easy to show that the stress and strain within the cavity are also uniform and that the matrix stress, $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\varepsilon}$, at the interface are given by (c.f. Laws [3, equations (28) and (29)])

$$\boldsymbol{\varepsilon} = \{I + D(L^* - L)\}\{I + P(L^* - L)\}^{-1}\boldsymbol{\varepsilon}(\infty), \tag{18}$$

$$\boldsymbol{\sigma} = \{I + F(M^* - M)\}\{I + Q(M^* - M)\}^{-1}\boldsymbol{\sigma}(\infty), \tag{19}$$

where

$$Q = L - LPL. \tag{20}$$

Now we already have an explicit formula for D and a simple integral formula for P whatever the degree of anisotropy. Thus equation (18) furnishes an elementary formula for the evaluation of strain concentrations at the face of an ellipsoidal inclusion in an infinite matrix for *arbitrary anisotropy*. Likewise (19) is a formula for the determination of stress concentration factors. The significant feature of these formulae is that they involve only algebraic manipulations together with the evaluation of a known surface integral. Thus we are able to calculate both stress and strain concentration factors at an ellipsoidal inclusion in an arbitrarily anisotropic matrix without solving any boundary value problems and without knowing the form of the Green's functions. Some examples of the use of (18) in special cases are given by Hill [2] and Laws [3].

We note that the present method of deriving stress and strain concentration factors is no less powerful if we consider polynomial loading at infinity. As shown by Kunin and Sosnina [7] the stress and strain within the ellipsoidal inclusion are also polynomial; the same conclusion can also be inferred from the recent work of Asaro and Barnett [9]. Thus once the stress and strain in the inclusion are known, we can again use equations (2) to determine the matrix stress and strain at the interface.

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Note added in proof: The recent paper by J. R. Willis, J. Mech. Phys. Solids 23 (1975) 129 contains some relevant discussion of the inclusion problem.