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Harmonic waves in elastic sandwich plates

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ABSTRACT

Motions of a sandwich plate with symmetric facings are studied in the framework of the three-dimensional equations of elasticity. Both the core and facings are assumed to be isotropic and linearly elastic.

Harmonic wave solutions, which satisfy traction-free face conditions and continuity conditions of tractions and displacements at the interfaces, are obtained for four cases: symmetric plane strain solutions for extensional motion, antisymmetric plane strain solutions for flexural motion, and solutions for the symmetric and antisymmetric SH-waves. The dispersion relation for each of these cases is obtained and computed. In order to exhibit the effect of the ratios of facing to core thicknesses, elastic stiffnesses and densities, on the dynamic behavior of sandwich plates, dispersion curves are computed and compared for plates with "thick, light, and soft" facings as well as for plates with "thin, heavy, and stiff" facings. Asymptotic expressions of dispersion relations for extensional, flexural, and symmetric SH-waves are obtained in explicit form, as the frequencies and wave numbers approach zero.

The thickness vibrations in sandwich plates are studied in detail. The resonance frequencies and modal functions of the thickness-shear and thickness-stretch motions are obtained. Simple algebraic formulas for predicting the lowest thickness-shear and the lowest thickness-stretch frequencies are deduced. The orthogonality of the thickness modal functions is established.

I. Introduction

Sandwich plates have been a subject of study for a long time. Most of the investigations are devoted to sandwiches in which the facings are thin, stiff, and heavy as compared with the core. The results of investigations for this type of sandwich plate are applied to bending and buckling problems of light weight structural constructions in civil and aerospace engineering [1], [2] and also to the vibrations of the elastic sandwich plates [3], [4] and piezoelectric crystal plates with metal electrodes [5].

On the other hand, very little has been found on the studies of sandwich plates in which the facings are thicker, softer, and lighter than the core. An elastic sandwich plate of this type is very useful to model a precipitator plate for studying its mechanical behavior. In industrial precipitators, electrostatically charged precipitator plates collect dust particles from a passing gas stream. The dust particles coagulate under the effect of various forces of adherence, and upon collection, form dust layers on both sides of the precipitator plates [6]. Depending upon the field situation, the mass of the dust layer can accumulate as much as the mass of the precipitator plate. The dust layers are removed, in most cases, by the impact of an applied force at the edge of the plate, either normal or parallel to the plane of the plate [7].

In order to study the effect of thickness, stiffness, and mass of the facings on the wave propagation and vibrations in an elastic, symmetric sandwich plate, the three-dimensional equations of elasticity are employed in the present paper. Since there are no limitations on the values of the ratios, between the facings and the core, of the thicknesses, elastic stiffnesses, and mass densities nor on the values of the frequencies of vibrations, the results of the present study are applicable to the sandwich plates with "thin, stiff, and heavy" facings as well as to those with "thick, soft, and light" facings. The exact dispersion relations and curves obtained from the three-dimensional theory of elasticity may also be used in evaluating the accuracy of two-dimensional approximate theories of vibrations of sandwich plates.

II. Three-dimensional equations and boundary conditions

Consider a sandwich plate which consists of three layers, namely a middle layer and two cover layers (shown in Figure 1). Each layer is homogeneous, isotropic, and linearly elastic. The sandwich plate is of symmetric construction, i.e., its facings are of the same material and of the same thickness. The Young's modulus, Poisson's ratio, density and the thickness of the core are denoted by E_1, ν_1, ρ_1, and $2b_1$, respectively. For the facings, the corresponding notations are E_2, ν_2, ρ_2, and b_2. The right-hand system of coordinates, x_1, x_2, and x_3, are as shown in Figure 1, where x_1, x_3 are in the middle plane of the plate and x_2 is in the thickness direction. The time is denoted by t, u_j is the displacement, and τ_{ij} the stress tensor, $i, j = 1, 2, 3$. In the absence of body forces, the equations of motion are:

$$\begin{aligned} \tau_{ij,i} &= \rho_1 \ddot{u}_j, \qquad |x_2| < b_1, \\ \tau_{ij,i} &= \rho_2 \ddot{u}_j, \qquad b_1 < |x_2| < b_1 + b_2, \end{aligned} \tag{1}$$

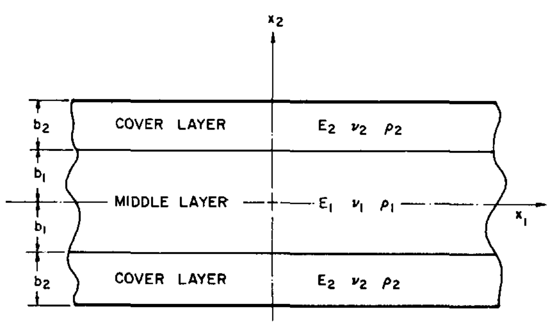

Figure 1. A symmetric sandwich plate.

where and henceforth $[\;]_{,j} \equiv \partial[\;]/\partial x_j$, $[\dot{\;}] \equiv \partial[\;]/\partial t$, and the summation convention is assumed unless specified otherwise. The constitutive relations:

$$\begin{aligned} \tau_{ij} &= \lambda_1 u_{k,k}\delta_{ij} + \mu_1(u_{i,j} + u_{j,i}), \qquad |x_2| < b_1, \\ \tau_{ij} &= \lambda_2 u_{k,k}\delta_{ij} + \mu_2(u_{i,j} + u_{j,i}), \qquad b_1 < |x_2| < b_1 + b_2, \end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta, and λ_1, μ_1, λ_2, and μ_2 are the Láme constants of the core and the facings, respectively. λ_l, μ_l are related to E_l and ν_l by

$$\begin{aligned} \lambda_l &= E_l\nu_l/[(1+\nu_l)(1-2\nu_l)], \\ \mu_l &= E_l/[2(1+\nu_l)], \qquad l = 1, 2. \end{aligned} \tag{3}$$

The sandwich plate is traction-free at the plate faces $|x_2| = b_1 + b_2$. The displacements and the tractions are continuous at the interfaces, $|x_2| = b_1$. Hence, the eighteen boundary conditions:

$$\begin{aligned} &[\tau_{2j}]_{|x_2| = b_1 + b_2} = 0, \\ &[u_j]_{|x_2| = b_1^+} = [u_j]_{|x_2| = b_1^-}, \\ &[\tau_{2j}]_{|x_2| = b_1^+} = [\tau_{2j}]_{|x_2| = b_1^-}. \end{aligned} \tag{4}$$

For an unbounded sandwich plate these are the only boundary conditions to be satisfied. Initial conditions are not needed for time-harmonic waves.

III. Harmonic wave solutions

In each layer of the sandwich plate there exist potential functions Φ and H_i such that

$$u_j = \Phi_{,j} + e_{ijk}H_{i,k}, \tag{5}$$

where $e_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$. The potential functions must satisfy the following wave equations [8]:

$$\begin{aligned} &v_{11}^2\nabla^2\Phi = \ddot{\Phi}, \qquad v_{21}^2\nabla^2 H_j = \ddot{H}_j, \qquad |x_2| < b_1, \\ &v_{12}^2\nabla^2\Phi = \ddot{\Phi}, \qquad v_{22}^2\nabla^2 H_j = \ddot{H}_j, \qquad b_1 < |x_2| < b_1 + b_2, \end{aligned} \tag{6}$$

where ∇^2 is a Laplacian operator, and v_{1l} and v_{2l} are the dilatational and shear wave velocities for the core ($l = 1$) and facings ($l = 2$)

$$\begin{aligned} v_{1l}^2 &= (\lambda_l + 2\mu_l)/\rho_l \\ v_{2l}^2 &= \mu_l/\rho_l, \qquad l = 1, 2. \end{aligned} \tag{7}$$

For straight-crested waves, the dependence of the potential functions on the x_3-coordinate is suppressed, and at the same time H_1 can be set equal to zero without loss of any physical significance from the interpretation of the results [9]. After such simplifications, the general solutions for straight-crested waves propagating in the x_1 direction may be written as:

$$\phi = \begin{cases} [a_1 \cos \alpha_1 x_2 + a_7 \sin \alpha_1 x_2] \exp i(\omega t - \xi x_1), & |x_2| < b_1, \\ [(a_2 \pm a_8) \cos \alpha_2(x_2 \pm b_1) + (\pm a_3 + a_9) \sin \beta_2(x_2 \mp b_1)] \exp i(\omega t - \xi x_1), & \\ & b_1 < \pm x_2 < b_1 + b_2, \end{cases}$$

$$H_3 = \begin{cases} [a_{10} \cos \beta_1 x_2 + a_4 \sin \beta_1 x_2] \exp i(\omega t - \xi x_1), & |x_2| < b_1, \\ [(\pm a_5 + a_{11}) \cos \beta_2(x_2 \mp b_1) + (a_6 \pm a_{12}) \sin \beta_2(x_2 \mp b_1)] \exp i(\omega t - \xi x_1), & \\ & b_1 < \pm x_2 < b_1 + b_2, \end{cases} \tag{8}$$

$$H_2 = \begin{cases} [a_{13} \cos \beta_1 x_2 + a_{16} \sin \beta_1 x_2] \exp i(\omega t - \xi x_1), & |x_2| < b_1, \\ [(a_{14} \pm a_{17}) \cos \beta_2(x_2 \mp b_1) + (\pm a_{15} + a_{18}) \sin \beta_2(x_2 \mp b_1)] \exp i(\omega t - \xi x_1), & \\ & b_1 < \pm x_2 < b_1 + b_2, \end{cases}$$

$$H_1 = 0,$$

where ω is the frequency, ξ is the wave number in the direction of wave propagation, and α_1, α_2, β_1, and β_2 are the wave numbers in the thickness direction such that they satisfy

$$\xi^2 + \alpha_l^2 = v_{1l}^{-2}\omega^2, \qquad \xi^2 + \beta_l^2 = v_{2l}^{-2}\omega^2, \qquad l = 1, 2. \tag{9}$$

One may verify that (8) are solutions of (6) by direct substitution. Also, one may obtain displacements and stresses from (5) and (2).

The displacements and the stresses derived from (8) must satisfy the eighteen boundary conditions of (4). This requirement leads to eighteen homogeneous, linear algebraic equations on $a_1, a_2 \cdots a_{18}$. In matrix notation the system of equations is

$$U\boldsymbol{a} = \mathbf{0}, \tag{10}$$

where $\mathbf{0}$ is a null vector and $\boldsymbol{a}$ is a vector having $a_1 \cdots a_{18}$ as its components. For a nontrivial solution the following relation must be satisfied

$$\det U = 0, \tag{11}$$

which relates the frequency ω and the wave number ξ and is called the dispersion relation. The nontrivial solution for vector $\boldsymbol{a}$ may be obtained from (10) for any $\omega - \xi$ pair that satisfies (11).

For the symmetric sandwich plate (see Figure 1), (10) has the following uncoupled form:

$$\left[\begin{array}{c|c|c|c} V & 0 & 0 & 0 \\ \hline 0 & W & 0 & 0 \\ \hline 0 & 0 & X & 0 \\ \hline 0 & 0 & 0 & Y \end{array}\right]\left[\begin{array}{c} a_1 \\ \vdots \\ a_6 \\ \hline a_7 \\ \vdots \\ a_{12} \\ \hline a_{13} \\ \vdots \\ a_{15} \\ \hline a_{16} \\ \vdots \\ a_{18} \end{array}\right]=0, \tag{12}$$

where the dimensions of the submatrices V, W, X, and Y are 6×6, 6×6, 3×3, and 3×3 respectively, and 0's are null matrices. The elements of X, Y are given in (13) and (14), and those of V and W in (15) and (16) on the following page.

$$X\equiv\begin{bmatrix} -\cos\beta_1 b_1 & 1 & 0 \\ \mu_1\beta_1\sin\beta_1 b_1 & 0 & \mu_2\beta_2 \\ 0 & -\sin\beta_2 b_2 & \cos\beta_2 b_2 \end{bmatrix} \tag{13}$$

and

$$Y\equiv\begin{bmatrix} -\sin\beta_1 b_1 & 1 & 0 \\ -\mu_1\beta_1\cos\beta_1 b_1 & 0 & \mu_2\beta_2 \\ 0 & -\sin\beta_2 b_2 & \cos\beta_2 b_2 \end{bmatrix}. \tag{14}$$

Because the system of equations in (12) is uncoupled into four groups, the solutions to (12) can also be classified into the following four types:

1. *Symmetric plane strain solution of extensional motion*

Set $a_7=a_8=\cdots=a_{18}=0$, and let

$$V[a_1\cdots a_6]^{\mathrm{T}}=\mathbf{0}, \tag{17}$$

$$V \equiv \begin{bmatrix} -i\xi\cos\alpha_1 b_1 & i\xi & 0 & \beta_1\cos\beta_1 b_1 & 0 & -\beta_2 \\ -\alpha_1\sin\alpha_1 b_1 & 0 & -\alpha_2 & i\xi\sin\beta_1 b_1 & -i\xi & 0 \\ 2\mu_1 i\xi\alpha_1\sin\alpha_1 b_1 & 0 & 2\mu_2 i\xi\alpha_2 & \mu_1(\xi^2-\beta_1^2)\sin\beta_1 b_1 & -\mu_2(\xi^2-\beta_2^2) & 0 \\ \mu_1(\xi^2-\beta_1^2)\cos\alpha_1 b_1 & -\mu_2(\xi^2-\beta_2^2) & 0 & 2\mu_1 i\xi\beta_1\cos\beta_1 b_1 & 0 & -2\mu_2 i\xi\beta_2 \\ 0 & 2i\xi\alpha_2\sin\alpha_2 b_2 & -2i\xi\alpha_2\cos\alpha_2 b_2 & 0 & (\xi^2-\beta_2^2)\cos\beta_2 b_2 & (\xi^2-\beta_2^2)\sin\beta_2 b_2 \\ 0 & (\xi^2-\beta_2^2)\cos\alpha_2 b_2 & (\xi^2-\beta_2^2)\sin\alpha_2 b_2 & 0 & -2i\xi\beta_2\sin\beta_2 b_2 & 2i\xi\beta_2\cos\beta_2 b_2 \end{bmatrix} \quad (15)$$

$$W \equiv \begin{bmatrix} -i\xi\sin\alpha_1 b_1 & i\xi & 0 & -\beta_1\sin\beta_1 b_1 & 0 & -\beta_2 \\ \alpha_1\cos\alpha_1 b_1 & 0 & -\alpha_2 & i\xi\cos\beta_1 b_1 & -i\xi & 0 \\ -\mu_1(\xi^2-\beta_1^2)\sin\alpha_1 b_1 & \mu_2(\xi^2-\beta_2^2) & 0 & 2\mu_1 i\xi\beta_1\sin\beta_1 b_1 & 0 & 2\mu_2 i\xi\beta_2 \\ -2\mu_1 i\xi\alpha_1\cos\alpha_1 b_1 & 0 & 2\mu_2 i\xi\alpha_2 & \mu_1(\xi^2-\beta_1^2)\cos\beta_1 b_1 & -\mu_2(\xi^2-\beta_2^2) & 0 \\ 0 & -(\xi^2-\beta_2^2)\cos\alpha_2 b_2 & -(\xi^2-\beta_2^2)\sin\alpha_2 b_2 & 0 & 2i\xi\beta_2\sin\beta_2 b_2 & -2i\xi\beta_2\cos\beta_2 b_1 \\ 0 & 2i\xi\alpha_2\sin\alpha_2 b_2 & -2i\xi\alpha_2\cos\alpha_2 b_2 & 0 & (\xi^2-\beta_2^2)\cos\beta_2 b_2 & (\xi^2-\beta_2^2)\sin\beta_2 b_2 \end{bmatrix} \quad (16)$$

where T represents the transpose of a matrix. To solve (17) for $a_1 \cdots a_6$, first solve the dispersion relation

$$\det V = 0. \tag{18}$$

Then, for any $\omega - \xi$ pair that satisfies (18) a nontrivial solution is obtained from (17). Since $[a_1 \cdots a_6]^T$ nontrivial but $a_7 = \cdots = a_{18} = 0$, the displacement field from (5) and (8) represents the symmetric, plane strain solution of extensional motion, i.e.

$$u_1(x_1, -x_2, t) = u_1(x_1, x_2, t),\ u_2(x_1, -x_2, t) = -u_2(x_1, x_2, t) \quad \text{and} \quad u_3 = 0.$$

2. *Antisymmetric plane strain solution of flexural motion*

Set $a_1 = \cdots a_6 = 0$, $a_{13} = \cdots = a_{18} = 0$, and let

$$W[a_7 \cdots a_{12}]^T = \mathbf{0}. \tag{19}$$

similarly, the corresponding dispersion relation is

$$\det W = 0, \tag{20}$$

and the displacement field from (5) and (8) represents the antisymmetric plane strain solution of flexural motion, i.e.,

$$u_1(x_1, -x_2, t) = -u_1(x_1, x_2, t),\ u_2(x_1, -x_2, t) = u_2(x_1, x_2, t), \quad \text{and} \quad u_3 = 0.$$

3. *Symmetric* SH-*wave solution*

Set $a_1 = \cdots = a_{12} = 0$, $a_{16} = a_{17} = a_{18} = 0$ and

$$X[a_{13} \quad a_{14} \quad a_{15}]^T = \mathbf{0}. \tag{21}$$

The associated dispersion relation is

$$\det X = 0. \tag{22}$$

The symmetric SH-wave solution from (5) and (8) has $u_1 = u_2 = 0$, and $u_3(x_1, -x_2, t) = u_3(x_1, x_2, t)$.

4. *Antisymmetric* SH-*wave solution*

Set $a_1 = \cdots = a_{15} = 0$ and

$$Y[a_{16} \quad a_{17} \quad a_{18}]^T = \mathbf{0}. \tag{23}$$

Dispersion relation becomes

$$\det Y = 0 \tag{24}$$

and the antisymmetric SH-wave solution from (5) and (8) has $u_1 = u_2 = 0$, and $u_3(x_1, -x_2, t) = -u_3(x_1, x_2, t)$.

The dispersion relations are now computed for two sandwich places, labeled SP4 and SP7, for which geometric and the material properties are:

$$\begin{aligned} &\text{SP4; } E_2/E_1 = 0.001, \quad \nu_1 = 0.3, \quad \nu_2 = 0.1, \quad \rho_2/\rho_1 = 0.3, \quad b_2/b_1 = 1.0, \\ &\text{SP7; } E_2/E_1 = 20.0, \quad \nu_1 = 0.3, \quad \nu_2 = 0.3, \quad \rho_2/\rho_1 = 2.0, \quad b_2/b_1 = 0.1. \end{aligned} \tag{25}$$

The elastic moduli and the density of typical dust layers are not well known at the moment. If the mechanical properties of dust layers are assumed to be similar to those of the fine and silty sands of inorganic soils, then $E_2 \approx 10^9$ dyne/cm^2, $\rho_2 \approx$ 2.65 gram/cm^3 [10]. For precipitator plates made of steel, $E_1 = 2.0 \times 10^{12}$ dyne/cm^2, $\nu_1 = 0.3$ and $\rho_1 = 7.7$ gram/cm^3. Thus, SP4 above is chosen to represent a reasonable model of a dust-covered precipitator plate. SP7 is a sandwich plate whose facings are heavier, thinner, and much stiffer than the core. It is chosen to model a light weight composite used typically in aerospace structures.

The dispersion curves are computed for SP4 and SP7 and are shown in Figures 2 and 3 for extensional waves, in Figures 4 and 5 for flexural waves, in Figures 6 and 7 for the symmetric SH waves, and in Figures 8 and 9 for the antisymmetric SH waves. In these figures, the nondimensional quantities $2b_1\omega/\pi v_{21}$, and $2b_1\xi/\pi$ are chosen as the ordinate and abscissa respectively. The nondimensionalizing factor $\bar{\omega}_1 = \pi v_{21}/2b_1$ is the first antisymmetric thickness-shear cut-off frequency for a single-layered plate of thickness $2b_1$ or a sandwich plate without facings ($b_2 = 0$). The corresponding dispersion curves for a single-layered plate, which are solutions of the Rayleigh–Lamb frequency equation, were studied in great detail by Mindlin and his co-workers [9], [11].

It can be seen from Figures 3, 5, 7, and 9 that the dispersion curves for SP7, a plate with thin, stiff, and heavy facings, are very similar to those of a single-layered plate. The first antisymmetric thickness-shear cut-off frequency (in Figure 5) is reduced to 0.836 from the unity for a single-layered plate. All the frequency branches have a similar downward shift. This frequency reduction phenomenon is due mostly to the mass and the stiffnesses of the facings and is known in the vibrations of crystal plates with electrodes [15].

From Figures 2, 4, 6, and 8 for SP4, which is a plate with thick, soft, and light facings, we see that the real frequency branches still resemble those of a single-layered plate, but the behavior of the imaginary branches has marked differences. The value of the first antisymmetric thickness-shear cut-off frequency is reduced to 0.0627 (in Figure 4) from the value of unity ($= 1$) for a single-layered plate. In this case, the frequency reduction is strongly affected by the strains developed in the facings, and, therefore, depends on the thickness and stiffnesses of the facings.

Of special interest and usefulness are the explicit expressions of asymptotic behavior of dispersion curves at low frequency, long wave length. When $\omega \to 0$ and $\xi \to 0$, (9) shows that $\alpha_l \to 0$, $\beta_l \to 0$, $l = 1, 2$. Then by a straightforward but quite laborious expansion of (18), (20), (22), and (24) after using the Taylor series expansion of trigonometric functions in (13)–(16), the asymptotic expressions of the

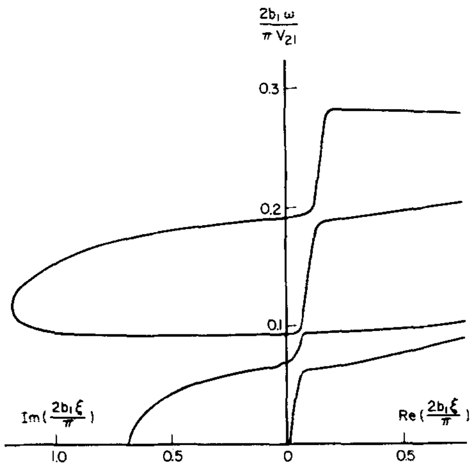

Figure 2. Dispersion curves of extensional waves in the sandwich plate SP4.

dispersion relations are obtained. For the symmetric plane strain solution of extensional motion,

$$[(1-\nu_1^2)^{-1}E_1b_1+(1-\nu_2^2)^{-1}E_2b_2]\xi^2=(\rho_1b_1+\rho_2b_2)\omega^2. \tag{26}$$

Letting $b_2=0$, we find that (26) reduces to the frequency equation in the classical extensional theory of vibration of single-layered plates [13]. For the antisymmetric

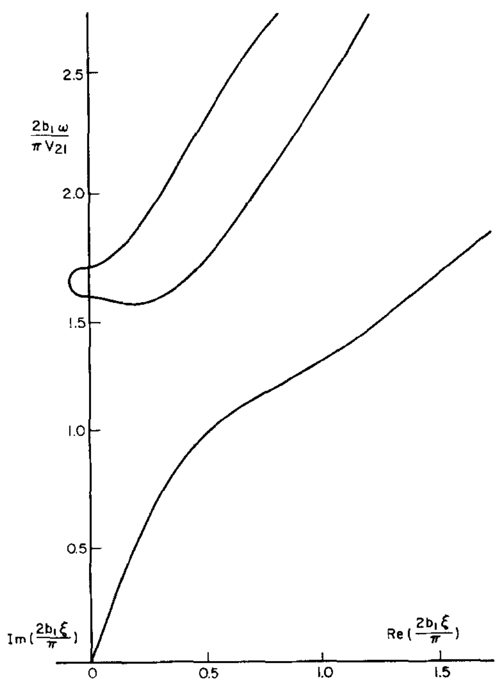

Figure 3. Dispersion curves of extensional waves in the sandwich plate SP7.

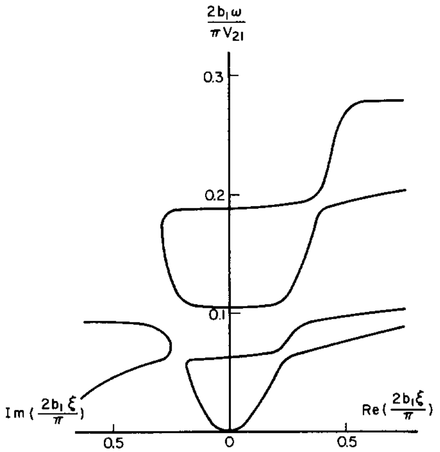

Figure 4. Dispersion curves of flexural waves in the sandwich plate SP4.

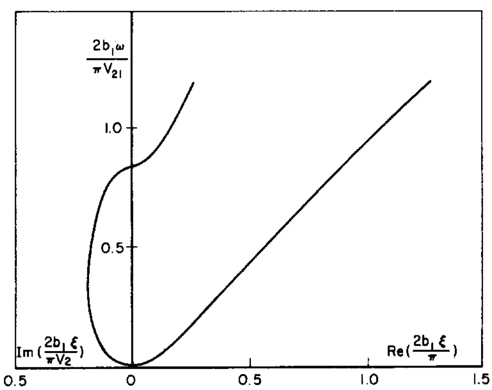

Figure 5. Dispersion curves of flexural waves in the sandwich plate SP7

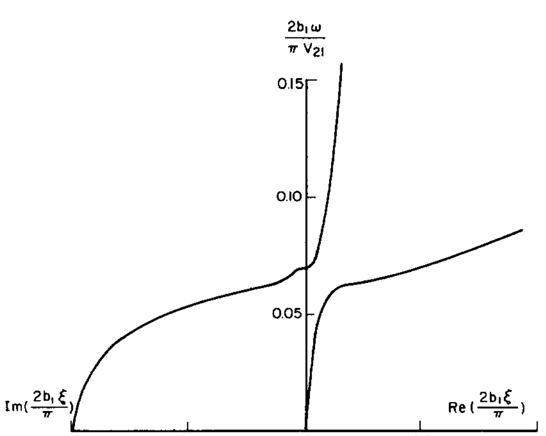

Figure 6. Dispersion curves of symmetric SH-waves in the sandwich plate SP4.

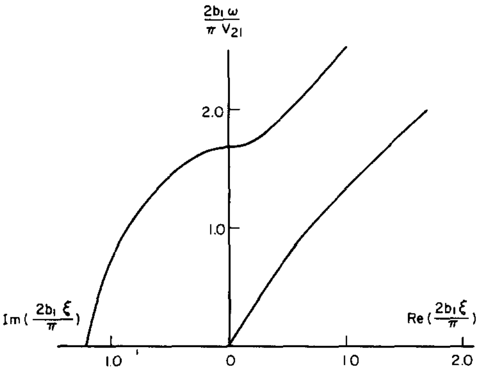

Figure 7. Dispersion curves of symmetric SH-waves in the sandwich plate SP7.

plane strain solution of flexural motion,

$$[(1-\nu_1^2)^{-1}E_1b_1^3+(1-\nu_2^2)^{-1}E_2\{(b_1+b_2)^3-b_1^3\}]\xi^4=3(\rho_1b_1+\rho_2b_2)\omega^2. \tag{27}$$

Setting $b_2=0$, we find that (27) reduced to the dispersion relation for the classical flexural theory of vibration of single-layered plates [13]. For the symmetric SH-wave solution,

$$(\mu_1b_1+\mu_2b_2)\xi^2=(\rho_1b_1+\rho_2b_2)\omega^2. \tag{28}$$

The dispersion curve for the antisymmetric SH-wave solution does not pass through the origin of the dispersion diagram, and so no asymptotic expression exists from (24) as ω and ξ approaching zero. It is interesting to note that in the asymptotic dispersion relations (26)–(28) the terms $(1-\nu_2^2)^{-1}E_2b_2$, $(1-\nu_2^2)^{-1}E_2\{(b_1+b_2)^3-b_1^3\}$,

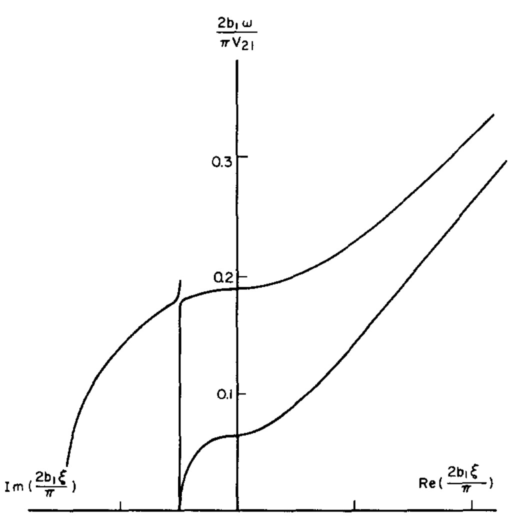

Figure 8. Dispersion curves of anti-symmetric SH-waves in the sandwich plate SP4.

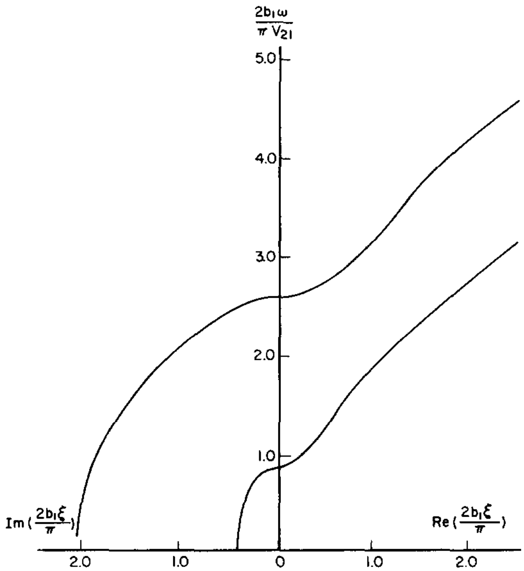

Figure 9. Dispersion curves of anti-symmetric SH-waves in the sandwich plate SP7.

$\mu_2 b_2$, and $\rho_2 b_2$ represent, respectively, the contribution of the cover layers to the extensional stiffness, flexural stiffness, shear stiffness and the mass of the sandwich plate. The asymptotic expressions for the slopes and curvatures of these dispersion curves can be obtained readily from (26)–(28) by differentiation.

IV. Thickness vibrations

In Figures 2–9 the points where the dispersion curves intersect the vertical, frequency axis are the cut-off frequencies at zero wave number of the propagating waves in the sandwich plate. When ξ approaches zero, the wave solutions of (8) become independent of the x_1 and x_3 coordinates and reduce to standing wave solutions in the x_2 direction. Since the motions depend only on the thickness coordinate, x_2, they are called simple thickness vibrations [9].

The solutions for the simple thickness vibrations are among the simplest, exact and closed-form solutions of the three-dimensional theory of elasticity. They often reveal clear understanding and offer simple interpretation of the dynamic behavior of the sandwich plates.

When the motion is independent of x_1 and x_3, the equations of motion (1) and the constitutive relations (2) reduce to

$$c_j u_{j,22} = \rho \ddot{u}_j \qquad j = 1, 2 \text{ (no sum)} \tag{29}$$

$$\tau_{2j} = c_j u_{j,2} \tag{30}$$

where

$$
\begin{aligned}
c_1 &= \begin{cases} \mu_1 & |x_2| < b_1 \\ \mu_2 & b_1 < |x_2| < b_1 + b_2 \end{cases} \\
c_2 &= \begin{cases} \lambda_1 + 2\mu_1 & |x_2| < b_1 \\ \lambda_2 + 2\mu_2 & b_1 < |x_2| < b_1 + b_2 \end{cases} \\
\rho &= \begin{cases} \rho_1 & |x_2| < b_1 \\ \rho_2 & b_1 < |x_2| < b_1 + b_2. \end{cases}
\end{aligned}
\tag{31}
$$

For isotropic plates, u_3 is suppressed here without loss of generality.

The differential equations (29) and the boundary conditions (4) form an eigenvalue problem for which the solutions may take the following form

$$u_j = a_j \phi_{jn}(x_2/b_1) \exp(i\omega_{jn} t), \quad j = 1, 2 \text{ (no sum)} \tag{32}$$

where ω_{jn} are the natural frequencies of the thickness vibrations, $\phi_{jn}(x_2/b_1)$ are the corresponding modal functions, and $n = 0, 1, 2, \ldots$.

Substitution of (32) into (29) gives the governing equations on ϕ_{jn}

$$c_j \phi''_{jn}(\eta) = -\rho b_1^2 \omega_{jn}^2 \phi_{jn}(\eta), \qquad 0 < |\eta| < 1 + b_0 \tag{33}$$

where the prime denotes differentiation with respect to η and

$$\eta = x_2/b_1, \qquad b_0 = b_2/b_1. \tag{34}$$

The boundary conditions (4) reduce to

$$
\begin{aligned}
&\phi_{jn}[\eta = 1^+] = \phi_{jn}[\eta = 1^-], \\
&c_j \phi'_{jn}[\pm\eta = 1^+] = c_j \phi'_{jn}[\pm\eta = 1^-], \\
&\phi'_{jn}[\eta = 1 + b_0] = 0.
\end{aligned}
\tag{35}
$$

We note that u_1 (or ϕ_{1n}) and u_2 (or ϕ_{2n}) may satisfy (33) and (35) independently. The solutions $\phi_{1n} \neq 0$ and $\phi_{2n} = 0$ correspond to displacements parallel to the faces of the plate and are called thickness-shear modes, while $\phi_{1n} = 0$ and $\phi_{2n} \neq 0$ correspond to displacements normal to the faces of the plate and are called thickness-stretch modes.

The solutions for thickness-shear and thickness-stretch vibrations, satisfying (33) and (35), are expressed in the following general, nondimensional form

$$
\phi_{jn}(\eta) = \begin{cases} \cos\frac{\pi}{2}(\Omega_{jn} - n) & |\eta| < 1 \\ \cos\frac{\pi}{2}(\Omega_{jn} - n)\cos\frac{\pi}{2}[v_{(3-j)0}^{-1}\Omega_{jn}(\eta - 1)] & \\ -[\rho_0 v_{(3-j)0}]^{-1}\sin\frac{\pi}{2}(\Omega_{jn} - n)\sin\frac{\pi}{2}[v_{(3-j)0}^{-1}\Omega_{jn}(\eta - 1)] & 1 < |\eta| < 1 + b_0 \end{cases}
\tag{36}
$$

and

$$\phi_{jn}(\eta) = (-1)^n \phi_{jn}(-\eta) \quad \text{for} \quad -(1+b_0) < \eta < 0,$$

where

$$\Omega_{jn} = \omega_{jn} \Big/ \left(\frac{2v_{(3-j)1}}{\pi b_1}\right), \qquad v_{j0} = v_{j2}/v_{j1}, \qquad \rho_0 = \rho_2/\rho_1.$$

For $j=1$, ϕ_{1n} and Ω_{1n} (or ω_{1n}) are the thickness-shear modal functions and frequencies, respectively, and $n = 0, 2, 4, \ldots$ are associated with symmetric thickness-shear vibrations while $n = 1, 3, 5, \ldots$ are associated with antisymmetric thickness-shear vibrations. For $j=2$, ϕ_{2n} and Ω_{2n} (or ω_{2n}) are the thickness-stretch modal functions and frequencies, respectively, and $n = 0, 2, 4, \ldots$ are associated with antisymmetric thickness-stretch and $n = 1, 3, 5, \ldots$ are associated with symmetric thickness-stretch.

The values of Ω_{jn} in (36) must satisfy the frequency equation

$$\rho_0 v_{(3-j)0} \tan \frac{\pi}{2} [b_0 v_{(3-j)0}^{-1} \Omega_{jn}] \pm \left[\tan \frac{\pi}{2} \Omega_{jn}\right]^{\pm 1} = 0. \tag{37}$$

Equation (37) is obtained from the boundary conditions (35) and is a condensed form for the following four cases. Set $j=1$ for thickness-shear vibrations; the upper and lower signs apply to the symmetric and antisymmetric deformations, respectively. Similarly, set $j=2$ for thickness-stretch vibrations; the upper and lower signs apply to the antisymmetric and symmetric deformations, respectively. Note that (36) and (37) may also be obtained directly from (8), (17), and (18) by reduction. We note that by setting $j=1$, (36) and (37) reduce to the Yu's results for simple thickness-shear vibrations [14].

The roots of (37), the resonance frequencies of the thickness vibrations, and their corresponding modes (36) are computed for the sandwich plates SP4 and SP7 which are defined in (25). The thickness-shear vibrational modes ϕ_{1n} and frequencies $\omega_{1n}/\tilde{\omega}_1$ are shown in Figure 10 for the first five modes ($n = 0, 1, 2, 3, 4$). Similarly, ϕ_{2n} and $\omega_{2n}/\tilde{\omega}_1$ for $n = 0, 1, 2, 3, 4$ for the thickness-stretch vibrations are shown in Figure 11. The nondimensional factor for the frequencies $\tilde{\omega}_1 = \pi v_{21}/2b_1$ is the lowest

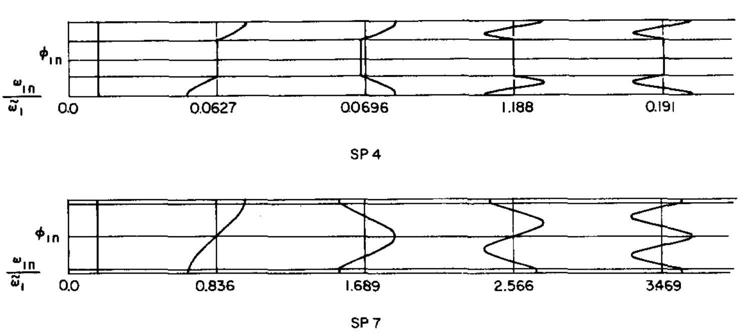

Figure 10. The thickness-shear modes and frequencies for the sandwich plates SP4 and SP7.

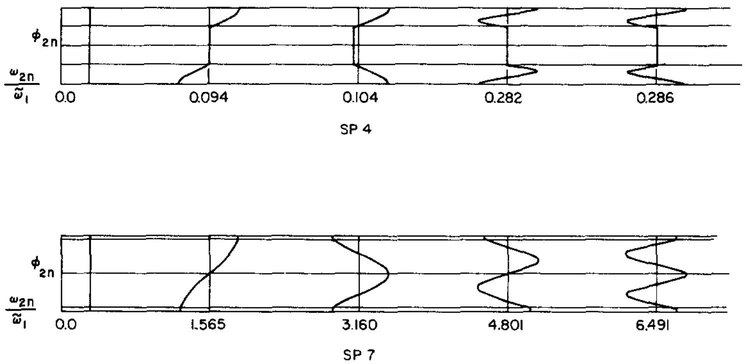

Figure 11. The thickness-stretch modes and frequencies for the sandwich plates SP4 and SP7.

antisymmetric thickness-shear frequency of the single layered plate which is reduced from a sandwich plate by letting $b_2 = 0$ [12].

From Figures 10 and 11, it can be seen that the modes ϕ_{10} and ϕ_{20} for $n = 0$ correspond to rigid body translations in the x_1 and x_2 directions, respectively. The integer n which is associated with the nth mode identifies the number of nodal planes parallel to the faces of the plate. Unlike the case of the single-layered plate, the higher resonance frequencies are not equal to integral multiples of the fundamental frequencies; therefore the higher-order thickness modes are anharmonic overtones of the fundamental modes in sandwich plates.

For SP4, which is a representative of the sandwich plates with "soft, light, and thick" facings, we see in Figures 10 and 11 that strains are developed mostly in the outer layers and the middle layer acts essentially like a rigid body. The values of the frequencies are given below each mode and distributed as follows

$$\omega_{11} \lesssim \omega_{12}, \quad \omega_{13} \lesssim \omega_{14}, \quad \omega_{21} \lesssim \omega_{22}, \quad \text{and} \quad \omega_{23} \lesssim \omega_{24}$$

where the symbol $\lesssim$ should read as "is less than but approximately equal to". The strains, in the cover layer, associated with these closely spaced frequencies are very similar.

The value of the first antisymmetric thickness-shear frequency for SP4 has been reduced from the unity $(\omega_{11}/\tilde{\omega}_1 = 1)$ obtained for a single layered plate to $\omega_{11}/\tilde{\omega}_1 = 0.063$ due to the effect of facings. Values of $\omega_{11}/\tilde{\omega}_1$ and $\omega_{21}/\tilde{\omega}_1$ are computed from (37) for $\rho_2/\rho_1 = 0.3$, $\nu_2/\nu_1 = 1/3$ and a range of values of E_2/E_1 and b_2/b_1 are are listed in Table 1.

TABLE 1
Values of the first thickness-shear and thickness-stretch frequencies in sandwich plates ($\rho_2/\rho_1 = 0.3$, $\nu_2/\nu_1 = \frac{1}{3}$)

E_2/E_1		$\frac{1}{500}$			$\frac{1}{1000}$			$\frac{1}{2000}$	
b_2/b_1	$\frac{1}{2}$	1	3	$\frac{1}{2}$	1	3	$\frac{1}{2}$	1	3
$\omega_{11}/\tilde{\omega}_1$	0.177	0.089	0.030	0.125	0.063	0.021	0.089	0.044	0.015
$\omega_{21}/\tilde{\omega}_1$	0.265	0.133	0.044	0.188	0.094	0.031	0.133	0.066	0.022

A useful simple formula for predicting the value of ω_{11} is obtained as follows. By realizing that for $E_2/E_1<1$, the mode shape ϕ_{11} in the cover layer has a shear wave length equal approximately to four times the thickness of the cover layer, and by the use of (9), we have

$$\omega_{11}\lesssim\frac{\pi v_{22}}{2b_2} \quad \text{or} \quad \frac{\omega_{11}}{\tilde{\omega}_1}\lesssim\frac{v_{22}/v_{21}}{b_2/b_1}. \tag{38}$$

The values computed from (38) differ by less than about 1% from those listed in Table 1.

In a similar manner, a formula for predicting the first thickness-stretch frequencies is obtained:

$$\omega_{21}\lesssim\frac{\pi v_{12}}{2b_2} \quad \text{or} \quad \frac{\omega_{21}}{\tilde{\omega}_1}\lesssim k_2\frac{v_{22}/v_{21}}{b_2/b_1} \tag{39}$$

where $k_2=v_{12}/v_{22}=[(2-2\nu_2)/(1-2\nu_2)]^{1/2}$, and $E_2/E_1\ll 1$.

For sandwich plate SP7, which is chosen as a representative of plates with "stiff, heavy, and thin" facings, we see in Figure 10 and 11 that the strains in the core are quite similar to those developed in a single layered plate ($b_2=0$) and the facing acts more like a rigid body, and the first thickness-shear frequency $\omega_{11}/\tilde{\omega}_1$ has been reduced from unity ($=1$) to 0.836, due to the effect of the mass and stiffnesses of the facings. The values of $\omega_{11}/\tilde{\omega}_1$ and $\omega_{21}/\tilde{\omega}_1$ for $b_2/b_1=0.1$, $\nu_2/\nu_1=1$ and a range of values of E_2/E_1 and ρ_2/ρ_1 are calculated from (37) and are listed in Table 2. From the values in the columns 1 and 2 of Table 2, we note that the frequency reductions remain constant for $E_2/E_1\geq 20$.

TABLE 2
Values of the first thickness-shear and thickness-stretch frequencies in sandwich plates ($b_2/b_1=0.1$, $\nu_2/\nu_1=1$)

E_2/E_1 ρ_2/ρ_1	200 2	20 2	10 1	3 0.3
$\omega_{11}/\tilde{\omega}_1$	0.836	0.836	0.909	0.971
$\omega_{21}/\tilde{\omega}_1$	1.564	1.564	1.702	1.816

A method employed by Mindlin and Lee [15] for calculating the mass effect of the electrode platings on the first anti-symmetric thickness-shear cut-off frequencies of crystal plates is adopted here to yield

$$R\beta_1 b_1 \tan(\beta_1 b_1)=1, \tag{40}$$

where $R=\rho_2 b_2/\rho_1 b_1$ is the ratio of the mass of the facings to the mass of the core per unit area of the plate, and $\beta_1=\omega/v_{21}$ is the shear wave number obtained from (9) by setting $\xi=0$.

For $R<1$, the smallest root of (40) is given, approximately, by

$$\frac{\omega_{11}}{\tilde{\omega}_1}=\frac{1}{1+R}\left[1+\frac{\pi^2}{12}\frac{R^3}{(1+R)^3}\right]. \tag{41}$$

Comparing (41) with equation (15) of Ref. [15], we see that the second term in (41) is an additional term to improve the accuracy of the formula when $R \leq 1$.

The value of $\omega_{11}/\tilde{\omega}_1$ predicted by (41) for SP7 is within 0.02% of the exact value listed in the column (2) of Table 2. A further check of the accuracy of (41) is made by considering a sandwich plate with $E_2/E_1 = 3.0$, $\rho_2/\rho_1 = 1.0$, $\rho_2/b_1 = 1.0$, and $\nu_2/\nu_1 = 1.0$ such that $R = 1.0$. The value computed from (41) has an error of 3.8% as compared to the exact value from (37).

A formula for calculating the lowest thickness-stretch frequency for $E_2/E_1 > 1$, $R < 1$ is similarly obtained as

$$\frac{\omega_{21}}{\tilde{\omega}_1} = \frac{k_1}{1+R}\left[1 + \frac{\pi^2}{12}\frac{R^3}{(1+R)^3}\right] \tag{42}$$

where $k_1 = v_{11}/v_{21} = [(2-2\nu_1)/(1-2\nu_1)]^{1/2}$.

V. Orthogonality conditions of thickness modes

The governing equation (33) of ϕ_{jn} and the boundary conditions (35) form a Sturm–Liouville problem with discontinuous material properties through the thickness of the sandwich plate as defined in (31). The orthogonality conditions for the modal functions ϕ_{jn} $(j = 1, 2)$ will be established as follows.

Let ω_{jm} and ω_{jn} be two distinct frequencies, and ϕ_{jm} and ϕ_{jn} be the corresponding solutions of (33), therefore

$$c_j\phi''_{jm} = -\rho b_1^2\omega_{jm}^2\phi_{jm} \tag{43}$$

$$c_j\phi''_{jn} = -\rho b_1^2\omega_{jn}^2\phi_{jn}. \tag{44}$$

By multiplying (43) by ϕ_{jn} and (44) by $-\phi_{jm}$, adding the result, and integrating both sides of the final equation over the thickness of the plate, we obtain

$$\int_{-1-b_0}^{1+b_0} (c_j\phi_{jn}\phi''_{jm} - c_j\phi_{jm}\phi''_{jn})\, d\eta = -b_1^2\int_{-1-b_0}^{1+b_0} \rho(\omega_{jm}^2 - \omega_{jn}^2)\phi_{jm}\phi_{jn}\, d\eta \tag{45}$$

By means of integration by parts, the left hand side of (45) becomes

$$\text{LHS} = [c_j\phi_{jn}\phi'_{jm} - c_j\phi_{jm}\phi'_{jn}]_{-1-b_0}^{1+b_0} + [c_j\phi_{jn}\phi'_{jm} - c_j\phi_{jm}\phi'_{jn}]_{-1}^{1}$$
$$- \int_{-1-b_0}^{1+b_0} (c_j\phi'_{jm}\phi'_{jn} - c_j\phi'_{jm}\phi'_{jn})\, d\eta.$$

The first two terms of the above equation vanish independently owing to the boundary conditions (35), and the third term is identically zero. Therefore

$$\int_{-1-b_0}^{1+b_0} (c_j\phi_{jn}\phi''_{jm} - c_j\phi_{jm}\phi''_{jn})\, d\eta = 0, \qquad \omega_{jm} \neq \omega_{jn}$$

which establishes that the system is self-adjoint [16]. From the right hand side of (45) we have

$$\int_{-1-b_0}^{1+b_0} \rho\phi_{jm}\phi_{jn}\, d\eta = 0, \qquad j=1,2, \qquad \omega_{jm} \neq \omega_{jn} \tag{46}$$

which is the orthogonality conditions for ϕ_{jn} with $\rho(\eta)$ defined in (31), as the weighting function.

Multiply (43) again by ϕ_{jn}, integrate both sides of the equation over the thickness, and make the use of (46); then

$$\int_{-1-b_0}^{1+b_0} c_j\phi_{jn}\phi''_{jm}\, d\eta = 0, \qquad \omega_{jm} \neq \omega_{jn}. \tag{47}$$

By integration by parts and boundary conditions (35), (47) is transformed to

$$\int_{-1-b_0}^{1+b_0} c_j\phi'_{jm}\phi'_{jn}\, d\eta = 0, \qquad j=1,2 \qquad \omega_{jm} \neq \omega_{jn} \tag{48}$$

which may be regarded as the orthogonality conditions of ϕ'_{jn} with respect to the weighting functions $c_j(\eta)$ defined in (31).

For the case of single-layered plates, $b_0 = 0$, $\rho(\eta)$, $c_j(\eta)$ become constant throughout the thickness, and the modal functions of (36) reduce to [9]

$$\phi_{jn}(\eta) = \cos\left[\frac{n\pi}{2}(\eta-1)\right] \qquad j=1,2, \qquad n=0,1,2,\ldots.$$

It is evident that the orthogonality conditions (46) and (48) are satisfied.

For a sandwich plate in general, the thickness-shear modes ϕ_{1n} $(j=1)$ and the thickness-stretch modes ϕ_{2n} $(j=2)$ are different and their orthogonality conditions, established in (48), contain different weighting functions $c_1(\eta)$ and $c_2(\eta)$. It should be noted that (46) and (48) represent four conditions and in which ϕ_{1n}, ϕ_{2n} are piecewise smooth and ρ, c_1, c_2 are piecewise continuous.

In addition to orthogonality, the completeness of the system of ϕ_{jn} can also be assured [17]. Hence the thickness modes are piecewise smooth and form an orthogonal and complete set, and may, therefore, be used as a natural basis for the series expansion of the displacements in sandwich plates.

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