Journal of Elasticity **22**: 157–183, 1989.

Nonlinear dynamics of oriented elastic solids
II. Propagation of solitons

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Received 9 February 1987; in revised form 20 January 1989

Abstract. Based on a continuum model for oriented elastic solids the set of nonlinear dispersive equations derived in Part I of this work allows one to investigate the nonlinear wave propagation of the soliton type. The equations govern the coupled rotation-displacement motions in connection with the linear elastic behavior and large-amplitude rotations of the director field. In the one-dimensional version of the equations and for two simple configurations an exhaustive study of solitons is presented. We show that the transverse and/or longitudinal elastic displacements are coupled to the rotational motion so that solitons, jointly in the rotation of the director and the elastic deformations, are exhibited. These solitons are solutions of a system of linear wave equations for the elastic displacements which are nonlinearly coupled to a sine-Gordon equation for the rotational motion. For each configuration, the solutions are numerically illustrated and the energy of the solitons is calculated. Finally, some applications of the continuum model and the related nonlinear dynamics to several physical situations are given and additional more complex problems are also evoked by way of conclusion.

1. Introduction

In a previous paper [1] we have constructed a continuum model for nonlinear oriented elastic media and deduced a set of nonlinear field equations governing both the usual deformation of the medium (at the macroscopic scale) and the evolution of the additional internal degrees of freedom (rotations) involving the microstructured description (at the microscopic scale). On the basis of this continuum approach we investigate the possible propagation of *nonlinear excitations* of the *soliton* type. Solitons emerge from a paradox, since, at first glance, it seems strange that a *nonlinear dispersive* equation could have soliton solutions. In fact, if either of these effects appear alone, there are no soliton solutions. It is only when the effects of nonlinearity are balanced by dispersion (in some cases but rarely, dissipation may play the role of dispersion) that solitons result. When only nonlinear terms are present, the wave steepens because of the continual supply of higher-frequency components. The profile of the wave then steepens until the function representing the wave profile is no longer single-valued and discontinuities are formed (shock waves). If only dispersion is present, different frequency components (of the Fourier expansion of the initial signal) of the wave propagate at different velocities with higher frequencies traveling more slowly so that the shape of the original signal spreads while propagating. Now, if both are present, the competition between the steepening due to *nonlinearities* and the spreading due to *dispersion* favors a traveling wave of constant profile and velocity [2, 3]. However, it would be better to refer to solitary wave than to a soliton since to have true solitons additional conditions must exist. A soliton is a solitary wave which preserves its shape and velocity in a collision with another such solitary wave. Nevertheless, in the forthcoming development the name of soliton will be used even if this is not a true soliton.*

The soliton concept occupies a key position in physics since solitons are elementary nonlinear excitations which allow one to model the nonlinear dynamics of the real world. Solitons are found in various areas of physics such as nonlinear optics, nonlinear plasma dynamics, nonlinear electronic transmission lines, hydrodynamics, dynamics of anharmonic atomic lattices, theory of commensurate-incommensurate structural phase transition, Josephson junctions as well as in biological materials and neurophysical models, to quote only a few examples [2, 4–7]. Solitons are exact propagative solutions of a large class of nonlinear dispersive partial differential equations such as the well-known Korteweg-de-Vries equation, the Boussinesq equation, the nonlinear Schrödinger equation, the sine-Gordon equation and the related double sine-Gordon equation, the Kadomtsev-Petviashvili equation, the Hirota equation, the reduced Maxwell-Bloch equations, the Toda lattice equation, etc. . . . [2, 8, 9]. Most of these equations are integrable by means of the inverse scattering transform (IST), or the Bäcklund transformation or also Hirota's method [8, 10–12]. In the present work, the equations of the model for oriented elastic media, in the one-dimensional version, are reduced to the classical wave equations for the elastic displacements (since a linear elastic behavior of the continuum is assumed) nonlinearity coupled to a sine-Gordon equation (a double sine-Gordon equation in more complex cases) governing the rotations of the director field. In the one-soliton case, the complete dynamical system, in spite of its greater or lesser complexity, admits an exact solution of the soliton type.

The present work is particularly devoted to nonlinear excitations connected with the nonlinear behavior in the internal degrees-of-freedom inherent in a

* The multicomponent soliton solutions of systems such as (8) and (37) below are not exact solitons. This can be proved only by studying the interaction between solitons (radiation may be generated during interaction). This was proved analytically and numerically by the authors (Second part of Ref. 16 below) for a system of the type (37), hence *a fortiori* for systems such as (8).

finer description of the microstructured medium. A good picture of such media can be given by a deformable lattice equipped, at each of its nodes, with a molecule considered as a small rigid body. If the existence of these molecules is not accounted for, then we recover the classical lattice, hence the usual elastic deformation of the continuum framework. If we now suppose that the molecules perform rigid-body rotational motions (which may have large amplitude) about the nodes of the lattice, the microgyration of the molecules provides additional degrees of freedom. If, moreover, an inertia is associated with the molecules, we can then define an angular momentum relating to the molecules and the dynamics of the coupled rotation–elastic displacements can be envisaged. The interactions involved in the microstructured media depend strongly on the physics of the micro-system and may be related to some phase-transition phenomena in particular cases. The interactions of special interest are those which permit nonlinear excitations of the soliton type (especially competitive interactions). The latter are found in media such as nematic liquid crystals [13], long chains of polymers [14, 15], molecular ferroelectric crystals of which we have already a microscopic model [16], and elastic chains of molecules (e.g. DNA) [17].

The basic equations developed in Part I are set forth in Section 2. The set of nonlinear equations governs the coupled rotation-displacement motions of an oriented medium with *one* director. In Section 3 an exhaustive study of solitons in the peculiar case of a simple configuration (so-called configuration A which is similar to the "Néel wall" in ferromagnets) is presented, and in this configuration the director rotation is effected about an axis which is perpendicular to the axis of propagation. A second simple configuration (configuration B identical to the "Bloch wall" in ferromagnets) is studied in Section 4. In this case the directors rotate about the axis of propagation. According to the configuration one or two elastic displacements (longitudinal and/or transverse displacements) are coupled to the rotational motion. In each situation the nonlinear dispersive equations thus obtained consist of a sine-Gordon equation for the rotational motion of the director and one or two linear wave equations for the elastic displacements, these equations being nonlinearly coupled. Exact solutions of the system are found, and coupled solitons in rotation and elastic deformation are placed in evidence. The stable solutions are numerically illustrated. For each configuration, comparisons with other approaches such as microscopic models and solitary waves in micropolar elastic media are made. The last section is devoted to the conclusions and remarks in which potential applications of this model and related nonlinear excitations are outlined. Finally, the extensions of the models to other problems (solitons in two-dimensions, or solitons with the composition of two rotations of the director, or the influence of an external field on the propagation of solitons) are evoked.

2. Basic equations

We recall the basic equations obtained in Ref. 1. These equations govern the classical deformation motion of an elastic body and the coupled rotational motion of the director field. The latter is associated with the internal degree of freedom of a deformable microstructure. These basic equations are [1]

$$\rho_0 \ddot{u}_i = \hat{C}_{ijpq}(\mathbf{d}) u_{p,qj} + \hat{\mathscr{C}}_{ijpq}(\mathbf{d}) \dot{u}_{p,qj} + \rho_0 D_{ij\ell}\, d_{\ell,j} + \rho_0^2 (Q_{ijpq} - \delta_{jp} B_{iq})(d_p d_q)_{,j}, \tag{1}$$

$$\rho_0 \dot{\mathscr{A}}_i = \varepsilon_{ipq} A_{pjmn}\, d_q d_{m,nj} - (\varepsilon_{ipq} \rho_0^2 B_{pj}) d_j d_q - \varepsilon_{ipq} \rho_0^2 \mathscr{N}_{pj}\, \dot{d}_j\, d_q + \varepsilon_{ipq} \rho_0^2 \mathscr{N}_{p\ell}\, d_k d_q \dot{u}_{\ell,k} + \rho_0^2 \varepsilon_{ipq} \hat{Q}_{\ell kpq} u_{\ell,k}, \tag{2}$$

where we have set

$$\hat{C}_{ijpq}(\mathbf{d}) = C_{ijpq} - \rho_0 D_{ijp} d_q - \rho_0 D_{pqi} d_j + \rho_0 \delta_{ip} D_{jq\ell} d_p - 2\rho_0^2 Q_{ijp\ell} d_p d_q - \rho_0^2 Q_{pqi\ell} d_\ell d_j + \rho_0^2 \delta_{ip} Q_{jq\ell k} d_k d_\ell + \rho_0^2 B_{ip} d_j d_q + \rho_0^2 \delta_{jp} B_{i\ell} d_\ell d_q + \rho_0^2 \delta_{ij} B_{p\ell} d_\ell d_q, \tag{3a}$$

$$\hat{\mathscr{C}}_{ijpq}(\mathbf{d}) = \mathscr{C}_{ijpq} + \rho_0^2 \mathscr{N}_{ip} d_q d_j, \tag{3b}$$

$$\hat{Q}_{\ell kpq}(\mathbf{d}) = B_{p\ell} d_k d_q - \rho_0^{-1} D_{\ell kp} d_q - Q_{\ell kmp} d_m d_q + \delta_{kp} B_{\ell m} d_m d_q, \tag{3c}$$

and the angular momentum per unit mass is given by

$$\mathbf{A} = I\, \dot{\mathbf{d}} \times \mathbf{d}. \tag{4}$$

The notation δ_{pq} is the Kronecker symbol and $\varepsilon_{ij\ell}$ is the permutation tensor. In Eqs. (1–4), ρ_0 is the mass density, C_{ijpq} is the tensor of elasticity coefficients, $\mathscr{C}_{ijpq}$ is the tensor of viscoelasticity coefficients, B_{ij} is the tensor denoting the phenomenological director interactions, A_{ijpq} is a tensor which represents the interactions between director gradients (this accounts for the spatial nonuniformity in the director field), $D_{ij\ell}$ is the linear coupling tensor between the elastic continuum and the director field, Q_{ijpq} is a nonlinear-coupling tensor between deformation and directors, and $\mathscr{N}_{ij}$ is the tensor of rotational relaxation. Also I is the inertial coefficient associated with the director. These equations have been obtained under some assumptions. This continuum model concerns a linear behavior with respect to classical elasticity but the nonlinearities in the director are kept. We have assumed that terms such as $(\nabla \mathbf{u})^2$, $\nabla \mathbf{u} \cdot \nabla \mathbf{d}$, $(\nabla \mathbf{d})^2$, $(\nabla \mathbf{u})\dot{\mathbf{d}}$, $\mathbf{d}(\nabla \dot{\mathbf{d}})$ either are negligible or they do not have any physical meaning.

Equation (1) is the motion equation of the classical continuum and is nonlinearly coupled to the director field. Eq. (2) governs the rotational motion of the director field. Note that Eq. (1) reduces to a classical elastic-wave equation if the director field is discarded. Eq. (2) is strongly nonlinear since the director field is present in all terms in this equation. With the view of examining the propagation of nonlinear waves, we somewhat simplify the above equations. We first discard the dissipative terms (in fact the latter might be considered as small perturbations), and next we only consider the case of centrosymmetric media, so that the linear-coupling tensor $D_{ij\ell}$ vanishes. In order to proceed further we consider only two special cases where the rotational motion of the director field is reduced to a pure finite rotation about one crystallographic axis. Furthermore, we consider that all the unknown quantities, that is, the elastic displacement **u** and the director **d**, depend on one spatial variable only and obviously on time. However, some involved cases can be considered in further works, for instance, the soliton propagation in the two dimensional (spatial) case, or the soliton propagation in the case of two rotation angles of the director field. Despite the simplifying hypotheses

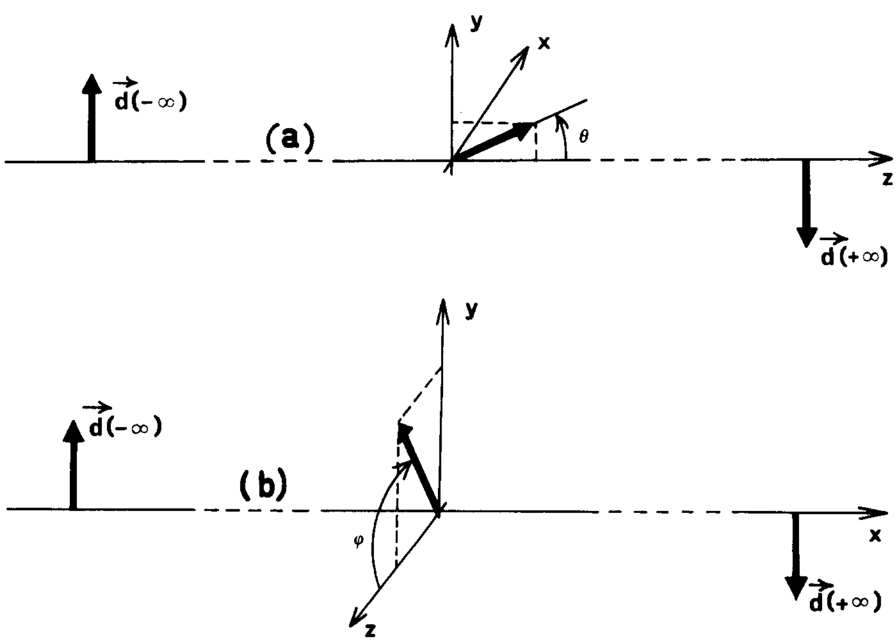

Fig. 1. Two special configurations in the one-dimensional case: (a) rotation about an axis perpendicular to the axis of propagation and (b) rotation about the axis of propagation.

considered, the model remains rich enough to offer propagation problems which are the subject of the paper. In one case we examine the soliton propagation in a configuration such that the rotation takes place about an axis perpendicular to the spatial axis (see Fig. 1a). The second studied configuration is distinguished from the first by the fact that the rotation takes place about the spatial axis (see Fig. 1b). In the framework of these two configurations, we examine the possible propagation of solitons in both the rotation of the director and the elastic deformation.

3. Solitons in the configuration A

A. *Equations in configuration A*

This configuration can be referred to as "Néel wall" by analogy with the moving domain wall in thin ferromagnetic films [18, 19]. Referring to Fig. 1a, the elastic displacement and the director field can be chosen as

$$\mathbf{u} = (0, u_2, u_3), \tag{5a}$$

$$\mathbf{d} = d_0(0, \sin\theta, \cos\theta), \tag{5b}$$

where u_2, u_3 and θ depend on the spatial variable z and time t. We consider a medium which possesses the hexagonal symmetry in class 6/m or 6/mmm [20]. The anisotropy axis is the z-axis. The system (1–4) is reduced to the following one (in dimensional notation),

$$\rho_0\ddot{u}_3 = \hat{C}_{33}u_{3,zz} + \tfrac{1}{2}\rho_0^2 d_0^2(Q_{33} - Q_{32} - 2B_{33})(\cos 2\theta + 1)_{,z}, \tag{6a}$$

$$\rho_0\ddot{u}_2 = \hat{C}_{44}u_{2,zz} + \tfrac{1}{2}\rho_0^2 d_0^2(Q_{44} - B_{22})(\sin 2\theta)_{,z}, \tag{6b}$$

$$\begin{aligned}\rho_0 I\ddot{\theta} = {} & A\theta_{,zz} + \tfrac{1}{2}\rho_0^2(B_{33} - B_{22})\sin 2\theta \\ & + \tfrac{1}{2}\rho_0^2(Q_{33} - Q_{22} - 2B_{33})(\sin 2\theta)u_{3,z} \\ & - \rho_0^2(Q_{44} - B_{22})(\cos 2\theta)u_{2,z},\end{aligned} \tag{6c}$$

where we have used the Voigt notation for the tensorial coefficients. These equations will be rewritten by using the following new functions and variables,

$$\phi = 2\theta, \quad U = \sqrt{(2/I)}u_3/d_0, \quad V = 2u_2/d_0\sqrt{I}, \tag{7a}$$

$$\tau = t/\omega_A, \quad Z = z/\delta_A, \tag{7b}$$

where we have set

$$\omega_A = (I/\rho_0\chi)^{1/2}, \quad \delta_A = (A_{44}/\rho_0^2\chi)^{1/2}, \tag{7c}$$

and we define

$$c_L = \sqrt{\hat{C}_{33}}/c_{RA}, \quad c_T = \sqrt{\hat{C}_{44}}/c_{RA}, \tag{7d}$$

$$\alpha = \frac{\rho_0 d_0}{c_{RA}}(2\rho_0\chi)^{-1/2}(Q_{33} - Q_{32} - 2B_{33}), \tag{7e}$$

$$\beta = \frac{\rho_0 d_0}{c_{RA}}(\rho_0\chi)^{-1/2}(Q_{44} - B_{22}), \tag{7f}$$

$$\hat{C}_{33} = [C_{33} + \tfrac{1}{2}\rho_0^2 d_0^2(Q_{32} - 2Q_{33} + 3B_{33})]/\rho_0, \tag{7g}$$

$$\hat{C}_{44} = [C_{44} + \tfrac{1}{2}\rho_0^2 d_0^2(Q_{33} - Q_{44} + Q_{32} + B_{22})]/\rho_0, \tag{7h}$$

$$c_{RA}^2 = A_{44}/\rho_0 I, \tag{7i}$$

$$\chi = B_{33} - B_{22}. \tag{7j}$$

In these notations, time and space are nondimensionalized with the help of a characteristic length δ_A and a characteristic frequency ω_A; c_L and c_T are the elastic longitudinal and transverse wave velocities, respectively, modified by the director field (terms depending on d_0^2).

In the above changes of variables we have intentionally supposed that $\chi > 0$ (or $B_{33} > B_{22}$); in the opposite case we would have a minus sign before the $\sin 2\theta$ term of Eq. (6c). The coefficient χ may depend on temperature if we study the problem of nonlinear phenomena connected with phase-transition. We then obtain the following system in dimensionless notation.

$$\frac{\partial^2 U}{\partial \tau^2} - c_L^2 \frac{\partial^2 U}{\partial Z^2} = \alpha \frac{\partial}{\partial Z}(1 + \cos\phi), \tag{8a}$$

$$\frac{\partial^2 V}{\partial \tau^2} - c_T^2 \frac{\partial^2 V}{\partial Z^2} = \beta \frac{\partial}{\partial Z}(\sin\phi), \tag{8b}$$

$$\frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial Z^2} = \sin\phi + \alpha \sin\phi \frac{\partial U}{\partial Z} - \beta \cos\phi \frac{\partial V}{\partial Z}. \tag{8c}$$

The structure of this system of dispersive nonlinear hyperbolic equations is of special interest. It consists of two wave equations for the two elastic displacements (U and V) and a sine-Gordon equation for the rotation ϕ (twice the physical rotation θ), each elastic displacement being nonlinearly coupled to ϕ through the coupling coefficient α or β (note the analogy with the electrostrictive coupling). If $\alpha = 0$, we recover the system of equations deduced in Ref. 16 which governs an anharmonic chain equipped with microscopic electric dipoles. The latter microscopic model, after passing to a continuum approximation, is reduced to a macroscopic model for oriented media with one director. Now, the next step of the study is the investigation of the solutions of the system (8a–c).

B. *Single-soliton solution*

We concentrate here on the case of one-soliton solutions of the system (8a–c). Some obvious properties of the system can be emphasized. Let (U, V, ϕ) be a solution. Then both (U, $-V$, $-\phi$) and (U, V, $\phi + 2k$) are also solutions. The system (9a–c) has the following uniform static elementary solution,

$$U = U_0 = \text{const.}, \quad V = V_0 = \text{const.}, \quad \phi = k\pi (k \in \mathbb{Z}). \tag{9}$$

Nevertheless, a more interesting solution can be looked for in the form of propagative waves, that is, functions of a single phase variable $\xi = QZ - \Omega\tau + \xi_0$, where Q and Ω may be referred to as a pseudo-wave number and a pseudo-circular frequency which must satisfy a certain "dispersion" relation. In this case the system (8) becomes

$$(\Omega^2 - \Omega_L^2)\frac{d^2U}{d\xi^2} = \alpha Q \frac{d}{d\xi}(1 + \cos\phi), \tag{10a}$$

$$(\Omega^2 - \Omega_T^2)\frac{d^2V}{d\xi^2} = \beta Q \frac{d}{d\xi}(\sin\phi), \tag{10b}$$

$$(\Omega^2 - Q^2)\frac{d^2\phi}{d\xi^2} = \sin\phi + \alpha Q \frac{dU}{d\xi}\sin\phi - \beta Q \frac{dV}{d\xi}\cos\phi, \tag{10c}$$

where we have set

$$\Omega_L^2 = c_L^2 Q^2, \quad \Omega_T^2 = c_T^2 Q^2. \tag{11}$$

Eqs. (10a) and (10b) integrate once with respect to ξ to produce

$$(\Omega^2 - \Omega_L^2)\frac{\mathrm{d}U}{\mathrm{d}\xi} = \alpha Q[\cos\phi - (-1)^k], \tag{12a}$$

$$(\Omega^2 - \Omega_T^2)\frac{\mathrm{d}V}{\mathrm{d}\xi} = \beta Q \sin\phi, \tag{12b}$$

where it has been assumed that the elastic deformations ($\partial U/\partial Z$ and $\partial V/dZ$) vanish when ϕ goes to $k\pi$, and Ω differs from $\pm\Omega_L$ and $\pm\Omega_T$. Now, on substituting $\mathrm{d}U/\mathrm{d}\xi$ and $\mathrm{d}V/\mathrm{d}\xi$ from Eqs. (12a) and (12b) into Eq. (10c), we obtain

$$(\hat{\Omega}^2 - \hat{Q}^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = \sin\phi + \gamma(\hat{\Omega}, \hat{Q}) \sin 2\phi, \tag{13}$$

where

$$\hat{\Omega} = \Omega/\sqrt{\mu}, \quad \hat{Q} = Q/\sqrt{\mu}, \tag{14a}$$

$$2\gamma(\hat{\Omega}, \hat{Q}) = \frac{\hat{Q}^2}{\mu}\left(\frac{\beta^2}{\hat{\Omega}_T^2 - \hat{\Omega}^2} - \frac{\alpha^2}{\hat{\Omega}_L^2 - \hat{\Omega}^2}\right), \tag{14b}$$

$$\mu = 1 + (-1)^k \frac{\alpha^2 \hat{Q}^2}{\hat{\Omega}_L^2 - \hat{\Omega}^2}. \tag{14c}$$

We see that the problem of one-soliton solutions of the somewhat complicated system (8a–c) is equivalent to the solution of the nonlinear ordinary differential equation (13), which can be formally deduced from a *double sine-Gordon equation* when a one-soliton solution is sought [21–24]. Eq. (13) possesses a first integral which can be written as

$$\tfrac{1}{2}(\hat{\Omega}^2 - \hat{Q}^2)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = \mathscr{V}(\phi) + E_0, \tag{15a}$$

$$\mathscr{V}(\phi) = -\cos\phi - \tfrac{1}{2}\gamma \cos 2\phi, \tag{15b}$$

where E_0 is the integration constant which can be related to the total energy of the system. Eq. (15a) stands for the equation governing the motion of a particle of mass $(\hat{\Omega}^2 - \hat{Q}^2)$ in a periodic potential $\mathscr{V}(\phi)$ (here the "mass" may be negative in some cases). In a general manner Eq. (15a) has periodic propagation solutions depending on the energy E_0 (cnoïdal waves) and which can be expressed with the help of Jacobian elliptic integrals [25]. Here, however, we consider the solution of Eq. (15) such that $\phi \to k\pi$ at infinity, and then $E_0 = \gamma/2 + (-1)^k$. In this case the first integral of Eq. (15a) can be written as

$$(\hat{\Omega}^2 - \hat{Q}^2)\left(\frac{d\phi}{d\xi}\right)^2 = 2[(-1)^k - \cos\phi] + \gamma(1 - \cos 2\phi). \tag{16}$$

With a view to seeking a solution of Eq. (13) or (16) which satisfies the boundary condition $\phi \to k\pi$ as $|\xi| \to +\infty$, we consider the following change of function,

$$\phi = 2\tan^{-1}\varphi, \quad (\lambda = \hat{\Omega}^2 - \hat{Q}^2). \tag{17}$$

Substituting ϕ into Eq. (13) and (16) and after some manipulations consisting in eliminating the undesirable terms between Eqs. (13) and (16), we arrive at

$$\lambda \frac{d^2\varphi}{d\xi^2} = \varphi[2\gamma + (-1)^k] + \varphi^3[1 + (-1)^k]. \tag{18}$$

We immediately notice that the type of solution in φ of Eq. (18) depends on whether k is odd or not. The boundary conditions are

$$\varphi \to 0 \quad \text{as } |\xi| \to +\infty \quad \text{for } k \text{ even}, \tag{19a}$$

$$|\varphi| \to +\infty \quad \text{as } |\xi| \to +\infty \quad \text{for } k \text{ odd}.$$

Note that the second condition can lead to $\varphi \to \pm\infty$ as $\xi \to \pm\infty$ or $\varphi \to +\infty$ as $\xi \to \pm\infty$, which correspond respectively to the conditions $\phi \to \pm\pi$ as $\xi \to \pm\infty$ or $\phi \to \pi$ as $\xi \to \pm\infty$. In the case of even k, the φ^3 term does not vanish and we have a "phi-four" equation. But, here, on account of the boundary condition (19a) and notation (14), it can be shown that, in these conditions, Eq. (18) does not possess a solution. For odd k, Eq. (18) is simpler since it is reduced to the ordinary *linear* differential equation

$$\lambda \frac{d^2\varphi}{d\xi^2} = (2\gamma - 1)\varphi. \tag{20}$$

The solution is subject to the boundary condition (19b) which implies that $\nu \equiv (2\gamma - 1)/\lambda > 0$. In addition, the compatibility for both φ and ϕ, related through eqn. (17), to satisfy simultaneously eqns. (20) and (13), requires that $\nu = 1$, thus providing the looked for "dispersion" relation

$$\lambda = 2\gamma - 1,$$

which can be rewritten as

$$[\Omega^2 - (Q^2 - 1)](\Omega^2 - \Omega_T^2) + \beta^2 Q^2 = 0. \tag{21}$$

We may notice that the dispersion relation (21) does not depend on the longitudinal quantities since neither the elastic longitudinal wave velocity nor the coupling coefficient α contribute in the dispersion relation, whereas the coupling between the longitudinal displacement and director rotation *does* exist in the system (8). The dispersion relation thus obtained is formally the conjugate relation of the dispersion relation for the linear case [26] (we have to change Ω and Q into $i\Omega$ and iQ, respectively). This situation is quite similar to that of solitons in elastic chains of dipoles [16]. The solutions of the dispersion relation (21) are represented in solid lines in Fig. 2; the branch (a) corresponds to the case $\Omega < Q$ while the branch (b) is very close to Ω_T (branch of elastic transverse waves), up to terms of order β^2, and corresponds to the soliton which propagates practically at the velocity of a transverse

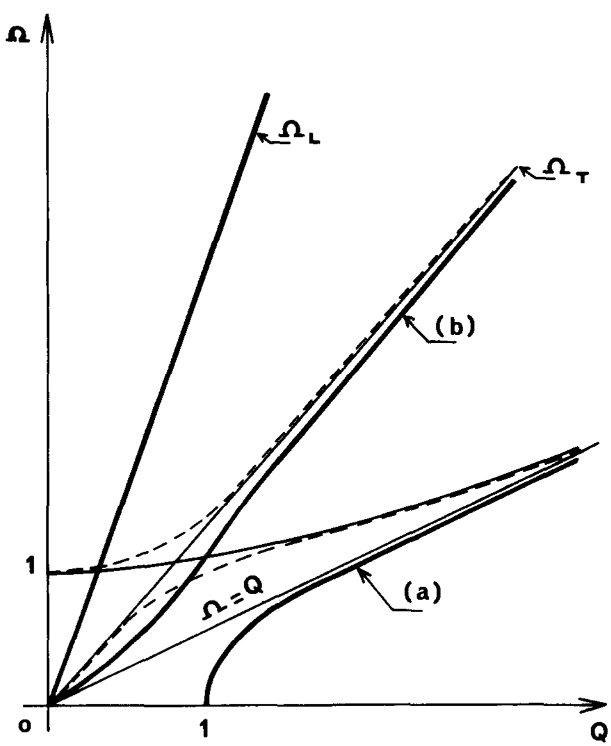

Fig. 2. Dispersion relation for single-soliton solution: branches (a) and (b) solutions of Eq. (21), Ω_L: longitudinal elastic mode, Ω_T: transverse elastic mode and coupled linear modes in broken lines.

elastic mode. In the same figure the linear case has been plotted in broken lines. However, two cases can be considered, either $\lambda > 0$ or $\lambda < 0$. Only the second case is interesting since it leads to a stable solution since we must have a subsonic soliton ($\Omega/Q < 1$) [27, 28]. The solution then corresponds to the branch (a) of the dispersion curves (Fig. 2). The solution is then given by

$$\phi = -2\tan^{-1}\left(\frac{\sinh \xi}{Q\sqrt{1-c^2}}\right). \tag{22}$$

The minus sign in Eq. (22) is only a matter of choice. In Eq. (22) $c = \Omega/Q$ is the phase velocity of the wave. The solution (22) represents the transition from the position $\phi = \pi$ to the position $\phi = -\pi$. Moreover, we can compute the strain state of the medium. This is characterized by

$$\frac{\partial U}{\partial Z} = \frac{2\alpha}{c^2 - c_L^2} \cdot \frac{(1-2\gamma)}{\cosh^2 \xi - 2\gamma}, \tag{23a}$$

$$\frac{\partial V}{\partial Z} = \frac{2\beta Q\sqrt{1-c^2}}{c^2 - c_L^2} \cdot \frac{\sinh \xi}{\cosh^2 \xi - 2\gamma}. \tag{23b}$$

It is possible to determine the solutions in longitudinal and transverse displacements by integrating the solutions (23a) and (24b), but here we only consider the strain state of the medium.

In Figs. 3 a, b and c we give numerical illustrations* of the solutions (22), (23a) and (23b) in space-time representation. The graph in Fig. 3a illustrates the rotation motion of the director from the state $\theta = \pi/2$ for $Z \to -\infty$ to the state $\theta = -\pi/2$ for $Z \to +\infty$; here we have a so-called "*kink*" *soliton*. In Fig. 3b we have the elongational deformation (Eq. (23a)) generated through the coupling coefficient α by the soliton in rotation; this is a "*hump*" *soliton* which is essentially nonzero in the thickness of the kink. Finally, the shear deformation (Eq. (23b)) is given in Fig. 3c; this deformation changes sign when the angle ϕ passes through zero. This is a sort of "*double hump*" *soliton* generated by the soliton in rotation as well.

C. *Soliton energy*

The system (9) can be derived from a Hamiltonian formulation of which the Hamiltonian for the whole system is given by

$$\mathscr{H} = \int_{-\infty}^{+\infty} \mathscr{h}(\phi, U, V)\, \mathrm{d}Z, \tag{24}$$

* Graphs are drawn without specific units and are obtained by solving the PDEs numerically and *not* by feeding in the analytic solution.

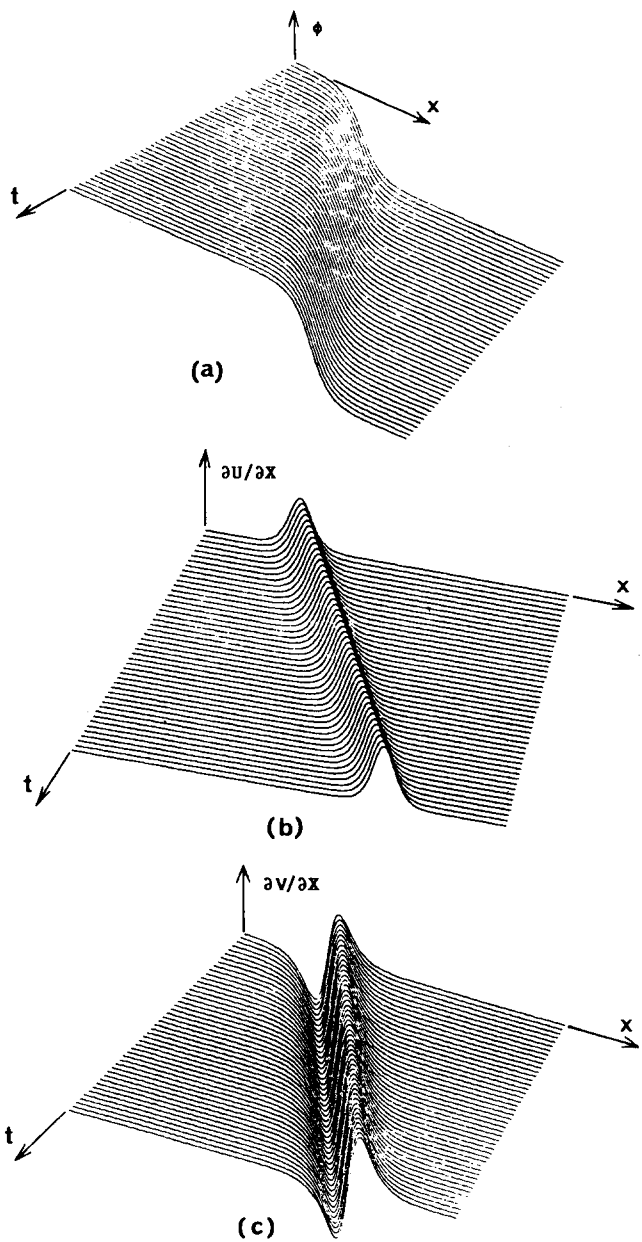

Fig. 3. Numerical illustrations for single-soliton solution in the case of configuration A: (a) soliton in rotation, (b) soliton in elongation and (c) soliton in shear.

with a density

$$\mathscr{h}(\phi, U, V) = \tfrac{1}{2}[(U_\tau)^2 + (V_\tau)^2 + (\phi_\tau)^2] + \tfrac{1}{2}[c_L^2(U_Z)^2 + c_T^2(V_Z)^2 + (\phi_Z)^2]$$

$$+ (1 + \cos\phi) + \alpha U_Z(1 + \cos\phi) + \beta V_Z \sin\phi. \tag{25}$$

On considering the relations (12a) and (13b) and the solution ϕ in terms of the phase variable ξ, the Hamiltonian (25) becomes

$$\mathscr{H} = \tfrac{1}{2}A \int_{-\infty}^{+\infty} [(\hat{\Omega}^2 - \hat{Q}^2)(\phi_\xi)^2 + 2(1 + \cos\phi) - \tilde{\gamma}(1 - \cos 2\phi)]\, \mathrm{d}\xi, \tag{26}$$

where we have set

$$\tilde{\gamma} = -B/A, \tag{27a}$$

$$B = \frac{1}{2}\left[\beta^2 \frac{3c^2 - c_T^2}{(c^2 - c_T^2)^2} - \alpha^2 \frac{3c^2 - c_L^2}{(c^2 - c_L^2)^2}\right], \tag{27b}$$

$$A = 1 + \alpha^2 \frac{2c^2 - c_L^2}{(c^2 - c_L^2)^2}, \tag{27c}$$

$$\hat{\Omega} = \Omega/A, \quad \hat{Q} = Q/A. \tag{27d}$$

The Hamiltonian (26) can be formally deduced from a system governed by a double sine-Gordon equation if we look for a solution depending only on $\xi = \hat{Q}Z - \hat{\Omega}\tau + \xi_0$. Now we substitute the solution (22) into the expression (26), and after a rather lengthy calculation we obtain

$$\mathscr{H}_A = 4\left\{\left[A + \frac{Q^2 - 1}{1 - 2\gamma} + \frac{1}{2}\left(1 + \frac{B}{\gamma}\right)\right]\left(\frac{1 - 2\gamma}{2\gamma}\right)^{1/2}\right.$$

$$\left. \times \sin^{-1}(\sqrt{2\gamma}) + Q^2 - 1 + \frac{1 - 2\gamma}{2}\left(1 - \frac{B}{\gamma}\right)\right\}, \tag{28}$$

where the various parameters introduced in this expression have already been defined. On account of the dispersion relation (21) the total energy (28) depends thus on the wave number Q, the elastic wave velocities c_L and c_T, and the coupling coefficients α and β. If the latter are neglected, then Eq. (28) simplifies to give the usual simple result (sine-Gordon model)

$$\mathscr{H}_{A_0} = 8Q. \tag{29}$$

The thickness of the soliton (defined similarly to the thickness of a structured shock) is given by

$$\Delta_A = \pi Q\sqrt{1-c^2}. \tag{30}$$

When the velocity of the soliton increases towards unity the thickness tends to zero. Going back to the real physical dimensions we have

$$\Delta_A = \pi\left(1-\frac{c^2}{c_{RA}^2}\right)^{1/2}\delta_A, \tag{31}$$

where c_{RA} and δ_A are defined in Eq. (8h) and (8c), respectively. The expression (31) may be useful if the model deals with the problem of the structure in domains and walls in ferroelectric crystals so that the director field has the physical dimension of an electric polarization and Δ_A represents the thickness of a domain wall.

D. *Remark*

Let us go back to the problem of the sign of χ in Eq. (6c). In the case where $\chi < 0$, we set $\chi = -\chi'$ and $\chi' > 0$ and only Eq. (6c) changes and becomes

$$\frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial^2\phi}{\partial Z^2} = -\sin\phi + \alpha\sin\phi\frac{\partial U}{\partial Z} - \beta\cos\phi\frac{\partial V}{\partial Z}. \tag{32}$$

In nondimensional notation, we then consider the following change of functions,

$$\phi = \pi + \phi', \tag{33a}$$

$$U = -U', \tag{33b}$$

$$V = -V'. \tag{33c}$$

Those new functions are substituted into Eqs. (6a–b) and Eq. (32) so that the new functions ϕ', U' and V' satisfy the system (6). The study of this case is the same as in the case $\chi > 0$. However, the topology of the soliton is different. Here ϕ decreases from 2π to 0 and the solitons in deformation have opposite signs. This means that the physical angle θ $(=\phi/2)$ of the director varies monotonously from π to 0. Fig. 4a and 4b show numerical pictures of the director orientation in the cases $\chi > 0$ (Fig. 4a) and $\chi < 0$ (Fig. 4b).

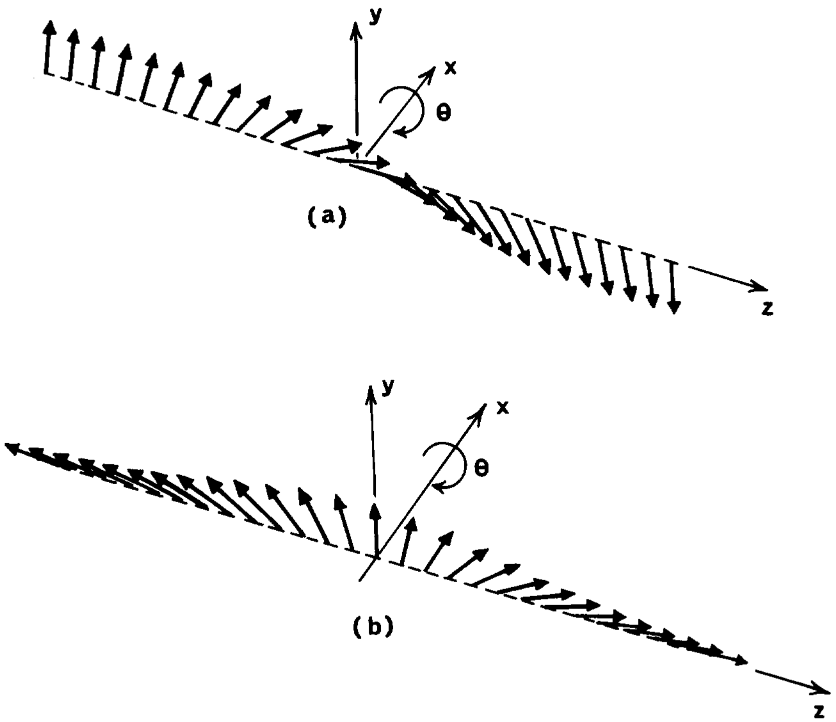

Fig. 4. Numerically obtained picture of the director orientation in the case of the configuration A: (a) for $\chi > 0$ and (b) $\chi < 0$.

4. Solitons in configuration B

A. *Equations in the configuration B*

Referring to Fig. 1b, we force the director to rotate about the propagation axis. This configuration can be referred to as a "Bloch wall" by analogy with the domain walls in ferromagnetic crystals [18, 19]. The elastic displacement and the director field are taken in the form,

$$\mathbf{u} = (u_1, u_2, 0), \tag{34a}$$

$$\mathbf{d} = d_0(0, \sin\varphi, \cos\varphi), \tag{34b}$$

where u_1, u_2 and φ depend only on the spatial variable x and time t. We consider a medium possessing the same crystalline symmetry as in the case of the configuration A. Under these conditions the system (1–4) is reduced to the following one,

$$\rho_0 \ddot{u}_1 = \hat{C}_{11} u_{1,xx} + \tfrac{1}{2}\rho_0^2 d_0^2 (Q_{13} - Q_{12})(1 + \cos 2\varphi)_{,x}, \tag{35a}$$

$$\rho_0 \ddot{u}_2 = \hat{C}_{66} u_{2,xx}, \tag{35b}$$

$$\rho_0 I \ddot{\varphi} = A_{66} \varphi_{,xx} + \tfrac{1}{2}\rho_0^2 d_0^2 (B_{33} - B_{22}) \sin 2\varphi$$

$$+ \tfrac{1}{2}\rho_0^2 d_0^2 (Q_{13} - Q_{12})(\sin 2\varphi) u_{1,x}. \tag{35c}$$

We have obviously used the same notation as in the previous configuration. The following change of functions and variables is effected,

$$\psi = 2\varphi, \quad U = \sqrt{2/I}\, u_1/d_0, \quad V = 2u_2/d_0\sqrt{I}, \tag{36a}$$

$$\tau = t/\omega_B, \quad X = x/\delta_B, \tag{36b}$$

where

$$\omega_B = (I/\rho_0 \chi)^{1/2}, \quad \delta_B = (A_{66}/\rho_0^2 \chi)^{1/2}. \tag{36c}$$

On using the same expressions as in Eqs. (7d) and (7e) but with the new coefficients

$$c_{RB}^2 = A_{66}/\rho_0 I, \tag{36d}$$

$$\alpha = \rho_0 d_0 (\rho_0 X)^{-1/2} (Q_{13} - Q_{12})/c_{RB}, \tag{36e}$$

$$\hat{C}_{11} = [C_{11} + \tfrac{1}{2}\rho_0^2 d_0^2 (Q_{12} + Q_{13})]/\rho_0, \quad c_L^2 = \hat{C}_{11}/c_{RB}, \tag{36f}$$

$$\hat{C}_{66} = [C_{66} + \tfrac{1}{2}\rho_0^2 d_0^2 (Q_{12} + Q_{13})]/\rho_0, \quad c_T^2 = \hat{C}_{66}/c_{RB}, \tag{36g}$$

the system (35) can be written in the following nondimensional form,

$$\frac{\partial^2 U}{\partial \tau^2} - c_L^2 \frac{\partial^2 U}{\partial X^2} = \alpha \frac{\partial}{\partial X}(1 + \cos \psi), \tag{37a}$$

$$\frac{\partial^2 V}{\partial \tau^2} - c_T^2 \frac{\partial^2 V}{\partial X^2} = 0, \tag{37b}$$

$$\frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial^2 \psi}{\partial X^2} = \sin \psi + \alpha \sin \psi \frac{\partial U}{\partial X}. \tag{37c}$$

We notice that the system (37) has the same structure as that obtained in the case of the configuration A, but in the present case $\beta = 0$. In Eqs (36b) and (36c) a characteristic length δ_B (connected with the thickness of the soliton) and a characteristic frequency ω_B have been introduced in order to have nondimensional time and space. Furthermore, in these changes of variables (36b) we have intentionally supposed that $\chi > 0$ – or $B_{33} > B_{22}$ – but in the opposite case we would have $(-\sin\psi)$ instead of $\sin\psi$ on the right hand side of Eq. (37c). The system (37) consists of two wave equations for longitudinal and transverse elastic displacements and a sine-Gordon equation for the rotations of the directors. Note that, whereas the equations of the system (8) are all three coupled in the case of the configuration A, here the transverse elastic displacement uncouples, only the longitudinal elastic displacement remaining nonlinearly coupled to the rotation.

Once the study of the configuration A is achieved, it is easy to examine the present configuration. Indeed, it is sufficient to set $\beta = 0$ in all the results obtained in Section 2 and make the necessary changes in notation.

B. *Single soliton solution*

The system (37) has the same properties as the system (8). As usual, we look for a solution in the form of propagative waves. The functions U, V and ψ depend thus on the phase variable $\xi = QX - \Omega\tau + \xi_0$. Eqs. (37a–c) become

$$(\Omega^2 - \Omega_L^2)\frac{d^2U}{d\xi^2} = \alpha Q\frac{d}{d\xi}(1 + \cos\psi), \tag{38a}$$

$$(\Omega^2 - \Omega_T^2)\frac{d^2V}{d\xi^2} = 0, \tag{38b}$$

$$(\Omega^2 - Q^2)\frac{d^2\psi}{d\xi^2} = \sin\psi + \alpha Q\frac{dU}{d\xi}\sin\psi, \tag{38c}$$

where Ω_L and Ω_T are defined by Eqs. (11). Eq. (38b) leads to the conditions that either $\Omega = \pm\Omega_T$ or $dV/d\xi = \text{const.}$; the definite choice is made in the forthcoming section. On eliminating $dU/d\xi$ between Eqs. (38a) and (38c), one obtains

$$(\hat{\Omega}^2 - \hat{Q}^2)\frac{d^2\psi}{d\xi^2} = \sin\psi + \bar{\gamma}(\Omega, Q)\sin 2\psi, \tag{39}$$

where

$$2\bar{\gamma} = -\frac{1}{\mu}\frac{\alpha^2 Q^2}{\Omega_L^2 - \Omega^2}, \tag{40}$$

with μ defined by Eq. (14c) (k odd) and α given by Eq. (36e). Since $\beta = 0$, the dispersion relation (21) is notably simplified to yield

$$(\Omega^2 - \Omega_T^2)[\Omega^2 - (Q^2 - 1)] = 0, \tag{41}$$

where we recover the condition $\Omega = \pm\Omega_T$. However, if $\Omega \neq \pm\Omega_T$, we must have $\mathrm{d}V/\mathrm{d}\xi = \mathrm{cst.}$ and the dispersion relation (41) gives

$$\Omega^2 = Q^2 - 1, \tag{42}$$

which is, in fact, the usual dispersion relation associated with the sine-Gordon soliton. The dispersion relation (42) is of interest since it is not altered by the elastic deformation, even if the coupling plays a predominant role in the wave

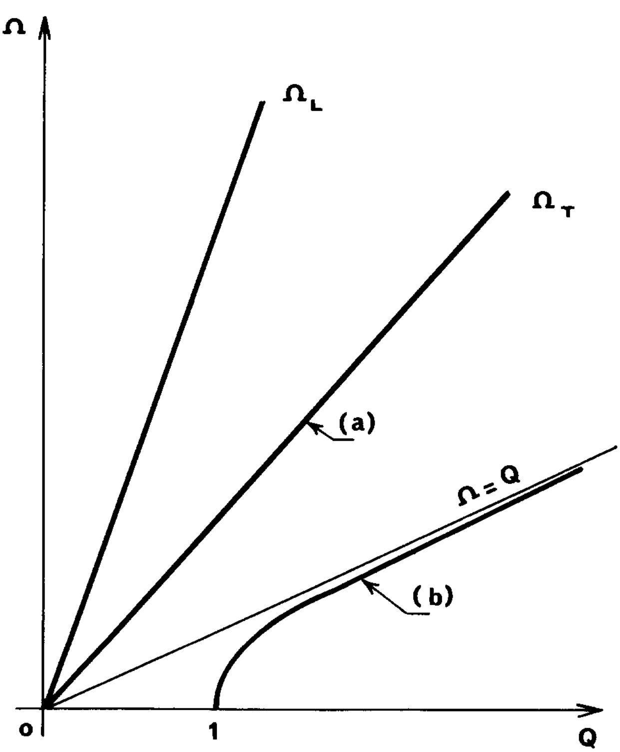

Fig. 5. Dispersion relation for single soliton solution for the configuration *B*: (a) the pure transversal elastic mode and (b) the pure subsonic soliton mode.

motion. In Fig. 5 the two solutions of the dispersion relation (41) are sketched: the wave (a) is a pure elastic transverse mode, and the branch (b) corresponds to the subsonic soliton stable solution given by

$$\phi = -2\tan^{-1}\left(\frac{\sinh \xi}{Q\sqrt{1-c^2}}\right). \tag{43}$$

This solution represents the rotation of the director field from the state $\psi = \pi$ (or $\varphi = \pi/2$) to the state $\psi = -\pi$ (or $\varphi = -\pi/2$). In addition, the strain state generated by the soliton in rotation is given by

$$\frac{\partial U}{\partial X} = \frac{2\alpha}{c^2 - c_L^2}\frac{1-2\gamma}{\cosh^2 \xi - 2\gamma}. \tag{44}$$

Figures 6a and 6b gather the numerical illustrations, in space-time representation, for the soliton in rotation (Fig. 6a) and the accompanying soliton in elongation deformation (Fig. 6b). These curves are practically not different from those obtained in the case of the configuration A.

C. *Soliton energy*

The calculations are the same as those of configuration A. It is sufficient to adjust the result (28) to the present case, which yields

$$\begin{aligned}\mathscr{H}_B = 4\Bigg\{&\left[A + \mu(Q^2-1) + \frac{1}{2}\left(1 - \mu\frac{1-A}{1-\mu}\right)\right](\mu-1)^{1/2}\tan^{-1}(\mu-1)^{1/2} \\ &+ (Q^2-1) + \tfrac{1}{2}\mu\left(1 + \frac{1-A}{1-\mu}\cdot\mu\right)\Bigg\},\end{aligned} \tag{45}$$

where A and μ have been defined by Eq. (27c) and Eq. (14c), respectively. Finally, the thickness of the soliton can be written in dimensional notation as

$$\Delta_B = \pi\left(1 - \frac{c^2}{c_{RB}^2}\right)^{1/2}\delta_B. \tag{46}$$

The parameters δ_B and C_{RB} are given by Eqs. (35c) and (35d). This situation can be compared to the moving domain wall in ferromagnetic crystals; this is the Bloch wall [19]. The microscopic model of this situation can be built from a nonlinear compressible chain of dipoles [29] of which the long-wave length limit leads to the same system as Eq. (35). The systems of compressible Heisenberg chains provide a comparable problem [30, 31].

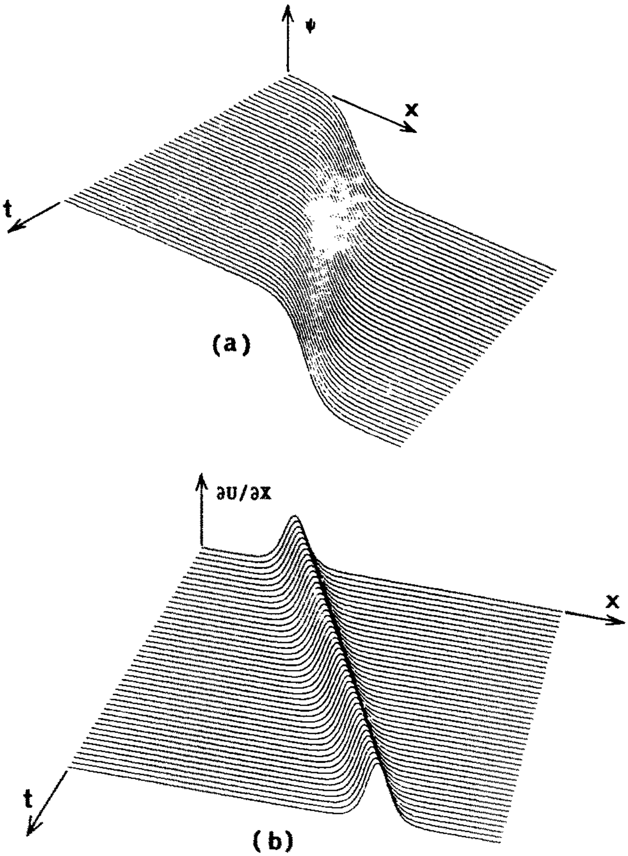

Fig. 6. Numerical illustrations for the single soliton solution in the case of the configuration *B*: (a) soliton in rotation and (b) soliton in elongation.

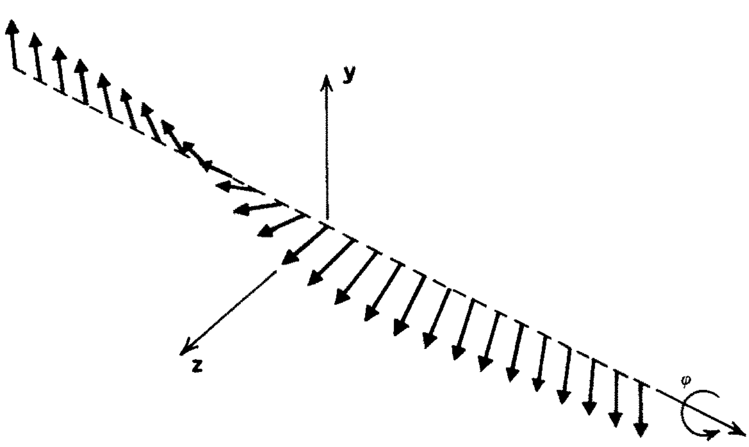

Fig. 7. Numerically obtained picture of the director orientation in the case of the configuration *B*.

D. *Remark*

As in configuration A, a digression can be made about the sign of χ. If $\chi < 0$, we have to change $\sin \psi$ into $-\sin \psi$ in Eq. (38), and then a similar change of functions to that of Eqs. (3a–c) (apart from V) is considered in order to recover the system (38a–c) but with the new functions. In this case, the angle of rotation ψ varies monotonically from 2π to 0 while the real angle decreases from π to 0, and a change in the soliton topology will ensue. In Fig. 7 we give the numerical simulation of the director orientation for $\chi > 0$. For $\chi < 0$ we have the same picture by rotating the frame by 90° about the x-axis.

5. Conclusions and remarks

The propagation of solitons for two simple configurations has been examined on the basis of a nonlinear continuum model for oriented media. The nonlinear excitations are intimately connected with the additional degrees of freedom modelled by a vector of constant length in interaction with the deformable continuum so that the theory accounts for the nonlinear coupling between the rotational motion of the directors and elastic deformation, hence the propagation of *solitons in director rotations and in* (*elongational and shear*) *elastic deformations*. As a general rule, the system, which admits soliton solutions, consists in two *wave equations* for the elastic displacement and a *sine-Gordon equation* for the rotations, the three equations being nonlinearly coupled by means of phenomenological coefficients which represents the interactions between the director field and the deformable medium. The first configuration, referred to as A, is much more involved than the second one since, in this case, both longitudinal and transverse elastic displacements polarized in the plane of rotation of the director field are coupled to the latter. This situation is identical to that of the Néel wall in thin elastic ferromagnetic films [19]. In the second situation, configuration B, the directors rotate about the axis of propagation. This in fact is the same situation as in the Bloch wall in ferromagnetic crystals apart from the fact that in the latter the wave equations for elastic displacements are not coupled to the equation governing the magnetic spin rotation [19]. We have an identical situation in the dynamics of a bar with a large-amplitude elastic twist [32]. On the other hand, in both cases we recover the results concerning solitary waves in micropolar media [33, 34]. For each situation, we have given the total energy in terms of characteristic parameters, that is, the elastic wave velocities, the coupling coefficients, the soliton velocity and characteristic length associated with the soliton thickness are obtained as well. Moreover, it has been shown that, in each configuration, the complete dynamical problem of the nonlinearly coupled system is equivalent to solving a *double sine-Gordon equation only* in the case of one-soliton solutions, which amounts to solving a single ordinary differential equation with respect to the phase variable.

The present nonlinear excitations occurring in the continuum model for oriented media can be compared to those obtained from the long-wavelength limit of a microscopic model made of a single atomic chain equipped with rotatory microscopic electric dipoles. The configuration A, where the director rotation occurs about an axis perpendicular to the plane of the displacements, can be compared to the microscopic model for molecular ferroelectric crystals [16]. This microscopic model has allowed us to study the stationary motion of a single-domain wall in these particular crystals of which sodium nitrite ($NaNO_2$) provides a good prototype (in addition, this crystal undergoes an incommensurate-commensurate phase transition [35, 36]). The second configuration can find its microscopic origin in the compressible atomic chain with dipoles [29]. In this situation the dipoles rotate about the axis of the longitudinal displacements. Thus the continuum model can be applied to describe the structure in domains and walls in molecular ferroelectric crystals if the director field is endowed with an electric polarization. The physical meaning of the soliton in rotation is then a moving domain wall coupled to the mechanical states, and solitons in elongation and shear deformations are accordingly generated through the electromechanical couplings [16]. The characteristic lengths Δ_A and Δ_B (see Eqs. (31) and (46)) or δ_A and δ_B for the static cases (Eqs. (7c) and (36c)) represent physically the wall thicknesses. It is these quantities which can be experimentally reached by means of various methods (electron microscopy, for instance) [37–39]. Insofar as the domain-wall structures problems are concerned, nonlinear oriented media can also be applied to molecular crystals and the director field then is associated with the rotation of a molecular group [40–42].

The nonlinear excitation of the soliton type studied in the present work can take place in various media when both the microstructure of the medium and the internal degrees of freedom are considered. In particular, we clearly account for the rigid-body rotational motion of the microstructure and this leads to the notion of angular momentum associated with the rotational motions. The microstructure can, in a way, be modelled by a small gyroscope of which the rotation about its mass center is characterized by a vector (director) of constant length. The underlying physics of the micro-motion corresponds to various media, and the micro-system possesses both nonlinearities related to the internal degrees of freedom (large amplitude rotational motions of the directors) and dispersion (long-range interactions between neighboring directors accounting for the spatial nonuniformity in the director field). Thus the above-developed model can be applied to the study of nonlinear excitations in nematic liquid crystals [13] where a bunch of rigid nematic molecules is well enough modelled by a director, in polymer media (poly-vinylidenefluoride) [14, 15, 43–45], or in long elastic chains of macromolecules (biological materials, e.g. DNA); in the latter case more complicated interactions between macromolecules may be involved [17, 46–48]. The macromolecules undergo relatively large amplitude rotational motions from a stable state to another one while passing through a metastable state. In such media one may associate a director with the orientation of a macromolecule. Moreover, the possible intermediate metastable state can be explained by introducing nonlinearities of higher order in the director field in the free energy of the oriented elastic medium (see Part I). For instance, if a nonlinear term of the fourth order in the director field is accounted for in the constitutive equations, the symmetry being still centro-symmetric, the equation governing the rotational motion can be written as (in nondimensional notation and in the case of the configuration A)

$$\ddot{\psi} - \psi_{,xx} = \sin\psi + \mu \sin 2\psi, \tag{47}$$

without considering the deformability of the medium. This is a double sine-Gordon equation. If the elastic deformation is again introduced, the results are slightly modified by the coupling, but this does not change the essence of the discussion. Nevertheless, the equilibrium states of Eq. (47) are given by setting its right-hand side equal to zero, from which there follow the solutions $\psi = k\pi$ and $\psi = (2k+1)\pi \pm \cos^{-1}(1/\mu)$ (only if $|\mu| > 1$). Accordingly the different solutions of this situation can be obtained by examining the extrema of the energy potential of the model described by Eq. (47) [23, 49, 50].

The influence of an external field on a soliton is a problem that can be studied. The external field acts on the internal degrees of freedom through a volume torque. Such a torque can easily be built if the micro system is ridigly endowed with an electric dipole performing the same rotation as the director does. Then, if an external field is applied the volume torque takes on the form $\mathbf{P} \times \mathbf{E}_0$ ($\mathbf{E}_0$ is the applied electric field and $\mathbf{P}$ is the electric polarization). This problem has already been discussed in the case of a microscopic model based on an anharmonic chain equipped with microscopic electric dipoles [51, 52]. If, in the case of configuration A (Fig. 1), the director carries an electric polarization in the same direction as the latter and an external electric field is applied in the y direction, the external torque is then given by $E_0 P \sin(\phi/2)$, and this term must be added to the right-hand side of Eq. (6c). This allows us to study the transient motion of a soliton from rest by means of perturbation methods [51, 52]. An identical problem can be studied if instead of an electric polarization we have a magnetization; however, in this case, the applied field is a magnetic field [19]. This problem can be extended to a time dependent external field which allows us to investigate the effects of a periodic driving field in the presence of rotational relaxation on a soliton, hence the possible transition to deterministic chaos [53–55].

Two interesting problems regarding the continuum model for oriented media deserve careful attention. The first problem concerns solitons in two spatial dimensions in the presence, or not, of applied fields, which leads to the notion of vortices. This problem is met in magnetic structures (Josephson junctions [56–58]) or in nematic liquid crystals. The second problem brings into play the combination of both configurations A and B, and two rotational angles of the director field are therefore considered. In this case we may have a complex behavior of the nonlinear excitations. Indeed, a soliton of the configuration A may be transformed while traveling into a soliton of the configuration B and *vice versa.* Such situations can model the nonlinear dynamics in DNA, in which both bending and twisting of the molecular chain must be taken into account (see also, for instance, the same situation in ferromagnetic media [59–62]).

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