Journal of Elasticity **22**: 121–133, 1989.

An exact derivation of the thin plate equation

P. PODIO-GUIDUGLI
Dipartimento di Ingegneria Civile Edile, Università di Roma 2, Via O.Raimondo, 00173 Roma, Italy

Received 15 December 1987

Abstract. It is shown that, when the traditional assumptions of thin plate theory are taken as exact mathematical hypotheses, the desired field and boundary equations can be obtained by mere integration over the thickness of the corresponding equations for a three-dimensional cylindrical body made of a homogeneous, linearly elastic *transversely isotropic, constrained* material, yet avoiding some inconsistencies usually to be found in textbooks of structural mechanics.

1. Introduction

On p. 1 of their well-known book, Timoshenko and Woinowsky-Krieger [14] so describe the assumptions on which the theory of small deflections of thin plates is based (the quotation is *verbatim*):

1. There is no deformation in the middle plane of the plate. This plane remains *neutral* during bending.
2. Points of the plate lying initially on a normal-to-the-middle plane of the plate remain on the normal-to-the-middle surface of the plate after bending.
3. The normal stresses in the direction transverse to the plate can be disregarded.

According to Love [7], the first two assumptions were first stated by Kirchhoff (1850, [6]). Novozhilov [8], in his presentation of those "specific simplifications which are possible in the study of the deformation of flexible bodies", accepts **2**, but replaces **3** with

3′. The distance of every point of the plate from the middle surface remains unchanged by the deformation.[1]

Interestingly, Novozhilov expresses **2** and **3′** as exact kinematical constraints on the possible deformations of the plate. Precisely, in terms of the right

[1] Cf. [8], pp. 177–178; again the quotation is *verbatim*; later, on p. 194, Novozhilov refers to **2** and **3′** as to "Kirchhoff's assumptions". The omission of **1** allows Novozhilov to encompass in his presentation v. Kármán's nonlinear theory for large deflections of a thin plate, where the deformations in the middle plane of the plate are taken into account.

Cauchy-Green strain tensor **C**. Novozhilov writes **2** as

$$C_{13} = C_{23} = 0, \tag{1.1}$$

and **3**′ as

$$C_{33} = 1 \tag{1.2}$$

(for these notations, *vid.* formula (2.1) in the next section). However, in his discussion of the nature of Kirchhoff's assumptions in §49, Novozhilov finds a serious difficulty with (1.1) and (1.2):

> "Interpreted as mathematically exact relations, [they] are absurd, since they lead, in general, to contradictions in the formulation of the condition of equilibrium of an element of the plate."

Novozhilov–and with him the bulk of the engineering literature on structural mechanics, whose treatment of these matters is well exemplified by his–seems content to interpret Kirchhoff's as purely geometrical simplifying assumptions, whose correctedness should be estimated *a posteriori*, meaning that "...elongations and shears are neglected in comparison with rotations in determining the direction of fibers of the strained body" which were straight and perpendicular to the middle surface of the plate before the deformation. However, within the framework of a treatment *à la* Novozhilov, the contradictions alluded at above remain, and could not possibly be removed by offering an appropriate interpretation for (**1**) and (**2**).

Inconsistencies of this sort are not uncommon in the mechanics of elastic structures,[2] and contribute to give the subject, in the view of many, the aspect of an unattractive superfetation over the general theory of elasticity (cf. e.g. [8], §54). Indeed, structural mechanics is born when in the three-dimensional theory assumptions are introduced reflecting the peculiar 'thinness' of a class of bodies, so as to reduce the problem to a bidimensional one (membranes, plates, shells) or a monodimensional one (strings, rods, jets). A current method of attack, as old as Poisson's (1829, [13]) and Cauchy's (1828, [2]) first researches on plates and still popular nowadays, consists in "...proceeding from the general equations of Elasticity, and supposing that all the quantities can be expanded in powers of the distance from the

[2] E.g., Washizu ([15], §8.1) assumes, in addition to **2**, both **3**, i.e., $T_{33} = 0$, and **3**′, i.e., $E_{33} = 0$; he also assumes, as is done in all other textbooks, that the material is isotropic, so that, in particular, $T_{33} = 2\mu E_{33} + \lambda(E_{11} + E_{22} + E_{33})$, (cf. equation (3.4) below). It follows that, necessarily $\lambda(E_{11} + E_{22}) = -\lambda x_3 \Delta u_3 = 0$ (here $u_3 = u_3(x_1, x_2)$ is the transversal displacement). Thus, for consistency, either $\lambda = 0$ (or, which is the same, the Poisson's ratio $\nu = 0$)–a very special instance–or u_3 has to be a harmonic function–a trivial instance.

middle-surface" ([7], p. 27). The tacit understanding is that, after some bookkeeping, the thinness of the structure under examination will be automatically accounted for by retaining only low order terms in the power expansions.

In this paper, I show that assumptions **1**, **2** and **3** can indeed be formulated as mathematically exact relations, and yet they do not yield any contradiction, provided those relations are consistently seen as internal constraints, i.e., provided that (i) an additive decomposition of the stress measure in a reactive and an active part is accepted, and (ii) the constitutive dependence of active stresses on the deformation is such as to reflect the maximal material symmetry compatible with the assumed internal constraints. It should be noted that, in the usual derivations of the thin plate equation, contradictions to equilibrium follow from the introduction of kinematical hypotheses of Kirchhoff-type precisely because of both failure to consider reactive stresses and the undue assumption of isotropy for the response function delivering the active stresses.

In my present approach, the theory of small deflections of thin plates is made fully consistent with the principles and methods of three-dimensional linear elasticity; in a sense, the imposition of appropriate internal constraints, together with integration over the thickness, builds neatly into the model that thinness which is universally considered to be peculiar of plates, dispensing one from use of the brute force of power expansions. The merits of this approach for shells and rods, or in the nonlinear case, remain to be investigated.[3]

2. Internal constraints and reactive stresses

Let (x_1, x_2, x_3) be a rectangular coordinate system with origin at $\mathbf{0}$; let P be a bounded regular region in the plane $x_3 = 0$, with $\mathbf{0} \in P$, and let ∂P be its boundary; finally, let $C = P \times]-h, +h[$ be a cylinder with generators parallel to the x_3-axis and with end faces at $x_3 = \pm h$. Later, I shall obtain boundary conditions over ∂P from boundary conditions over the mantle $M = \{\mathbf{x} \in \partial C | x_3 \neq \pm h\}$ of C *via* integration over the thickness. In preparation for that, as is done in the study of the plane problem of linear elasticity (cf. [3], §45), I assume that whenever the boundary conditions are not the same ones on the whole of M then M is so partitioned that the intersection of the closures of any two elements of such partition consists at most of line segments parallel to the x_3-axis and of length $2h$.

[3] An attempt to account for rod thinness by introducing *ad hoc* internal constraints was made in [1].

Now, let $\mathbf{u} = \mathbf{u}(\mathbf{x}) \equiv (u_1, u_2, u_3)$ be the displacement of a point $\mathbf{x} \equiv (x_1, x_2, x_3)$ of C, and let

$$\mathbf{E} = \mathbf{E}(\mathbf{u}) := \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \quad \mathbf{C} = \mathbf{C}(\mathbf{u}) := \mathbf{I} + 2\mathbf{E}(\mathbf{u}) + \nabla\mathbf{u}^T\nabla\mathbf{u} \tag{2.1}$$

be, respectively, the linear and nonlinear strain measures (here, $\nabla\mathbf{u}$ is the displacement gradient and $\mathbf{I}$ is the identity tensor).

A completely standard interpretation of the components of $\mathbf{E}$ and $\mathbf{T}$, the Cauchy stress measure, shows that the assumptions listed in the introduction can be mathematically phrased as follows, in the context of the linear theory:

1 $\Rightarrow$ (H1) $E_{11}(x_1, x_2, 0) = E_{22}(x_1, x_2, 0) = E_{12}(x_1, x_2, 0) = 0$;
for all $\mathbf{x} \in C$,
2 $\Rightarrow$ (H2) $E_{13} = E_{23} = 0$;
3 $\Rightarrow$ (H3) $T_{33} = 0$;
3′ $\Rightarrow$ (H3′) $E_{33} = 0$.

Moreover, correspondence of (1.1) with (H2), and of (1.2) with (H3′), is obvious in the light of definitions (2.1). It should also be noticed that, at variance with the other ones, **3** and, consequently, (H3) are not kinematical assumptions.

As anticipated in the introduction, I here choose to regard assumptions (H2) as internal constraints.

In continuum mechanics, an internal constraint is a constitutive prescription restricting the class of possible deformations (cf. [5]; [4], §16; [11]). In the linear theory of elasticity (cf. [9]; [10]) such a prescription is expressed by the assignment of a subspace

$$S_{\mathbf{A}} := \{\mathbf{E} \in \mathrm{Sym} \mid \mathbf{A} \cdot \mathbf{E} = 0\} \tag{2.2}$$

of the space Sym of all symmetric tensors (in (2.2) a dot designates the inner product of two tensors: $\mathbf{L} \cdot \mathbf{M} = \mathrm{trace}(\mathbf{L}\mathbf{M}^T)$). In particular, $(\mathrm{H2})_1$ is obtained for $\mathbf{A} = \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1$, and $(\mathrm{H2})_2$ is obtained for $\mathbf{A} = \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2$.

An internal constraint is accompanied by reactive stresses $\mathbf{T}^{(R)}$, whose role is to maintain the constraint itself. For simplicity, the reactions $\mathbf{T}^{(R)}$ are assumed to contribute nothing to the stress power in any admissible motion; it follows that $\mathbf{T}^{(R)}$ must be orthogonal to $S_{\mathbf{A}}$, or rather, parallel to $\mathbf{A}$:

$$\mathbf{T}^{(R)} = \tau\mathbf{A}, \quad \tau \in \mathbb{R}, \tag{2.3}$$

where τ is a scalar multiplier, at this stage arbitrary. On othe other hand, an expedient normalization consistent with the arbitrariness of τ suggests that the active stresses

$$\mathbf{T}^{(A)} = \mathbf{T} - \mathbf{T}^{(R)} \tag{2.4}$$

are taken as elements of S_A; if more than one constraint subspace S_A is to be considered, and if D denotes the intersection of all constraint subspaces, then $\mathbf{T}^{(A)} \in D$. In the light of the above, (H2) imply that

$$[\mathbf{T}] = \begin{bmatrix} T_{11}^{(A)} & T_{12}^{(A)} & 0 \\ \cdot & T_{22}^{(A)} & 0 \\ \cdot & \cdot & T_{33}^{(A)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & T_{13}^{(R)} \\ \cdot & 0 & T_{23}^{(R)} \\ \cdot & \cdot & 0 \end{bmatrix}.$$

3. The constitutive equation for active stresses

In the linear theory, the active stresses are delivered by a linear mapping $\mathbb{C}$ from D into itself:

$$\mathbf{T}^{(A)} = \mathbb{C}[\mathbf{E}], \tag{3.1}$$

where $\mathbb{C}$, the elasticity tensor, associates a null stress only to the null strain, and is such that

$$\mathbb{C}_{ijhk} = \mathbb{C}_{jihk} = \mathbb{C}_{ijkh} = \mathbb{C}_{hkij}. \tag{3.2}$$

Let Rot denote the group of all rotations, i.e., of all orthogonal tensors with positive determinant. The response symmetries of the constrained material described by $\mathbb{C}$ are reflected in its symmetry group, i.e., the collection of all $\mathbf{Q} \in$ Rot such that

$$\mathbf{QEQ}^T \in D \quad \text{and} \quad \mathbf{QT}^{(A)}\mathbf{Q}^T = \mathbb{C}[\mathbf{QEQ}^T] \quad \text{for all } \mathbf{E} \in D. \tag{3.3}$$

Here I have adapted to constrained materials the classical notions of an elasticity tensor and its symmetry group,[4] which are obtained when the material is unconstrained, or rather, $D \equiv \text{Sym}$ and $(3.3)_1$ is trivially satisfied.

In the classical constitutive theory, an interesting problem is to find a representation formula for all elasticity tensors having an assigned symmetry group: e.g., if $(3.3)_2$ has to be satisfied by all orthogonal tensors (isotropic materials), then the Lamé constitutive equation follows:

$$\mathbf{T} = \mathbb{C}[\mathbf{E}] = 2\mu\mathbf{E} + \lambda(\mathbf{I} \cdot \mathbf{E})\mathbf{I}. \tag{3.4}$$

[4] The latter are masterfully presented by Gurtin [3], §§20–22.

For a constrained material ($D \subset\subset \mathrm{Sym}$), not any assignment of a subgroup of Rot is compatible with $(3.3)_1$. In particular, an isotropic material may be incompressible ($D = \{\mathbf{E} \in \mathrm{Sym} | \mathbf{E} \cdot \mathbf{I} = 0\}$), but cannot be inextensible in the direction $\mathbf{e}_3$ ($D = \{\mathbf{E} \in \mathrm{Sym} | \mathbf{e}_3 \cdot \mathbf{E}\mathbf{e}_3 = 0\}$).

I here assume that the elastic material which comprises the cylinder C has the maximal response symmetry compatible with the internal constraints imposed by (H2). A representation formula for $\mathbb{C}$ then follows from (3.3):

$$\mathbf{T}^{(A)} = 2\mu\mathbf{E} + \lambda((\mathbf{I} - \mathbf{P}) \cdot \mathbf{E})(\mathbf{I} - \mathbf{P}) + \lambda'(\mathbf{P} \cdot \mathbf{E})\mathbf{P}, \quad \mathbf{P} = \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{3.5}$$

where $\mathbf{P}$ is the perpendicular projection onto the direction $\mathbf{e}_3$ of the x_3-axis; in components,

$$[\mathbf{T}^{(A)}] = 2\mu[\mathbf{E}] + \lambda(E_{11} + E_{22})\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} + \lambda' E_{33}\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} (2\mu + \lambda)E_{11} + \lambda E_{22} & 2\mu E_{12} & 0 \\ \cdot & (2\mu + \lambda)E_{22} + \lambda E_{11} & 0 \\ \cdot & \cdot & (2\mu + \lambda')E_{33} \end{bmatrix}. \tag{3.6}$$

Formula (3.6) describes a transversely isotropic material (cf. e.g. [12]), with $\mathbf{e}_3$ its anisotropicity axis, constrained as required by (H2).

4. Kirchhoff elastic states

In view of $(2.1)_1$, hypotheses (H2) can be written as

$$u_{1,3} + u_{3,1} = 0, \quad u_{2,3} + u_{3,2} = 0. \tag{4.1}$$

Moreover, (H3) and (3.6) imply that

$$E_{33} = u_{3,3} = 0. \tag{4.2}$$

The solution of system (4.1), (4.2) is

$$\begin{aligned} &u_1 = -x_3 u_{3,1} + u_1^0, \quad u_2 = -x_3 u_{3,2} + u_2^0, \\ &u_3 = u_3(x_1, x_2), \quad u_1^0 = u_1^0(x_1, x_2), \quad u_2^0 = u_2^0(x_1, x_2), \end{aligned} \tag{4.3}$$

for all $\mathbf{x} \in C$. Furthermore, (H1) implies that

$$u_1^0 = \alpha_1 - \beta x_2, \quad u_2^0 = \alpha_2 + \beta x_1, \tag{4.4}$$

a rigid displacement in the middle plane of the plate.

In the technical literature, (4.3) and (4.4) are sometimes referred to as describing a Kirchhoff displacement field in C. The corresponding strain and (active) stress fields are, respectively,

$$[\mathbf{E}] = -x_3 \begin{bmatrix} u_{3,11} & u_{3,12} & 0 \\ \cdot & u_{3,22} & 0 \\ \cdot & \cdot & 0 \end{bmatrix} \tag{4.5}$$

and

$$[\mathbf{T}^{(A)}] = -x_3 \begin{bmatrix} 2\mu u_{3,11} + \lambda\,\Delta u_3 & 2\mu u_{3,12} & 0 \\ \cdot & 2\mu u_{3,22} + \lambda\,\Delta u_3 & 0 \\ \cdot & \cdot & 0 \end{bmatrix}. \tag{4.6}$$

For the cylinder C, consider the triplet $\{\mathbf{u}, \mathbf{E}, \mathbf{T} = \mathbf{T}^{(A)} + \mathbf{T}^{(R)}\}$, with $\mathbf{u}$ as in (4.2) and (4.3), $\mathbf{E}$ as in (4.5), $\mathbf{T}^{(A)}$ as in (4.6) and $\mathbf{T}^{(R)}$ as in (2.5). Given a body force field $\mathbf{b} = (0, 0, b(\mathbf{x}))$ over C, I call such a triplet a Kirchhoff state if it satisfies the equilibrium equation

$$\operatorname{Div} \mathbf{T} + \mathbf{b} = \mathbf{0} \quad \text{in } C \tag{4.7}$$

and the boundary condition of null tangential traction on the end faces of C

$$T_{13} = T_{23} = 0 \quad \text{for } x_3 = \pm h. \tag{4.8}$$

It is easily found that $(4.7)_{1,2}$ and (4.8) yield

$$\begin{aligned} T_{13}^{(R)} &= \frac{h^2}{2}(2\mu + \lambda)(\Delta u_3)_{,1}\left(\frac{x_3^2}{h^2} - 1\right) \\ T_{23}^{(R)} &= \frac{h^2}{2}(2\mu + \lambda)(\Delta u_3)_{,2}\left(\frac{x_3^2}{h^2} - 1\right), \end{aligned} \tag{4.9}$$

whereas from $(4.7)_3$ and (4.9) one obtains

$$\frac{h^2}{2}(2\mu + \lambda)\left(\frac{x_3^2}{h^2} - 1\right)\Delta\Delta u_3 + b = 0. \tag{4.10}$$

For any given Kirchhoff state, the shear forces are

$$Q_1 = \int_{-h}^{+h} T_{13}\,\mathrm{d}x_3 = \int_{-h}^{+h} T_{13}^{(R)}\,\mathrm{d}x_3 = -D(\Delta u_3)_{,1}$$
$$Q_2 = \int_{-h}^{+h} T_{23}\,\mathrm{d}x_3 = \int_{-h}^{+h} T_{23}^{(R)}\,\mathrm{d}x_3 = -D(\Delta u_3)_{,2}, \tag{4.11}$$

where the flexural rigidity D is defined to be

$$D = \frac{2h^3}{3}(2\mu + \lambda); \tag{4.12}$$

the moments are

$$M_1 = \int_{-h}^{+h} x_3 T_{11}\,\mathrm{d}x_3 = \int_{-h}^{+h} x_3 T_{11}^{(A)}\,\mathrm{d}x_3 = -D\left(u_{3,11} + \frac{\lambda}{2\mu+\lambda}u_{3,22}\right),$$
$$M_2 = \int_{-h}^{+h} x_3 T_{22}\,\mathrm{d}x_3 = \int_{-h}^{+h} x_3 T_{22}^{(A)}\,\mathrm{d}x_3 = -D\left(u_{3,22} + \frac{\lambda}{2\mu+\lambda}u_{3,11}\right), \tag{4.13}$$
$$M_{12} = -\int_{-h}^{+h} x_3 T_{12}\,\mathrm{d}x_3 = -\int_{-h}^{+h} x_3 T_{12}^{(A)}\,\mathrm{d}x_3 = D\frac{2\mu}{2\mu+\lambda}u_{3,12}\,.$$

These formulae make evident the reactive nature of shears and the constitutive nature of moments.

Shears and moments are linked by two direct consequences of the equilibrium equations. In view of the identity

$$x_k T_{ij,j} = (x_k T_{ij})_{,j} - T_{ik}, \tag{4.14}$$

it follows from $(4.7)_{1,2}$ and (4.8), after integration over the thickness of C, that

$$Q_1 - M_{1,1} + M_{12,2} = 0 \quad \text{and} \quad Q_2 - M_{2,2} + M_{12,1} = 0. \tag{4.15}$$

Moreover, again by integration over the thickness, one gets from $(4.7)_3$

$$Q_{1,1} + Q_{2,2} + q = 0, \tag{4.16}$$

with

$$q = \int_{-h}^{+h} b \, dx_3. \tag{4.17}$$

Combining (4.13), (4.15) and (4.16), one obtains the classical equilibrium equation for the small deflections u_3 of a thin plate of flexural rigidity D, subject to transversal loads of surface density q:

$$\Delta\Delta u_3 = \frac{q}{D} \quad \text{in } P. \tag{4.18}$$

Of course, (4.18) is also arrived at by direct integration of (4.10).

5. Boundary conditions

(4.18) is the strong form of the Euler-Lagrange equation associated with the energy functional

$$\Sigma(\mathbf{u}) = \int_C [\sigma(\mathbf{u}) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}] \, d\mathbf{x}, \tag{5.1}$$

where the stored energy density σ is defined to be

$$\sigma = \tfrac{1}{2}\mathbf{T}^{(A)} \cdot \mathbf{E}, \tag{5.2}$$

with $\mathbf{T}^{(A)}$ and $\mathbf{E}$ given by (4.6) and (4.5), respectively, for a Kirchhoff displacement field $\mathbf{u}$.

In view of the hypothesis on M, the admissible boundary conditions on ∂P can be obtained by performing in either order the operations of integrating over the thickness and of taking the first variation of the energy functional. E.g., after integration over the thickness, (5.1) becomes

$$\Sigma(\mathbf{u}) = \tfrac{1}{2}D \int_P \left\{ (\Delta u_3)^2 - \frac{4\mu}{2\mu + \lambda} [u_{3,11} u_{3,22} - (u_{3,12})^2] \right\} dx_1 \, dx_2 - \int_P q u_3 \, dx_1 \, dx_2 \tag{5.3}$$

(cf. formula (117) on p. 88 of [14]). Notice that the terms

$$[u_{3,11}u_{3,22} - (u_{3,12})^2]$$

compose a null Lagrangean, and thus contribute nothing to the field equation. The variational format now allows for a straightforward derivation of those boundary conditions which may be imposed along ∂P. When use is made of the divergence theorem, one obtains:

$$0 = D\int_{\partial P}\left[\Delta u_3 v_{,1} - (\Delta u_3)_{,1}v - \frac{2\mu}{2\mu+\lambda}(u_{3,22}v_{,1} - u_{3,12}v_{,2})\right]n_1 + \left[\Delta u_3 v_{,2} - (\Delta u_3)_{,2}v - \frac{2\lambda}{2\mu+\lambda}(u_{3,11}v_{,2} - u_{3,12}v_{,1})\right]n_2, \tag{5.4}$$

where $v = v(x_1, x_2)$ denotes the variation of u_3, and $\mathbf{n} \equiv (n_1, n_2, 0)$ is the unit outer normal to M along ∂P. On denoting by $\partial_n v = \nabla v \cdot \mathbf{n}$ and $\partial_t v = \nabla v \cdot \mathbf{t}$ the derivatives of v in the directions of $\mathbf{n}$ and the tangent $\mathbf{t}$ to ∂P, respectively, one has that

$$v_{,1} = n_1\,\partial_n v - n_2\,\partial_t v, \quad v_{,2} = n_1\,\partial_t v + n_2\,\partial_n v. \tag{5.5}$$

Substituting (5.5) into (5.4) yields

$$0 = -\int_{\partial P}(M_1 n_1^2 + M_2 n_2^2 - 2M_{12}n_1n_2)\,\partial_n v + \int_{\partial P}(Q_1 n_1 + Q_2 n_2)v + [(M_{12}(n_1^2 - n_2^2) + (M_1 - M_2)n_1n_2]\,\partial_t v, \tag{5.6}$$

or rather,

$$0 = -\int_{\partial P} M_n\,\partial_n v - (Q_n + \partial_t M_{nt})v, \tag{5.7}$$

where

$$M_n = \mathbf{n}\cdot\mathbf{M}\cdot\mathbf{n}, \quad M_{nt} = \mathbf{t}\cdot\mathbf{M}\cdot\mathbf{n}, \quad Q_n = \mathbf{Q}\cdot\mathbf{n} \tag{5.8}$$

and, with slight abuse of notations,

$$[\mathbf{M}] = \begin{bmatrix} M_1 & -M_{12} \\ -M_{12} & M_2 \end{bmatrix}, \quad [\mathbf{Q}] = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{5.9}$$

Thus, as is well-known, on ∂P the geometrical boundary conditions may involve u_3 and $\partial_n u_3$, whereas the natural boundary conditions may involve the bending moment M_n and Kirchhoff's shear force $V_n = Q_n - \partial_t M_{nt}$.

Appendix 1

The interpretation of the material moduli of transverse isotropy

In standard textbooks of structural mechanics, where the material of which the plate is made is supposed to be isotropic, the flexural rigidity is defined to be

$$D = \frac{2h^3}{3}\frac{E}{1-\nu^2}, \tag{A1.1}$$

where E is Young's modulus and ν in Poisson's ratio.

As is well-known, for a uniaxial state of stress

$$\mathbf{T} = \tau \mathbf{e}_1 \otimes \mathbf{e}_1, \tag{A1.2}$$

where $\mathbf{e}_1$ is, say, the unit vector of the x_1-axis, E is the ratio of the only non-vanishing stress component T_{11} and the corresponding strain E_{11}, whereas ν is the inverse ratio of E_{11} and any transversal strain, say, E_{22}:

$$E = \frac{\tau}{E_{11}} = \frac{\mu(3\lambda+\mu)}{\lambda+\mu}, \quad \nu = \frac{E_{22}}{E_{11}} = \frac{\lambda}{2(\lambda+\mu)}. \tag{A1.3}$$

For the transversely isotropic material of constitutive equation (3.5) the same interpretation holds. One has

$$E_{(1)} = \frac{\tau}{E_{11}} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}, \quad \nu_{(1)} = \frac{E_{22}}{E_{11}} = \frac{\lambda}{\lambda+2\mu}, \tag{A1.4}$$

so that

$$\frac{E_{(1)}}{1-\nu_{(1)}^2} = 2\mu + \lambda, \tag{A1.5}$$

and (4.12) and (A1.1) can be brought to formally coincide. However, it should be noticed that a transversely isotropic material has more than one "Young's modulus" and "Poisson's ratio" (just to continue to use the familiar terminology, in the same vein as the "Lamé constants" of a transversely isotropic material have been denoted here by the same letters λ and μ used in the isotropic case); e.g., for the material described by (3.5), one has (with self-explanatory notation)

$$E_{(3)} = \frac{\tau}{E_{33}} = \lambda' + 2\mu, \quad \nu_{(3)} = \frac{E_{11}}{E_{33}} = \frac{E_{22}}{E_{33}} = 0. \tag{A1.6}$$

Appendix 2

Null transverse strains in place of null transverse stresses

In the present derivation of the thin plate equation, the vanishing of transverse strains,

$$E_{33} = u_{3,3} = 0, \tag{4.2}$$

has not the logical position of an independent hypothesis, as (4.2) follows from (H3), the hypothesis of vanishing transverse stresses, and the constitutive equation (3.6). If, in place of (H3), (H3′) were accepted, (4.2) would express an internal constraint, and accordingly, (2.5) would feature an active stress

$$T_{33}^{(A)} = 0$$

and a reactive stress

$$T_{33}^{(R)} \neq 0.$$

Of course, the Kirchhoff displacement field, together with the associated strain and active stress fields in C, would not change.

At this stage, in order to complete a derivation of the plate equations conceptually equivalent to the one presented in the text, one would confront two main alternatives:

either one might choose to stick to the given notion of a Kirchhoff state, supplementing (4.8) with a boundary condition of null normal traction on the end faces, and integrating (4.10), which would now look as follows:

$$\frac{h^2}{2}(2\mu + \lambda)\left(\frac{x_3^2}{h^2} - 1\right)\Delta\Delta u_3 + T_{33,3}^{(R)} + b = 0, \tag{A2.1}$$

to get

$$T_{33}^{(R)} = -\int_{-h}^{x_3} b\, \mathrm{d}x_3 + \left(\frac{1}{2} + \frac{3}{4}\frac{x_3}{h} - \frac{1}{4}\frac{x_3^3}{h^3}\right) D\, \Delta\Delta u_3 \tag{A2.2}$$

together with (4.18);
or one might choose to set $\mathbf{b} = \mathbf{0}$ in (4.7) and supplement (4.8) with appropriate boundary conditions for the normal tractions:

$$T_{33}^{(R)}(x_1, x_2, -h) = -q(x_1, x_2) \quad \text{and} \quad T_{33}^{(R)}(x_1, x_2, +h) = 0, \tag{A2.3}$$

etc.. Which course to follow is a matter of taste.

Acknowledgment

I wish to thank G. Del Piero for a lively after-dinner discussion which prompted me not only to transform my scribbled notes into a formal paper, but also, in particular, to add Appendix 2.

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